A LETTER TO ANDRÉ WEIL, PART 2

Contents

1. Introduction

This is the rest of the letter I promised. After making the necessary apologies for its length, the style in which it is written, and the delay in sending it let me tell you what is in it and what is not in it. There are also one or two matters about which you should be concerned.

Of course the goal is to extend the theorem of your paper to all number fields and to function fields. If I have made no mistakes such an extension is obtained in paragraph 7. (Although I am not really at home with function fields I do not think I made any blunders.) Moreover as I said I do have to assume the existence of an Euler product.

If you want to see quickly what the basic idea of the proof is you should probably concentrate on function fields. For these only paragraphs 6 and 7 are necessary. Indeed in this letter the only difference between a number field and a function field is that a function field has no archimedean primes. The reason that so much space is devoted to archimedean fields is that, at the moment, I know more about the representations of $\mathrm{GL}(2, K)$ for such fields. As soon as I understand the representation theory of $\mathrm{GL}(2, K)$ for non-archimedean fields I should be able to avoid the assumption, which appears in both the letter and your paper, about the character χ. Of course ignorance of the representation theory of $\mathrm{GL}(2, K)$ for a non-archimedean field is not fatal. The same ignorance for an archimedean field would be.

Perhaps it will help when you read paragraph 7 if I give some idea of the relation between the notation of the letter and your paper. Associate to the function Γ of your paper the function

$$F_0(g) = \frac{(ad-bc)^{k/2}}{(ci+d)^k} F\left(\frac{ai+b}{ci+d}\right) \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_+(2, \mathbf{R}).$$

Date: 1967.

If $K = \mathbf{Q}$, as we now assume, the divisor D of the letter is just the number A of your paper. Let ϵ' be the ϵ of your paper and let δ be a character modulo A. If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ lies in $U^D_{K_\mathfrak{p}}$ set

$$\epsilon_\mathfrak{p}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \epsilon'(a)\delta(ad)$$

if $\mathfrak{p}|D$ and set

$$\epsilon_\mathfrak{p}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$$

if $\mathfrak{p} \nmid D$. Then, for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in U^D

$$\epsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \prod_{\mathfrak{p}\neq\mathfrak{p}_\infty} \epsilon_\mathfrak{p}\begin{pmatrix} a_\mathfrak{p} & b_\mathfrak{p} \\ c_\mathfrak{p} & d_\mathfrak{p} \end{pmatrix}$$

is the ϵ of the letter. The relation $F|\gamma = \epsilon(\gamma)^{-1}F$ for $\gamma = \left(\begin{smallmatrix} r & s \\ t & u \end{smallmatrix}\right)$ in $F_0(A)$ is equivalent to

$$F_0(\gamma g) = \left\{\prod_{\mathfrak{p}|D} \epsilon_\mathfrak{p}(\gamma_\mathfrak{p})\right\} F_0(g)$$

for γ in $G_K \cap G_{K_{\mathfrak{p}\infty}} \times U^D$, $\det\gamma > 0$. Define a function φ on $G^D_\mathbf{A}$ by

$$\varphi(\gamma g) = \varphi(g) = F_0(g_\infty)\prod_{\mathfrak{p}|D} \epsilon_\mathfrak{p}(g_\mathfrak{p})$$

if γ belongs to $G_K \cap G^D_\mathbf{A}$, g belongs to $G_{K_{\mathfrak{p}_\infty}} \times U^D$, and $\det(g_\infty) > 0$. φ is well-defined and is the φ of my letter. If we want to indicate its independence on δ we should write $\varphi = \varphi_\delta$.

Now let me show that the assumption

$$F|\omega(A) = C^{-1}i^{-k}F$$

implies that $\widehat{\varphi} = \frac{i^k}{C}\left\{\prod_{\mathfrak{p}|D} \epsilon_\mathfrak{p}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\right\}\varphi_{\epsilon'\delta}$. Since $(\epsilon')^2 = 1$

$$\widetilde{\epsilon}_\mathfrak{p}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \epsilon'(a)\delta(ad) = \epsilon'(a)(\epsilon'\delta)(ad)$$

if $\mathfrak{p}|D$. If g belongs to $G_{K_\mathfrak{p}} \times U^D$ and $\det g_\infty > 0$

$$\begin{aligned}
\widehat{\varphi}(g) &= \varphi\left(\begin{pmatrix} 0 & A^{-1} \\ 1 & 0 \end{pmatrix} g \prod_{\mathfrak{p}|D} \begin{pmatrix} 0 & 1 \\ A_\mathfrak{p} & 0 \end{pmatrix}\right) \\
&= \varphi\left(\begin{pmatrix} 0 & A^{-1} \\ -1 & 0 \end{pmatrix} g \prod_{\mathfrak{p}|D} \begin{pmatrix} 0 & -1 \\ A_\mathfrak{p} & 0 \end{pmatrix}\right) \prod_{\mathfrak{p}|D} \epsilon_\mathfrak{p}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \\
&= F_0\left(\begin{pmatrix} A_\infty^{-1} & 0 \\ 0 & -A_\infty^{-1} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ A_\infty & 0 \end{pmatrix} g_\infty\right)\left\{\prod_{\mathfrak{p}|D} \widetilde{\epsilon}_\mathfrak{g}(g_\mathfrak{p})\right\}\left\{\prod_{\mathfrak{p}|D} \epsilon_\mathfrak{p}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \\
&= \frac{i^k}{C}\left\{\prod_{\mathfrak{p}|D} \epsilon_\mathfrak{p}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} F_0(g_\infty)\left\{\prod_{\mathfrak{p}|D} \widetilde{\epsilon}_\mathfrak{p}(g_\mathfrak{p})\right\} \\
&= \frac{i^k}{C}\left\{\prod_{\mathfrak{p}|D} \epsilon_\mathfrak{p}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right\} \varphi_{\epsilon' d}(g).
\end{aligned}$$

If

$$\omega\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\right)\left|\frac{\alpha_1}{\alpha_2}\right|^{\frac{k-1}{2}} (\operatorname{sgn} \alpha_2)^k \qquad \alpha_1, \alpha_2 \in \mathbf{R}^k$$

the representation $\pi_{\mathfrak{p}_\infty}$ is the infinite-dimensional quasi-simple irreducible representation deducible from π_ω. $\varphi_{\mathfrak{p}_\infty}$ will have to lie in $L(\xi_{\mathfrak{p}_\infty}, \pi_{\mathfrak{p}_\infty})_k$. ξ will of course be the character

$$\xi(x) = e^{2\pi i x_\infty} \prod_\mathfrak{p} e^{-2\pi i x_\mathfrak{p}}.$$

Let χ' be one of the χ of your paper. χ' determines a homomorphism of $\prod_{\mathfrak{p}|m} O_\mathfrak{p}^\times$ into $\mathbf{C}^\times$. m is of course the conductor of χ. Let χ' also denote the character of $K^\times \backslash I$ which satisfies

$$\chi'\left(\prod_\mathfrak{p} \beta_\mathfrak{p}\right) = \chi'\left(\prod_{\mathfrak{p}|m} \beta_\mathfrak{p}\right)$$

if $\beta_{\mathfrak{p}_\infty} > 0$ and $\beta_\mathfrak{p} \in O_\mathfrak{p}^\times$ for $\mathfrak{p} \neq \mathfrak{p}_\infty$. Then $\chi = (\epsilon' \delta \chi')^{-1}$ is one of the characters of the letter.

If g is

$$I \times \prod_{\mathfrak{p} \neq \mathfrak{p}_\infty} \begin{pmatrix} 1 & \frac{1}{m_\mathfrak{p}} \\ 0 & 1 \end{pmatrix}$$

the value of the integral of Lemma 7.3 is, in your notation,[1]

$$\frac{1}{\varphi(m)} \sum_{a \bmod m} \overline{\chi}'(a) \int_0^\infty \sum_n c_n e^{2\pi i n\left(t - \frac{a}{m}\right)} t^{s+\frac{k}{2}} \frac{dt}{t}.$$

[1]The second formula from the bottom on p. 150 of your paper does not look correct.

This equals

$$\frac{\overline{g(\chi')}}{\varphi(m)}\Gamma\left(s+\frac{k}{2}\right)\sum_{n=1}^{\infty}\frac{\chi(n)c_n}{(2\pi n)^{s+\frac{k}{2}}}=\frac{\overline{g(\chi')}}{\varphi(m)m^{s+\frac{k}{2}}}\Lambda_{\chi'}\left(s+\frac{k}{2}\right).$$

On the other hand it is equal to the product of $\Xi(s,\chi)$ and the expression (D) on page 7.20[2]. If $\varphi_{\mathfrak{p}_\infty}$ is suitably normalized then, for the g chosen, this expression equals

$$\frac{1}{(2\pi)^{s+\frac{k}{2}}}\prod_{\mathfrak{p}\in R}\int_{O_{\mathfrak{p}}^{\times}} e^{-2\pi i\frac{\alpha}{m}}\zeta_{\mathfrak{p}}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)d\alpha.$$

This is equal to

$$\frac{1}{(2\pi)^{s+\frac{k}{2}}}\frac{\overline{g(\chi')}}{\varphi(m)}.$$

Thus

$$\Xi(s,\chi)=\left(\frac{2\pi}{m}\right)^{s+\frac{k}{2}}\Lambda\chi'\left(s+\frac{k}{2}\right).$$

Moreover

$$\widehat{\Xi}\big(s,(\chi\eta)^{-1}\big)=\frac{i^k}{c}\left(\prod_{\mathfrak{p}|D}\epsilon_{\mathfrak{p}}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)\left(\frac{2\pi}{m}\right)^{s+\frac{k}{2}}\Lambda_{\overline{\chi}}\left(s+\frac{k}{2}\right).$$

The letter and the paper will be consistent if

$$\left(\frac{2\pi}{m}\right)^{s+\frac{k}{2}}A^{-s}C\epsilon'(m)\frac{g(\chi')}{g(\overline{\chi}')}\chi^{-1}(-A)$$

is equal to

$$\frac{i^k}{C}\left\{\prod_{\mathfrak{p}|D}\epsilon_{\mathfrak{p}}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)\right\}\left(\frac{2\pi}{m}\right)^{-s+\frac{k}{2}}\left\{\prod_{\mathfrak{p}|D}\zeta'_{\mathfrak{p}}\left(\begin{pmatrix}-A_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}\right)\right\}$$
$$\times\,\epsilon(\zeta_{\mathfrak{p}_\infty}\xi_{\mathfrak{p}_\infty},\pi_{\mathfrak{p}_\infty})\left\{\prod_{\substack{\mathfrak{p}\nmid D\\ \mathfrak{p}\neq\mathfrak{p}_\infty}}\epsilon(\zeta_{\mathfrak{p}},\xi_{\mathfrak{p}},\omega_{\mathfrak{p}})\right\}.$$

[2]Added—this is pagination of the original letter.

