

December 9, 1975

Dear Serre,

Thanks for your letter. About the second question I have nothing to say, but the first I find very suggestive. Let me sketch a proof of the relation.  $\pi_{\text{pseudo}}(\rho) = \pi(\rho)$ , and hence of the Artin conjectures, for representations of tetrahedral type. It is based on your observation.

- (a)
- $\rho$ : two-dimensional representation of  $\mathfrak{S}(K/F)$  of tetrahedral type.
  - $\rho_v$ : restriction of  $\rho$  to decomposition group at  $v$ . It is also regarded as a representation of the local Weil group.
  - $\pi = \pi_{\text{pseudo}}(\rho)$ : for almost all  $v$  choose  $\rho'_v$  so that  $\pi_v = \pi(\rho'_v)$ —see p. 19 of the notes. If  $E$  is the cubic extension of  $F$  used to define  $\pi$  and  $w$  is a place of  $E$  dividing  $v$  then  $P_w = P'_w$  if  $P_w, P'_w$  are the restrictions of  $\rho_v, \rho'_v$  to the Weil group over  $E_w$ .

Think of [2]

$$\begin{array}{ccc} \text{GL}(2) & \xrightarrow{\psi} & \text{GL}(3) \\ & \searrow \text{dashed} & \nearrow \text{dashed} \\ & \text{PGL}(2) & \end{array}$$

as a map from one associate group to another. Set  $\sigma = \varphi \circ \rho$  and define  $\sigma_v, \sigma'_v$  in a similar manner. Clearly  $\Sigma_w = \Sigma'_w$ , if  $\Sigma_w, \Sigma'_w$  are the restrictions of  $\sigma_v, \sigma'_v$ . As you observe, there is a character  $\theta$  of  $\mathfrak{S}(K/E)$ , or of  $E^\times \setminus I_E$  so that  $\sigma = \text{Ind}(\mathfrak{S}(K/F), \mathfrak{S}(K/E), \theta)$ .

- (i) Results of Piatetski-Shapiro presumably imply that  $\pi_1 = \pi(\sigma)$  exists as an automorphic representation of  $\text{GL}(3)$ .
- (ii) Results of Jacquet, P-S, and Shalika presumably imply that the map  $\varphi_*$  from automorphic representations of  $\text{GL}(2)$  to automorphic representations of  $\text{GL}(3)$  demanded by the philosophy exists. Let  $\pi'_1 = \varphi_* \pi$ .
- (iii) There is one possible way to show that  $\pi_1 = \pi'_1$ . Let  $\tilde{\pi}_1$  be the representation contragradient to  $\pi_1$ . According to Jacquet one may be able [3] to show that the analytic behaviour of  $L(s, \pi_1 \times \tilde{\pi}_1)$  and of  $L(s, \pi'_1 \times \tilde{\pi}_1)$  are different, that the first has a pole at  $s = 1$  and the second does not, unless  $\pi_1 = \pi'_1$ . We need to show that

$$(*) \quad L_v(s, \pi \times \tilde{\pi}_1) = L_v(s, \pi' \times \tilde{\pi}_1)$$

for almost all  $v$ . The left side is

$$L(s, \sigma_v \otimes \tilde{\sigma}_v)$$

and

$$\sigma_v = \bigoplus_{w|v} \text{Ind}(F_v, E_w, \theta_w)$$

$$\sigma_v \otimes \tilde{\sigma}_v = \bigoplus_{w|v} \text{Ind}(F_v, E_w, \Sigma_w \otimes \theta_w^{-1}).$$

$\Sigma_w$  is the restriction of  $\sigma_v$  from  $F_v$  to  $E_w$ . Moreover the right side is

$$L(s, \sigma'_v \otimes \tilde{\sigma}_v)$$

and

$$\sigma'_v \otimes \tilde{\sigma}_v = \bigoplus_{w|v} \text{Ind}(F_v, E_w, \Sigma'_v \otimes \theta_w^{-1}).$$

Since  $\Sigma'_w = \Sigma_w$  we deduce the equation (\*). [4] Then one argues as on pp. 9.21–9.22 of the notes and concludes that  $\pi'_1 = \pi_1$ .

