APPENDIX TO LETTER TO SERGE LANG—DECEMBER 17, 1970

It is enough to prove the relation

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^{\langle 2\rho, \widehat{\mu}\rangle}$$

when $\widetilde{G}$, and therefore also $\widehat{G}_0$ is simple. ϵ^2 certainly lies in the Cartan subgroup of $\widehat{G}_0$ corresponding to $\widehat{\mathfrak{h}}$. $\widehat{G}$ acts as a group of automorphisms of $\widehat{g}$ (of course not faithfully). Let $(\cdot,\cdot)$ be the Killing form on $\widehat{\mathfrak{g}}$. $\widehat{G}$ preserves the Killing form. Let $e_{\widehat{\alpha}}$ be a root vector. Then

$$(e_{\widehat{\alpha}}, \epsilon e_{\widehat{\alpha}}) \neq 0$$

and

$$(e_{\widehat{\alpha}}, \epsilon e_{\widehat{\alpha}}) = (\epsilon e_{\widehat{\alpha}}, \epsilon^2 e_{\widehat{\alpha}}) = \xi_{\widehat{\alpha}}(\epsilon^2)(\epsilon e_{\widehat{\alpha}}, e_{\widehat{\alpha}}) = \xi_{\widehat{\alpha}}(\epsilon^2)(e_{\widehat{\alpha}}, \epsilon e_{\widehat{\alpha}})$$

because the Killing form is symmetric. Thus $\xi_{\widehat{\alpha}} = 1$. Since $\langle \rho, \widehat{\alpha}\rangle$ is integral and in fact equal to 1 for simple $\widehat{\alpha}$,

$$\xi_{\widehat{\alpha}} = (-1)^{\langle 2\rho, \widehat{\alpha}\rangle}.$$

In particular the assertion is true in those cases that $\widehat{L}_- = \widehat{L}_+$. Thus when $\widehat{\mathfrak{g}}$ is of type E_8, F_4, or G_2.

In the other cases let $\widehat{\beta}$ be the top root (as defined in Freudenthal-de Vries). Then $\omega e_{\widehat{\beta}} = \eta e_{\widehat{\beta}}$, with $\eta = \pm 1$, and $\omega e_{-\widehat{\beta}} = \eta e_{-\widehat{\beta}}$. Thus, taking the standard isomorphism of the group corresponding to

$$\mathrm{Span}\Big\{e_{\widehat{\beta}}, e_{-\widehat{\beta}}, [e_{\widehat{\beta}}, e_{-\widehat{\beta}}]\Big\}$$

with $\mathrm{SL}(2, \mathbf{C})$, ω corresponds to the trivial automorphism if $\eta = 1$ and to the inner automorphism determined by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

if $\eta = -1$. The reflection corresponding to $\widehat{\beta}$ is determined by

$$\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus $\omega\delta\omega^{-1} = \delta$ if $\eta = +1$ and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

if $\eta = -1$. We regard δ as an element of $\widehat{G}_0$. It takes every root which is not orthogonal to $\widehat{\beta}$ to a negative root. The roots orthogonal to $\widehat{\beta}$ are linear combinations of the simple roots orthogonal to $\widehat{\beta}$.

We consider the various types separately. $\widehat{H}_0$ will be the group corresponding to the algebra generated by root vectors belonging to roots orthogonal to $\widehat{\beta}$. $\widehat{H}_0$ is invariant under ω. $\widehat{H}$ will be $\widehat{H}_0 \cup \omega\widehat{H}_0$. If $\widehat{H}_0$ is simple we may suppose our assertion is proved for $\widehat{H}$. We use the tables in Fr.-De Vries.

(i) A_ℓ: We identify L_- and $\widehat{L}_-$ and L_+ and $\widehat{L}_+$ as in the previous letter.

$$2\rho = \big(n-1, n-3, \dots, -(n-1)\big), \qquad n = \ell + 1.$$

Here ω, acting on $\widehat{G}_0$ which we may take as simply connected, is

$$A \to \begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ \pm 1 & & & & \end{pmatrix} {}^tA^{-1} \begin{pmatrix} & & & \pm 1 \\ & & \cdot^{\cdot^{\cdot}} & \\ & 1 & & \\ -1 & & & \\ 1 & & & \end{pmatrix} = \gamma\, {}^tA^{-1}\gamma^{-1}$$

and ϵ is $\gamma\omega$. It is clear that $\omega(\gamma) = \gamma$ so that $\epsilon^2 = \gamma\omega\gamma\omega = \gamma^2$ which is $(-1)^{n-1}I$. It is enough to check the relation

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^{\langle\widehat{\mu},2\rho\rangle}$$

for $\widehat{\mu}$ fundamental. Let

$$\widehat{\mu} = \Bigl(\overbrace{\frac{n-k}{n}, \dots, \frac{n-k}{n}}^{k \text{ times}}, \overbrace{-\frac{k}{n}, \dots, -\frac{k}{n}}^{n-k \text{ times}}\Bigr)$$

Then

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^{k(n-1)}$$

and

$$\langle 2\rho, \widehat{\mu}\rangle \equiv k(n-1) \pmod 2.$$

The result follows.