This is a consequence of the following relations.

$$\prod_{\mathfrak{p}|D} \epsilon_{\mathfrak{p}}\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right) = \delta(-1)$$

$$\prod_{\mathfrak{p}|D} \zeta_{\mathfrak{p}}\left(\begin{pmatrix}-A_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}\right) = \prod_{\mathfrak{p}\nmid D} \zeta_{\mathfrak{p}}^{-1}\left(\begin{pmatrix}-A_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}\right) = \chi'(A)\delta(-1)(-1)^k A^{-s}$$

$$\epsilon(\zeta_{\mathfrak{p}_\infty}\xi_{\mathfrak{p}_\infty}, \pi_{\mathfrak{p}_\infty}) = i^k(2\pi)^{2s}$$

$$\epsilon(\zeta_{\mathfrak{p}}, \xi_{\mathfrak{p}}, \pi_{\mathfrak{p}}) = 1 \text{ if } \mathfrak{p}\nmid D \text{ and } \mathfrak{p}\nmid m$$

$$\prod_{\mathfrak{p}|m} \epsilon(\zeta_{\mathfrak{p}}, \xi_{\mathfrak{p}}, \omega_{\mathfrak{p}}) = \frac{g(\chi')}{g(\overline{\chi}')}\prod_{\mathfrak{p}|m} \zeta_{\mathfrak{p}}\left(\begin{pmatrix}m_{\mathfrak{p}} & 0\\ 0 & -m_{\mathfrak{p}}^{-1}\end{pmatrix}\right)$$

$$\begin{aligned}\prod_{\mathfrak{p}|m} \zeta_{\mathfrak{p}}\left(\begin{pmatrix}m_{\mathfrak{p}} & 0\\ 0 & -m_{\mathfrak{p}}^{-1}\end{pmatrix}\right) &= \prod_{\mathfrak{p}|m}(\epsilon'\delta^2)^{-1}(-m_{\mathfrak{p}}^{-1})\prod_{\mathfrak{p}|m}(\epsilon'\delta\chi')^{-1}(-m_{\mathfrak{p}}^2)\prod_{\mathfrak{p}|m}|m_{\mathfrak{p}}|^{2s}\\ &= \left\{\prod_{\mathfrak{p}|m}(\epsilon')^{-1}(m_{\mathfrak{p}})\right\}\left\{\prod_{\mathfrak{p}|m}\chi'(-m_{\mathfrak{p}}^{-2})\right\}m^{-2s}\\ &= \left\{\prod_{\mathfrak{p}\nmid m}\epsilon'(m_{\mathfrak{p}})\right\}\left\{\prod_{\mathfrak{p}\nmid m}\chi'(-m_{\mathfrak{p}}^{2})\right\}m^{-2s}\\ &= \epsilon'(m)\chi'(-1)m^{-2s}.\end{aligned}$$

Of course all these formulae will be meaningless to you until you have read the letter.

For lemmas 2.4 and 4.3 I have referred to a paper of Harish-Chandra. These lemmas are not stated explicitly in that paper. It has been a long time since I looked at that paper and I should read it again to see that the lemmas are really implicit in it. I will do so as soon as possible. The appendix to paragraph 7 is not relevant to the rest of the paper. You should not read it. I include it only because the footnotes contain corrections to paragraph 5.

With so many formulae there are bound to be some small errors. They should show up as soon as one starts to apply the theorem.

2. Representations of $\mathrm{GL}(2,\mathbf{R})$

In this paragraph the next $G_{\mathbf{R}}$ will be $\mathrm{GL}(2,\mathbf{R})$ and $G^0_{\mathbf{R}}$ will be the group of matrices in $G_{\mathbf{R}}$ with determinant ± 1. U will be $O(2,\mathbf{R})$ and U^0 will be $\mathrm{SO}(2,\mathbf{R})$. $\mathfrak{g}$ will be the Lie algebra of $G_{\mathbf{R}}$ and $\mathfrak{g}_{\mathbf{C}}$ its complexification. $\mathfrak{g}^0$ will be the Lie algebra of $G^0_{\mathbf{R}}$ and $\mathfrak{g}^0_{\mathbf{C}}$ its complexification. $\mathfrak{A}$ and $\mathfrak{A}^0$ will be the universal enveloping algebras of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{g}^0_{\mathbf{C}}$ respectively. Since neither $G_{\mathbf{R}}$ nor $G^0_{\mathbf{R}}$ is connected it is not sufficient for us to study representations of $\mathfrak{A}$ or $\mathfrak{A}^0$. Let

$$\sigma = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

A representation π of $\{\sigma, \mathfrak{A}\}$ on a vector space W assigns to each X in $\mathfrak{A}$ a linear transformation $\pi(X)$ of W. It also assigns to σ a linear transformation $\pi(\sigma)$. We demand not only that $X \to \pi(X)$ be a representation of $\mathfrak{A}$ but also that $\big(\pi(\sigma)\big)^2 = I$, and $\pi(\sigma)\pi(X)\pi(\sigma^{-1}) = \pi\big(\operatorname{ad}\sigma(X)\big)$ for all X in $\mathfrak{A}$. A representation of $\{\sigma, \mathfrak{A}^0\}$ is defined in a similar manner. If π is a representation of $\{\sigma, \mathfrak{A}\}$, π^0 will denote its restriction to $\{\sigma, \mathfrak{A}^0\}$

Two bases of $\mathfrak{g}^0_{\mathbf{C}}$ are

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad V = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \qquad W = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

U is contained in the Lie algebra of the one-dimensional group U. If π is a representation of $\{\sigma, \mathfrak{A}\}$ on W let $W_n = \big\{\, w \in W \bigm| \pi(U)w = inw \,\big\}$. We shall always assume that $W_n = \{0\}$ if n is not an integer. The representation π will be called quasi-simple[3] if $W = \sum_n W_n$ and $\pi(Z)$ is a scalar for all Z in the centre of $\mathfrak{A}$. If π_1 and π_2 are two representations of $\{\sigma, \mathfrak{A}\}$ on W_1 and W_2 respectively π_2 will be said to be deducible from π_1 if there are two invariant subspaces $W_3 \supseteq W_4$ of W_1 and π_2 is equivalent to the representation of $\{\sigma, \mathfrak{A}\}$ on W_3/W_4. Similar notions can be introduced for representations of $\{\sigma, \mathfrak{A}^0\}$.

If Z lies in the centre of $\mathfrak{A}$ then $\operatorname{ad}\sigma(Z) = Z$. The centre of $\mathfrak{A}^0$ is generated by

$$D = XY + YX + \frac{1}{2}Z^2 = 2YX + Z + \frac{1}{2}Z^2 = 2XY - Z + \frac{1}{2}Z^2.$$

The centre of $\mathfrak{A}$ is generated by D and $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

If G is any Lie group and X lies in its Lie algebra $\rho(X)$ is the left-invariant vector field defined by $\rho(X)\varphi(g) = \frac{d}{dt}\varphi(g \exp tX)\Big|_{t=0}$ and $\lambda(X)$ is the right-invariant vector field defined by $\lambda(X)\varphi(g) = \frac{d}{dt}\varphi\big(\exp(-tX)g\big)\Big|_{t=0}$. The maps $X \to \rho(X)$ and $X \to \lambda(X)$ extend to representations of the complex universal enveloping algebra.

Let ω be a continuous homomorphism of $A_{\mathbf{R}}$, the group of diagonal matrices in $G_{\mathbf{R}}$, into $\mathbf{C}^\times$. Let ω_1 and ω_2 be the homomorphisms of $\mathbf{R}^\times$ into $\mathbf{C}^\times$ defined by

$$\omega_1(t) = \omega\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right)$$

and

$$\omega_2(t) = \left(\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}\right).$$

Let $\omega_i(t) = |t|^{s_i}\left(\frac{t}{|t|}\right)^{m_i}$, $m_i = 0$ or 1, and set $s = s_1 - s_2$, $m = m_1 - m_2$. If $N_{\mathbf{R}}$ is the group of all matrices of the form

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

[3]I use the expression in a slightly different sense than Harish-Chandra.

let $L(\omega)$ be the space of all infinitely differentiable U-finite functions on $N_{\mathbf{R}}\backslash G_{\mathbf{R}}$ satisfying $\varphi(ag) \equiv \omega(a)\left|\frac{\alpha_1}{\alpha_2}\right|^{1/2}\varphi(g)$ for all

$$a = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}$$

in $A_{\mathbf{R}}$. If φ belongs to $L(\omega)$ and X belongs to $\mathfrak{A}$ then $\rho(X)\varphi$ also belongs to $L(\omega)$. Of course $\rho(\sigma)\varphi$ which is defined by $\big(\rho(\sigma)\varphi\big)(g) = \varphi(g\sigma)$ also belongs to $L(\omega)$ and we obtain a representation π_ω of $\{\sigma, \mathfrak{A}\}$ on $L(\omega)$.

Because of the Iwasawa decomposition $G_{\mathbf{R}} = N_{\mathbf{R}}A_{\mathbf{R}}U^0$, the functions in $L(\omega)$ are determined by their restrictions to U^0. The functions φ_n with $\frac{n-m}{2} \in \mathbf{Z}$, which are defined by

$$\varphi_n(g) = \omega(a)\left|\frac{\alpha_1}{\alpha_2}\right|^{1/2} e^{in\theta}$$

if $g = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) a \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix}\right)$ and $a = \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{smallmatrix}\right)$, form a basis of $L(\omega)$.

Lemma 2.1.

(i) $\pi_\omega(\sigma)\varphi_n = (-1)^{m_2}\varphi_{-n}$
(ii) $\pi_\omega(U)\varphi_n = in\varphi_n$
(iii) $\pi_\omega(V)\varphi_n = (s+1+n)\varphi_{n+2}$
(iv) $\pi_\omega(W)\varphi_n = (s+1-n)\varphi_{n-2}$
(v) $\pi_\omega(D) = \frac{s^2-1}{2}I$
(vi) $\pi_\omega(J) = (s_1+s_2)I$

The relations (i), (ii), and (vi) are clear. To prove (iv) we observe that $\rho(D) = \lambda(D)$ and that if $\varphi \in L(\omega)$,

$$\lambda(D)\varphi = \lambda(Z)\varphi + \frac{1}{2}\lambda(Z^2)\varphi = \left[-(s+1) + \frac{1}{2}(s+1)^2\right]\varphi = \frac{s^2-1}{2}\varphi.$$

Since $[U,V] = 2iV$ and $[U,W] = 2iW$, $\pi_\omega(V)\varphi_n$ is a multiple of φ_{n+2} and $\pi_\omega(W)\varphi_n$ is a multiple of φ_{n-2}. It is easily seen that $\big(\pi_\omega(V)\varphi_n\big)(1) = (s+1+n)$ and $\big(\pi_\omega(W)\varphi_n\big)(1) = (s+1-n)$. The relations (ii) and (iii) follow.

Corollary.

(i) *If $s-m$ is not an even integer the restriction of π_ω to $\mathfrak{A}^0$ is irreducible.*
(ii) *If $s-m$ is an odd integer and $s \geqslant 0$ the only subspaces of $L(\omega)$ invariant under $\mathfrak{A}^0$ are*

$$M_1(\omega) = \sum_{\substack{n \geqslant s+1 \\ \frac{n-m}{2} \in \mathbf{Z}}} \mathbf{C}\varphi_n, \quad M_2(\omega) = \sum_{\substack{n \leqslant -(s+1) \\ \frac{n-m}{2} \in \mathbf{Z}}} \mathbf{C}\varphi_n,$$

and $M(\omega) = M_1(\omega) + M_2(\omega)$. The spaces $M_1(\omega)$, $M_2(\omega)$, and $L(\omega)/M(\omega)$ are irreducible under $\mathfrak{A}^0$. The only subspace invariant under $\{\sigma, \mathfrak{A}^0\}$ is $M(\omega)$. The representations of $\{\sigma, \mathfrak{A}^0\}$ on $M(\omega)$ and $L(\omega)/M(\omega)$ are irreducible.
(iii) *If $s-m$ is an odd integer and $s < 0$ the only subspaces of $L(\omega)$ invariant under $\mathfrak{A}^0$ are*

$$M_1(\omega) = \sum_{\substack{n \geqslant s+1 \\ \frac{n-m}{2} \in \mathbf{Z}}} \mathbf{C}\varphi_n, \quad M_2(\omega) = \sum_{\substack{n \leqslant -(s+1) \\ \frac{n-m}{2} \in \mathbf{Z}}} \mathbf{C}\varphi_n,$$

and $M(\omega) = M_1(\omega) \cap M_2(\omega)$. The only subspace invariant under $\{\sigma, \mathfrak{A}^0\}$ is $M(\omega)$ and the representations of $\mathfrak{A}^0$ on $M(\omega)$ and $L(\omega)/M(\omega)$ are irreducible.

This follows immediately from the lemma and the observation that an invariant subspace of $L(\omega)$ is spanned by the φ_n it contains.

