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Dear Singer,

I will tell you what I can about the alternating sum in which you are interested, sometimes in my own idiom and sometimes in the classical. Because of this you may well wish to pass over some parts of the letter very quickly. I shall first give an expression for the alternating sum in terms of class numbers and then a second expression in terms of the values of L-functions.

The formulae I shall give, special cases of which are already to be found in Hecke, are consequences of some joint work with Jean-Pierre Labesse which, because of my dilatoriness, is not yet written up. It is therefore a little reckless of me to state them in an apodictic form. I recommend in fact that they be treated with some circumspection. I would be glad to give you an idea of the proof orally now and then, if you wanted them, could send you the details later. They are however rather complicated and involve considerable computation. I suspect that it would be easier for you just to sit down and apply the trace formula to your particular case and let the result fall out—it is not really very difficult.

Take F to be a totally real field and let G be the group SL(2) over F. If S_∞ is the set of infinite places of F and K_∞ is a maximal compact subgroup of

$$\prod_{v\in S_\infty} G(F_v) = \mathrm{SL}(2,\mathbf{R})^{[F:\mathbf{Q}]}$$

then to specify a complex structure on

$$\prod_{v\in S_\infty} G(F_v)/K_\infty$$

we have to give for each v an imbedding of the circle group

$$S^1 = \{z\in\mathbf{C}^\times \mid |z|=1\}$$

in $G(F_v)$. The complex structure as defined by the condition that the eigenvalues of $\prod_v \psi_v(e^{ix_v})$ on the complex tangent space to the point fixed by $\prod_v S^1$ under $\prod_v \psi_v$ are the numbers e^{2ix_v}.

On the other hand given ψ_v and a character θ_v of S^1 one can define two infinite-dimensional unitary representations $\pi^1(\theta_v)$ and $\pi^{-1}(\theta_v)$ of $G(F_v)$. If

$$\theta(e^{ix_v}) = e^{in_v x_v}$$

and $n_v \neq 0$ then $\pi^1(\theta_v)$ and $\pi^{-1}(\theta_v)$ are the holomorphic and anti-holomorphic discrete series of parameter $|n_v|$. If $T_\infty \subseteq S_\infty$ is given the dimension of the space of cusp forms on $G(O_F)\backslash\prod G(F_v)/K_\infty$ of weight $|n_v|+1$ at v which are holomorphic for $v\in T_\infty$ and anti-holomorphic for $v\in S-T_\infty$ is the multiplicity $m(\{\epsilon_v\},\{\theta_v\})$ with which $\bigotimes_v \pi^{\epsilon_v}(\theta_v)$, $\epsilon_v=1$,

$v \in T_\infty$, $\epsilon_v = -1$, $v \in S_\infty - T_\infty$ occurs in the space of cusp forms in $L^2(G(O_f)\backslash \prod G(F_v))$. I believe you want a formula for

$$\sum_{\{\epsilon_v\}} \left(\prod_v \epsilon_v \right) m(\{\epsilon_v\}, \{\theta_v\}). \tag{1}$$

O_F is of course the ring of integers in F.

Let $\mathbf{A}$ be the group of adèles of F, $\mathbf{A}_f$ the adèles which are 1 at every infinite place, and $\widehat{O}_F$ the integral adèles in $\mathbf{A}_f$. Then

$$G(O_F) \Big\backslash \prod_v G(F_v) = G(F)\backslash G(\mathbf{A})/G(\widehat{O}_F).$$

Every irreducible representation π of $G(A)$ which occurs in the space cusp forms in $L^2(G(F)\backslash G(\mathbf{A}))$ is of the form

$$\left(\bigotimes_{v \in S_\infty} \pi_v \right) \otimes \pi_f$$

where π_v is a representation of $G(F_v)$ and π_f a representation of $G(\mathbf{A}_f)$. It is a fact that π_f contains the trivial representation of $G(\widehat{O}_F)$ either once or not at all. The representation of $G(A)$ on the space of cusp forms is a discrete direct sum

$$\bigoplus m(\pi)\pi \qquad m(\pi) > 0 \text{ in } \mathbf{Z}. \tag{2}$$

For a given collection of θ_v, $m(\{\epsilon_v\}, \{\theta_v\})$ is the number of π counted with multiplicity for which π_v is equivalent to $\pi^{\epsilon_v}(\theta_v)$ for each $v \in S_\infty$ and π_f contains the trivial representation of $G(\widehat{O}_f)$.

