June 2, 1965

Dear Professor Ono,

I am writing to ask if it is known that on the basis of the theory of Eisenstein series it is possible to give a formula for the volume of the fundamental domain of a discrete group associated to a Chevalley group defined over a number field. I shall give you the formula and sketch a proof of it below but first some preliminary remarks.

Suppose $\mathfrak{G}_K$ is a split semi-simple Lie algebra over the number field K; suppose $\mathfrak{h}_K$ is a splitting Cartan subalgebra; θ is an automorphism of $\mathfrak{G}_K$ with $\theta(X) = -X$ for $X \in \mathfrak{h}_K$; and suppose for each root α X_α is a root vector and $\theta(X_\alpha) = -X_{-\alpha}$ and $\alpha(H_\alpha) = 2$ if $H_\alpha = [X_\alpha, X_{-\alpha}]$. Suppose also that $\mathfrak{G}_K$ is actually a Lie algebra of linear transformations of a finite-dimensional vector space V_K over K. There is of course a connected algebraic group G of linear transformations on V associated to $\mathfrak{G}_K$.

Let $\alpha_1, \ldots, \alpha_p$ be the simple positive roots of $\mathfrak{h}_K$. Let $A_1, \ldots, A_p$ be p fractional ideals in K and if $\alpha = \sum_{i=1}^p m_i \alpha_i$ is a root set $A_\alpha = \prod_{i=1}^p A_i^{m_i}$. If M is an O-module in V_K which contains a basis of V_K then M is said to be admissible (with respect to $\mathfrak{h}_K$, $\{X_\alpha\}$, $\{A_\alpha\}$) if, for all α and all $n \geqslant 0$

$$\frac{1}{n!} A_\alpha^n (M X_\alpha^n) \subseteq M$$

and

$$M = \sum_{\lambda \in L} M \cap V_K(\lambda)$$

L is the set of integral linear functions and, for $\lambda \in L$, $V_K(\lambda)$ is the associated weight space. Then the group of all γ in G_K such that $M\gamma = M$ is independent of the choice of the admissible m adelic M. (cf. Sem. Bourbaki, #219). Let G_O be this group.

If $\mathfrak{G}_p$ is the direct product of p copies of $\mathfrak{G}$, the ideal of class group of K and $\lambda \in L$, let $\pi(\lambda)$ be the homomorphism of $\mathfrak{G}_p$ into $\mathfrak{G}$ defined by

$$\pi(\lambda)(\mathfrak{a}_1, \ldots, \mathfrak{a}_p) = \prod_{i=1}^p \mathfrak{a}_i^{\lambda(H_{\alpha_i})}$$

and let $\mathfrak{G}(V)$ be the intersection over those λ such that $V_K(\lambda) \neq \{0\}$ of the kernel of $\pi(\lambda)$. Then one can show that the number of conjugacy classes of Borel subgroups, defined over K, with respect to G_O is $\left[\mathfrak{G}_p : \mathfrak{G}(V)\right]$. Indeed there is a simple correspondence between the conjugacy classes of Borel subgroups and the cosets of $\mathfrak{G}(V)$ in $\mathfrak{G}_p$. It is probably possible to show that the number $c(G)$ introduced by Borel in his paper on Adèle groups is $\left[\mathfrak{G}(V) : 1\right]$ in the present case but I have not done this.

Let G_r be the set of real points on the connected group associated to $\mathfrak{G}_K$ considered as an algebra over $\mathbf{Q}$ acting on V_K considered as a vector space over $\mathbf{Q}$. For each α let $\left\{ X_\alpha^i \,\middle|\, 1 \leqslant i \leqslant m \right\}$ be a basis for A_α; let $\{X^i\}$ be a basis for O over $\mathbf{Z}$ and let H_i, $1 \leqslant i \leqslant p$, be a basis over $\mathbf{Z}$ of $\{\, H \in \mathfrak{h}_K \mid \lambda(H) \in \mathbf{Z} \text{ if } V_K(\lambda) \neq \{0\} \,\}$. Then

$$\left\{\, X^i_\alpha X_\alpha \,\middle|\, \alpha \text{ a root},\ 1 \leqslant i \leqslant n \right\} \cup \left\{\, X^i H_j \,\middle|\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant p \right\}$$

is a basis for the Lie algebra of G_r. Let dg_r be the measure on G_r obtained from the form dual to this basis, then

$$(*) \qquad \int_{G_O \backslash G_r} dg_r = \frac{\varepsilon}{[\mathfrak{G}(V):1]} \prod_{i=1}^{p} f(a_i, K)$$

with ε equal to the order of the fundamental group of G and $a_1, \ldots, a_p$ as in formula (8) of your recent paper. $f(a_i, K)$ is the value of the zeta function of K at a_i.

In the proof which is sketched below the references are to some mimeographed notes of mine on Eisenstein series, a copy of which I am sending to you.