If π is a quasi-simple representation of $\{\sigma, \mathfrak{A}^0\}$ on H then $\pi(V)H_n \subseteq H_{n+2}$ and $\pi(W)H_n \subseteq H_{n-2}$. Consequently $H^0 = \sum_{n \text{ even}} H_n$ and $H^1 = \sum_{n \text{ odd}} H_n$ are invariant subspaces of H. We shall say that π is of type 0 if $H^1 = \{0\}$ and that π as of type 1 if $H^0 = \{0\}$.

Lemma 2.2. *Suppose π is a quasi-simple irreducible representation of $\{\sigma, \mathfrak{A}^0\}$ on H which is of type m. Suppose moreover that $\pi(D) = \frac{s^2-1}{2} I$ and $s - m$ is not an odd integer. If $n \geqslant 0$ let A_n be the restriction of*

$$\frac{\pi(\sigma)\pi(W)^n}{\prod_{k=0}^{m-1} \big(s + 2k - (n-1)\big)}$$

to H_n. If $n \leqslant 0$ let A_n be the restriction of

$$\frac{\pi(\sigma)\pi(V)^{|n|}}{\prod_{k=0}^{|n|-1} \Big(s + 2k - \big(|n|-1\big)\Big)}$$

to H_n. Then $A_n^2 = I$ for all n. Let $A(\pi)$ be the operator H whose restriction to H_n is A_n. $A(\pi)$ commutes with $\pi(\sigma)$ and with $\pi(X)$ if X is in $\mathfrak{A}^0$.

Using the relations $Z = \frac{V+W}{2}$, $2X = U - i\frac{(V-W)}{2}$, $2Y = -U - i\frac{(V-W)}{2}$ one shows easily that

$$D = \frac{VW}{4} + \frac{WV}{4} - \frac{U^2}{2} = \frac{VW}{2} + iU - \frac{U^2}{2} = \frac{WV}{2} - iU - \frac{U^2}{2}.$$

Thus, if φ lies in H_n,

$$\pi(V)\pi(W)\varphi = \pi(2D - 2iU + U^2)\varphi = (s^2 - 1 + 2n - n^2)\varphi = \big[s^2 - (n-1)^2\big]\varphi$$
$$\pi(W)\pi(V)\varphi = \pi(2D + 2iU + U^2)\varphi = (s^2 - 1 - 2n - n^2)\varphi = \big[s^2 - (n+1)^2\big]\varphi.$$

In particular if $0 \leqslant j < |n|$ and $\varphi \in W_n$

$$\pi(V)^{j+1}\pi(W)^{j+1}\varphi = \big[s^2 - (n-2j-1)^2\big]\pi(V)^j\pi(W)^j\varphi \qquad \text{if } n \geqslant 0$$
$$\pi(W)^{j+1}\pi(V)^{j+1}\varphi = \Big[s^2 - \big(|n|-2j-1\big)^2\Big]\pi(W)^j\pi(V)^j\varphi \qquad \text{if } n \leqslant 0.$$

Since $\pi(\sigma)\pi(W)\pi(\sigma) = \pi(V)$ and $\pi(\sigma)\pi(V)\pi(\sigma) = \pi(W)$ it follows that

$$A_n^2\varphi = \frac{\prod_{j=0}^{|n|-1}\Big[s^2 - \big(|n|-2j-1\big)^2\Big]}{\Big\{\prod_{k=0}^{|n|-1}\Big(s + 2k - \big(|n|-1\big)\Big)\Big\}^2}\varphi = \varphi$$

It is easy to see that $A(\pi)$ commutes with $\pi(\sigma)$ and $\pi(U)$. Thus to prove the last assertion of the lemma we need only show that it commutes with $\pi(V)$ and $\pi(W)$ or that $A_{n+2}\pi(V) = \pi(V)A_n$ and $A_{n-2}\pi(W) = \pi(W)A_n$. We must study various cases separately.

Suppose that $n \geqslant 0$ and φ belongs to H_n.

$$\begin{aligned} A_{n+2}\pi(V)\varphi &= \frac{1}{\prod_{k=0}^{n+1}\big(s+2k-(n+1)\big)}\pi(\sigma)\pi(W)^{n+2}\pi(V)\varphi \\ &= \frac{\pi(V)}{\prod_{k=0}^{n+1}\big(s+2k-(n+1)\big)}\cdot\big(s^2-(n+1)^2\big)\pi(\sigma)\pi(W)^n\varphi \\ &= \pi(V)A_n\varphi. \end{aligned}$$

If $n \geqslant 2$

$$\begin{aligned} \pi(W)A_n\varphi &= \frac{1}{\prod_{k=0}^{n-1}\big(s+2k-(n-1)\big)}\pi(W)\pi(\sigma)\pi(W)^n\varphi \\ &= \frac{\pi(\sigma)}{\prod_{k=0}^{n-1}\big(s+2k-(n-1)\big)}\big(s^2-(-n+1)^2\big)\pi(W)^{n-1}\varphi \\ &= A_{n-2}\pi(W)\varphi. \end{aligned}$$

If $n = 1$

$$\pi(W)A_n\varphi = \frac{1}{s}\pi(W)\pi(\sigma)\pi(W)\varphi = \frac{1}{s}\pi(\sigma)\pi(V)\pi(W)\varphi = A_{n-2}\pi(W)\varphi.$$

If $n = 0$

$$\begin{aligned} A_{n-2}\pi(W)\varphi &= \frac{1}{s^2-1}\pi(\sigma)\pi(V)^2\pi(W)\varphi \\ &= \frac{\pi(W)}{s^2-1}\pi(\sigma)\pi(V)\pi(W)\varphi = \pi(W)\pi(\sigma)\varphi = \pi(W)A_n\varphi. \end{aligned}$$

There is no need to discuss the case $n \leqslant 0$ because $\pi(\sigma)A_{n'}\pi(\sigma) = A_{-n}$, $\pi(\sigma)\pi(W)\pi(\sigma) = \pi(V)$, and $\pi(\sigma)\pi(V)\pi(\sigma) = \pi(W)$.

Lemma 2.3. *A quasi-simple representation π of $\{\sigma, \mathfrak{A}\}$ is irreducible if and only if π^0 is irreducible. If π is an irreducible quasi-simple representation of $\{\sigma, \mathfrak{A}\}$ on H there are two possibilities.*

(*i*) *The restriction $\overline{\pi}$ of π to $\mathfrak{A}$ is irreducible and the two representations $X \to \overline{\pi}(X)$ and $X \to \overline{\pi}\big(\operatorname{ad}\sigma(X)\big)$ are equivalent.*

(*ii*) *H is the direct sum of two subspaces H_1 and H_2 invariant under $\mathfrak{A}$. The representations $\overline{\pi}_1$ and $\overline{\pi}_2$ of $\mathfrak{A}$ on H_1 and H_2 are inequivalent but $\overline{\pi}_2$ is equivalent to $X \to \overline{\pi}_1\big(\operatorname{ad}\sigma(X)\big)$ and $\pi(\sigma)H_1 = H_2$.*

The first assertion is a matter of definition. Suppose π is irreducible. Either H is irreducible under $\mathfrak{A}$, when the first possibility occurs, or it is not. Suppose it is not. Let H_1 be a proper subspace of H invariant under $\mathfrak{A}$ and let $H_2 = \pi(\sigma)H_1$. Since $H_1 + H_2$ and $H_1 \cap H_2$ are invariant under $\{\sigma, \mathfrak{A}\}$, $H_1 \cap H_2 = \{0\}$ and $H = H_1 \oplus H_2$. If H_1' were a proper subspace of H_1 invariant under $\mathfrak{A}$ then $H_1' \oplus H_2'$, with $H_2' = \pi(\sigma)H_1'$, would be a proper invariant subspace of H. $\overline{\pi}_2$ is certainly equivalent to $X \to \overline{\pi}_1\big(\operatorname{ad}\sigma(X)\big)$. To complete the proof of the lemma we have merely to show that $\overline{\pi}_1$ and $X \to \overline{\pi}_1\big(\operatorname{ad}\sigma(X)\big)$ are not equivalent. To do this we use the following lemma which is a special case of a theorem of Harish-Chandra (*Representations of semi-simple Lie groups,* II, T.A.M.S. v. 16, 1954).

Lemma 2.4. *Let $\overline{\sigma}$ be an irreducible quasi-simple representation of $\mathfrak{A}$ on W. There is at least one continuous homomorphism ω of $A_{\mathbf{R}}$ into $\mathbf{C}^{\times}$ such that $\overline{\sigma}$ is of type $|m|$ and $\overline{\sigma}(D) = \frac{s^2-1}{2}I$ and $\overline{\sigma}(J) = (s_1+s_2)I$. Moreover if ω is any such homomorphism, $\overline{\sigma}$ is deducible from $\overline{\pi}_\omega$, the restriction of π_ω to $\mathfrak{A}$.*

As usual $\omega_1(t) = \omega\left(\left(\begin{smallmatrix} t & 0 \\ 0 & 1\end{smallmatrix}\right)\right)$, $\omega_2(t) = \omega\left(\left(\begin{smallmatrix} 1 & 0 \\ 0 & t\end{smallmatrix}\right)\right)$, $\omega_i(t) = |t|^{s_i}\left(\frac{t}{|t|}\right)^{m_i}$, $s = s_1 - s_2$, and $m = m_1 - m_2$. Although the adjectives of the lemma have only been defined for representations of $\{\sigma, \mathfrak{A}\}$ their meaning for representations of $\mathfrak{A}$ is clear. The lemma implies that W_n is of dimension at most 1. Consequently any linear transformation leaving W_n invariant has an eigenvector and any linear transformation commuting with $\overline{\sigma}(X)$ for all X in $\mathfrak{A}$ is a scalar.

If $\overline{\pi}_1$ and $X \to \overline{\pi}_1\big(\operatorname{ad}\sigma(X)\big)$ were equivalent there would be an operator A such that $A^{-1}\overline{\pi}_1(X)A = \overline{\pi}_1\big(\operatorname{ad}\sigma(X)\big)$ for all X. Thus $A^2\overline{\pi}_1(X)A^{-2} = A\Big(\overline{\pi}_1\big(\operatorname{ad}\sigma(X)\big)\Big)A^{-1} = \overline{\pi}_1(X)$ and A^2 is a scalar. We may suppose that $A^2 = I$. If x lies in H_1 and X lies in $\mathfrak{A}$ then $\pi(X)\big(x \oplus \pi(\sigma)Ax\big) = y \oplus \pi(\sigma)\pi(A)y$ if $y = \pi(X)x$ and $\pi(\sigma)\big(x \oplus \pi(\sigma)Ax\big) = y \oplus \pi(\sigma)Ay$ if $y = Ax$ so that $\big\{x \oplus \pi(\sigma)Ax\big\}$ is a proper invariant subspace.

Lemma 2.5. *Suppose π is an irreducible quasi-simple representation of $\{\sigma, \mathfrak{A}\}$ on H. There is a continuous homomorphism ω of $A_{\mathbf{R}}$ into $\mathbf{C}^{\times}$ such that π is of type $|m|$, $\pi(D) = \frac{s^2-1}{2}I$, $\pi(J) = (s_1+s_2)I$ and, if $s-m$ is not an odd integer, $A(\pi) = (-1)^{m_2}I$. If ω is any such homomorphism and π is infinite-dimensional then π is deducible from π_ω.*

Choose s so that $\pi(D) = \frac{s^2-1}{2}I$ and define s_1 and s_2 by $s_1 - s_2 = s$ and $\pi(J) = (s_1+s_2)I$. Choose m_2 to be 0 or 1 and define m_1, which is 0 or 1, by the condition that π is of type $|m|$ if $m = m_1 - m_2$. If $s-m$ is not an odd integer $A(\pi)$ is defined and commutes with $\pi(\sigma)$ and all $\pi(X)$. By the previous two lemmas H_n is finite-dimensional. Consequently $A(\pi)$ is a scalar. Since $A^2(\pi) = I$, $A(\pi) = \pm I$. Choose m_2 so that $A(\pi) = (-1)^{m_2}I$. If $s-m$ is an odd integer m_2 may be chosen to be either 0 or 1. It follows from Lemma 2.1 that if $s-m$ is not an odd integer then $A(\pi_\omega) = (-1)^{m_2}I$.

