

December 9, 1975

Dear Serre,

Thanks for your letter. About the second question I have nothing to say, but the first I find very suggestive. Let me sketch a proof of the relation. $\pi_{\text{pseudo}}(\rho) = \pi(\rho)$, and hence of the Artin conjectures, for representations of tetrahedral type. It is based on your observation.

- (a)
- ρ : two-dimensional representation of $\mathfrak{G}(K/F)$ of tetrahedral type.
 - ρ_v : restriction of ρ to decomposition group at v . It is also regarded as a representation of the local Weil group.
 - $\pi = \pi_{\text{pseudo}}(\rho)$: for almost all v choose ρ'_v so that $\pi_v = \pi(\rho'_v)$ —see p. 19 of the notes. If E is the cubic extension of F used to define π and w is a place of E dividing v then $P_w = P'_w$ if P_w, P'_w are the restrictions of ρ_v, ρ'_v to the Weil group over E_w .

Think of [2]

$$\begin{array}{ccc} \text{GL}(2) & \xrightarrow{\psi} & \text{GL}(3) \\ & \searrow \text{dashed} & \nearrow \text{dashed} \\ & \text{PGL}(2) & \end{array}$$

as a map from one associate group to another. Set $\sigma = \varphi \circ \rho$ and define σ_v, σ'_v in a similar manner. Clearly $\Sigma_w = \Sigma'_w$, if Σ_w, Σ'_w are the restrictions of σ_v, σ'_v . As you observe, there is a character θ of $\mathfrak{G}(K/E)$, or of $E^\times \setminus I_E$ so that $\sigma = \text{Ind}(\mathfrak{G}(K/F), \mathfrak{G}(K/E), \theta)$.

- (i) Results of Piatetski-Shapiro presumably imply that $\pi_1 = \pi(\sigma)$ exists as an automorphic representation of $\text{GL}(3)$.
- (ii) Results of Jacquet, P-S, and Shalika presumably imply that the map φ_* from automorphic representations of $\text{GL}(2)$ to automorphic representations of $\text{GL}(3)$ demanded by the philosophy exists. Let $\pi'_1 = \varphi_* \pi$.
- (iii) There is one possible way to show that $\pi_1 = \pi'_1$. Let $\tilde{\pi}_1$ be the representation contragradient to π_1 . According to Jacquet one may be able [3] to show that the analytic behaviour of $L(s, \pi_1 \times \tilde{\pi}_1)$ and of $L(s, \pi'_1 \times \tilde{\pi}_1)$ are different, that the first has a pole at $s = 1$ and the second does not, unless $\pi_1 = \pi'_1$. We need to show that

$$(*) \quad L_v(s, \pi \times \tilde{\pi}_1) = L_v(s, \pi' \times \tilde{\pi}_1)$$

for almost all v . The left side is

$$L(s, \sigma_v \otimes \tilde{\sigma}_v)$$

and

$$\sigma_v = \bigoplus_{w|v} \text{Ind}(F_v, E_w, \theta_w)$$

$$\sigma_v \otimes \tilde{\sigma}_v = \bigoplus_{w|v} \text{Ind}(F_v, E_w, \Sigma_w \otimes \theta_w^{-1}).$$

Σ_w is the restriction of σ_v from F_v to E_w . Moreover the right side is

$$L(s, \sigma'_v \otimes \tilde{\sigma}_v)$$

and

$$\sigma'_v \otimes \tilde{\sigma}_v = \bigoplus_{w|v} \text{Ind}(F_v, E_w, \Sigma'_v \otimes \theta_w^{-1}).$$

Since $\Sigma'_w = \Sigma_w$ we deduce the equation (*). [4] Then one argues as on pp. 9.21–9.22 of the notes and concludes that $\pi'_1 = \pi_1$.

