March 15, 1968
Ankara, Turkey

Dear Serre,

Thank you very much for your letter and the various reprints. I should have anticipated the trouble you would have with the functional equation but I did not. Please accept my apologies. To clear the matter up let me say a few words in general about the problem of attaching Euler products to automorphic forms.

Suppose G is a reductive algebraic group over a number (or even function field) K with adèle ring Λ. Implicit in the theory of Eisenstein series is an exact notion of what it means for an irreducible representation π_β of $G_{\mathbf{B}}$ to occur in $L^2(G_K\backslash G_{\mathbf{A}})$. Moreover all the evidence indicates that any representation that occurs occurs with multiplicity 1. One wants to associate to G another reductive, but not necessarily connected, group $\widehat{G}$ over $\mathbf{C}$ and to each complex representation ρ of $\widehat{G}$ and to each π that occurs in $L^2(G_K\backslash G_{\mathbf{A}})$ an L-series $\xi(s,\rho,\pi)$ (I abuse the customary language by allowing $\xi(s,\rho,\pi)$ to contain factors coming from the infinite primes). Now there are some conditions to be satisfied. One supposes that π is an infinite tensor product $\bigotimes_{\mathfrak{p}} \pi_{\mathfrak{p}}$ (the product is taken over all primes) where $\pi_{\mathfrak{p}}$ is a representation of $G_{K_{\mathfrak{p}}}$. One should first define for *all* reasonable irreducible representations of $G_{K_{\mathfrak{p}}}$ (regardless of their relation to automorphic forms) and each ρ a local factor $\xi(s,\rho,\pi_\gamma)$ and for each additive character $\psi_{\mathfrak{p}}$ of $K_{\mathfrak{p}}$ a factor $\xi(s,\rho,\pi_{\mathfrak{p}},\psi_{\mathfrak{p}})$ such that

$$\xi(s,\rho,\pi) = \prod_{\mathfrak{p}} \xi(s,\rho,\pi_{\mathfrak{p}}).$$

Then if a character ψ of $K\backslash\mathbf{A}$ is given and $\psi_{\mathfrak{p}}$ is its restrictive to $K_{\mathfrak{p}}$ one wants the functional equation to be

$$\xi(s,\rho,\pi) = \xi(s,\rho,\pi)\xi(1-s,\widetilde{\rho},\pi).$$

$\widetilde{\rho}$ is the contragradient representation. The factors $\epsilon(s,\rho,\pi_{\mathfrak{p}},\Psi_{\mathfrak{p}})$ must be 1 for almost all $\mathfrak{p}$ and

$$\prod_{\mathfrak{p}} \xi(s,\rho,\pi_{\mathfrak{p}},\psi_{\mathfrak{p}}) = \xi(s,\rho,\pi)$$

must be independent of $\psi_{\mathfrak{p}}$. In the situations encountered the last condition is an immediate consequence of the product formula.

When starting out to define these local factors one can proceed in the traditional manner. That is one makes the definition at the unramified primes and then when one learns how to prove something, one fills in at the ramified primes. Now what does unramified mean in the present context? I think it should mean the following

(i) $G_{K_{\mathfrak{p}}}$ should be quasi-split and should split over an unramified extension of K_γ

(ii) $\pi_{\mathfrak{p}}$ when restricted to a standard maximal compact subgroup of $G_{K_{\mathfrak{p}}}$ should contain the identity representation

(iii) If $K_{\mathfrak{p}} = \mathbf{R}$ then $\psi_{\mathfrak{p}}(x) = e^{2\pi i x}$, if $K_{\mathfrak{p}} = \mathbf{C}$ then $\psi_{\mathfrak{p}}(z) = e^{4\pi i \operatorname{Re} z}$, if $K_{\mathfrak{p}}$ is non-archimedean the largest ideal on which $\psi_{\mathfrak{p}}$ is trivial is $\mathcal{O}_{\mathfrak{p}}$.

It is fair to assume that in the unramified case $\xi(s,\rho,\pi_{\mathfrak{p}},\psi_{\mathfrak{p}})$ is 1. When $K_{\mathfrak{p}}$ is non-archimedean the second condition allows the Hecke operators to come into play and I have made tentative suggestions in my letter to Weil about how they can be used to define $\xi(s,\rho,\pi_{\mathfrak{p}})$ (in the characteristic 0 case). If pressed I think I could also define factors at the infinite primes. In order to carry out the calculations of my Yale lectures I had to know the factor at each prime and thus I had to restrict myself to a situation in which there was ramification nowhere or, at worst, only condition (iii) could be violated. Because I was pressed for time and everything was so tentative I took the easy way out to insure condition (ii) and took G to be a Chevalley group and K to be the field of rational numbers. For these I did define a factor at the infinite prime.

Now to answer your question. An automorphic (cusp) form in the ordinary sense which is an eigenvalue of all the Hecke operators corresponds to a function on $L^2\big(\mathrm{GL}(2,\mathbf{Q})\backslash\mathrm{GL}(2,\mathbf{A})\big)$ which lies in an irreducible subspace. Let this subspace transform under the representation $\pi=\bigotimes_{\mathfrak{p}}\pi_{\mathfrak{p}}$. If the automorphic form is of level 1 there is no ramification at the finite primes and the factors you write down are the ones I would write down. However the restrictions of π (which is determined by the weight of the automorphic form) to the orthogonal group does not contain the identity representation so there is ramification at infinity and the conditions I have imposed in my Yale notes are not satisfied. The condition at infinity means essentially that I am considering the non-analytic automorphic forms of Maass. If you look at his paper I believe you will see my Γ-factor.

I have been trying in the case that $G=\mathrm{GL}(2,K)$, so that $\widehat{G}=\mathrm{GL}(2,\mathbf{C})$ and ρ is the standard two-dimensional representation to find out what the factors $\xi(s,\rho,\pi_{\mathfrak{p}})$ and $\xi(s,\rho,\pi_{\mathfrak{p}},\psi_{\mathfrak{p}})$ are and to prove, following Hecke, the functional equation. Jacquet has been doing the same thing. The results are not yet complete. When they are I shall write and explain them to you and suggest some relations with your Clermont-Ferrand paper.

Yours,

R. Langlands

Compiled on November 12, 2024.