Suppose first that $s-m$ is not an odd integer. Lemmas 2.3, 2.4 and the corollary to Lemma 2.1 imply that $\overline{\pi}$, the restriction of π to $\mathfrak{A}$, is irreducible and equivalent to $\overline{\pi}_\omega$. Let B be a map from H to $L(\omega)$ such that $B\pi(X) = \pi_\omega(X)B$ for all X. I claim that $B\pi(\sigma) = \pi_\omega(\sigma)B$. It is enough to verify that $B\pi(\sigma)x = \pi_\omega(\sigma)Bx$ for x in H_n. Clearly $BA(\pi) = A(\pi_\omega)B$. Since $A^2(\pi) = I$, $B\pi(\sigma)x = B\pi(\sigma)A^2x$. If $n \geqslant 0$

$$\begin{aligned} B\pi(\sigma)x &= \frac{B\pi(\sigma)A(\pi)\pi(\sigma)\pi(W)^n x}{\prod_{k=0}^{n-1}\big(s+2k-(n-1)\big)} \\ &= \frac{bA(\pi)\pi(W)^n x}{\prod_{k=0}^{n-1}\big(s+2k-(n-1)\big)} \\ &= \frac{A(\pi_\omega)\pi_\omega(W)^n Bx}{\prod_{k=0}^{n-1}\big(s+2k-(n-1)\big)} \\ &= \pi_\omega(\sigma)Bx \end{aligned}$$

and if $n \leqslant 0$

$$\begin{aligned}B\pi(\sigma)x &= \frac{B\pi(\sigma)A(\pi)\pi(\sigma)\pi(V)^{|n|}x}{\prod_{k=0}^{|n|-1}\Big(s+2k-\big(|n|-1\big)\Big)}\\ &= \frac{BA(\pi)\pi(V)^{|n|}x}{\prod_{k=0}^{|n|-1}\Big(s+2k-\big(|n|-1\big)\Big)}\\ &= \frac{A(\pi_\omega)\pi_\omega(V)^{|n|}Bx}{\prod_{k=0}^{|n|-1}\Big(s+2k-\big(|n|-1\big)\Big)}\\ &= \pi_\omega(\sigma)Bx.\end{aligned}$$

If $s-m$ is an odd integer and π is infinite-dimensional it follows from Lemmas 2.3, 2.4 and the corollary to Lemma 2.1 that $H = H_1 \oplus H_2$. Let $V' \supseteq V''$ be subspaces of $L(\omega)$ invariant under $\mathfrak{A}$ such that $\overline{\pi}_1$ is equivalent to the representation of $\mathfrak{A}$ on V'/V''. Let W' be the intersection of all subspaces of $L(\omega)$ which contain V' and are invariant under $\{\sigma, \mathfrak{A}\}$. Let W'' be the union of all subspaces of $L(\omega)$ which are contained in V'' and are invariant under $\{\sigma, \mathfrak{A}\}$. By the corollary to Lemma 2.1 the representation $\widetilde{\pi}_\omega$ of $\{\sigma, \mathfrak{A}\}$ on $W = W'/W''$ is irreducible. By Lemma 2.3, W is the direct sum of two subspaces W_1 and W_2 invariant under $\mathfrak{A}$. We may suppose that the representation of $\mathfrak{A}$ on W_1 is equivalent to $\overline{\pi}_1$. Let B_1 be a map of H_1 to W_1 such that $B_1\pi(X) = \pi_\omega(X)B_1$ for X in $\mathfrak{A}$. Let $B_2 = \widetilde{\pi}_\omega(\sigma)B_1\pi(\sigma)$ and set $B = B_1 \oplus B_2$. It is immediate that $B\pi(\sigma) = \widetilde{\pi}_\omega(\sigma)B$ and $B\pi(X) = \widetilde{\pi}_\omega(X)B$ for all X.

It is not difficult to see that every finite-dimensional representation of $\{\sigma, \mathfrak{A}\}$ is deducible from some π_ω. As a consequence $A(\pi)$ can be defined by the formulae of Lemma 2.6. If π is deducible from π_ω then $A(\pi) = (-1)^{m_2}I$.

Corollary. *Suppose $\lambda(D)$, $\lambda(J)$, and m, which is to be 0 or 1, are given numbers. Let $\lambda(D) = \frac{s^2-1}{2}$. If $s-m$ is not an odd integer there are two irreducible quasi-simple representations π of $\{\sigma, \mathfrak{A}\}$ of type m for which $\pi(D) = \lambda(D)I$ and $\pi(J) = \lambda(J) = \lambda(J)I$. For one $A(\pi) = I$ and for the other $A(\pi) = -I$. If $s-m$ is an odd integer there are three such representations. One is infinite-dimensional. The other two are finite-dimensional. For one of these $A(\pi) = I$ and for the other $A(\pi) = -I$.*

Since s is not unambiguously determined neither is $A(\pi)$. However once a representation π_ω from which π is deducible is specified s can be taken to be $s_1 - s_2$. Such a choice was implicit at various places in the preceding paragraph.

3. The local functional equation for $\mathrm{GL}(2, \mathbf{R})$

If π is an irreducible quasi-simple representation of $\{\sigma, \mathfrak{A}\}$ and π_1 is a representation of $\{\sigma, \mathfrak{A}\}$ on W we shall say that π is contained in π_1 if there is an invariant subspace V of W such that the restriction of π_1 to V is equivalent to π. We shall say that π is contained at most once in π_1 if there is at most one such subspace. If V' were another such subspace either $V \cap V' = \{0\}$ or $V = V'$; thus to show that π is contained at most once in π_1 one has merely to show that two such subspaces must have a non-zero element in common. Similar notions can be introduced for representations of $\{\sigma, \mathfrak{A}^0\}$.

If η is a continuous homomorphism of $A_{\mathbf{R}}$ into $\mathbf{C}^\times$ let $L(\eta)$ be the space of all infinitely differentiable U-finite functions on $G_{\mathbf{R}}$ satisfying $\varphi(ag) \equiv \eta(a)\varphi(g)$ for all a in $A_{\mathbf{R}}$. If φ lies in $L(\eta)$ so does $\rho(\sigma)\varphi$ and $\rho(X)\varphi$ for X in $\mathfrak{A}$. Thus we have a representation $\rho(\eta)$ of $\{\sigma, \mathfrak{A}\}$ on $L(\eta)$.

Lemma 3.1. *No irreducible quasi-simple representation of $\{\sigma, \mathfrak{A}\}$ is contained more than once in $\rho(\eta)$.*

Let π be an irreducible quasi-simple representation of $\{\sigma, \mathfrak{A}\}$ and let π^0 be its restriction to $\{\sigma, \mathfrak{A}^0\}$. Suppose π is deducible from π_ω. Let $L^0(\eta)$ be the space of infinitely differentiable U-finite functions on $G^0_{\mathbf{R}}$ satisfying $\varphi(ag) \equiv \eta(a)\varphi(g)$ for all a in $A_{\mathbf{R}} \cap G^0_{\mathbf{R}}$ and let $\rho^0(\eta)$ be the representation of $\{\sigma, \mathfrak{A}^0\}$ on $L^0(\eta)$. It is enough to show that π^0 is contained at most once in $\rho^0(\eta)$.

Suppose $H \subseteq L^0(\eta)$ and the restriction of $\rho^0(\eta)$ to H is equivalent to π^0. The integers n for which $H_n \neq \{0\}$ are determined by π. To prove the lemma we need only show that, for some such n, H_n is uniquely determined by π. Let $\eta_1(t) = \eta\big(\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)$, $\eta_2(t) = \eta\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)\big)$, and let $\eta_i(t) = |t|^{r_i}\left(\frac{g}{|t|}\right)^{\ell_i}$ with $\ell_i = 0$ or 1. If φ lies in H_n set $\psi(x) = \varphi\big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\big)$; then

$$\varphi(g) = \eta(a)\psi(x)e^{in\theta}$$

if

$$g = a\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

with a in $A_{\mathbf{R}} \cap G^0_{\mathbf{R}}$. Consequently φ is uniquely determined by ψ. Let $\varphi_1 = \rho(V)\varphi$, $\varphi_2 = \rho(W)\varphi$, and let ψ_1 and ψ_2 be the corresponding functions on $\mathbf{R}$. Since

$$\rho(U)\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = in\psi(x)$$
$$\rho(Z)\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = r\psi(x) - 2x\frac{d\psi}{dx}$$
$$\rho(X)\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \frac{d\psi}{dx}$$

and

$$V = Z + 2iX - iU$$
$$W = Z - 2iX + iU$$

one has

$$\psi_1(x) = -2(x-i)\frac{d\psi}{dx} + (r+n)\psi$$
$$\psi_2(x) = -2(x+i)\frac{d\psi}{dx} + (r-n)\psi.$$

Moreover $\rho(D)\varphi = \rho\Big(\frac{WV}{2} - iU - \frac{U^2}{2}\Big)\varphi$ corresponds to the function

$$2(x^2+1)\frac{d^2\psi}{dx^2} + (4x - 2rx - 2in)\frac{d\psi}{dx} + \frac{(r-1)^2 - 1}{2}\psi.$$

Consequently

$$2(x^2+1)\frac{d^2\psi}{dx^2}+(4x-2rx-2in)\frac{d\psi}{dx}+\frac{\left[(r-1)^2-s^2\right]}{2}\psi=0. \tag{A}$$

Finally $\rho(\sigma)\varphi$ corresponds to $(-1)^{\ell_2}\psi(-x)$.

There are a number of separate cases to consider. If $s-m$ is an odd integer and π is infinite-dimensional take $n_0=|s|+1$. Then $H_{n_0}\neq\{0\}$ and $\rho(W)\varphi=0$ if $\varphi\in H_{n_0}$. Thus

$$-2(x+i)\frac{d\psi}{dx}+(r-n_0)\psi=0.$$

This equation determines ψ up to a scalar factor.

If $s-m$ is not an odd integer or π is finite-dimensional and if $m=0$ then $H_0\neq\{0\}$. If φ lies in H_0 then ψ must satisfy equation (A) and the condition $\psi(-x)=(-1)^{\ell_2+m_2}\psi(x)$ because $A(\pi^0)=(-1)^{m_2}I$. Thus ψ is determined up to a scalar factor.

If $s-m$ is not an odd integer or π is finite-dimensional and if $|m|=1$ then $H_1\neq\{0\}$. Referring to the definition of $A(\pi)$ in Lemma 2.2 we see that ψ satisfies equation (A) and the equation

$$-2(x+i)\frac{d\psi}{dx}+(r-1)\psi(x)=(-1)^{\ell_2+m_2}s\psi(-x).$$

This equation implies a non-trivial linear relation between the values of ψ and its first derivative at $x=0$. Thus ψ is determined up to a scalar factor.

If $\xi(x)=e^{iux}$, with $u\neq 0$, is a non-trivial character of $\mathbf{R}$ let $L(\xi)$ be the space of all infinitely differentiable U-finite functions on $G_{\mathbf{R}}$ satisfying

(i) $\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)\equiv\xi(x)\varphi(g)$ for all x in $\mathbf{R}$,

(ii) if g belongs to $G_{\mathbf{R}}$ and X belongs to $\mathfrak{A}$ there is a constant M such that

$$\left|\rho(X)\varphi\left(\begin{pmatrix}t_1 & 0\\ 0 & t_2\end{pmatrix}g\right)\right|\leqslant M\left\{|t_1|^M+|t_2|^M\right\}$$

for $|t_1|\geqslant|t_2|$.

Let $\rho(\xi)$ be the representation of $\{\sigma,\mathfrak{A}\}$ on $L(\xi)$.

