Bonn
December 17, 1970

Dear Lang,

It occurred to me after I sent you the last letter that it would be worthwhile to see if the considerations in it agreed with Serre's suggestions about the form of the Γ-factor. I just want to explain now that as far as the matter can be checked formally everything is alright.

I had better explain first in more detail, at least for a group over $\mathbf{R}$, how one goes about defining the dual group and, in the cases of interest to us, local zeta-functions corresponding to representations of this group.

$\widetilde{G}$ will be a semi-simple Lie group over $\mathbf{C}$ with Lie algebra $\widetilde{\mathfrak{g}}$ and fixed Cartan subalgebra $\widetilde{\mathfrak{h}}$. Fix also an order on the roots and hence a positive Weyl chamber. Choosing a Chevalley basis with root vectors e_α we obtain a real (even rational) structure on $\widetilde{\mathfrak{g}}$ and hence on $\widetilde{G}$. Let C_0 be the complex anti-linear involution on $\widetilde{\mathfrak{g}}$ with respect to this structure. C_0 determines the split real structure on $\widetilde{\mathfrak{g}}$ of $\widetilde{G}$. Any other real structure can be obtained by an involution C which maps $\widetilde{\mathfrak{h}}$ into itself. $C = IC_0$ where I is an automorphism of $\widetilde{\mathfrak{g}}$. The real structure is inner or outer according as I is inner or outer. Let L_+ be the lattice of weights of $\widetilde{\mathfrak{h}}$—(or $\widetilde{\mathfrak{g}}$), L_- the sublattice generated by the roots and L, where $L_- \subseteq L \subseteq L_+$, the weights of $\widetilde{\mathfrak{g}}$ which actually define weights of $\widetilde{G}$. α denotes a root. We have the following scheme:

$$
\begin{array}{ccc}
\alpha & \widehat{\alpha} & \\
\Cap & \Cap & \\
L_- & \widehat{L}_- = \mathrm{Hom}(L_+, \mathbf{Z}) & \\
\text{\rotatebox{90}{$\subseteq$}} & \text{\rotatebox{90}{$\subseteq$}} & \\
L & L = \mathrm{Hom}(L, \mathbf{Z}) & \\
\text{\rotatebox{90}{$\subseteq$}} & \text{\rotatebox{90}{$\subseteq$}} & \\
L_+ & L_+ = \mathrm{Hom}(L_-, \mathbf{Z}) &
\end{array}
$$

$\widehat{\alpha}(\lambda)$ (or $\langle \lambda, \widehat{\alpha} \rangle$) is $2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}$ if $\lambda \in L_+$. $(\cdot,\cdot)$ is the Killing form on $\widetilde{\mathfrak{h}}$.

We may suppose that $\widehat{\alpha}$ is simple or positive only when α is. There is a semi-simple Lie algebra $\widehat{\mathfrak{g}}$ and a corresponding group $\widehat{G}_0$ and Cartan subalgebra $\widehat{\mathfrak{h}}$ for which the roofed objects have the same significance as the unroofed for $\widetilde{\mathfrak{g}}$, $\widetilde{G}$, and $\widetilde{\mathfrak{h}}$. The Weyl groups of $\widehat{\mathfrak{h}}$ in $\widehat{\mathfrak{g}}$ and $\widetilde{\mathfrak{h}}$ in $\widetilde{\mathfrak{g}}$ are isomorphic ($\sigma \leftrightarrow \widehat{\sigma}$) in such a way that $\langle \sigma\lambda, \widehat{\sigma}\widehat{\lambda} \rangle = \langle \lambda, \widehat{\lambda} \rangle$. $\widehat{G}_0$ is the connected component of the dual group for any real form G of $\widetilde{G}$. Let G be determined by the involution $C = IC_0$. I is the product of an inner automorphism normalizing $\widetilde{\mathfrak{h}}$ and an outer automorphism permuting the positive simple roots of $\widetilde{\mathfrak{h}}$. Every such permutation naturally determines a permutation of the positive simple $\widehat{\alpha}$ and hence an automorphism of $\widehat{G}_0$ (called the straight extension in Freudenthal-de Vries). We take $\widehat{G}$ to be the corresponding split extension of $\widehat{G}_0$ so that $[\widehat{G} : \widehat{G}_0] = 2$ and $\widehat{G} = \widehat{G}_0 \cup \omega \widehat{G}_0$ where $\omega^2 = 1$ (I note that two forms which differ by an inner twisting have the same dual group. I also observe that, according to the definitions of my Washington lecture, $\widehat{G}$ is not uniquely determined—if G is inner one is free to take $\widehat{G}_0 = \widehat{G}$—however the present choice is here convenient.) ω corresponds to an automorphism of the Dynkin diagram and therefore also to a permutation of the highest weights $\widehat{\lambda} \to \omega\widehat{\lambda}$. Let $\widehat{\sigma}_0$ be an irreducible representation of $\widehat{G}_0$ with highest weight $\widehat{\lambda}$. Then $\widehat{\sigma}_0(\omega g \omega^{-1})$ has highest weight $\omega\widehat{\lambda}$. If $\widehat{\lambda} = \omega\widehat{\lambda}$ there is an A such that