- (iv) We conclude that $\sigma'_v = \sigma_v$ for almost all v . If v splits in E we know that $\rho'_v = \rho_v$. Otherwise let ρ_v take the Frobenius to

$$\begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix}$$

and ρ'_v take it to

$$\begin{pmatrix} \xi a_v & 0 \\ 0 & \xi^2 b_v \end{pmatrix} \quad \xi^3 = 1.$$

We need to show that $\xi = 1$. Since $\sigma'_v = \sigma_v$ either

$$\xi a_v = \lambda a_v \quad \xi^2 b_v = \lambda b_v \implies \xi = 1, \lambda = 1$$

or

$$\xi a_v = \lambda b_v \quad \xi^2 b_v = \lambda a_v \implies \lambda^2 = 1.$$

Thus

$$\begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix} = a_v \begin{pmatrix} 1 & \\ & \lambda \xi \end{pmatrix}.$$

If $\lambda = -1$ then σ_v takes the Frobenius to

$$\begin{pmatrix} -\xi^2 & \\ & 1 \\ & & -\xi \end{pmatrix}$$

which has order 6, and that is impossible. [5]

Unfortunately, Jacquet estimates the amount of work necessary to establish the assertions used above as several hundred pages. For representations of octahedral type one is even worse off.

- (b) ρ representation of octahedral type. Assume representations of tetrahedral type have been handled. Let E be the quadratic extension of F used to define $\pi_{\text{pseudo}}(\rho) = \pi$. Introduce ρ_v, ρ'_v as before. If w is a place of E dividing v then $P_w = P'_w$. Let

$$\varphi : \text{GL}(2) \rightarrow \text{GL}(4)$$

be defined by the representations on the symmetric tensors of degree 3. Let $\sigma = \varphi \circ \rho$. I claim that it is monomial. The relevant sub group consists of those group elements, which take a given diagonal of the square into itself. Grant [text cut off] for the

moment. I assume also, even though no proofs are in sight, that $\pi_1 = \pi(\tau)$ and that $\pi'_1 = \varphi_*\pi$ exist. Step (iii) can probably be [6] [illegible] out again. However the argument used to show that

$$(*) \quad L_v(s, \pi_1 \times \tilde{\pi}_1) = L_v(s, \pi'_1 \times \tilde{\pi}'_1)$$

has to be a little different since the group of elements which take a given diagonal to itself is not contained in $\mathfrak{G}(K/E)$. If v splits on E then the relation is clear. Otherwise ρ_v takes the Frobenius to

$$a \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$$

with $\zeta^4 = 1$ and $\zeta \neq 1$, where ρ'_v takes it to

$$a' \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$$

with $a' = \pm a$, $\xi = \pm \zeta$. The eigenvalues of σ_v applied to the Frobenius are therefore

$$a^3, a^3\zeta, a^3\zeta^3, a^3\zeta^3$$

and those of σ'_v applied to it are

$$a'^3, a'^3\zeta, a'^3\zeta^3, a'^3\zeta^3$$

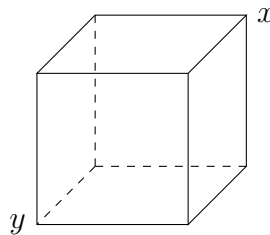
[7] these two sets are the same unless $\xi = 1, \zeta = -1$. Then the numbers $\frac{a^3\zeta^i}{a^3\zeta^j}$ and $\frac{a'^3\xi^i}{a'^3\xi^j}$, $0 \leq i, j \leq 3$ each consist of $+1$ and -1 counted with multiplicity 8; so the equality again follows.

We conclude that $\sigma_v \sim \sigma'_v$ and hence that either

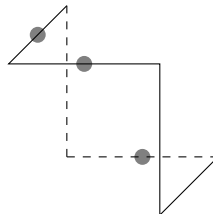
$$a' = \lambda a, \xi a' = \lambda \xi a, \lambda^3 = 1 \implies \lambda = 1 \text{ and } \rho_v \sim \rho'_v$$

$$\xi a' = \lambda a, a' = \lambda \zeta a, \lambda^3 = 1 \implies \lambda = \pm \xi \implies \lambda = 1 \text{ and } \rho_v \sim \rho'_v.$$