Lemma 3.2. *No irreducible quasi-simple representation of $\{\sigma,\mathfrak{A}\}$ is contained more than once in $\rho(\xi)$.*

Let π be such a representation and let π be deducible from π_ω. Let $L^0(\xi)$ be the space of all infinitely differentiable U-finite functions on $G^0_{\mathbf{R}}$ satisfying

(i) $\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)\equiv\xi(x)\varphi(g)$,

(ii) if g lies in $G^0_{\mathbf{R}}$ and X lies in $\mathfrak{A}^0$ there is a constant M such that

$$\left|\rho(X)\varphi\left(\begin{pmatrix}t^{1/2} & 0\\ 0 & t^{-1/2}\end{pmatrix}g\right)\right|\leqslant Mt^M$$

for $t\geqslant 1$.

Let $\rho^0(\xi)$ be the representation of $\{\sigma, \mathfrak{A}^0\}$ on $L^0(\xi)$. It is enough to show that π^0 is contained at most once in $\rho^0(\eta)$. The proof of this will be similar to the proof of the previous lemma.

Suppose H is an invariant subspace of $L^0(\xi)$ and the restriction of $\rho^0(\xi)$ to H is equivalent to π^0. If φ lies in H_n set

$$\psi(t) = \varphi\left(\begin{pmatrix} \frac{t}{|t|^{1/2}} & 0 \\ 0 & \frac{1}{|t|^{1/2}} \end{pmatrix}\right), \qquad t \in \mathbf{R}^\times.$$

Since $\varphi(g) = \xi(x)\psi(t)e^{in\theta}$ if

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{t}{|t|^{1/2}} & 0 \\ 0 & \frac{1}{|t|^{1/2}} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

the function φ is determined by ψ. Let $\varphi_1 = \rho(V)\varphi$, $\varphi_2 = \rho(W)\rho$, and let ψ_1 and ψ_2 be the corresponding functions on $\mathbf{R}^\times$. Since

$$\rho(U)\varphi\left(\begin{pmatrix} \frac{t}{|t|^{1/2}} & 0 \\ 0 & \frac{1}{|t|^{1/2}} \end{pmatrix}\right) = in\psi(t)$$

$$\rho(Z)\varphi\left(\begin{pmatrix} \frac{t}{|t|^{1/2}} & 0 \\ 0 & \frac{1}{|t|^{1/2}} \end{pmatrix}\right) = 2t\frac{d\psi}{dt}$$

$$\rho(X)\varphi\left(\begin{pmatrix} \frac{t}{|t|^{1/2}} & 0 \\ 0 & \frac{1}{|t|^{1/2}} \end{pmatrix}\right) = iut\psi(t)$$

one has

$$\psi_1(t) = 2t\frac{d\psi}{dt} - (2ut - n)\psi$$
$$\psi_2(t) = 2t\frac{d\psi}{dt} + (2ut - n)\psi.$$

Moreover $\rho(D)\varphi$ corresponds to $2t\frac{d}{dt}\left(t\frac{d\psi}{dt}\right) - 2t\frac{d\psi}{dt} + (2nut - 2u^2t^2)\psi$ so that

$$2t\frac{d}{dt}\left(t\frac{d\psi}{dt}\right) - 2t\frac{d\psi}{dt} + (2nut - 2u^2t^2)\psi = \frac{s^2-1}{2}\psi. \tag{B}$$

Finally $\rho(\sigma)\varphi$ corresponds to $(-1)^n\psi(-t)$.

Suppose that $s - m$ is an odd integer and π is infinite-dimensional. Take $n_0 = |s| + 1$. Then $H_{n_0} \neq \{0\}$ and $\rho(W)\varphi = 0$ if φ belongs to H_{n_0}. Consequently

$$2t\frac{d\psi}{dt} + (2ut - n_0)\psi = 0.$$

If ψ is to satisfy this equation and the growth condition it must vanish for $ut < 0$ and be a multiple of $|t|^{n_0/2}e^{-ut}$ for $ut > 0$. Thus it is determined up to a scalar factor.

Before discussing the remaining cases we should comment on equation (B). It may be written as

$$\frac{d^2\psi}{dt^2} + \left(-u^2 + \frac{nu}{t} + \frac{(1-s^2)}{4t^2}\right)\psi = 0.$$

Dropping the terms in $\frac{1}{t}$ and $\frac{1}{t^2}$ we obtain the equation $\frac{d^2\psi}{dt^2} - u^2\psi = 0$. As a consequence the original equation has one solution on the positive real axis of the form $t^\mu e^{-|u|t}\Big(1 + O\big(\frac{1}{t}\big)\Big)$ and one of the form $t^\nu e^{|u|t}\Big(1 + O\big(\frac{1}{t}\big)\Big)$. Only the first will satisfy the growth conditions. On the negative real axis it has solutions of the forms $t^{\mu'} e^{|u|t}\Big(1 + O\big(\frac{1}{t}\big)\Big)$ and $t^{\nu'} e^{-|u|t}\Big(1 + O\big(\frac{1}{t}\big)\Big)$. Only the first satisfies the required growth conditions. Thus the space of solutions of equation (B) which satisfy the growth conditions has dimension two.

If $s - m$ is not an odd integer or π is finite-dimensional and if $m = 0$ then $H_0 \neq \{0\}$. If φ belongs to H_0 then $\psi(-t) = (-1)^{m_2}\psi(t)$ because $A(\pi^0) = (-1)^{m_2} I$. This supplementary condition will determine ψ up to a scalar factor.

If $s - m$ is not an odd integer or π is finite-dimensional and if $|m| = 1$ then $H_1 \neq \{0\}$. If φ belongs to H_1 then

$$2t\frac{d\psi}{dt} + (2ut - 1)\psi(t) = (-1)^{(m_2+1)} s\psi(-t).$$

This supplementary condition determines ψ up to a scalar factor.

Suppose $\psi(t)$ satisfies equation (B) with $n = 1$ and

$$\psi'(t) = \frac{1}{s}(-1)^{m_2+1}\left\{-2t\frac{d\psi}{dt}(-t) - (2ut + 1)\psi(-t)\right\}.$$

Then

$$\begin{aligned}\frac{(-1)^{m_2+1}}{s}&\left\{-2t\frac{d\psi'}{dt}(-t) - (2ut+1)\psi'(-t)\right\}\\ &= \frac{(-1)^{m_2+1}}{s}\left\{2t\frac{d}{dt}\big(\psi'(-t)\big) - (2ut+1)\psi'(-t)\right\}\end{aligned}$$

which equals

$$\frac{2}{s^2}\left\{t\frac{d}{dt}\left[2t\frac{d\psi}{dt} + (2ut-1)\psi\right] - \frac{(2ut+1)}{2}\left[2t\frac{d\psi}{dt} + (2ut-1)\psi(t)\right]\right\}.$$

Simplifying we obtain

$$\frac{2}{s^2}\left\{2t\frac{d}{dt}\left(t\frac{d\psi}{dt}\right) - 2t\frac{d\psi}{dt} + \left(-2u^2t^2 + 2ut + \frac{1}{2}\right)\psi\right\}$$

which is just ψ itself.

Corollary. *Let π be an irreducible quasi-simple representation of $\{\sigma, \mathfrak{A}\}$. π is contained in $\rho(\xi)$ if and only if π is infinite-dimensional.*

It is enough to show that π^0 is contained in $\pi^0(\xi)$ if and only if π^0 is infinite-dimensional. Suppose H is a non-trivial finite-dimensional subspace of $L^0(\xi)$. Let τ be the representation of $\{\sigma, \mathfrak{A}\}$ on H and let $\widetilde{\tau}$ be the contragredient representation. If

$$X_x = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

the only eigenvalue of $\widetilde{\tau}(X_x)$ is zero because $\widetilde{\tau}$ is finite-dimensional. Let $\widetilde{\varphi}$ be the element in the dual of H defined by $\widetilde{\varphi}(\varphi) = \varphi(1)$. $\widetilde{\varphi}$ is not zero and

$$\big(\widetilde{\tau}(X_x)\widetilde{\varphi}\big)(\varphi) = -\widetilde{\varphi}\big(\tau(X_x)\varphi\big) = -\big(\tau(X_x)\varphi\big)(1) = -iu\varphi(1)$$

so $-iu$ is an eigenvalue of $\widetilde{\tau}(X_x)$. This is a contradiction.

Suppose π is infinite-dimensional and deducible from π_ω. Let $L^0(\xi, s)$ be the space of functions in $L^0(\xi)$ satisfying $\rho(D)\varphi = \frac{s^2-1}{2}\varphi$. The dimension of $L^0(\xi, s)_n$ is two. Let

$$L^0(\xi, s, m) = \sum_{\frac{n-m}{2} \in \mathbf{Z}} L^0(\xi, s)_n$$

and let $\rho^0(\xi, s, m)$ be the representation of $\{\sigma, \mathfrak{A}\}$ on $L^0(\xi, s, m)$.

Suppose $W_1 \supsetneqq W_2$ are two invariant subspaces of $L^0(\xi, s, m)$ and $W = W_1/W_2$. The representation of $\{\sigma, \mathfrak{A}\}$ on W is quasi-simple. Choose n so that W_n is not empty. The dimension of W_n is at most two. Among all the non-zero subspaces of W_n obtained by intersecting W_n with an invariant subspace of W there is a minimal one W_n^0. Let W' be the intersection of all invariant subspaces containing W_n^0 and let W'' be the sum of all invariant subspaces of W' which do not contain W_n^0. W'' does not contain W_n^0 and the representation of $\{\sigma, \mathfrak{A}\}$ on $V = W'/W''$ is irreducible.

If $s-m$ is not an odd integer Lemma 2.1 and Lemma 2.5 and its corollary imply that $V_n \neq \{0\}$ if $\frac{n-m}{2}$ is an integer. Because the dimension of $L^0(\xi, s)_n$ is two we conclude that there is no chain $L^0(\xi, s, m) \supsetneqq W_1 \supsetneqq W_2 \supsetneqq \{0\}$ of invariant subspaces. The operator $A\big(\rho^0(\xi, s, m)\big)$ is defined and $L^0(\xi, s, m)$ is the direct sum of $L^+ = \left\{ \varphi \;\middle|\; A\big(\rho^0(\xi, s, m)\big)\varphi = \varphi \right\}$ and $L^- = \left\{ \varphi \;\middle|\; A\big(\rho^0(\xi, s, m)\big)\varphi = -\varphi \right\}$. We have seen that neither of these is empty. Consequently they are both irreducible and the corollary to Lemma 2.5 implies that the restriction of $\rho^0(\xi, s, m)$ to one of them is equivalent to π^0.

If $s - m$ is an odd integer the same kind of argument shows that there is no chain $L^0(\xi, s, m) \supsetneqq W_1 \supsetneqq W_2 \supsetneqq W_3 \supsetneqq W_4 \supsetneqq \{0\}$ of invariant subspaces. As a consequence $L^0(\xi, s, m)$ must contain an invariant irreducible subspace. The restriction of $\rho^0(\xi, s, m)$ to this subspace will be equivalent to π^0 which is the only infinite-dimensional irreducible representation deducible from π^0_ω.