The expression (2) decomposes into three parts.

(3) the sum over all π such that, for some v, π_v is equivalent to no $\pi^{\epsilon_v}(\theta_v)$

$$\bigoplus m(\pi)\pi.$$

(4) A sum over all π such that, for each v, π_v is equivalent to some $\pi^{\epsilon_v}(\theta_v)$

$$\bigoplus m'(\pi)\pi.$$

Here $m'(\pi) \in \mathbf{R}$ and

$$m'(\left(\bigotimes_v \pi^{\epsilon_v}(\theta_v) \right) \otimes \pi_f)$$

is independent of the ϵ_v.

(5) A sum over the same π as in (4)

$$\bigoplus m''(\pi)\pi.$$

Here $m''(\pi) \in \mathbf{R}$ and

$$\sum_{\{\epsilon_v\}} m''(\left(\bigotimes_v \pi^{\epsilon_v}(\theta_v) \right) \otimes \pi_f) = 0.$$

You will agree I hope that the alternating sum (1) is affected by (5) alone. What I do now is give an explicit formula for (5) in terms of totally imaginary quadratic extensions of F. Then from this explicit form I single out the part corresponding to those π for which π_f contains the trivial representation of $G(\widehat{O}_F)$. The next part of the letter you will very likely prefer not to read. None the less it's safer not to omit it.

The sum (5) will break up further into a sum over the totally imaginary quadratic extensions E of F. I consider the term corresponding to a single E. Fix a basis λ_1, λ_2 of E over F. For each $v \in S_\infty$ this gives a basis of $E_v = E \otimes F_v$ over F_v. If $a \in E_v$ let

$$\begin{pmatrix} \alpha\lambda_1 \\ \alpha\lambda_2 \end{pmatrix} = \begin{pmatrix} a\lambda_1 + c\lambda_2 \\ b\lambda_1 + d\lambda_2 \end{pmatrix}.$$

We identify E_v with $\mathbf{C}$ in such a way that if $\alpha \leftrightarrow e^{ix}$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is conjugate to $\psi_v(e^{ix})$ under $G(F_v) \sim \mathrm{SL}(2, \mathbf{R})$. Choose $t_0 \in E$ so that $\operatorname{Im} t_0 > 0$ in each E_v, $v \in S_\infty$.

λ_1, λ_2 and t_0 are now fixed. let v be a finite place of F, let $E_v = E \otimes F_v$, $H_v = \{\alpha \in E_v \mid N_{E_v/F_v}\alpha = 1\}$ and let θ_v be a character of H_v.

(i) If v splits in E so that $E_v \simeq F_v \oplus F_v$ it is possible to define a representation $\pi(\theta_v)$ of $G(F_v) = \mathrm{SL}(2, F_v)$ with the following properties:
   (a) If $\theta_v^2 = 1$, $\theta_v \neq 1$ then $\pi(\theta_v)$ is irreducible. Otherwise it is the direct sum of two irreducible representations.
   (b) $\pi(\theta_v)$ contains the trivial representation of $G(O_F)$ if and only if θ_v is unramified, that is, trivial in $\{(\alpha, \alpha^{-1}) \mid \alpha \in O_{F_v}^\times\} \subseteq F_v^v \oplus F_v \simeq E_v$ and then it contains it exactly once.

