December 9, 1975

Dear Serre,

Thanks for your letter. About the second question I have nothing to say, but the first I find very suggestive. Let me sketch a proof of the relation. $\pi_{\text{pseudo}}(\rho) = \pi(\rho)$, and hence of the Artin conjectures, for representations of tetrahedral type. It is based on your observation.

(a)
- $\rho$: two-dimensional representation of $\mathfrak{G}(K/F)$ of tetrahedral type.
- $\rho_v$: restriction of $\rho$ to decomposition group at $v$. It is also regarded as a representation of the local Weil group.
- $\pi = \pi_{\text{pseudo}}(\rho)$: for almost all $v$ choose $\rho_v^i$ so that $\pi_v = \pi(\rho_v')$—see p. 19 of the notes. If $E$ is the cubic extension of $F$ used to define $\pi$ and $w$ is a place of $E$ dividing $v$ then $P_w = P_w'$ if $P_w$, $P_w'$ are the restrictions of $\rho_v$, $\rho_v'$ to the Weil group over $E_w$.

Think of [**2**]

$$\begin{array}{ccc} \mathrm{GL}(2) & \xrightarrow{\ \psi\ } & \mathrm{GL}(3) \\ & \searrow \quad \nearrow & \\ & \mathrm{PGL}(2) & \end{array}$$

as a map from one associate group to another. Set $\sigma = \varphi \circ \rho$ and define $\sigma_v$, $\sigma_v'$ in a similar manner. Clearly $\Sigma_w = \Sigma_w'$, if $\Sigma_w$, $\Sigma_w'$ are the restrictions of $\sigma_v$, $\sigma_v'$. As you observe, there is a character $\theta$ of $\mathfrak{G}(K/E)$, or of $E^\times \backslash I_E$ so that $\sigma = \mathrm{Ind}\big(\mathfrak{G}(K/F), \mathfrak{G}(K/E), \theta\big)$.

(i) Results of Piatetski-Shapiro presumably imply that $\pi_1 = \pi(\sigma)$ exists as an automorphic representation of GL(3).

(ii) Results of Jaquet, P-S, and Shalika presumably imply that the map $\varphi_*$ from automorphic representations of GL(2) to automorphic representations of GL(3) demanded by the philosophy exists. Let $\pi_1' = \varphi_* \pi$.

(iii) There is one possible way to show that $\pi_1 = \pi_1'$. Let $\widetilde{\pi}_1$ be the representation contragradient to $\pi_1$. According to Jacquet one may be able [**3**] to show that the analytic behaviour of $L(s, \pi_1 \times \widetilde{\pi}_1)$ and of $L(s, \pi_1' \times \widetilde{\pi}_1)$ are different, that the first has a pole at $s = 1$ and the second does not, unless $\pi_1 = \pi_1'$. We need to show that

$$L_v(s, \pi \times \widetilde{\pi}_1) = L_v(s, \pi' \times \widetilde{\pi}_1) \qquad (*)$$

for almost all $v$. The left side is

$$L(s, \sigma_v \otimes \widetilde{\sigma}_v)$$

and

$$\sigma_v = \bigoplus_{w|v} \operatorname{Ind}(F_v, E_w, \theta_w)$$

$$\sigma_v \otimes \widetilde{\sigma}_v = \bigoplus_{w|v} \operatorname{Ind}(F_v, E_w, \Sigma_w \otimes \theta_w^{-1}).$$

$\Sigma_w$ is the restriction of $\sigma_v$ from $F_v$ to $E_w$. Moreover the right side is

$$L(s, \sigma_v' \otimes \widetilde{\sigma}_v)$$

and

$$\sigma_v' \otimes \widetilde{\sigma}_v = \bigoplus_{w|v} \operatorname{Ind}(F_v, E_w, \Sigma_v' \otimes \theta_w^{-1}).$$

Since $\Sigma_w' = \Sigma_w$ we deduce the equation $(*)$. [**4**] Then one argues as on pp. 9.21–9.22 of the notes and concludes that $\pi_1' = \pi_1$.