We return to the study of the functions $\psi(t)$. The Mellin transforms

$$\theta^+(z) = \int_{\mathbf{R}^\times} \psi(t)|t|^{z-1}\, dt$$

$$\theta^-(z) = \int_{\mathbf{R}^\times} \psi(t)(\operatorname{sgn} t)|t|^{z-1}\, dt$$

are defined for $\operatorname{Re} z$ sufficiently large. Equations (B) are equivalent to the difference equations

$$\begin{aligned}\left[(2z+1)^2-s^2\right]\theta^+(z)+4nu\theta^-(z+1)-4u^2\theta^+(z+2)&=0\\ \left[(2z+1)^2-s^2\right]\theta^-(z)+4nu\theta^+(z+1)-4u^2\theta^-(z+2)&=0.\end{aligned}$$

If, as before, ψ corresponds to φ, ψ_1 corresponds to $\varphi_1=\rho(V)\varphi$, and ψ_2 corresponds to $\varphi_2=\rho(W)\varphi$ let θ_i^+ and θ_i^- be the Mellin transforms of ψ_i. Then

$$\text{(C)}\qquad\begin{aligned}\theta_1^+(z)&=-2z\theta^+(z)-2u\theta^-(z+1)+n\theta^+(z)\\ \theta_1^-(z)&=-2z\theta^-(z)-2u\theta^+(z+1)+n\theta^-(z)\\ \theta_2^+(z)&=-2z\theta^+(z)+2u\theta^-(z+1)-n\theta^+(z)\\ \theta_2^-(z)&=-2z\theta^-(z)+2u\theta^+(z+1)-n\theta^-(z).\end{aligned}$$

If φ is replaced by $\rho(\sigma)\varphi$ then $\theta^+(z)$ is replaced by $(-1)^n\theta^+(z)$ and $\theta^-(z)$ is replaced by $(-1)^{n+1}\theta^-(z)$.

If π is an infinite-dimensional irreducible quasi-simple representation of $\{\sigma,\mathfrak{A}\}$ let $L^0(\xi,\pi)$ be the unique subspace of $L^0(\xi)$ which transforms according to π^0.

Lemma 3.3. *Suppose π is an infinite-dimensional irreducible quasi-simple representation of $\{\sigma,\mathfrak{A}\}$ which is deducible from π_ω. If $L^0(\xi,\pi)_n\neq 0$ let $\theta_n^+(z)$ and $\theta_n^-(z)$ be the Mellin transforms corresponding to some non-zero element in $L^0(\xi,\pi)_n$.*

(*i*) *If $s-m$ is not an odd integer, $m=0$, and $m_2=0$, then*

$$\begin{aligned}\theta_0^+(z)&=\alpha_0\left(\frac{2}{|u|}\right)^z\Gamma\left(\frac{z+\frac12+\frac s2}{2}\right)\Gamma\left(\frac{z+\frac12-\frac s2}{2}\right)\\ \theta_0^-(z)&=0\\ \theta_2^+(z)&=\alpha_1\left(\frac{2}{|u|}\right)^z z\Gamma\left(\frac{z+\frac12+\frac s2}{2}\right)\Gamma\left(\frac{z+\frac12-\frac s2}{2}\right)\\ \theta_2^-(z)&=2\alpha_1\operatorname{sgn}u\left(\frac{2}{|u|}\right)^z\Gamma\left(\frac{z+\frac32+\frac s2}{2}\right)\Gamma\left(\frac{z+\frac32-\frac s2}{2}\right).\end{aligned}$$

(*ii*) *If $s-m$ is not an odd integer, $m=0$, and $m_2=1$, then*

$$\begin{aligned}\theta_0^+(z)&=0\\ \theta_0^-(z)&=\beta_0\left(\frac{2}{|u|}\right)^z\Gamma\left(\frac{z+\frac12+\frac s2}{2}\right)\Gamma\left(\frac{z+\frac12-\frac s2}{2}\right)\\ \theta_2^+(z)&=2\beta_2\operatorname{sgn}u\left(\frac{2}{|u|}\right)^z\Gamma\left(\frac{z+\frac32+\frac s2}{2}\right)\Gamma\left(\frac{z+\frac32-\frac s2}{2}\right)\\ \theta_2^-(z)&=\beta_1\left(\frac{2}{|u|}\right)^z z\Gamma\left(\frac{z+\frac12+\frac s2}{2}\right)\Gamma\left(\frac{z+\frac12-\frac s2}{2}\right).\end{aligned}$$

(iii) If $s-m$ is not an odd integer, $|m|=1$, and $m_2=0$ then

$$\theta_1^+(z)=\gamma_0\left(\frac{2}{|u|}\right)^z\Gamma\left(\frac{z+\frac{3}{2}+\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+\frac{1}{2}-\frac{s}{2}}{2}\right)$$

$$\theta_1^-(z)=\gamma_0\operatorname{sgn}u\left(\frac{2}{|u|}\right)^z\Gamma\left(\frac{z+\frac{1}{2}+\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+\frac{3}{2}-\frac{s}{2}}{2}\right).$$

(iv) If $s-m$ is not an odd integer, $|m|=1$, and $m_2=1$ then

$$\theta_1^+(z)=\gamma_1\left(\frac{2}{|u|}\right)^z\Gamma\left(\frac{z+\frac{1}{2}+\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+\frac{3}{2}-\frac{s}{2}}{2}\right)$$

$$\theta_1^-(z)=\gamma_1\operatorname{sgn}u\left(\frac{2}{|u|}\right)^z\Gamma\left(\frac{z+\frac{3}{2}+\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+\frac{1}{2}-\frac{s}{2}}{2}\right).$$

(v) If $s-m$ is an odd integer and $n_0=|s|+1$ then

$$\theta_{n_0}^+(z)=\frac{\delta_0}{|u|^{z+\frac{n_0}{2}}}\Gamma\left(z+\frac{1}{2}+\frac{|s|}{2}\right)$$

$$\theta_{n_0}^-(z)=\frac{\delta_0}{|u|^{z+\frac{n_0}{2}}}\operatorname{sgn}u\Gamma\left(z+\frac{1}{2}+\frac{|s|}{2}\right).$$

The letters α_0, α_1, β_0, β_1, γ_0, γ_1, δ_0 denote constants.

If $s-m$ is not an odd integer and $m=0$ the supplementary conditions on $\theta_0^+(z)$ and $\theta_0^-(z)$ corresponding to $A(\pi^0)=(-1)^{m_2}I$ are $\theta_0^+(z)=(-1)^{m_2}\theta_0^+(z)$, $\theta_0^-(z)=(-1)^{m_2+1}\theta_0^-(z)$. The first and second functions in parts (i) and (ii) of the lemma satisfy these conditions as well as the difference equations. Taking the inverse Mellin transform we obtain a function $\psi(t)$ which satisfies the growth condition as well as the differential equation (B). The function defined by $\varphi(g)=\xi(x)\psi(t)$ if

$$g=\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}\frac{t}{|t|^{1/2}}&0\\0&\frac{1}{|t|^{1/2}}\end{pmatrix}\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}$$

will lie in $L^0(\xi,s,m)_0$. Moreover $A\big(\rho^0(\xi,s,m)\big)\varphi$ will equal $(-1)^{m_2}\varphi$ so that φ will lie in $L^0(\xi,\pi)_0$. Thus the first two equations of parts (i) and (ii) are valid. The last two can be obtained from the first two by applying relations (C).

In the first four cases π is equivalent to π_ω. It follows from Lemma 2.5 that π is equivalent to $\pi_{\widetilde{\omega}}$ if

$$\widetilde{\omega}\left(\begin{pmatrix}\alpha_1&0\\0&\alpha_2\end{pmatrix}\right)=\omega\left(\begin{pmatrix}\alpha_2&0\\0&\alpha_1\end{pmatrix}\right).$$

Replacing ω by $\widetilde{\omega}$ interchanges cases (iii) and (iv) so we need discuss case (iii) alone.

Substituting the function $\left(\frac{2}{|u|}\right)^z \Gamma\left(\frac{z+\frac{3}{2}\pm\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+\frac{1}{2}\mp\frac{s}{2}}{2}\right)$ (taking only all the upper signs or all the lower signs) into the expression $\left[(2z+1)^2 - s^2\right]\theta(z) - 4u^2\theta(z+2)$ one obtains

$$\left[\frac{(2z+1)^2-s^2}{2} - 2\left(z+\frac{3}{2}\pm s\right)\left(z+\frac{1}{2}\mp s\right)\right]$$
$$\times |u|\left(\frac{2}{|u|}\right)^{z+1}\Gamma\left(\frac{z+\frac{3}{2}\pm\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+\frac{1}{2}\mp\frac{s}{2}}{2}\right)$$

which equals

$$-4u\left\{\operatorname{sgn} u\left(\frac{2}{|u|}\right)^{z+1}\Gamma\left(\frac{(z+1)+\frac{1}{2}\pm\frac{s}{2}}{2}\right)\Gamma\left(\frac{(z+1)+\frac{3}{2}\mp\frac{s}{2}}{2}\right)\right\}.$$

Consequently the functions of part (iii) satisfy the difference equations. The supplementary conditions on $\theta_1^+(z)$ and $\theta_1^-(z)$ corresponding to the relation $A(\pi) = I$ are

$$s\theta_1^+(z) = 2z\theta_1^+(z) - 2u\theta_1^-(z+1) + \theta_1^+(z)$$
$$-s\theta_1^-(z) = 2z\theta_1^-(z) - 2u\theta_1^+(z+1) + \theta_1^-(z).$$

These will be satisfied by the functions of part (iii) because

$$(2z+1\mp s)\left(\frac{2}{|u|}\right)^z\Gamma\left(\frac{z+\frac{1}{2}\mp\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+\frac{3}{2}\pm\frac{s}{2}}{2}\right)$$
$$= 2u\operatorname{sgn} u\left(\frac{2}{|u|}\right)^{z+1}\Gamma\left(\frac{(z+1)+\frac{1}{2}\pm\frac{s}{2}}{2}\right)\Gamma\left(\frac{(z+1)+\frac{3}{2}\mp\frac{s}{2}}{2}\right).$$

The formulae of part (iii) can now be proved in the same way as those of parts (i) and (ii).

The simplest way to prove part (v) is to appeal to the explicit form for the corresponding function $\psi(t)$ found during the proof of Lemma 3.2.

Lemma 3.4. *Suppose $\psi(t)$ corresponds to φ in $L^0(\xi,\pi)_n$ and π is deducible from π_ω.*

(*i*) *If $s-m$ is not an integer, $m=0$, $m_2=0$ then, in a neighbourhood of 0, $\psi(t)$ has a convergent expansion of the form*

$$|t|^{\frac{s+1}{2}}\sum_{p=0}^{\infty} a_p t^p + |t|^{\frac{-s+1}{2}}\sum_{p=0}^{\infty} b_p t^p.$$

(*ii*) *If $s-m$ is an even integer, $m=0$, and $m_2=0$ then, in a neighbourhood of 0, $\psi(t)$ has a convergent expansion of the form*

$$|t|^{\frac{-|s|+1}{2}}\sum_{p=0}^{\infty} a_p t^p + \big(\log|t|\big)t^{\frac{|s|+1}{2}}\sum_{p=0}^{\infty} b_p t^p.$$

(*iii*) *If $s-m$ is not an integer, $m=0$, and $m_2=1$ then, in a neighbourhood of 0, $\psi(t)$ has a convergent expansion of the form*

$$(\operatorname{sgn} t)|t|^{\frac{s+1}{2}}\sum_{p=0}^{\infty} a_p t^p + (\operatorname{sgn} t)|t|^{\frac{-s+1}{2}}\sum_{p=0}^{\infty} b_p t^p.$$

(*iv*) *If* $s-m$ *is an even integer,* $m=0$*, and* $m_2=1$*, then, in a neighbourhood of* 0*,* $\psi(t)$ *has a convergent expansion of the form*

$$(\operatorname{sgn} t)|t|^{\frac{-|s|+1}{2}}\sum_{p=0}^{\infty} a_p t^p + (\operatorname{sgn} t)|t|^{\frac{|s|+1}{2}}\log|t|\sum_{p=0}^{\infty} b_p t^p.$$

(*v*) *If* $s-m$ *is not an integer,* $|m|=1$*, and* $m_2=0$ *then, in a neighbourhood of* 0*,* $\psi(t)$ *has a convergent expansion of the form*

$$(\operatorname{sgn} t)|t|^{\frac{s+1}{2}}\sum_{p=0}^{\infty} a_p t^p + |t|^{\frac{-s+1}{2}}\sum_{p=0}^{\infty} b_p t^p.$$