If N_r is the connected group whose Lie algebra is spanned by $\{\, X'_\alpha X_\alpha \mid \alpha > 0,\ 1 \leqslant i \leqslant n \,\}$, with a measure dn dual to this basis, if A_r is the connected group in G_r whose Lie algebra is spanned by $\{X^i H_j\}$ with a measure da dual to this basis, and if $K = \{\, k \in G_r \mid \theta(k) = k \,\}$ with a measure dk so that $\int_K dk = 1$ let $d_1 g$ be such that

$$\int_G \phi(g)\, d_1 g = \int_{N_r} \int_{A_r} \int_K dn_r\, da_r\, dk\, \omega^2(a) \phi(nak).$$

Here $d(an_r a^{-1}) = \omega^{-2}(a)\, dn_r$ is the definition of $\omega(a)$. Let

$$\Phi(s, K) = (2^{-r_2} \pi^{-n/2} \Delta^{1/2})^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \delta(s, K).$$

Using an integral formula due to Gindikin and Karpelevich one sees that $(*)$ is equivalent to

$$(**) \qquad \int_{\Gamma \backslash G} d_1 g = \frac{c}{[\mathfrak{G}(V):1]} \prod_{i=1}^{p} \Phi(a_i, K) \left(\prod_{a>0} \operatorname{Nm} A_\alpha \right)$$

Let $P_1, \ldots, P_r$ be a set of representatives for the conjugacy classes of Borel subgroups and let P be the Borel subgroup with the Lie algebra spanned by $\{\, X_\alpha \mid \alpha > 0 \,\} \cup \{H_j\}$. If $g_r \in G_r$ and $g = (g_1, \ldots, g_n)$ where $n = [K : \mathbf{Q}]$, and $g_1 \in G_{\mathbf{C}}$ let $g_i = a_i n_i u_i$, $a_i \in A_{\mathbf{R}}$, $n_i \in N_{\mathbf{C}}$, $u_i \in$ unitary restriction of $G_{\mathbf{C}}$. If $a_i = \exp H_i$ and λ is a linear function on $\mathfrak{a}_{\mathbf{C}}$ set

$$F(g, H) = \exp\left(\lambda\left(\sum_{i=1}^{n} H_i \right) + \rho\left(\sum_{i=1}^{n} H_i \right) \right).$$

ρ is one-half the sum of the positive roots. For each j one can choose ideals $A^j_1, \ldots, A^j_p$ in a special way and an element h_j in a special way so that $h_j^{-1} P h_j = P_j$. If $\lambda + \rho = \sum \mu_i \alpha_i$ set

$$F(g, \lambda, P_j) = \prod_{i=1}^{p} (\operatorname{Nm} A_i)^{M_i} \prod_{i=1}^{p} (\operatorname{Nm} A^j_i)^{-H_i} F(h_j g, P)$$

If $\phi_j(\cdot)$ is a measurable function on $N_j(G_O \cap P_j)\backslash G/K$, then $\phi_j(\cdot)$ can be represented as an integral

$$\phi_j(g) = \frac{1}{(2\pi)^p} \int_{\operatorname{Re}\lambda = \lambda_0} \Phi_i(\lambda) F(g, \lambda, P_j)\, (d\lambda).$$

where $\Phi_i(\lambda)$ is an entire function. It is easy to see that if $\widehat{\phi}_j(g) = \sum_{\Delta_j\backslash\Gamma} \phi_j(\gamma g)$ then

$$\left(\widehat{\phi}_j(\cdot), 1\right) = \int_{\Gamma\backslash G} \widehat{\phi}(g)\, d_1 g = R^p \Phi_j(\rho)$$

$$R = \frac{\Delta^{-1/2} 2^n \pi^{r_2} M}{w d_{r+1}}$$

(M is defined as in Landau's book on Algebraic Numbers. Let $\widehat{\phi} = \sum_{j=1}^{p} \widehat{\phi}_j$ and let Π be the orthogonal projection on the space of constant functions. If $\widehat{\psi} = \sum \widehat{\psi}_j$ then

$$(1,1)(\Pi\widehat{\phi}, \Pi\widehat{\psi}) = (\widehat{\phi}, 1)\overline{(\widehat{\psi}, 1)}$$

Since

$$\int_{\Gamma\backslash G} d_1 g = (1,1)$$

we can compute the volume of the fundamental domain if we can find a formula for $(\Pi\widehat{\phi}, \Pi\widehat{\psi})$. Such a formula can be obtained from the main theorem of my notes on Eisenstein series together with a little computation. The result is

$$\frac{1}{c}(hR)^p \frac{1}{\prod_{i=1}^{p} \Phi(a_i, K)} \frac{\left(\sum_i \Phi_i(\rho)\right)\overline{\left(\sum_j \Phi_j(\rho)\right)}}{\left[\mathfrak{G}_p : \mathfrak{G}(V)\right]} \left\{\prod_{\alpha>0} \mathrm{Nm}\, A_\alpha\right\} R^p$$

which one sees leads to the desired result.

If the formula $(*)$ is of any interest we will perhaps have an opportunity to discuss the proof in more detail at Boulder if you are attending the Institute there this summer.

Yours truly,

R. Langlands

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