June 2, 1965

Dear Professor Ono,

I am writing to ask if it is known that on the basis of the theory of Eisenstein series it is possible to give a formula for the volume of the fundamental domain of a discrete group associated to a Chevalley group defined over a number field. I shall give you the formula and sketch a proof of it below but first some preliminary remarks.

Suppose $\mathfrak{G}_K$ is a split semi-simple Lie algebra over the number field K; suppose $\mathfrak{h}_K$ is a splitting Cartan subalgebra; θ is an automorphism of $\mathfrak{G}_K$ with $\theta(X) = -X$ for $X \in \mathfrak{h}_K$; and suppose for each root α X_α is a root vector and $\theta(X_\alpha) = -X_{-\alpha}$ and $\alpha(H_\alpha) = 2$ if $H_\alpha = [X_\alpha, X_{-\alpha}]$. Suppose also that $\mathfrak{G}_K$ is actually a Lie algebra of linear transformations of a finite-dimensional vector space V_K over K. There is of course a connected algebraic group G of linear transformations on V associated to $\mathfrak{G}_K$.

Let $\alpha_1, \ldots, \alpha_p$ be the simple positive roots of $\mathfrak{h}_K$. Let $A_1, \ldots, A_p$ be p fractional ideals in K and if $\alpha = \sum_{i=1}^p m_i\alpha_i$ is a root set $A_\alpha = \prod_{i=1}^p A_i^{m_i}$. If M is an O-module in V_K which contains a basis of V_K then M is said to be admissible (with respect to $\mathfrak{h}_K$, $\{X_\alpha\}$, $\{A_\alpha\}$) if, for all α and all $n \geqslant 0$

$$\frac{1}{n!} A_\alpha^n (M X_\alpha^n) \subseteq M$$

and

$$M = \sum_{\lambda \in L} M \cap V_K(\lambda)$$

L is the set of integral linear functions and, for $\lambda \in L$, $V_K(\lambda)$ is the associated weight space. Then the group of all γ in G_K such that $M\gamma = M$ is independent of the choice of the admissible m adelic M. (cf. Sem. Bourbaki, #219). Let G_O be this group.

If $\mathfrak{G}_p$ is the direct product of p copies of $\mathfrak{G}$, the ideal of class group of K and $\lambda \in L$, let $\pi(\lambda)$ be the homomorphism of $\mathfrak{G}_p$ into $\mathfrak{G}$ defined by

$$\pi(\lambda)(\mathfrak{a}_1, \ldots, \mathfrak{a}_p) = \prod_{i=1}^p \mathfrak{a}_i^{\lambda(H_{\alpha_i})}$$

and let $\mathfrak{G}(V)$ be the intersection over those λ such that $V_K(\lambda) \neq \{0\}$ of the kernel of $\pi(\lambda)$. Then one can show that the number of conjugacy classes of Borel subgroups, defined over K, with respect to G_O is $\left[\mathfrak{G}_p : \mathfrak{G}(V)\right]$. Indeed there is a simple correspondence between the conjugacy classes of Borel subgroups and the cosets of $\mathfrak{G}(V)$ in $\mathfrak{G}_p$. It is probably possible to show that the number $c(G)$ introduced by Borel in his paper on Adèle groups is $\left[\mathfrak{G}(V) : 1\right]$ in the present case but I have not done this.

Let G_r be the set of real points on the connected group associated to $\mathfrak{G}_K$ considered as an algebra over $\mathbf{Q}$ acting on V_K considered as a vector space over $\mathbf{Q}$. For each α let $\left\{X_\alpha^i \;\middle|\; 1 \leqslant i \leqslant m\right\}$ be a basis for A_α; let $\{X^i\}$ be a basis for O over $\mathbf{Z}$ and let H_i, $1 \leqslant i \leqslant p$, be a basis over $\mathbf{Z}$ of $\{H \in \mathfrak{h}_K \mid \lambda(H) \in \mathbf{Z}$ if $V_K(\lambda) \neq \{0\}\}$. Then

$$\left\{X^i_\alpha X_\alpha \;\middle|\; \alpha \text{ a root, } 1 \leqslant i \leqslant n\right\} \cup \left\{X^i H_j \;\middle|\; 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant p\right\}$$

is a basis for the Lie algebra of G_r. Let dg_r be the measure on G_r obtained from the form dual to this basis, then

$$(*) \qquad \int_{G_O\backslash G_r} dg_r = \frac{\varepsilon}{[\mathfrak{G}(V):1]} \prod_{i=1}^{p} f(a_i, K)$$

with ε equal to the order of the fundamental group of G and $a_1, \dots, a_p$ as in formula (8) of your recent paper. $f(a_i, K)$ is the value of the zeta function of K at a_i.

In the proof which is sketched below the references are to some mimeographed notes of mine on Eisenstein series, a copy of which I am sending to you.