(*vi*) *If* $s-m$ *is an even integer,* $|m|=1$*, and* $m_2=0$ *then, in a neighbourhood of* 0*,* $\psi(t)$ *has a convergent expansion of the form*

$$|t|^{\frac{-s+1}{2}}\sum_{p=0}^{\infty} a_p t^p + (\operatorname{sgn} t)|t|^{\frac{s+1}{2}}\log|t|\sum_{p=0}^{\infty} b_p t^p$$

if s *is positive and one of the form*

$$(\operatorname{sgn} t)|t|^{\frac{s+1}{2}}\sum_{p=0}^{\infty} a_p t^p + |t|^{\frac{-s+1}{2}}\log|t|\sum_{p=0}^{\infty} b_p t^p$$

if s *is negative.*

(*vii*) *If* $s-m$ *is not an integer,* $|m|=1$*, and* $m_2=1$*, then, in a neighbourhood of* 0*,* $\psi(t)$ *has a convergent expansion of the form*

$$|t|^{\frac{s+1}{2}}\sum_{p=0}^{\infty} a_p t^p + (\operatorname{sgn} t)|t|^{\frac{-s+1}{2}}\sum_{p=0}^{\infty} b_p t^p.$$

(*viii*) *If* $s-m$ *is an even integer,* $|m|=1$*, and* $m_2=1$ *then, in a neighbourhood of* 0*,* $\psi(t)$ *has a convergent expansion of the form*

$$(\operatorname{sgn} t)|t|^{\frac{-s+1}{2}}\sum_{p=0}^{\infty} a_t t^p + |t|^{\frac{s+1}{2}}\log|t|\sum_{p=0}^{\infty} b_p t^p$$

if s *is positive and one of the form*

$$|t|^{\frac{s+1}{2}}\sum_{p=0}^{\infty} a_p t^p + (\operatorname{sgn} t)|t|^{\frac{-s+1}{2}}\log|t|\sum_{p=0}^{\infty} b_p t^p$$

if s *is negative.*

(*ix*) *If* $s-m$ *is an odd integer then* $\psi(t)$ *is zero unless* $nut>0$ *and in this region* $\psi(t)$ *has a convergent expansion of the form*

$$|t|^{\frac{|s|+1}{2}}\sum_{p=0}^{\infty} a_t t^p.$$

We know that if $\psi(t)$ corresponds to φ then $2t\frac{d\psi}{dt}-(2ut-n)\psi$ corresponds to $\rho(V)\varphi$, $2t\frac{d\psi}{dt}+(2ut-n)\psi$ corresponds to $\rho(W)\varphi$, and $(-1)^n\psi(-t)$ corresponds to $\rho(\sigma)\varphi$. Because each of these operations take a function with an expansion of one of the given forms to a function with an expansion of the same form and π^0 is irreducible it will be enough to show that there is at least one n for which the lemma is valid. If $s-m$ is not an odd integer we shall take $n=|m|$ and if $s-m$ is an odd integer we shall take $n=|s|+1$.

The indicial equation of the equation (B) does not depend on n. It is $(2\lambda-1)^2-s^2=0$ and has the roots $\lambda_1=\frac{s+1}{2}$, $\lambda_2=\frac{-s+1}{2}$ with difference $\lambda_1-\lambda_2=s$. If $n=0$ the series $t^{\lambda_i}\sum_{p=0}^{\infty}c_pt^p$, $t>0$ satisfies the equation if and only if

$$\Big[\big(2(\lambda_i+p)-1\big)^2-s^2\Big]c_p=4u^2c_{p-2}.$$

Thus if s is not an integer $\psi(t)$ has an expansion of the form

$$t^{\frac{s+1}{2}}\sum_{p=0}^{\infty}a_{2p}t^{2p}+t^{\frac{-s+1}{2}}\sum_{p=0}^{\infty}b_{2p}t^{2p}$$

valid for t positive and close to 0. If s is an even integer one of the two linearly independent solutions given by the method of Frobenius must contain a logarithmic term because it will not be possible to solve these equations recursively when λ_i is the smaller of the roots. Since the equation is invariant under the substitution $t\to -t$ the logarithmic solution must be of the form

$$t^{\frac{-|s|+1}{2}}\sum_{p=0}^{\infty}c_{2p}t^{2p}+t^{\frac{|s|+1}{2}}\log t\sum_{p=0}^{\infty}d_{2p}t^{2p}$$

and $\psi(t)$ has an expansion of the form

$$t^{\frac{-|s|+1}{2}}\sum_{p=0}^{\infty}a_{2p}t^{2p}+t^{\frac{|s|+1}{2}}\log t\sum_{p=0}^{\infty}b_{2p}t^{2p}$$

valid for t positive and close to 0. Cases (i) to (iv) of the lemma follow immediately because, since $n=0$, $\psi(t)$ is even in the first two and odd in the second two.

Just as in the previous lemma, (vii) and (viii) are redundant since they are covered already by (v) and (vi) which we now treat. If $n=1$, $t^{\frac{\pm s+1}{2}}\sum_{p=0}^{\infty}c_pt^p$, $t>0$, satisfies equation (B) if and only if

$$\frac{1}{2}\big[(\pm s+2p)^2-s^2\big]c_p+2uc_{p-1}-2u^2c_{p-2}=0$$

or

$$(\pm 2ps+2p^2)c_p+2uc_{p-1}-2u^2c_{p-2}=0.$$

For convenience let $c_p=0$ if $p<0$. If s is not an integer choose $c_0^{\pm}$ and define $c_p^{\pm}$ inductively by $(\pm s+2p)c_p^{\pm}+2uc_{p-1}^{\pm}=\pm s(-1)^pc_p^{\pm}$ or, equivalently, $(\pm s+p)c_p^{\pm}+uc_{p-1}^{\pm}=0$ when p is odd and $pc_p^{\pm}+uc_{p-1}^{\pm}=0$ when p is even. This equation will be satisfied for all p if $c_p^{\pm}=0$ when p is negative. If p is odd

$$(\pm ps+p^2)c_p^{\pm}+uc_{p-1}^{\pm}-u^2c_{p-2}^{\pm}=-u\Big[(p-1)c_{p-1}^{\pm}+uc_{p-2}^{\pm}\Big]=0$$

and if p is even

$$(\pm ps+p^2)c_p^{\pm}+uc_{p-1}^{\pm}-u^2c_{p-2}^{\pm}=-u\Big[\big(\pm s+(p-1)\big)c_{p-1}^{\pm}+uc_{p-2}^{\pm}\Big]=0.$$

Thus, if s is not an integer, $\psi(t)$ will have an expansion of the form

$$t^{\frac{s+1}{2}}\sum_{p=0}^{\infty}c_p^{+}t^p+t^{\frac{-s+1}{2}}\sum_{p=0}^{\infty}c_p^{-}t^p$$

valid for t positive and close to zero. Since $m_2 = 0$

$$-s\psi(-t) = 2t\frac{d\psi}{dt} + (2ut-1)\psi(t).$$

The expression

$$2t\frac{d}{dt}\left\{t^{\frac{\pm s+1}{2}}\sum_{p=0}^{\infty} c_t^{\pm} t^p\right\} + (2ut-1)t^{\frac{\pm s+1}{2}}\sum_{p=0}^{\infty} c_p^{\pm} t^p$$

is equal to

$$t^{\frac{\pm s+1}{2}}\sum_{p=0}^{\infty}\Big[(\pm s+2p)c_p^{\pm} + 2uc_{p-1}^{\pm}\Big]t^p = \pm st^{\frac{\pm s+1}{2}}\sum_{p=0}^{\infty} c_p^{\pm}(-t)^p.$$

Case (v) of the lemma for $n = 1$ follows immediately.

Since

$$t\frac{d}{dt}\big(\log tA(t)\big) = t\frac{dA}{dt}\log t + A(t)$$
$$t\frac{d}{dt}\left(t\frac{d}{dt}\big(\log tA(t)\big)\right) = t\frac{d}{dt}\left(t\frac{dA}{dt}\right) + 2t\frac{dA}{dt}$$

the series

$$t^{\frac{-|s|+1}{2}}\sum_{p=0}^{\infty} c_p t^p + t^{\frac{|s|+1}{2}}\log t\sum_{p=0}^{\infty} d_p t^p, \qquad t > 0,$$

will satisfy equation (B) when s is an odd integer and $n = 1$ if and only if

$$\frac{\Big[\big(|s|+2p\big)^2 - s^2\Big]}{2}d_p + 2ud_{p-1} - 2u^2d_{p-2} = 0$$

or

$$\big(|s|p + p^2\big)d_p + ud_{p-1} - u^2d_{p-2} = 0$$

and

$$\big(-|s|p + p^2\big)cp + uc_{p-1} - u^2c_{p-2} + \big(-|s| + 2p\big)d_{p-|s|} = 0.$$

Choose c_0 and $c_{|s|}$ and define the other coefficients by $\big(|s|+2p\big)d_p + 2ud_{p-1} = (-1)^p|s|d_p$ or $pd_p + ud_{p-1} = 0$ if p is even and $\big(|s|+p\big)d_p + ud_{p-1} = 0$ if p is odd and

$$\big(-|s|+2p\big)c_p + 2uc_{p-1} + 2d_{p-|s|} = (-1)^{p+1}|s|c_p$$

or

$$pc_p + uc_{p-1} + d_{p-|s|} = 0$$

if p is even and

$$\big(-|s|+p\big)c_p + uc_{p-1} + d_{p-|s|} = 0$$

if p is odd. Take c_p and d_p to be 0 if p is negative. These equations are consistent and determine the remaining c_p and all d_p uniquely. We have already seen that the coefficients d_p will satisfy

$$\big(|s|p + p^2\big)d_p + ud_{p-1} - u^2d_{p-2} = 0.$$

If p is even

$$\big(-|s|p + p^2\big)c_p + uc_{p-1} - u^2c_{p-2} + \big(-|s| + 2p\big)d_{p-|s|}$$

equals

$$\big[|s|-(p-1)\big]uc_{p-1}-u^2c_{p-2}+pd_{p-|s|}=ud_{p-|s|-1}+\Big(|s|+\big(p-|s|\big)\Big)d_{p-|s|}=0$$

and if p is odd

$$\big(-|s|p+p^2\big)c_p+uc_{p-1}-u^2c_{p-2}+\big(-|s|+2p\big)d_{p-|s|}$$

equals

$$-u\big[(p-1)c_{p-1}+uc_{p-1}\big]+\big(p-|s|\big)d_{p-|s|}=ud_{p-|s|-1}+\big(p-|s|\big)d_{p-|s|}=0.$$

Thus if c_0 and $c_{|s|}$ are suitably chosen

$$\psi(t)=t^{\frac{-|s|+1}{2}}\sum_{p=0}^{\infty}c_pt^p+t^{\frac{|s|+1}{2}}\log t\sum_{p=0}^{\infty}d_pt^p$$

for t positive and close to 0.

Since $m_2=0$

$$-s\psi(-t)=2t\frac{d\psi}{dt}+(2ut-1)\psi.$$

The right hand side is equal to

$$t^{\frac{-|s|+1}{2}}\sum_{p=0}^{\infty}c_p't^p+t^{\frac{|s|+1}{2}}\sum_{p=0}^{\infty}d_p't^p$$

with

$$\begin{aligned}c_p'&=\big(-|s|+2p\big)c_p+2uc_{p-1}+2d_{p-|s|}=-|s|(-1)^pc_p\\ d_p'&=\big(|s|+2p\big)d_p+2ud_{p-1}=(-1)^p|s|d_p.\end{aligned}$$

Case (vi) of the lemma follows.

The assertion for case (ix) with $n=|s|+1$ was established while proving Lemma 3.2.

If $\psi(t)$ is the function of the lemma and x is a real number the functions

$$\begin{aligned}\theta^+(z,x)&=\int_{\mathbf{R}^\times}e^{itx}\psi(t)|t|^{z-1}\,dt\\ \theta^-(z,x)&=\int_{\mathbf{R}^\times}e^{itx}\psi(t)|t|^{z-1}\operatorname{sgn}t\,dt\end{aligned}$$

are defined for $\operatorname{Re}z$ sufficiently large.