$$A\widehat{\sigma}_0(g)A^{-1} = \widehat{\sigma}_0(\omega g \omega^{-1}).$$

A takes the vector corresponding to λ to a scalar multiple of itself. We may suppose that this scalar is ± 1 so that $A^2 = 1$. Setting $\widehat{\sigma}(g) = \widehat{\sigma}_0(g)$ and $\widehat{\sigma}(\omega) = A$ we obtain a representation $\widehat{\sigma}$ of $\widehat{G}$ (that with highest weight $\widehat{\lambda}$.) If $\widehat{\lambda} \neq \omega\widehat{\lambda}$ set

$$\widehat{\sigma}(g) = \begin{pmatrix} \widehat{\sigma}_0(g) & 0 \\ 0 & \widehat{\sigma}_0(\omega g \omega^{-1}) \end{pmatrix} \quad g \in \widehat{G}_0,$$

$$\widehat{\sigma}(\omega) = \pm \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

$\widehat{\sigma}$ is irreducible and is said to have the highest weight $\widehat{\lambda}$ (or $\omega\widehat{\lambda}$). These are the only representations of $\widehat{G}$ we need to consider.

We are interested at first in the case that $G = G_{\text{un}}$ is the compact form. Then the effect of I is to take every root into its negative. G_{un} is inner if and only if this can be effected by an element of the Weyl group. When $G = G_{\text{un}}$ then $\widehat{\mu}$ is a root of $\widehat{\sigma}$ only when $-\widehat{\mu}$ is also. Moreover $\omega\widehat{G}_0$ contains an element ϵ which normalizes $\widehat{\mathfrak{h}}$ and sends roots to their negatives. ϵ is determined up to conjugacy by an element of $\exp\widehat{\mathfrak{h}} = \widehat{T}_0$.

I want to describe how in accordance with the suggestions of my Washington lecture the irreducible representations of G_{un} correspond to certain homomorphisms of the Weil group $W_{\mathbf{C}/\mathbf{R}}$ into $\widehat{G}$. Let $W_{\mathbf{C}/\mathbf{R}} = \mathbf{C}^\times \cup \delta\mathbf{C}^\times$ where $\delta^2 = -1$. I consider only homomorphisms τ which map $\mathbf{C}^\times$ into $\widehat{T}_0$ and δ into ϵ. If $\widehat{\mu}$ belongs to $\widehat{L}$ and $\xi_{\widehat{\mu}}$ is the corresponding rational character of $\widehat{T}_0$ there is an $m(\widehat{\mu}) \in \mathbf{Z}$ such that

$$\xi_{\widehat{\mu}}\big(\tau(z)\big) = \left(\frac{z}{|z|}\right)^{m(\widehat{\mu})}.$$

Define $2\Lambda \in L$ by $m(\widehat{\mu}) = \langle 2\Lambda, \widehat{\mu}\rangle$ for every $\widehat{\mu}$. Then

$$\xi_{\widehat{\mu}}\big(\tau(-1)\big) = (-1)^{\langle 2\Lambda, \widehat{\mu}\rangle} = \xi_{\widehat{\mu}(\epsilon^2)}.$$