If N_r is the connected group whose Lie algebra is spanned by $\{X'_\alpha X_\alpha \mid \alpha > 0,\ 1 \leqslant i \leqslant n\}$, with a measure dn dual to this basis, if A_r is the connected group in G_r whose Lie algebra is spanned by $\{X^i H_j\}$ with a measure da dual to this basis, and if $K = \{k \in G_r \mid \theta(k) = k\}$ with a measure dk so that $\int_K dk = 1$ let $d_1 g$ be such that

$$\int_G \phi(g)\, d_1 g = \int_{N_r} \int_{A_r} \int_K dn_r\, da_r\, dk\, \omega^2(a) \phi(nak).$$

Here $d(an_r a^{-1}) = \omega^{-2}(a)\, dn_r$ is the definition of $\omega(a)$. Let

$$\Phi(s, K) = (2^{-r_2} \pi^{-n/2} \Delta^{1/2})^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \delta(s, K).$$

Using an integral formula due to Gindikin and Karpelevich one sees that $(*)$ is equivalent to

$$(**) \qquad \int_{\Gamma\backslash G} d_1 g = \frac{c}{[\mathfrak{G}(V):1]} \prod_{i=1}^{p} \Phi(a_i, K) \left(\prod_{a>0} \operatorname{Nm} A_\alpha\right)$$

Let $P_1, \dots, P_r$ be a set of representatives for the conjugacy classes of Borel subgroups and let P be the Borel subgroup with the Lie algebra spanned by $\{X_\alpha \mid \alpha > 0\} \cup \{H_j\}$. If $g_r \in G_r$ and $g = (g_1, \dots, g_n)$ where $n = [K:\mathbf{Q}]$, and $g_1 \in G_\mathbf{C}$ let $g_i = a_i n_i u_i$, $a_i \in A_\mathbf{R}$, $n_i \in N_\mathbf{C}$, $u_i \in$ unitary restriction of $G_\mathbf{C}$. If $a_i = \exp H_i$ and λ is a linear function on $\mathfrak{a}_\mathbf{C}$ set

$$F(g, H) = \exp\left(\lambda\left(\sum_{i=1}^{n} H_i\right) + \rho\left(\sum_{i=1}^{n} H_i\right)\right).$$

ρ is one-half the sum of the positive roots. For each j one can choose ideals $A^j_1, \dots, A^j_p$ in a special way and an element h_j in a special way so that $h_j^{-1} P h_j = P_j$. If $\lambda + \rho = \sum \mu_i \alpha_i$ set

$$F(g, \lambda, P_j) = \prod_{i=1}^{p} (\operatorname{Nm} A_i)^{M_i} \prod_{i=1}^{p} (\operatorname{Nm} A^j_i)^{-H_i} F(h_j g, P)$$

If $\phi_j(\cdot)$ is a measurable function on $N_j(G_O \cap P_j)\backslash G/K$, then $\phi_j(\cdot)$ can be represented as an integral

$$\phi_j(g) = \frac{1}{(2\pi)^p} \int_{\operatorname{Re}\lambda = \lambda_0} \Phi_i(\lambda) F(g, \lambda, P_j)\, (d\lambda).$$

where $\Phi_i(\lambda)$ is an entire function. It is easy to see that if $\widehat{\phi}_j(g) = \sum_{\Delta_j\backslash\Gamma} \phi_j(\gamma g)$ then

$$\left(\widehat{\phi}_j(\cdot), 1\right) = \int_{\Gamma\backslash G} \widehat{\phi}(g)\, d_1 g = R^p \Phi_j(\rho)$$

$$R = \frac{\Delta^{-1/2} 2^n \pi^{r_2} M}{w d_{r+1}}$$

(M is defined as in Landau's book on Algebraic Numbers. Let $\widehat{\phi} = \sum_{j=1}^p \widehat{\phi}_j$ and let Π be the orthogonal projection on the space of constant functions. If $\widehat{\psi} = \sum \widehat{\psi}_j$ then

$$(1,1)(\Pi\widehat{\phi}, \Pi\widehat{\psi}) = (\widehat{\phi}, 1)\overline{(\widehat{\psi}, 1)}$$

Since

$$\int_{\Gamma\backslash G} d_1 g = (1,1)$$

we can compute the volume of the fundamental domain if we can find a formula for $(\Pi\widehat{\phi}, \Pi\widehat{\psi})$. Such a formula can be obtained from the main theorem of my notes on Eisenstein series together with a little computation. The result is

$$\frac{1}{c}(hR)^p \frac{1}{\prod_{i=1}^p \Phi(a_i, K)} \frac{\left(\sum_i \Phi_i(\rho)\right)\overline{\left(\sum_j \Phi_j(\rho)\right)}}{\left[\mathfrak{G}_p : \mathfrak{G}(V)\right]} \left\{\prod_{\alpha>0} \mathrm{Nm}\, A_\alpha\right\} R^p$$

which one sees leads to the desired result.

If the formula $(*)$ is of any interest we will perhaps have an opportunity to discuss the proof in more detail at Boulder if you are attending the Institute there this summer.

Yours truly,

R. Langlands

Compiled on May 7, 2024.