(iv) We conclude that $\sigma_v' = \sigma_v$ for almost all $v$. If $v$ splits in $E$ we know that $\rho_v' = \rho_v$. Otherwise let $\rho_v$ take the Frobenius to

$$\begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix}$$

and $\rho_v'$ take it to

$$\begin{pmatrix} \xi a_v & 0 \\ 0 & \xi^2 b_v \end{pmatrix} \qquad \xi^3 = 1.$$

We need to show that $\xi = 1$. Since $\sigma_v' = \sigma_v$ either

$$\xi a_v = \lambda a_v \qquad \xi^2 b_v = \lambda b_v \implies \xi = 1,\ \lambda = 1$$

or

$$\xi a_v = \lambda b_v \qquad \xi^2 b_v = \lambda a_v \implies \lambda^2 = 1.$$

Thus

$$\begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix} = a_v \begin{pmatrix} 1 & \\ & \lambda\xi \end{pmatrix}.$$

If $\lambda = -1$ then $\sigma_v$ takes the Frobenius to

$$\begin{pmatrix} -\xi^2 & & \\ & 1 & \\ & & -\xi \end{pmatrix}$$

which has order 6, and that is impossible. [**5**]

Unfortunately, Jacquet estimates the amount of work necessary to establish the assertions used above as several hundred pages. For representations of octahedral type one is even worse off.

(b) $\rho$ representation of octahedral type. Assume representations of tetrahedral type have been handled. Let $E$ be the quadratic extension of $F$ used to define $\pi_{\text{pseudo}}(\rho) = \pi$. Introduce $\rho_v$, $\rho_v'$ as before. If $w$ is a place of $E$ dividing $v$ then $P_w = P_w'$. Let

$$\varphi : \mathrm{GL}(2) \to \mathrm{GL}(4)$$

be defined by the representations on the symmetric tensors of degree 3. Let $\sigma = \varphi \circ \rho$. I claim that it is monomial. The relevant sub group consists of those group elements, which take a given diagonal of the square into itself. Grant [text cut off] for the moment. I assume also, even though no proofs are in sight, that $\pi_1 = \pi(\tau)$ and that $\pi'_1 = \varphi_* \pi$ exist. Step (iii) can probably be **[6]** [illegible] out again. However the argument used to show that

$$(*) \qquad L_v(s, \pi_1 \times \widetilde{\pi}_1) = L_v(s, \pi'_1 \times \widetilde{\pi}'_1)$$

has to be a little different since the group of elements which take a given diagonal to itself is not contained in $\mathfrak{G}(K/E)$. If $v$ splits on $E$ then the relation is clear. Otherwise $\rho_v$ takes the Frobenius to

$$a \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$$

with $\zeta^4 = 1$ and $\zeta \neq 1$, where $\rho'_v$ takes it to

$$a' \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$$

with $a' = \pm a$, $\xi = \pm \zeta$. The eigenvalues of $\sigma_v$ applied to the Frobenius are therefore

$$a^3, \; a^3\zeta, \; a^3\zeta^3, a^3\zeta^3$$

and those of $\sigma'_v$ applied to it are

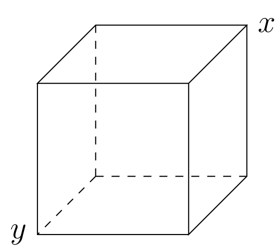
$$a'^3, \; a'^3\zeta, \; a'^3\zeta^3, a'^3\zeta^3$$

**[7]** these two sets are the same unless $\xi = 1$, $\zeta = -1$. Then the numbers $\frac{a^3\zeta^i}{a^3\zeta^j}$ and $\frac{a^3\xi^i}{a'^3\xi^j}$, $0 \leqslant i, \; j \leqslant 3$ each consist of $+1$ and $-1$ counted with multiplicity 8; so the equality again follows.

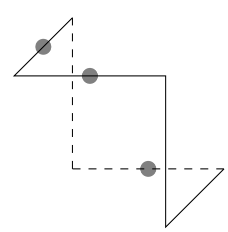
We conclude that $\sigma_v \sim \sigma'_v$ and hence that either

$$a' = \lambda a, \; \xi a' = \lambda \xi a, \; \lambda^3 = 1 \implies \lambda = 1 \text{ and } \rho_v \sim \rho'_v$$
$$\xi a' = \lambda a, \; a' = \lambda \zeta a, \; \lambda^3 = 1 \implies \lambda = \pm \xi \implies \lambda = 1 \text{ and } \rho_v \sim \rho'_v.$$