- (iv) We conclude that  $\sigma'_v = \sigma_v$  for almost all  $v$ . If  $v$  splits in  $E$  we know that  $\rho'_v = \rho_v$ . Otherwise let  $\rho_v$  take the Frobenius to

$$\begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix}$$

and  $\rho'_v$  take it to

$$\begin{pmatrix} \xi a_v & 0 \\ 0 & \xi^2 b_v \end{pmatrix} \quad \xi^3 = 1.$$

We need to show that  $\xi = 1$ . Since  $\sigma'_v = \sigma_v$  either

$$\xi a_v = \lambda a_v \quad \xi^2 b_v = \lambda b_v \implies \xi = 1, \lambda = 1$$

or

$$\xi a_v = \lambda b_v \quad \xi^2 b_v = \lambda a_v \implies \lambda^2 = 1.$$

Thus

$$\begin{pmatrix} a_v & 0 \\ 0 & b_v \end{pmatrix} = a_v \begin{pmatrix} 1 & \\ & \lambda \xi \end{pmatrix}.$$

If  $\lambda = -1$  then  $\sigma_v$  takes the Frobenius to

$$\begin{pmatrix} -\xi^2 & \\ & 1 \\ & & -\xi \end{pmatrix}$$

which has order 6, and that is impossible. [5]

Unfortunately, Jacquet estimates the amount of work necessary to establish the assertions used above as several hundred pages. For representations of octahedral type one is even worse off.

- (b)  $\rho$  representation of octahedral type. Assume representations of tetrahedral type have been handled. Let  $E$  be the quadratic extension of  $F$  used to define  $\pi_{\text{pseudo}}(\rho) = \pi$ . Introduce  $\rho_v, \rho'_v$  as before. If  $w$  is a place of  $E$  dividing  $v$  then  $P_w = P'_w$ . Let

$$\varphi : \text{GL}(2) \rightarrow \text{GL}(4)$$

be defined by the representations on the symmetric tensors of degree 3. Let  $\sigma = \varphi \circ \rho$ . I claim that it is monomial. The relevant sub group consists of those group elements, which take a given diagonal of the square into itself. Grant [text cut off] for the

moment. I assume also, even though no proofs are in sight, that  $\pi_1 = \pi(\tau)$  and that  $\pi'_1 = \varphi_*\pi$  exist. Step (iii) can probably be [6] [illegible] out again. However the argument used to show that

$$(*) \quad L_v(s, \pi_1 \times \tilde{\pi}_1) = L_v(s, \pi'_1 \times \tilde{\pi}'_1)$$

has to be a little different since the group of elements which take a given diagonal to itself is not contained in  $\mathfrak{G}(K/E)$ . If  $v$  splits on  $E$  then the relation is clear. Otherwise  $\rho_v$  takes the Frobenius to

$$a \begin{pmatrix} 1 & 0 \\ 0 & \zeta \end{pmatrix}$$

with  $\zeta^4 = 1$  and  $\zeta \neq 1$ , where  $\rho'_v$  takes it to

$$a' \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$$

with  $a' = \pm a$ ,  $\xi = \pm \zeta$ . The eigenvalues of  $\sigma_v$  applied to the Frobenius are therefore

$$a^3, a^3\zeta, a^3\zeta^3, a^3\zeta^3$$

and those of  $\sigma'_v$  applied to it are

$$a'^3, a'^3\zeta, a'^3\zeta^3, a'^3\zeta^3$$

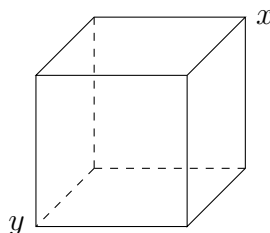
[7] these two sets are the same unless  $\xi = 1$ ,  $\zeta = -1$ . Then the numbers  $\frac{a^3\zeta^i}{a^3\zeta^j}$  and  $\frac{a'^3\xi^i}{a'^3\xi^j}$ ,  $0 \leq i, j \leq 3$  each consist of  $+1$  and  $-1$  counted with multiplicity 8; so the equality again follows.