ϵ^2 does not depend on the choice of ϵ. By some messy calculations involving classification one can show that

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^{\langle 2\rho, \widehat{\mu}\rangle}$$

if 2ρ is the sum of the positive roots of $\widetilde{\mathfrak{h}}$. Thus $2\Lambda - 2\rho$ is even for all $\widehat{\mu}$ in $\widehat{L}$ and $\Lambda - \rho$ is in L. Of course any Λ satisfying this condition arises from a suitable τ. The irreducible representation $\pi(\tau)$ of G_{un} corresponding to τ is (this is a definition) the representation with natural parameter Λ. To obtain this representation one chooses w in the Weyl group so that $w\Lambda$ lies in the positive Weyl chamber. The representation with "natural" parameter Λ is that with highest weight $w\Lambda - \rho = w(\Lambda - \rho) + w\rho - \rho$ ($w\rho - \rho \in L_- \subseteq L$). Thus $\pi(\tau)$

is only defined if Λ is non-singular. (This fits perfectly well with my Washington lecture since G_{un} is not quasi-split.)

If G is any form with compact Cartan subgroup then G is an inner form of G_{un}; so $\widehat{G}$ is defined as above. In this case the representations of G corresponding to τ are those members of the discrete series for G with a parameter Λ' which equals $w\Lambda$ for some w in the (complex) Weyl group of G. We set (a definition)

$$L\big(s, \pi(\tau), \widehat{\sigma}\big) = L(s, \widehat{\sigma} \circ \tau)$$

if $\widehat{\sigma}$ is a complex-analytic representation of $\widehat{G}$. $L(s, \widehat{\sigma} \circ \tau)$ is the Γ-factor (defined as in my notes on Artin L-functions) for the representation $\widehat{\sigma} \circ \tau$ of $W_{\mathbf{C}/\mathbf{R}}$. We consider those $\widehat{\sigma}$ introduced above. Every time that a non-zero weight $\widehat{\mu}$ occurs in $\widehat{\sigma}$ so does $-\widehat{\mu}$. Thus the representation

$$\operatorname{Ind}(W_{\mathbf{C}/\mathbf{R}}, W_{\mathbf{C}/\mathbf{C}}, \delta_{\langle\widehat{\mu}, 2\Lambda\rangle})$$

with

$$\delta_m(z) = \left(\frac{z}{|z|}\right)^m : \mathbf{C}^\times \to \mathbf{C}^\times$$

is included in $\widehat{\sigma} \circ \tau$ as often as $\widehat{\mu}$ is a weight of $\widehat{\sigma}$. Therefore each time that $\widehat{\mu}$ occurs one gets a factor

$$2(2\pi)^{-\left(s+|\langle\widehat{\mu},\Lambda\rangle|\right)}\Gamma\Big(s + \big|\langle\widehat{\mu}, \Lambda\rangle\big|\Big)$$

in $L(s, \widehat{\sigma} \circ \tau)$. On the spaces corresponding to the weight 0, $\widehat{\sigma} \circ \tau$ kills $\mathbf{C}^\times$ and is a representation of $\mathfrak{G}(\mathbf{C}/\mathbf{R}) = W_{\mathbf{C}/\mathbf{R}}/\mathbf{C}^\times$. Every time that this representation contains the trivial representation we get a factor

$$\pi^{-\frac{1}{2}s}\Gamma\left(\frac{s}{2}\right)$$

and every time that it contains the non-trivial representation we get a factor

$$\pi^{-\frac{1}{2}(s+1)}\Gamma\left(\frac{s+1}{2}\right)$$

Notice that if the weight 0 does not occur in $\widehat{\sigma}_0$ the function $L\big(s, \pi(\tau), \widehat{\sigma}\big)$ depends only on $\widehat{\sigma}_0$.