(ii) If v does not split then it is possible to define two representations $\pi^1(\theta_v)$ and $\pi^{-1}(\theta_v)$ with the following properties:
   (a) If $\theta_v^2 = 1$, $\theta_v \neq 1$ then $\pi^1(\theta_v)$, $\pi^{-1}(\theta_v)$ are irreducible. Otherwise they are the direct sum of two one-dimensional representations.
   (b) $\pi^1(\theta_v) + \pi^{-1}(\theta_v)$ contains the trivial representation of $G(O_{F_v})$ if and only if E is unramified at v and θ_v is trivial. Then it contains it exactly once. If ϖ is a generator of the prime ideal in O_{F_v} and if ℓ_v is defined by $|t_0 - \bar{t}_0|_v = |\varpi_v|^{\ell_v}$, if μ_1, μ_2 form a basis of O_{E_v} over O_{F_v}, if

$$\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

   if

$$|ad - bc|_v = |\varpi_v|^{k_v}$$

   then the trivial representation of $G(O_{F_v})$ is contained in

$$\pi^{(-1)^{k_v+\ell_v}}(1).$$

   Thus the distinction between $\pi^1(\theta_v)$ and $\pi^{-1}(\theta_v)$ depends on the choice of λ_1, λ_2 and t_0.

Let $H(\mathbf{A})$ be the idèles of E which have norm 1 over F and let $H(\mathbf{Q})$ be the elements of E which have norm 1 in F. We have

Theorem: *The sum (5) is a sum over the total imaginary quadratic extensions of*

$$\frac{1}{2}\sum_{\substack{\{\theta,\theta^{-1}\}\\ \theta^2\neq 1}}\bigotimes_v\begin{bmatrix}\pi(\theta_v)\\ \pi^1(\theta_v)-\pi^{-1}(\theta_v)\end{bmatrix}+\frac{1}{4}\sum_{\substack{\{\theta,\theta^{-1}\}\\ \theta^2=1\\ \theta\neq 1}}\bigotimes_v\begin{bmatrix}\pi(\theta_v)\\ \pi^1(\theta_v)-\pi^{-1}(\theta_v)\end{bmatrix}$$

Explanation. Here the sum is over characters of the compact group $H(F)\backslash H(\mathbf{A})$. In the first part θ and θ^{-1} are grouped together. Moreover the tensor product is taken over all places, finite and infinite. At a finite split place we put $\pi(\theta_v)$ and at an infinite place or a finite place which does not split we put $\pi^1(\theta_v)-\pi^{-1}(\theta_v)$. Here θ_v is the restriction of θ to $H_v\subseteq H(\mathbf{A})$. Notice that if $v\in S_\infty$ then $H_v\simeq\psi_v(S^1)\simeq S^1$ so that $\pi^1(\theta_v)$ and $\pi^{-1}(\theta_v)$ are defined. The literal meaning of the tensor products is as follows. One interchanges the tensor product with the sums, expands the result into an infinite sum of tensor products, discards those for which an infinite number of minus signs occur, and keeps the rest, each provided with a well-determined sign. Since and given representation will only occur a finite number of times in the sum one can then collect terms.

Application. In order to evaluate the sum (1) we are only interested in those representations such that

$$\pi_f=\bigotimes_{v\text{ finite}}\pi_v$$

contains the trivial representation of $G(\widehat{O}_F)$, that is, each π_v contains the trivial representation of $G(O_F)$. By (ii, b) a given E yields a non-zero contribution only if it is unramified everywhere. In particular if F has a unit of norm -1, there is no such E and we have the trivial fact:

If F contains a unit of norm -1 the sum (1) is 0.

Let's calculate the contribution of a given unramified field E to

$$\sum_{\{\epsilon_v\}}\left(\prod_v\epsilon_v\right)m(\{\epsilon_v\},\{\theta_v^0\})\tag{1}$$

where n_v^0 are given and

$$\theta_v^0(e^{ix_v})=e^{in_v^0x_v}\qquad v\in S_\infty.$$

To avoid complications which are irrelevant for your purposes suppose $n_v^0\neq 0$ for at least one v. Let $\{n_v\}$ at first run over all collections such that $n_v=\pm n_v^0$. However if ω is a generator of the roots of unity and in E we then restrict to those $\{n_v\}$ such that

$$\prod_v\omega_v^{n_v}=1.\tag{6}$$

Here ω_v is the image of ω under $E\to E_v\to\mathbf{C}$. (Recall that we had earlier identified E_v with $\mathbf{C}$ in a certain way.) Let the number of such collections $\{n_v\}$ be $\delta_E(\{\theta_v^0\})$. Let the quotient of the ideal class group of E by the image of the ideal class group of F have order $\epsilon(E/F)$. If $\{n_v\}$ satisfies (6) the number of characters θ of $H(F)\backslash H(\mathbf{A})$ such that θ restricted to H_v is θ_v, with $\theta_v(e^{ix_v})=e^{in_vx_v}$, for $v\in S_\infty$ is $\epsilon(E/F)$.

