

June 2, 1965

Dear Professor Ono,

I am writing to ask if it is known that on the basis of the theory of Eisenstein series it is possible to give a formula for the volume of the fundamental domain of a discrete group associated to a Chevalley group defined over a number field. I shall give you the formula and sketch a proof of it below but first some preliminary remarks.

Suppose \mathfrak{G}_K is a split semi-simple Lie algebra over the number field K ; suppose \mathfrak{h}_K is a splitting Cartan subalgebra; θ is an automorphism of \mathfrak{G}_K with $\theta(X) = -X$ for $X \in \mathfrak{h}_K$; and suppose for each root α X_α is a root vector and $\theta(X_\alpha) = -X_{-\alpha}$ and $\alpha(H_\alpha) = 2$ if $H_\alpha = [X_\alpha, X_{-\alpha}]$. Suppose also that \mathfrak{G}_K is actually a Lie algebra of linear transformations of a finite-dimensional vector space V_K over K . There is of course a connected algebraic group G of linear transformations on V associated to \mathfrak{G}_K .

Let $\alpha_1, \dots, \alpha_p$ be the simple positive roots of \mathfrak{h}_K . Let A_1, \dots, A_p be p fractional ideals in K and if $\alpha = \sum_{i=1}^p m_i \alpha_i$ is a root set $A_\alpha = \prod_{i=1}^p A_i^{m_i}$. If M is an O -module in V_K which contains a basis of V_K then M is said to be admissible (with respect to $\mathfrak{h}_K, \{X_\alpha\}, \{A_\alpha\}$) if, for all α and all $n \geq 0$

$$\frac{1}{n!} A_\alpha^n (M X_\alpha^n) \subseteq M$$

and

$$M = \sum_{\lambda \in L} M \cap V_K(\lambda)$$

L is the set of integral linear functions and, for $\lambda \in L$, $V_K(\lambda)$ is the associated weight space. Then the group of all γ in G_K such that $M\gamma = M$ is independent of the choice of the admissible m adelic M . (cf. Sem. Bourbaki, #219). Let G_O be this group.

If \mathfrak{G}_p is the direct product of p copies of \mathfrak{G} , the ideal of class group of K and $\lambda \in L$, let $\pi(\lambda)$ be the homomorphism of \mathfrak{G}_p into \mathfrak{G} defined by

$$\pi(\lambda)(\mathfrak{a}_1, \dots, \mathfrak{a}_p) = \prod_{i=1}^p \mathfrak{a}_i^{\lambda(H_{\alpha_i})}$$

and let $\mathfrak{G}(V)$ be the intersection over those λ such that $V_K(\lambda) \neq \{0\}$ of the kernel of $\pi(\lambda)$. Then one can show that the number of conjugacy classes of Borel subgroups, defined over K , with respect to G_O is $[\mathfrak{G}_p : \mathfrak{G}(V)]$. Indeed there is a simple correspondence between the conjugacy classes of Borel subgroups and the cosets of $\mathfrak{G}(V)$ in \mathfrak{G}_p . It is probably possible to show that the number $c(G)$ introduced by Borel in his paper on Adèle groups is $[\mathfrak{G}(V) : 1]$ in the present case but I have not done this.

Let G_r be the set of real points on the connected group associated to \mathfrak{G}_K considered as an algebra over \mathbf{Q} acting on V_K considered as a vector space over \mathbf{Q} . For each α let $\{X_\alpha^i \mid 1 \leq i \leq m\}$ be a basis for A_α ; let $\{X^i\}$ be a basis for O over \mathbf{Z} and let $H_i, 1 \leq i \leq p$,

be a basis over \mathbf{Z} of $\{H \in \mathfrak{h}_K \mid \lambda(H) \in \mathbf{Z} \text{ if } V_K(\lambda) \neq \{0\}\}$. Then

$$\left\{ X_\alpha^i X_\alpha \mid \alpha \text{ a root, } 1 \leq i \leq n \right\} \cup \left\{ X^i H_j \mid 1 \leq i \leq n, 1 \leq j \leq p \right\}$$

is a basis for the Lie algebra of G_r . Let dg_r be the measure on G_r obtained from the form dual to this basis, then

$$(*) \quad \int_{G_O \backslash G_r} dg_r = \frac{\epsilon}{[\mathfrak{G}(V) : 1]} \prod_{i=1}^p f(a_i, K)$$

with ϵ equal to the order of the fundamental group of G and a_1, \dots, a_p as in formula (8) of your recent paper. $f(a_i, K)$ is the value of the zeta function of K at a_i .

In the proof which is sketched below the references are to some mimeographed notes of mine on Eisenstein series, a copy of which I am sending to you.