To verify that $\sigma$ is monomial I use the correspondence between points on $S^2 \subseteq \mathbf{R}^3$ and lines on $\mathbf{C}^2$ compatible with $U(2) \to \mathrm{SO}(3, \mathbf{R})$



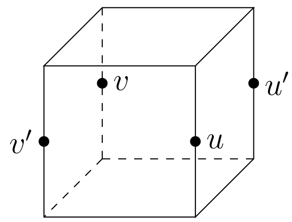

Take the diagonal to be $xy$ and the corresponding group to be $\mathfrak{G}(K/E')$. The edges not containing $x$ or $y$ are



[**8**] Let $x_1$, $x_2$, $x_3$ be vectors in $\mathbf{C}^2$ in the directions determined by the three marked points. In the symmetric algebra $x_1x_2x_3$ is an eigenvector of each element of $\mathfrak{G}(K/E')$ and defines a character $\theta$. To show that

$$\sigma \sim \operatorname{Ind}\big(\mathfrak{G}(K/F), \mathfrak{G}(K/E'), \theta\big)$$

we need only verify that the representation on the right is irreducible or that its restriction to $\mathfrak{G}(K/E')$ contains $\theta$ only once. If $\mathfrak{G}(K/E')$ is the group fixing each point of the axis $uv$, if $r$ is the rotation through $\pi$ about the axis $u'v'$,




and if $\mu(x) = \theta(rxr^{-1})$, $x \in \mathfrak{G}(K/E'')$ then the restriction is

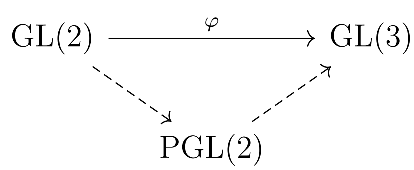
$$\theta \oplus \operatorname{Ind}\big(\mathfrak{G}(K/E'), \mathfrak{G}(K/E''), \mu\big).$$

Since $\mu$ is not equal to the restriction of $\theta$ to $\mathfrak{G}(K/E')$ the second summand does not contain $\theta$.

Yours,
R. Langlands

[**9**]
PS. I read through your letter again, and after opening up a text to recall which was the alternating and which was the symmetric group, I saw that I misinterpreted the first question. You want to show in addition that if $\rho$ is of octahedral type and $\pi = \pi_{\text{pseudo}}(\rho)$ then $\det\rho = \omega_\pi$, the restriction of $\pi$ [cut off] the centre, and therefore that Deligne-Serre gives $\pi = \pi(\rho)$ when $F = \mathbf{Q}$ and $\det\rho$ is the sign character at infinity. This seems feasible.

Define $\varphi$ by




and let $\sigma = \varphi \circ \rho$ as on case (a). Let $E$ be the quadratic extension over which $\rho$ becomes tetrahedral.

Define $\pi_1$ and $\pi_1'$ as before. This is done by steps (i) and (ii). The step (iii) of (a) [is] accomplished by results from the theory of base change for GL(3). It may be permissible to extend the other parts of the theory to GL(3) for the difficulties are not overwhelming. Suppose, for the sake of the present argument, that it can be done, then $\pi_1$ and $\pi_1'$ have the same lifting to automorphic representations of $\mathrm{GL}(3, \mathbf{A}_E)$. Since $\det\sigma_v = \det\sigma_v' = 1$, $\pi_1$ and $\pi_1'$ agree on the centre and are the same. It follows that $\sigma_v \simeq \sigma_v'$ for almost all $v$ and that $\rho_v \simeq \pm\rho_v'$. Hence $\det\rho_v = \det\rho_v'$ and $\det\rho = \omega_\pi$.

**Added**: Feb. 10, 1976. Choose $v$ so that $\pi_v$ is unramified and so that $\rho(\Phi_v)$ and $\rho(\Phi_\infty)$ are conjugate. It is enough to show that $\rho_v(\Phi_v) = \rho(\Phi_v)$ and $\rho_v'(\Phi_v)$ have the same eigenvalues.

Now

$$\rho_v(\Phi_v) \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so

$$\rho'_v(\Phi_v) \simeq \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

If the conclusion is not satisfied then $\rho'_v(\Phi_v) = \pm I$ and $\sigma'_v(\Phi_v) = 1$. This is impossible.

Compiled on December 22, 2023.