Finally if k_v and ℓ_v are defined as before in terms of λ_1, λ_2, and t_0 then $k_v=\ell_v=0$ for almost all v which do not split in E and we may set $\mu(E/F)$ equal to the product over all finite places v which do not split in E of $(-1)^{k_v+\ell_v}$.

The sum (1) is equal to

$$\boxed{2^{[F:\mathbf{Q}]-2}\sum_E \mu(E/F)\epsilon(E/F)\delta_E(\{\theta_v^0\})}$$

The sum, which is finite, is taken over all unramified quadratic extensions E of F.

Example. Take F to be the real quadratic extension $\mathbf{Q}(\sqrt{p})$, p a prime. When can sometimes use information from the *Zahlbericht* to simplify the expression above. If $p \equiv 1 \pmod 4$ and the fundamental unit has norm $+1$ there is one totally imaginary quadratic extension. If $p \equiv 3 \pmod 4$ then the fundamental unit necessarily has norm $+1$ and there is again exactly one totally imaginary quadratic extension. It is in fact

$$\mathbf{Q}(\sqrt{-p}, \sqrt{p}) = \mathbf{Q}(\sqrt{-1}, \sqrt{p}).$$

In either case if an ideal class $\mathfrak{Q}$ of F becomes trivial in E then $\mathfrak{Q}\overline{\mathfrak{Q}} = \mathfrak{Q}^2$ is principal. Since F has no ambiguous classes $\mathfrak{Q}$ itself must be trivial so

$$\epsilon(E/F) = \frac{\epsilon(E)}{\epsilon(F)}$$

if $\epsilon(E)$ and $\epsilon(F)$ are the class members of E and F respectively.

Suppose $p \equiv 3 \pmod 4$. Let $(2) = \mathfrak{a}^2$ where $\mathfrak{a}$ is an ideal of F. Since F has no non-trivial ambiguous classes $\mathfrak{a}$ is principal. From §10 of Hilbert's *Über den Dirichletschen biquadratischen Zahlkörper* we conclude that

$$\frac{\epsilon(E)}{\epsilon(F)} = \epsilon(\mathbf{Q}(\sqrt{p})).$$

If $p \neq 3$ the generator of the roots of unity in E is $\sqrt{-1}$. Thus if $n_v^0 = 1$ for both infinite v, the case in which you are principally interested,

$$\delta_E(\{\theta_v^0\}) = 2.$$

The choice of t_0 depends on the choice of the ψ_v, that is on the choice of complex structure. For simplicity suppose that the ψ_v have been so chosen that t_0 may be taken as $\sqrt{-1}$. Then all ℓ_v are either 0 or 2, in fact ℓ_v is 0, if v does not divide 2 and 2 if v is the unique place dividing 2. We may take λ_1, λ_2 to be 1, $\sqrt{-1}$. Hilbert, in the paper to which I just referred, gives an integral basis for E from which it follows that k_v is 0 if v does not divide 2 but is 2 if v divides 2. Thus the sum (1) is equal to

$$\boxed{2\epsilon(\mathbf{Q}(\sqrt{-p}))}$$

Evaluation in terms of L-functions. You are primarily interested in evaluating (1) in terms of the values of L-functions. For this one should, in order to disencumber oneself of the fixed points, work not with $G(O_F)$ but with a congruence subgroup $\Gamma(N)$, for N sufficiently large. The same two interpretations of (1) are still valid. Of course

$$G(O_F)\Big\backslash \prod_{v\in S_\infty} G(F_v)$$

is to be replaced by

$$\Gamma(N)\Big\backslash \prod_{v\in S_\infty} G(F_v) = G(F)\backslash G(\mathbf{A})/K(N)$$

where

$$K(N) = \{g \in G(\widehat{O}_F) \mid g \equiv 1 \pmod{N}\}.$$

We assume that no element of $\Gamma(N)$ has a fixed point in

$$\prod_{v \in S_\infty} G(F_v)/K_\infty.$$

In particular $2 \not\equiv 0 \pmod{N}$.

