

APPENDIX TO LETTER TO SERGE LANG—DECEMBER 17, 1970

It is enough to prove the relation

$$\xi_{\hat{\mu}}(\epsilon^2) = (-1)^{\langle 2\rho, \hat{\mu} \rangle}$$

when $\tilde{G}$, and therefore also $\hat{G}_0$ is simple. ϵ^2 certainly lies in the Cartan subgroup of $\hat{G}_0$ corresponding to $\hat{\mathfrak{h}}$. $\hat{G}$ acts as a group of automorphisms of $\hat{\mathfrak{g}}$ (of course not faithfully). Let $(\cdot, \cdot)$ be the Killing form on $\hat{\mathfrak{g}}$. $\hat{G}$ preserves the Killing form. Let $e_{\hat{\alpha}}$ be a root vector. Then

$$(e_{\hat{\alpha}}, \epsilon e_{\hat{\alpha}}) \neq 0$$

and

$$(e_{\hat{\alpha}}, \epsilon e_{\hat{\alpha}}) = (\epsilon e_{\hat{\alpha}}, \epsilon^2 e_{\hat{\alpha}}) = \xi_{\hat{\alpha}}(\epsilon^2)(\epsilon e_{\hat{\alpha}}, e_{\hat{\alpha}}) = \xi_{\hat{\alpha}}(\epsilon^2)(e_{\hat{\alpha}}, \epsilon e_{\hat{\alpha}})$$

because the Killing form is symmetric. Thus $\xi_{\hat{\alpha}} = 1$. Since $\langle \rho, \hat{\alpha} \rangle$ is integral and in fact equal to 1 for simple $\hat{\alpha}$,

$$\xi_{\hat{\alpha}} = (-1)^{\langle 2\rho, \hat{\alpha} \rangle}.$$

In particular the assertion is true in those cases that $\hat{L}_- = \hat{L}_+$. Thus when $\hat{\mathfrak{g}}$ is of type E_8 , F_4 , or G_2 .

In the other cases let $\hat{\beta}$ be the top root (as defined in Freudenthal-de Vries). Then $\omega e_{\hat{\beta}} = \eta e_{\hat{\beta}}$, with $\eta = \pm 1$, and $\omega e_{-\hat{\beta}} = \eta e_{-\hat{\beta}}$. Thus, taking the standard isomorphism of the group corresponding to

$$\text{Span}\{e_{\hat{\beta}}, e_{-\hat{\beta}}, [e_{\hat{\beta}}, e_{-\hat{\beta}}]\}$$

with $\text{SL}(2, \mathbf{C})$, ω corresponds to the trivial automorphism if $\eta = 1$ and to the inner automorphism determined by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

if $\eta = -1$. The reflection corresponding to $\hat{\beta}$ is determined by

$$\delta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus $\omega \delta \omega^{-1} = \delta$ if $\eta = +1$ and

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

if $\eta = -1$. We regard δ as an element of $\hat{G}_0$. It takes every root which is not orthogonal to $\hat{\beta}$ to a negative root. The roots orthogonal to $\hat{\beta}$ are linear combinations of the simple roots orthogonal to $\hat{\beta}$.

We consider the various types separately. $\hat{H}_0$ will be the group corresponding to the algebra generated by root vectors belonging to roots orthogonal to $\hat{\beta}$. $\hat{H}_0$ is invariant under ω . $\hat{H}$ will be $\hat{H}_0 \cup \omega \hat{H}_0$. If $\hat{H}_0$ is simple we may suppose our assertion is proved for $\hat{H}$. We use the tables in Fr.-De Vries.

$\widehat{\beta}$ is $2\omega_1$. ω is again the trivial automorphism. $\widehat{H}_0$ is now the group obtained by removing ω_1 from the Dynkin diagram. If δ_1 is the element of $\widehat{H}_0$ taking roots to their negatives we may take $\epsilon = \omega\delta\delta_1$ so that $\epsilon^2 = \delta^2\delta_1^2$. By induction

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^x$$

with

$$\begin{aligned} x &= \langle \widehat{\mu}, \omega_1 \rangle + \langle \widehat{\mu}, (0, 2\ell - 3, \dots, 1) \rangle \\ &= \langle \widehat{\mu}, (1, 2\ell - 3, \dots, 1) \rangle \end{aligned}$$

The difference between 2ρ and $(1, 2\ell - 3, \dots, 1)$ is $(2(\ell - 1), 0, \dots, 0)$. The value of this element of L_+ at any $\widehat{\mu}$ in $\widehat{L}_+$ is even.