(ii) $B_\ell(=\widehat{\mathfrak{g}})$. Again the notation is that of my previous letter.

$$2\rho = \bigl(2\ell, 2(\ell-1), \dots, 2\bigr),$$

$\widehat{\beta}$ is $\omega_1 + \omega_2$. The automorphism ω is trivial; so $\eta = 1$. $\widehat{H}_0$ corresponds to the diagram of B_ℓ with ρ_3 omitted. Thus if δ_1 is the reflection corresponding to ρ_2 and δ_2 is an automorphism of the group obtained by removing ρ_1 and ρ_3 from the diagram of B_ℓ we may take $\epsilon = \omega\delta\delta_1\delta_2$ and, since δ, δ_1, and δ_2 commute,

$$\epsilon^2 = \delta^2\delta_1^2\delta_2^2.$$

By induction

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^x$$

with

$$\begin{aligned} x &= \langle\widehat{\mu}, \omega_1 + \omega_2\rangle + \langle\widehat{\mu}, \omega_1 - \omega_2\rangle + \Bigl\langle\widehat{\mu}, \bigl(0, 0, 2(\ell-2), \dots, 2\bigr)\Bigr\rangle \\ &= \Bigl\langle\widehat{\mu}, \bigl(2, 0, 2(\ell-2), \dots, 2\bigr)\Bigr\rangle. \end{aligned}$$

The difference between 2ρ and the element of $\widehat{L}_+$ appearing here is

$$\bigl(2(\ell-1), 2(\ell-1), 0, \dots, 0\bigr).$$

The value of this element at any $\widehat{\mu}$ is even.

(iii) $C_\ell(=\widehat{\mathfrak{g}})$

$$2\rho = (2\ell-1, 2\ell-3, \dots, 1)$$

$\widehat{\beta}$ is $2\omega_1$. ω is again the trivial automorphism. $\widehat{H}_0$ is now the group obtained by removing ω_1 from the Dynkin diagram. If δ_1 is the element of $\widehat{H}_0$ taking roots to their negatives we may take $\epsilon = \omega\delta\delta_1$ so that $\epsilon^2 = \delta^2\delta_1^2$. By induction

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^x$$

with

$$\begin{aligned} x &= \langle\widehat{\mu}, \omega_1\rangle + \big\langle\widehat{\mu}, (0, 2\ell-3, \ldots, 1)\big\rangle \\ &= \big\langle\widehat{\mu}, (1, 2\ell-3, \ldots, 1)\big\rangle \end{aligned}$$

The difference between 2ρ and $(1, 2\ell-3, \ldots, 1)$ is $\big(2(\ell-1), 0, \ldots, 0\big)$. The value of this element of L_+ at any $\widehat{\mu}$ in $\widehat{L}_+$ is even.

(iv) $D_\ell(=\widehat{\mathfrak{g}})$. Here

$$\rho = (2\ell-2, 2\ell-4, \ldots, 0).$$

$\widehat{\beta}$ is $\omega_1 + \omega_2$. We take $\ell > 3$. We take $\widehat{H}_0$ by removing ρ_4 from the Dynkin diagram. We may take

$$e_{\widehat{\beta}} = [e_{\rho_4}, [e_{\rho_5}, [\ldots, [e_{\rho_\ell}, [e_{\rho_3}, [\ldots, [e_{\rho_{\ell-1}}, [e_{\rho_2}, [e_{\rho_\ell}, e_{\rho_1}] \ldots]$$

Now

$$\big[e_{\rho_2}, [e_{\rho_\ell}, e_{\rho_1}]\big] + \big[e_{\rho_1}, [e_{\rho_2}, e_{\rho_\ell}]\big] + \big[e_{\rho_\ell}, [e_{\rho_1}, e_{\rho_2}]\big] = 0.$$

the last term is 0; so

$$\big[e_{\rho_2}, [e_{\rho_\ell}, e_{\rho_1}]\big] = \big[e_{\rho_1}, [e_{\rho_\ell}, e_{\rho_2}]\big].$$