If N_r is the connected group whose Lie algebra is spanned by $\{X'_\alpha X_\alpha \mid \alpha > 0, 1 \leq i \leq n\}$, with a measure dn dual to this basis, if A_r is the connected group in G_r whose Lie algebra is spanned by $\{X^i H_j\}$ with a measure da dual to this basis, and if $K = \{k \in G_r \mid \theta(k) = k\}$ with a measure dk so that $\int_K dk = 1$ let d_1g be such that

$$\int_G \phi(g) d_1g = \int_{N_r} \int_{A_r} \int_K dn_r da_r dk \omega^2(a) \phi(nak).$$

Here $d(an_r a^{-1}) = \omega^{-2}(a) dn_r$ is the definition of $\omega(a)$. Let

$$\Phi(s, K) = (2^{-r_2} \pi^{-n/2} \Delta^{1/2})^s \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \delta(s, K).$$

Using an integral formula due to Gindikin and Karpelevich one sees that $(*)$ is equivalent to

$$(**) \quad \int_{\Gamma \backslash G} d_1g = \frac{c}{[\mathfrak{G}(V) : 1]} \prod_{i=1}^p \Phi(a_i, K) \left(\prod_{a>0} \text{Nm } A_\alpha \right)$$

Let P_1, \dots, P_r be a set of representatives for the conjugacy classes of Borel subgroups and let P be the Borel subgroup with the Lie algebra spanned by $\{X_\alpha \mid \alpha > 0\} \cup \{H_j\}$. If $g_r \in G_r$ and $g = (g_1, \dots, g_n)$ where $n = [K : \mathbf{Q}]$, and $g_1 \in G_{\mathbf{C}}$ let $g_i = a_i n_i u_i$, $a_i \in A_{\mathbf{R}}$, $n_i \in N_{\mathbf{C}}$, $u_i \in$ unitary restriction of $G_{\mathbf{C}}$. If $a_i = \exp H_i$ and λ is a linear function on $\mathfrak{a}_{\mathbf{C}}$ set

$$F(g, H) = \exp \left(\lambda \left(\sum_{i=1}^n H_i \right) + \rho \left(\sum_{i=1}^n H_i \right) \right).$$

ρ is one-half the sum of the positive roots. For each j one can choose ideals A_1^j, \dots, A_p^j in a special way and an element h_j in a special way so that $h_j^{-1} P h_j = P_j$. If $\lambda + \rho = \sum \mu_i \alpha_i$ set

$$F(g, \lambda, P_j) = \prod_{i=1}^p (\text{Nm } A_i)^{M_i} \prod_{i=1}^p (\text{Nm } A_i^j)^{-H_i} F(h_j g, P)$$

If $\phi_j(\cdot)$ is a measurable function on $N_j(G_O \cap P_j) \backslash G/K$, then $\phi_j(\cdot)$ can be represented as an integral

$$\phi_j(g) = \frac{1}{(2\pi)^p} \int_{\text{Re } \lambda = \lambda_0} \Phi_i(\lambda) F(g, \lambda, P_j) (d\lambda).$$

where $\Phi_i(\lambda)$ is an entire function. It is easy to see that if $\widehat{\phi}_j(g) = \sum_{\Delta_j \backslash \Gamma} \phi_j(\gamma g)$ then

$$\begin{aligned} \left(\widehat{\phi}_j(\cdot), 1 \right) &= \int_{\Gamma \backslash G} \widehat{\phi}_j(g) d_1 g = R^p \Phi_j(\rho) \\ R &= \frac{\Delta^{-1/2} 2^n \pi^{r_2} M}{w d_{r+1}} \end{aligned}$$

(M is defined as in Landau's book on Algebraic Numbers. Let $\widehat{\phi} = \sum_{j=1}^p \widehat{\phi}_j$ and let Π be the orthogonal projection on the space of constant functions. If $\widehat{\psi} = \sum \widehat{\psi}_j$ then

$$(1, 1)(\Pi \widehat{\phi}, \Pi \widehat{\psi}) = (\widehat{\phi}, 1) \overline{(\widehat{\psi}, 1)}$$

Since

$$\int_{\Gamma \backslash G} d_1 g = (1, 1)$$

we can compute the volume of the fundamental domain if we can find a formula for $(\Pi \widehat{\phi}, \Pi \widehat{\psi})$. Such a formula can be obtained from the main theorem of my notes on Eisenstein series together with a little computation. The result is

$$\frac{1}{c} (hR)^p \frac{1}{\prod_{i=1}^p \Phi(a_i, K)} \frac{(\sum_i \Phi_i(\rho)) \overline{(\sum_j \Phi_j(\rho))}}{[\mathfrak{G}_p : \mathfrak{G}(V)]} \left\{ \prod_{\alpha > 0} \text{Nm } A_\alpha \right\} R^p$$

which one sees leads to the desired result.

If the formula (*) is of any interest we will perhaps have an opportunity to discuss the proof in more detail at Boulder if you are attending the Institute there this summer.

Yours truly,

R. Langlands

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