(iv) $D_\ell(=\widehat{\mathfrak{g}})$. Here

$$\rho = (2\ell - 2, 2\ell - 4, \dots, 0).$$

$\widehat{\beta}$ is $\omega_1 + \omega_2$. We take $\ell > 3$. We take $\widehat{H}_0$ by removing ρ_4 from the Dynkin diagram. We may take

$$e_{\widehat{\beta}} = [e_{\rho_4}, [e_{\rho_5}, [\dots, [e_{\rho_\ell}, [e_{\rho_3}, [\dots, [e_{\rho_{\ell-1}}, [e_{\rho_2}, [e_{\rho_\ell}, e_{\rho_1}] \dots]]]]]]]$$

Now

$$[e_{\rho_2}, [e_{\rho_\ell}, e_{\rho_1}]] + [e_{\rho_1}, [e_{\rho_2}, e_{\rho_\ell}]] + [e_{\rho_\ell}, [e_{\rho_1}, e_{\rho_2}]] = 0.$$

the last term is 0; so

$$[e_{\rho_2}, [e_{\rho_\ell}, e_{\rho_1}]] = [e_{\rho_1}, [e_{\rho_\ell}, e_{\rho_2}]].$$

If ℓ is even, ω is trivial and η is 1. If ℓ is odd it follows from these calculations that η is also 1, because $\omega e_{\rho_2} = e_{\rho_1}$ and $\omega e_{\rho_1} = e_{\rho_2}$. Let δ_1 be the reflection corresponding to ρ_3 and let δ_2 be that element of the Weyl group of $\widehat{H}_0$ taking positive roots to negative roots. Take $\epsilon = \omega\delta\delta_1\delta_2$. Then by induction

$$\xi_{\widehat{\mu}}(\epsilon^2) = (-1)^x$$

with

$$x = \langle \widehat{\mu}, \omega_1 + \omega_2 \rangle + \langle \widehat{\mu}, \omega_1 - \omega_2 \rangle + \langle \widehat{\mu}, (0, 0, 2\ell - 6, \dots, 0) \rangle = \langle \widehat{\mu}, (2, 0, 2\ell - 6, \dots, 0) \rangle.$$

The difference between ρ and $(2, 0, 2\ell - 6, \dots, 0)$ is $(2\ell - 4, 2\ell - 4, 0, \dots, 0)$ which takes an even value at every $\widehat{\mu}$. If $\ell = 4$ the induction argument cannot be used. In this case δ_2 is the product of the reflection corresponding to ρ_1 and that corresponding to ρ_2 . Thus

$$\xi_{\widehat{\mu}}(\delta_2^2) = (-1)^y$$

with

$$y = \langle \widehat{\mu}, \omega_3 - \omega_4 \rangle + \langle \widehat{\mu}, \omega_3 + \omega_4 \rangle.$$

This justifies the use of the induction.

E_6 : The toproot is

$$\begin{array}{cccccc} & & 2 & & & \\ 1 & 2 & 3 & 2 & 1 & \end{array}$$

$$4 - 3 = 1$$

$$\begin{array}{ccccccccc} & \cdot & & & \cdot & & & \cdot & \\ 2-2=0 & 4-3-1=0 & 6-2-2-2=0 & 4-3-1=0 & 2-2=0 & & & & \end{array}$$

$$e_{\hat{\beta}} = [e_{\rho_2}, [e_{\rho_6}, [e_{\rho_4}, [e_{\rho_5}, [e_{\rho_6}, [e_{\rho_2}, [e_{\rho_3}, [e_{\rho_1}, [e_{\rho_5}, [e_{\rho_4}, [e_{\rho_6}], \dots]]]]]]]]$$
$$[e_{\rho_2}, [e_{\rho_6}, [e_{\rho_5}, [e_{\rho_4}, [e_{\rho_6}, [e_{\rho_2}, [e_{\rho_1}, [e_{\rho_3}, [e_{\rho_4}, [e_{\rho_5}, [e_{\rho_6}], \dots].$$
$$[e_{\rho_4}, [e_{\rho_5}, e_{\rho_6}]] + [e_{\rho_6}, [e_{\rho_4}, e_{\rho_5}]] + [e_{\rho_5}, [e_{\rho_6}, e_{\rho_4}]] = 0,$$
$$[e_{\rho_4}, [e_{\rho_5}, e_{\rho_6}]] = [e_{\rho_5}, [e_{\rho_4}, e_{\rho_6}]].$$
$$\xi_{\hat{\mu}(\epsilon^2)} = (-1)^{\langle \hat{\mu}, \beta + 2\rho_0 \rangle},$$