If ℓ is even, ω is trivial and η is 1. If ℓ is odd it follows from these calculations that η is also 1, because $\omega e_{\rho_2} = e_{\rho_1}$ and $\omega e_{\rho_1} = e_{\rho_2}$. Let δ_1 be the reflection corresponding to ρ_3 and let δ_2 be that element of the Weyl group of $\widehat{H}_0$ taking positive roots to negative roots. Take $\epsilon = \omega\delta\delta_1\delta_2$. Then by induction

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^x$$

with

$$x = \langle\widehat{\mu}, \omega_1 + \omega_2\rangle + \langle\widehat{\mu}, \omega_1 - \omega_2\rangle + \big\langle\widehat{\mu}, (0, 0, 2\ell-6, \ldots, 0)\big\rangle = \big\langle\widehat{\mu}, (2, 0, 2\ell-6, \ldots, 0)\big\rangle.$$

The difference between ρ and $(2, 0, 2\ell-6, \ldots, 0)$ is $(2\ell-4, 2\ell-4, 0, \ldots, 0)$ which takes an even value at every $\widehat{\mu}$. If $\ell = 4$ the induction argument cannot be used. In this case δ_2 is the product of the reflection corresponding to ρ_1 and that corresponding to ρ_2. Thus

$$\xi_{\widehat{\mu}}(\delta_2^2) = (-1)^y$$

with

$$y = \langle\widehat{\mu}, \omega_3 - \omega_4\rangle + \langle\widehat{\mu}, \omega_3 + \omega_4\rangle.$$

This justifies the use of the induction.

E_6: The toproot is

$$\begin{matrix} & & 2 & & \\ 1 & 2 & 3 & 2 & 1 \end{matrix}$$

The scheme $\langle \widehat{\beta}, \widehat{\alpha} \rangle$ is given by

$$\begin{array}{ccccc} & & 4-3=1 & & \\ & & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 2-2=0 & 4-3-1=0 & 6-2-2-2=0 & 4-3-1=0 & 2-2=0 \end{array}$$

Thus $\widehat{H}_0$ corresponds to A_5. Using the table of roots in Freudenthal-de Vries, we see that

$$e_{\widehat{\beta}} = [e_{\rho_2}, [e_{\rho_6}, [e_{\rho_4}, [e_{\rho_5}, [e_{\rho_6}, [e_{\rho_2}, [e_{\rho_3}, [e_{\rho_1}, [e_{\rho_5}, [e_{\rho_4}, [e_{\rho_6}], \dots]$$

Applying ω we get

$$[e_{\rho_2}, [e_{\rho_6}, [e_{\rho_5}, [e_{\rho_4}, [e_{\rho_6}, [e_{\rho_2}, [e_{\rho_1}, [e_{\rho_3}, [e_{\rho_4}, [e_{\rho_5}, [e_{\rho_6}], \dots].$$

Now

$$\big[e_{\rho_4}, [e_{\rho_5}, e_{\rho_6}]\big] + \big[e_{\rho_6}, [e_{\rho_4}, e_{\rho_5}]\big] + \big[e_{\rho_5}, [e_{\rho_6}, e_{\rho_4}]\big] = 0,$$

so that

$$\big[e_{\rho_4}, [e_{\rho_5}, e_{\rho_6}]\big] = \big[e_{\rho_5}, [e_{\rho_4}, e_{\rho_6}]\big].$$

For the same reason the interchange of ρ_1 and ρ_3 has no effect. Neither has the other interchange of ρ_4 and ρ_5. Thus $\eta = 1$. By induction

$$\xi_{\widehat{\mu}(\epsilon^2)} = (-1)^{\langle \widehat{\mu}, \beta + 2\rho_0 \rangle},$$

where β is the root dual to $\widehat{\beta}$. ρ_0 is one-half the sum of the positive roots for the A_5 obtained by throwing out the root ρ_2. From the table of fundamental wts. (ρ is the sum of these) we find that

$$\begin{array}{ccccccc} & & & & \cdot\, 22 & & \\ 2\rho & \longleftrightarrow & \cdot\, 16 & \cdot\, 30 & \cdot\, 42 & \cdot\, 30 & \cdot\, 16 \end{array}$$

$$\begin{array}{ccccccc} & & & & \cdot\, 0 & & \\ 2\rho_0 & \longleftrightarrow & \cdot\, 5 & \cdot\, 8 & \cdot\, 9 & \cdot\, 8 & \cdot\, 5 \end{array}$$

Thus $2\rho - \beta - 2\rho_0$ corresponds to

$$\begin{array}{ccccc} & & \cdot\, 20 & & \\ \cdot\, 10 & \cdot\, 20 & \cdot\, 30 & \cdot\, 20 & \cdot\, 10 \end{array}$$

and takes even values on every $\widehat{\mu}$.