We conclude that  $\sigma_v \sim \sigma'_v$  and hence that either

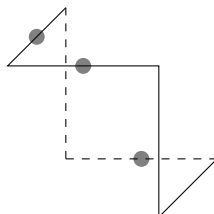
$$a' = \lambda a, \xi a' = \lambda \xi a, \lambda^3 = 1 \implies \lambda = 1 \text{ and } \rho_v \sim \rho'_v$$

$$\xi a' = \lambda a, a' = \lambda \zeta a, \lambda^3 = 1 \implies \lambda = \pm \xi \implies \lambda = 1 \text{ and } \rho_v \sim \rho'_v.$$

To verify that  $\sigma$  is monomial I use the correspondence between points on  $S^2 \subseteq \mathbf{R}^3$  and lines on  $\mathbf{C}^2$  compatible with  $U(2) \rightarrow SO(3, \mathbf{R})$



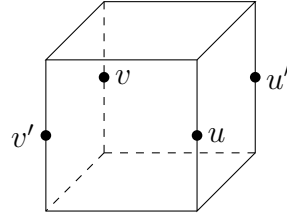
Take the diagonal to be  $xy$  and the corresponding group to be  $\mathfrak{G}(K/E')$ . The edges not containing  $x$  or  $y$  are



[8] Let  $x_1, x_2, x_3$  be vectors in  $\mathbf{C}^2$  in the directions determined by the three marked points. In the symmetric algebra  $x_1x_2x_3$  is an eigenvector of each element of  $\mathfrak{S}(K/E')$  and defines a character  $\theta$ . To show that

$$\sigma \sim \text{Ind}(\mathfrak{S}(K/F), \mathfrak{S}(K/E'), \theta)$$

we need only verify that the representation on the right is irreducible or that its restriction to  $\mathfrak{S}(K/E')$  contains  $\theta$  only once. If  $\mathfrak{S}(K/E')$  is the group fixing each point of the axis  $uv$ , if  $r$  is the rotation through  $\pi$  about the axis  $u'v'$ ,



and if  $\mu(x) = \theta(rxr^{-1})$ ,  $x \in \mathfrak{S}(K/E'')$  then the restriction is

$$\theta \oplus \text{Ind}(\mathfrak{S}(K/E'), \mathfrak{S}(K/E''), \mu).$$

Since  $\mu$  is not equal to the restriction of  $\theta$  to  $\mathfrak{S}(K/E')$  the second summand does not contain  $\theta$ .

Yours,  
R. Langlands

[9]

PS. I read through your letter again, and after opening up a text to recall which was the alternating and which was the symmetric group, I saw that I misinterpreted the first question. You want to show in addition that if  $\rho$  is of octahedral type and  $\pi = \pi_{\text{pseudo}}(\rho)$  then  $\det \rho = \omega_\pi$ , the restriction of  $\pi$  [cut off] the centre, and therefore that Deligne-Serre gives  $\pi = \pi(\rho)$  when  $F = \mathbf{Q}$  and  $\det \rho$  is the sign character at infinity. This seems feasible.

Define  $\varphi$  by

$$\begin{array}{ccc} \text{GL}(2) & \xrightarrow{\varphi} & \text{GL}(3) \\ & \searrow \text{dashed} & \nearrow \text{dashed} \\ & & \text{PGL}(2) \end{array}$$

and let  $\sigma = \varphi \circ \rho$  as on case (a). Let  $E$  be the quadratic extension over which  $\rho$  becomes tetrahedral.

Define  $\pi_1$  and  $\pi'_1$  as before. This is done by steps (i) and (ii). The step (iii) of (a) [is] accomplished by results from the theory of base change for  $\text{GL}(3)$ . It may be permissible to extend the other parts of the theory to  $\text{GL}(3)$  for the difficulties are not overwhelming. Suppose, for the sake of the present argument, that it can be done, then  $\pi_1$  and  $\pi'_1$  have the same lifting to automorphic representations of  $\text{GL}(3, \mathbf{A}_E)$ . Since  $\det \sigma_v = \det \sigma'_v = 1$ ,  $\pi_1$  and  $\pi'_1$  agree on the centre and are the same. It follows that  $\sigma_v \simeq \sigma'_v$  for almost all  $v$  and that  $\rho_v \simeq \pm \rho'_v$ . Hence  $\det \rho_v = \det \rho'_v$  and  $\det \rho = \omega_\pi$ .

**Added:** Feb. 10, 1976. Choose  $v$  so that  $\pi_v$  is unramified and so that  $\rho(\Phi_v)$  and  $\rho(\Phi_\infty)$  are conjugate. It is enough to show that  $\rho_v(\Phi_v) = \rho(\Phi_v)$  and  $\rho'_v(\Phi_v)$  have the same eigenvalues.

Now

$$\rho_v(\Phi_v) \simeq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so

$$\rho'_v(\Phi_v) \simeq \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

If the conclusion is not satisfied then  $\rho'_v(\Phi_v) = \pm I$  and  $\sigma'_v(\Phi_v) = 1$ . This is impossible.

Compiled on November 12, 2024.