We have to apply this to the representation $\widehat{\sigma}_0 = \sigma_0$ introduced in my last letter. According to the vague suggestions of that letter a certain part of the cohomology group in the middle dimension is broken up into d-dimensional subspaces. To compare with Serre's suggestion we have to see how much of this d-dimensional part corresponds to a given $H^{p,q}$. We arrange things so that $\widetilde{\mathfrak{h}}$ corresponds to a compact Cartan subgroup of G and so that, with respect to the order chosen, every non-compact positive root is totally positive. (To fix ideas I have in mind the case that G is simple and defined over $\mathbf{Q}$. Presumably $\dim\widehat{\sigma}_0 = \dim\widehat{\sigma}$ (with the given $\widehat{\sigma}_0 = \sigma_0$) corresponds to the case that we should think of the Shimura variety $\widetilde{V}$ as defined over $\mathbf{Q}$ and $\dim\widehat{\sigma}_0 = \frac{1}{2}\dim\widehat{\sigma}$ to the case that we should think of $\widetilde{V}$ as defined over a certain imaginary quadratic extension of $\mathbf{Q}$—see remark in previous letter.)

The tangent space T at $K \in K\backslash G = X$ breaks up into the direct sum of the holomorphic tangent vectors T^+ and the anti-holomorphic T^-. If $t = \frac{1}{2}\,\text{Real-dim}\,X$ then

$$\bigwedge^t T = \bigoplus_{p+q=t} \bigwedge^p (T^+) \otimes \bigwedge^q (T^-).$$

Let ρ_+ be one-half the sum of the positive non-compact roots (with respect to the given order). This order determines a positive (or for us standard) Weyl chamber W. Take another Weyl chamber W' and let ρ' be one-half the sum of the corresponding positive roots and ρ'_+ one-half the sum of the corresponding non-compact positive roots. Choose w in W_G, the Weyl group of $G_{\mathbf{C}}$, so that $wW = W'$ and let $T_{\rho'}$ be the element of the discrete series corresponding to ρ'. $T_{\rho'}$ depends only on the coset $W_K w$, if $W_K \subseteq W_G$ is the Weyl group of $K_{\mathbf{C}}$. Let p positive non-compact roots with respect to W' be positive with respect to W and let q be negative. Then according to Matsushima's discussion (at least formally) and the remarks at the end of the last letter the representation of K with extreme weight $2\rho'_+$ occurs with multiplicity one in $\bigwedge^p T^+ \otimes \bigwedge^q T^-$ so that the one-dimensional part of the d-dimensional space contributed by $T_{\rho'}$ lies in $H^{p,q}$.

However $\dim \widehat{\sigma}_0 = [W_G : W_K] = d$ so that the weights of $\widehat{\sigma}_0 (= \sigma_0)$ are $\widehat{w}^{-1}\widehat{\mu}_0$, w varying over representations of $W_K\backslash W_G$. In particular 0 is not a weight. Finally

$$\langle \widehat{w}^{-1}\widehat{\mu}_0, 2\rho\rangle = \big\langle \widehat{\mu}_0, w(2\rho)\big\rangle = \langle \widehat{\mu}_0, 2\rho'\rangle = \langle \widehat{\mu}_0, 2\rho'_+\rangle$$

because $\widehat{\mu}_0$ annihilates compact roots. Because of the definition of $\widehat{\mu}_0$

$$\langle \widehat{\mu}_0, 2\rho'_+\rangle = p - q.$$

Moreover the relevant part of the Hasse-Weil(?) zeta-function should be

$$L\left(s - \frac{t}{2}, T_\rho, \widehat{\sigma}\right) \qquad \text{with } t = \frac{1}{2}\dim X = p + q$$

if σ is the representation constructed from $\widehat{\sigma}_0$ as above. Since

$$-\frac{p+q}{2} + \frac{|p-q|}{2} = -\inf(p,q),$$

we are in perfect agreement with Serre (Sem. Delange-Pisot-Poitou—May 1970). Of course this supposes that the previous parenthetical remarks are O.K. When $\dim \widehat{\sigma}_0 = \dim \widehat{\sigma}$ so that, under the circumstances mentioned, $\widetilde{V}$ is to be thought of as defined over $\mathbf{Q}$ one has also to check the effect of Serre's σ on $H^{\frac{t}{2},\frac{t}{2}}$ (when t is even). This one can of course do only in those cases where Shimura has actually defined $\widetilde{V}$. I have not yet checked anything.

Yours truly,

Bob Langlands

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