To verify that σ is monomial I use the correspondence between points on $S^2 \subseteq \mathbf{R}^3$ and lines on \mathbf{C}^2 compatible with $U(2) \rightarrow \text{SO}(3, \mathbf{R})$



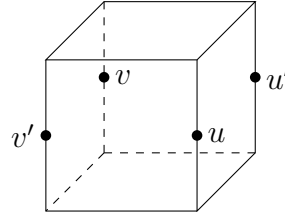
Take the diagonal to be xy and the corresponding group to be $\mathfrak{G}(K/E')$. The edges not containing x or y are



[8] Let x_1, x_2, x_3 be vectors in \mathbf{C}^2 in the directions determined by the three marked points. In the symmetric algebra $x_1x_2x_3$ is an eigenvector of each element of $\mathfrak{G}(K/E')$ and defines a character θ . To show that

$$\sigma \sim \text{Ind}(\mathfrak{G}(K/F), \mathfrak{G}(K/E'), \theta)$$

we need only verify that the representation on the right is irreducible or that its restriction to $\mathfrak{G}(K/E')$ contains θ only once. If $\mathfrak{G}(K/E')$ is the group fixing each point of the axis uv , if r is the rotation through π about the axis $u'v'$,



and if $\mu(x) = \theta(rxr^{-1})$, $x \in \mathfrak{G}(K/E'')$ then the restriction is

$$\theta \oplus \text{Ind}(\mathfrak{G}(K/E'), \mathfrak{G}(K/E''), \mu).$$

Since μ is not equal to the restriction of θ to $\mathfrak{G}(K/E')$ the second summand does not contain θ .

Yours,
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[9]

PS. I read through your letter again, and after opening up a text to recall which was the alternating and which was the symmetric group, I saw that I misinterpreted the first question. You want to show in addition that if ρ is of octahedral type and $\pi = \pi_{\text{pseudo}}(\rho)$ then $\det \rho = \omega_\pi$, the restriction of π [cut off] the centre, and therefore that Deligne-Serre gives $\pi = \pi(\rho)$ when $F = \mathbf{Q}$ and $\det \rho$ is the sign character at infinity. This seems feasible.

Define φ by

$$\begin{array}{ccc} \text{GL}(2) & \xrightarrow{\varphi} & \text{GL}(3) \\ & \searrow \text{dashed} & \nearrow \text{dashed} \\ & & \text{PGL}(2) \end{array}$$

and let $\sigma = \varphi \circ \rho$ as on case (a). Let E be the quadratic extension over which ρ becomes tetrahedral.

Define π_1 and π'_1 as before. This is done by steps (i) and (ii). The step (iii) of (a) [is] accomplished by results from the theory of base change for $\text{GL}(3)$. It may be permissible to extend the other parts of the theory to $\text{GL}(3)$ for the difficulties are not overwhelming. Suppose, for the sake of the present argument, that it can be done, then π_1 and π'_1 have the same lifting to automorphic representations of $\text{GL}(3, \mathbf{A}_E)$. Since $\det \sigma_v = \det \sigma'_v = 1$, π_1 and π'_1 agree on the centre and are the same. It follows that $\sigma_v \simeq \sigma'_v$ for almost all v and that $\rho_v \simeq \pm \rho'_v$. Hence $\det \rho_v = \det \rho'_v$ and $\det \rho = \omega_\pi$.

Added: Feb. 10, 1976. Choose v so that π_v is unramified and so that $\rho(\Phi_v)$ and $\rho(\Phi_\infty)$ are conjugate. It is enough to show that $\rho_v(\Phi_v) = \rho(\Phi_v)$ and $\rho'_v(\Phi_v)$ have the same eigenvalues.

Now

$$\rho_v(\Phi_v) \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so

$$\rho'_v(\Phi_v) \simeq \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

If the conclusion is not satisfied then $\rho'_v(\Phi_v) = \pm I$ and $\sigma'_v(\Phi_v) = 1$. This is impossible.

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