We again want an expression for

$$\sum_{\{\epsilon_v\}} \left(\prod_v \epsilon_v \right) m(\{\epsilon_v\}, \{\theta_v^0\}). \tag{1}$$

Now for simplicity I assume all $n_v^0 \neq 0$. You would take all $n_v^0 = 1$, so this is, at the moment, not an important restriction.

Set

$$M(n_v^0) = \frac{\int_{-\infty}^{\infty} (1+ui)^{-n_v^0-1} \operatorname{sign} u \, du}{\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} du \left| \frac{t+1/t}{2} + \frac{u}{t} i \right|^{-2(n_v^0+1)}}$$

and let

$$\Theta(\{n_v^0\}) = \frac{1}{2} \left\{ \prod_{v \in S_\infty} 2i \operatorname{Im} M(n_v^0) \right\} \lim_{s \to 1} \frac{\zeta_{\mathbf{Q}}(s)}{\zeta_F(s)}. \tag{7}$$

$\zeta_{\mathbf{Q}}$ and ρ_F are the zeta-functions of $\mathbf{Q}$ and F respectively. The constant $\Theta(\{n_v^0\})$ will occur in the formula I shall give for (1). I have done as well as I could on extracting the explicit expression (7) for it from my notes. However when computing it one must keep careful track of factors which arise from different possible normalizations of Haar measures, so (7) is to be regarded as *completely unreliable.*

If v is a finite place of F we let f_v^N be the characteristic function of

$$K(N) \cap G(F_v) = \{g \in G(F_v) \mid g \equiv 1 \pmod{N}\}.$$

f_v^N is a function in $G(F_v)$. Let χ be a character of the idèles of F which is 1 on $F^\times$. If v is a place of F, let χ_v be the restriction of χ to $F_v^\times$. We suppose that $\chi^2 = 1$, $\chi \neq 1$, and that χ is the sign character for each $v \in S_\infty$. If v is a finite place

$$\int_{F_v^\times} f_v^N \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \chi_v(u) |u|_v^s \, d^\times u$$

converges for $\operatorname{Re} s > 0$. If v divides p and $(p, N) = 1$ this integral is 0 if χ_v is ramified but if χ_v is unramified it equals

$$\frac{1}{1 - \chi_v(\varpi_v)|\varphi_v|^s} = L(s, \chi_v).$$

Set

$$\lambda(\chi, N) = \lim_{s \to 1} \prod_{v \text{ finite}} \int_{F_v^\times} f_v^N \left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \chi_v(u) |u|^s \, d^\times u.$$

Then $\lambda(\chi, N)$ is 0 except for a finite number of χ, those which are unramified outside the places dividing N, and for such χ

$$\lambda(\chi, N) = L(1,\chi)\prod_{v|N}\left\{\frac{\int_{F_v^\times} f_v^N\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\right)\chi_v(u)|u|\,d^\times u}{L(1,\chi_v)}\right\}.$$

The product on the right is elementary.

Each χ determines, by class field theory, a totally imaginary quadratic extension E of F. Choose t_0, λ_1, and λ_2 as before. Let the matrix of t_0 with respect to λ_1, λ_2 be

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}.$$

Then $\chi_v(b) = 1$ for almost all v. Set

$$\nu(E/F) = \prod_{v \text{ finite}} \chi_v(b) = \prod_{v\in S_\infty} \operatorname{sign} b.$$

The alternating sum (1) is equal to

$$\boxed{\Theta(\{n_v^0\})\sum_\chi \nu(E/F)\lambda(\chi, N)}.$$

The sum is taken over the χ described above.

Yours truly,
R. Langlands

RL:MMM

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