E_7 : In this case ω is trivial. The toproot is

$$\begin{array}{cccccc} & & & \cdot\, 2 & & \\ \cdot\, 1 & \cdot\, 2 & \cdot\, 3 & \cdot\, 4 & \cdot\, 3 & \cdot\, 2 \end{array}$$

We give $(\widehat{\beta}, \widehat{\alpha})$, α simple, schematically. Note that in this case $\mathfrak{g}$ is also of type E_7. The normalization is such that $(\widehat{\alpha}, \widehat{\alpha}) = 1$.

$$\begin{array}{cccccc} & & & \cdot & & \\ & & & 4-4=0 & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2-2=0 & 4-4=0 & 6-6=0 & 8-8=0 & 6-6=0 & 4-3=1 \end{array}$$

Thus $\widehat{H}_0$ is of type D_6. Let $2\rho_0$ be the sum of the positive roots for this D_6. We have to show that $\rho - \beta - 2\rho_0$ is a sum of simple roots in which each root enters an even number of times. Using the tables as before, we obtain

$$\begin{array}{cccccccc} & & & & & \cdot 49 & & \\ 2\rho & \longleftrightarrow & \cdot 27 & \cdot 52 & \cdot 75 & \cdot 96 & \cdot 66 & \cdot 34 \end{array}$$

$$\begin{array}{cccccccc} & & & & & \cdot 15 & & \\ 2\rho_0 & \longleftrightarrow & \cdot 10 & \cdot 18 & \cdot 24 & \cdot 28 & \cdot 15 & \cdot 0 \end{array}$$

Everything checks.

As an extra check let's work out G_2 and F_4 along these lines. Again the Lie algebras and their duals are the same.

F_4: Toproot is

$$2 \;\text{——}\; 3 \Rrightarrow 4 \;\text{——}\; 2$$

ω is of course trivial. The diagram is

$$\begin{array}{ccccccc} 2 & & 2 & & 1 & & 1 \\ \bullet & \text{———} & \bullet & \Rrightarrow & \bullet & \text{———} & \bullet \\ & -1 & & -1 & & -1/2 & \end{array}$$

Thus $(\widehat{\beta}, \widehat{\alpha})$ is given by

$$\begin{array}{ccccccc} 2 & & 2 & & 1 & & 1 \\ \bullet & \text{———} & \bullet & \Rrightarrow & \bullet & \text{———} & \bullet \\ 4-3=1 & & 6-2-4=0 & & 4-1-3=0 & & 2-4/2=0 \end{array}$$

$\widehat{H}_0$ is in this case C_3 and its dual is B_3.

$$\begin{array}{ccccccccc} 2\rho_0 & \longleftrightarrow & \bullet & \text{———} & \bullet & \Lleftarrow & \bullet & \text{———} & \bullet \\ & & 0 & & 9 & & 8 & & 5 \end{array}$$

Note that in passing to the dual the direction of the arrow is reversed.

$$\begin{array}{ccccccccc} & & 22 & & 42 & & 30 & & 16 \\ 2\rho & \longleftrightarrow & \bullet & \text{———} & \bullet & \Lleftarrow & \bullet & \text{———} & \bullet \\ & & 2 & & 3 & & 2 & & 1 \\ \beta & \longleftrightarrow & \bullet & \text{———} & \bullet & \Lleftarrow & \bullet & \text{———} & \bullet \end{array}$$

because $(\beta, \rho_2) = (\beta, \rho_4) = (\beta, \rho_3) = 0$. Note again that arrows are reversed. Note also that there is only one positive root orthogonal to ρ_2, ρ_3 and ρ_4. In any case it checks.

G_2: Toproot is

$$\begin{array}{ccc} 2 & & 3 \\ \bullet & \Rrightarrow & \bullet \end{array}$$

The diagram is

$$\overset{3}{\bullet} \quad \underset{-3/2}{\Lleftarrow} \quad \overset{1}{\bullet}$$

$(\widehat{\beta}, \widehat{\alpha})$ is given by

$$\underset{6-\frac{9}{2}=\frac{3}{2}}{\bullet} \quad \Rrightarrow \quad \underset{3-\frac{3}{2}\cdot 2=0}{\bullet}$$

Thus, in this case,

$$2\rho_0 \quad \longleftrightarrow \quad \underset{0}{\bullet} \Lleftarrow \underset{1}{\bullet}$$

$$2\rho \quad \longleftrightarrow \quad \underset{10}{\bullet} \Lleftarrow \underset{6}{\bullet}$$

and, since β annihilates the root on the right

$$\beta \quad \longleftrightarrow \quad 2 \Lleftarrow 1.$$

Everything checks in this case also.

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