				.22		
2ρ	$\longleftrightarrow$	.16	.30	.42	.30	.16
				.0		
$2\rho_0$	$\longleftrightarrow$	.5	.8	.9	.8	.5

$$\begin{array}{ccccc} & & .20 & & \\ .10 & .20 & .30 & .20 & .10 \end{array}$$

E_7 : In this case ω is trivial. The toproot is

$$\begin{array}{cccccc} & & & .2 & & \\ .1 & .2 & .3 & .4 & .3 & .2 \end{array}$$

$$\begin{array}{cccccc} & \cdot & & & \cdot & \\ 2-2=0 & & 4-4=0 & & 6-6=0 & & 8-8=0 & & 6-6=0 & & 4-3=1 \\ & \cdot & & & \cdot & & \cdot & & \cdot & & \end{array}$$

Thus $\widehat{H}_0$ is of type D_6 . Let $2\rho_0$ be the sum of the positive roots for this D_6 . We have to show that $\rho - \beta - 2\rho_0$ is a sum of simple roots in which each root enters an even number of times. Using the tables as before, we obtain

$$\begin{array}{ccccccc}
 & & & & & .49 & \\
 2\rho & \longleftrightarrow & .27 & .52 & .75 & .96 & .66 & .34 \\
 & & & & & .15 & \\
 2\rho_0 & \longleftrightarrow & .10 & .18 & .24 & .28 & .15 & .0
 \end{array}$$

Everything checks.

As an extra check let's work out G_2 and F_4 along these lines. Again the Lie algebras and their duals are the same.

F_4 : Toproot is

$$2 \text{ --- } 3 \Rightarrow 4 \text{ --- } 2$$

ω is of course trivial. The diagram is

$$\begin{array}{ccc}
 2 & & 2 \\
 \bullet & \text{---} & \bullet \\
 & -1 & \\
 & \Rightarrow & \\
 1 & & 1 \\
 \bullet & \text{---} & \bullet \\
 & -1/2 & \\
 & -1 &
 \end{array}$$

Thus $(\widehat{\beta}, \widehat{\alpha})$ is given by

$$\begin{array}{ccc}
 2 & & 2 \\
 \bullet & \text{---} & \bullet \\
 4-3=1 & & 6-2-4=0 \\
 & \Rightarrow & \\
 1 & & 1 \\
 \bullet & \text{---} & \bullet \\
 4-1-3=0 & & 2-4/2=0
 \end{array}$$

$\widehat{H}_0$ is in this case C_3 and its dual is B_3 .

$$2\rho_0 \longleftrightarrow \begin{array}{cc} \bullet & \text{---} & \bullet \\ 0 & & 9 \end{array} \Leftarrow \begin{array}{cc} \bullet & \text{---} & \bullet \\ 8 & & 5 \end{array}$$

Note that in passing to the dual the direction of the arrow is reversed.

$$\begin{array}{ccc}
 22 & & 42 \\
 2\rho & \longleftrightarrow & \begin{array}{cc} \bullet & \text{---} & \bullet \end{array} & \Leftarrow & \begin{array}{cc} 30 & & 16 \\ \bullet & \text{---} & \bullet \end{array} \\
 2 & & 3 \\
 \beta & \longleftrightarrow & \begin{array}{cc} \bullet & \text{---} & \bullet \end{array} & \Leftarrow & \begin{array}{cc} 2 & & 1 \\ \bullet & \text{---} & \bullet \end{array}
 \end{array}$$

because $(\beta, \rho_2) = (\beta, \rho_4) = (\beta, \rho_3) = 0$. Note again that arrows are reversed. Note also that there is only one positive root orthogonal to ρ_2 , ρ_3 and ρ_4 . In any case it checks.

G_2 : Toproot is

$$\begin{array}{ccc}
 2 & & 3 \\
 \bullet & \Rightarrow & \bullet
 \end{array}$$

The diagram is

$$\begin{array}{ccc} 3 & & 1 \\ \bullet & \Leftarrow & \bullet \\ & -3/2 & \end{array}$$

$(\widehat{\beta}, \widehat{\alpha})$ is given by

$$\begin{array}{ccc} \bullet & \Rightarrow & \bullet \\ 6 - \frac{9}{2} = \frac{3}{2} & & 3 - \frac{3}{2} \cdot 2 = 0 \end{array}$$

Thus, in this case,

$$\begin{array}{ccc} 2\rho_0 & \longleftrightarrow & \begin{array}{ccc} \bullet & \Leftarrow & \bullet \\ 0 & & 1 \end{array} \\ 2\rho & \longleftrightarrow & \begin{array}{ccc} \bullet & \Leftarrow & \bullet \\ 10 & & 6 \end{array} \end{array}$$

and, since β annihilates the root on the right

$$\beta \longleftrightarrow 2 \Leftarrow 1.$$

Everything checks in this case also.

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