Lemma 3.5. *$\theta^{\pm}(z,x)$ are meromorphic in the whole complex plane and bounded in regions of the form $|\operatorname{Re}u|\leqslant$ constant, $|\operatorname{Im}u|\geqslant$ constant $\gg 0$*

(*i*) *If $s-m$ is not an odd integer, $m=0$, and $m_2=0$ then* $\dfrac{\theta^+(z,x)}{\Gamma\left(\frac{z+\frac12+\frac s2}{2}\right)\Gamma\left(\frac{z+\frac12-\frac s2}{2}\right)}$ *and* $\dfrac{\theta^-(z,x)}{\Gamma\left(\frac{z+\frac32+\frac s2}{2}\right)\Gamma\left(\frac{z+\frac32-\frac s2}{2}\right)}$ *are entire functions of z.*

(*ii*) *If $s-m$ is not an odd integer, $m=0$, and $m_2=1$ then* $\dfrac{\theta^+(z,x)}{\Gamma\left(\frac{z+\frac32+\frac s2}{2}\right)\Gamma\left(\frac{z+\frac32-\frac s2}{2}\right)}$ *and* $\dfrac{\theta^-(z,x)}{\Gamma\left(\frac{z+\frac12+\frac s2}{2}\right)\Gamma\left(\frac{z+\frac12-\frac s2}{2}\right)}$ *are entire functions of z.*

(*iii*) *If* $s-m$ *is not an odd integer,* $|m|=1$ *and* $m_2=0$ *then* $\frac{\theta^+(z,x)}{\Gamma\left(\frac{z+\frac{3}{2}+\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+\frac{1}{2}-\frac{s}{2}}{2}\right)}$ *and* $\frac{\theta^-(z,x)}{\Gamma\left(\frac{z+\frac{1}{2}+\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+\frac{3}{2}-\frac{s}{2}}{2}\right)}$ *are entire functions of* z.

(*iv*) *If* $s-m$ *is not an odd integer,* $|m|=1$, *and* $m_2=1$ *then* $\frac{\theta^+(z,x)}{\Gamma\left(\frac{z+\frac{1}{2}+\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+\frac{3}{2}-\frac{s}{2}}{2}\right)}$ *and* $\frac{\theta^-(z,x)}{\Gamma\left(\frac{z+\frac{3}{2}+\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+\frac{1}{2}-\frac{s}{2}}{2}\right)}$ *are entire functions of* z.

(*v*) *If* $s-m$ *is an odd integer then* $\frac{\theta^\pm(z,x)}{\Gamma\left(z+\frac{1}{2}+\frac{|s|}{2}\right)}$ *are entire functions of* z.

Let $m(t)$ be an infinitely differentiable function with compact support on the line which is even and equal to 1 in a neighbourhood of 0. $\theta^\pm(z,x)$ is the sum of

$$\widehat{\theta}^\pm(z,x)=\int_{\mathbf{R}^\times} e^{itx}\psi(t)|t|^{z-1}(\operatorname{sgn} t)^{\frac{1\pm 1}{2}}m(t)\,dt$$

and

$$\int_{\mathbf{R}^\times} e^{itx}\psi(t)|t|^{z-1}(\operatorname{sgn} t)^{\frac{1\pm 1}{2}}\big(1-m(t)\big)\,dt.$$

The second integral is an entire function of z which is bounded in vertical strips. Thus it is enough to prove the lemma with $\theta^\pm(z,x)$ replaced by $\widehat{\theta}^\pm(z,x)$. The function $e^{itx}\psi(t)+(-1)^{\frac{1\pm1}{2}}e^{-itx}\psi(-t)$ is, for $t>0$, a linear combination of convergent series of the form

$$t^\alpha(\log t)^\beta\sum_{p=0}^{\infty}c_pt^p$$

where α is $\frac{s+1}{2}$ or $\frac{-s+1}{2}$ and β is 0 or 1. Given a series of this form and a real number c there is a P such that

$$\int_0^\infty t^\alpha(\log t)^\beta\left\{\sum_{p\geqslant P}c_pt^p\right\}t^{z-1}m(t)\,dt$$

is analytic for $\operatorname{Re} z>c$ and bounded in vertical strips of finite width contained in this region.

The first assertion of the lemma is a consequence of the relations

$$\int_0^\infty t^{\alpha+p+z-1}m(t)\,dt=\frac{-1}{\alpha+p+z}\int_0^\infty t^{\alpha+p+z}m'(t)\,dt$$

$$\int_0^\infty t^{\alpha+p+z-1}\log t\,m(t)\,dt=\frac{-1}{(\alpha+p+z)^2}\int_0^\infty\big[(\alpha+p+z)t^{\alpha+p+z}\log t-t^{\alpha+p+z}\big]m'(t)\,dt$$

and the condition that $m'(t)$ vanish near zero. To prove the remaining assertions one shows that the zeros of the denominator on the right are cancelled by the poles of the Γ-factor. This is easy but the various cases of Lemma 3.4 must be examined separately. I leave it to the reader to do so.

If η is any homomorphism of $A_{\mathbf{R}}$ into $\mathbf{C}^\times$ then $\widetilde{\eta}$ will be the homomorphism defined by

$$\widetilde{\eta}\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)=\eta\left(\begin{pmatrix}\alpha_2 & 0\\ 0 & \alpha_1\end{pmatrix}\right).$$

If ζ is a homomorphism of $A_{\mathbf{R}}^{\times}$ into $\mathbf{C}^{\times}$ such that

$$\zeta\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right)\omega\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}\right)\equiv 1$$

and z and ℓ are defined by

$$\zeta\left(\begin{pmatrix}\frac{t}{|t|^{1/2}} & \\ & \frac{1}{|t|^{1/2}}\end{pmatrix}\right)=|t|^{z}(\operatorname{sgn} t)^{\ell},$$

with $\ell=0$ or 1, then ζ is determined by z and ℓ and we shall sometimes write $\zeta=\zeta(z,\ell)$.

Lemma 3.6. *Suppose π is an infinite-dimensional representation of $\{\sigma,\mathfrak{A}\}$ and π is deducible from π_ω. Let $L(\xi,\pi)$ be the unique subspace of $L(\xi)$ which transforms according to π. If φ belongs to $L(\xi,\pi)$ and $\zeta=\zeta(z,\ell)$ the function*

$$\Phi(g,\zeta,\varphi)=\int_{\mathbf{R}^{\times}}\varphi\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}g\right)\zeta\left(\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix}\right)d^{\times}t$$

is defined for $\operatorname{Re} z$ *sufficiently large.*

(*i*) *If $s-m$ is not an odd integer set*

$$\Phi'(g,\zeta,\varphi)=\frac{\Phi(g,\zeta,\varphi)}{\Gamma\left(\frac{z+|m_1-\ell|+\frac{1}{2}+\frac{s}{2}}{2}\right)\Gamma\left(\frac{z+|m_2-\ell|+\frac{1}{2}-\frac{s}{2}}{2}\right)}.$$

(*ii*) *If $s-m$ is an odd integer set*

$$\Phi'(g,\zeta,\varphi)=\frac{\Phi(g,\zeta,\varphi)}{\Gamma\left(z+\frac{1}{2}+\frac{|s|}{2}\right)}.$$

Then $\Phi'\big(g,\zeta(z,\ell),\varphi\big)$ is an entire function of z and $\Phi\big(g,\zeta(z,\ell),\varphi\big)$ is bounded in regions of the form $|\operatorname{Re} z|\leqslant$ constant, $|\operatorname{Im} z|\geqslant$ constant $\gg 0$. Moreover if $s-m$ is not an odd integer

$$\left(\frac{2}{|u|}\right)^{-z}\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g,\zeta,\varphi\right)=(i)^{|m_1-\ell|+|m_2-\ell|}(\operatorname{sgn} u)^{m}\left(\frac{2}{|u|}\right)^{z}\Phi'(g,\widetilde{\zeta},\varphi)$$

and if $s-m$ is an odd integer

$$\left(\frac{1}{|u|}\right)^{-z'}\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g,\zeta,\varphi\right)=(i)^{|s|+1}(\operatorname{sgn} u)^{m}\left(\frac{1}{|u|}\right)^{z'}\Phi'(g,\widetilde{\zeta},\varphi).$$

It is enough to prove the lemma for φ in $L(\xi,\pi)_n$. If $\widetilde{\varphi}$ is the restriction of φ to $G_{\mathbf{R}}^{0}$ let $\psi(t)$ be the function on $\mathbf{R}^{\times}$ corresponding to $\widetilde{\varphi}$. Then, if

$$g=\begin{pmatrix}t_1 & 0\\ 0 & t_2\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix},$$

$\varphi\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g\right)$ is equal to

$$e^{i\frac{tt_1}{t_2}ux}\omega\left(\begin{pmatrix} |tt_1t_2|^{1/2}\operatorname{sgn} t_2 & 0 \\ 0 & |tt_1t_2|^{1/2}\operatorname{sgn} t_2 \end{pmatrix}\right)\psi\left(\frac{tt_1}{t_2}\right)e^{in\theta}.$$

Thus

$$\varphi\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g\right)\zeta\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is equal to

$$\zeta^{-1}\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right)e^{ei\frac{tt_1}{t_2}ux}\zeta\left(\begin{pmatrix} \frac{tt_1}{t_2}\left|\frac{t_2}{tt_1}\right|^{1/2} & 0 \\ 0 & \left|\frac{t_2}{tt_1}\right|^{1/2} \end{pmatrix}\right)\psi\left(\frac{tt_1}{t_2}\right)e^{in\theta}$$

and $\Phi\big(g, \zeta(z,\ell), \varphi\big)$ is equal to

$$\zeta^{-1}\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right)\theta^+(z, ux)e^{in\theta}, \qquad \text{if } \ell = 0,$$
$$\zeta^{-1}\left(\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\right)\theta^-(z, ux)e^{in\theta}, \qquad \text{if } \ell = 1.$$

All assertions of the lemma except the functional equations follow immediately from Lemma 3.5.

If $\eta = \widetilde{\zeta}^{-1}$ the maps

$$\varphi \to \Phi'(g, \widetilde{\zeta}, \varphi),$$
$$\varphi \to \Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, \zeta, p\right)$$

are $\{\sigma, \mathfrak{A}\}$ invariant maps of $L(\xi, \pi)$ into $L(\eta)$. According to Lemma 3.1 one must be a scalar multiple of the other. To see what the multiple is we choose $g = 1$ so that $\Phi(g, \widetilde{\zeta}, \varphi)$ is equal to $\theta_n^+(-z)$ if $\ell - |m| = 0$ and is equal to $\theta_n^-(-z)$ if $\big|\ell - |m|\big| = 1$ and choose n in such a way that Lemma 3.3 can be applied.

$$\Phi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, \zeta, \varphi\right)$$

is equal to $(i)^n\theta_n^+(z)$ if $\ell = 0$ and to $(i)^n\theta_n^-(z)$ is $\ell = 1$. In the first column below we write the values of $\Phi'(1, \widetilde{\zeta}, \varphi)$ for the values of n and ℓ in the last column; in the second column we write the values of $\Phi'\big(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\zeta, \varphi\big)$. Comparing them we obtain the lemma. In all but the last line $s-m$ is not an odd integer.

$$\begin{array}{lll}
\Phi'(1,\widetilde{\zeta},\varphi) & \Phi'\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix},\zeta,\varphi\right) & \\
\alpha_0\left(\frac{2}{|u|}\right)^{-z} & \alpha_0\left(\frac{2}{|u|}\right)^{z} & \ell=0,\ m=0,\ m_2=0,\ n=0. \\
2\alpha_1\operatorname{sgn}u\left(\frac{2}{|u|}\right)^{-z} & -2\alpha_1\operatorname{sgn}u\left(\frac{2}{|u|}\right)^{z} & \ell=1,\ m=0,\ m_2=0,\ n=2. \\
2\beta_1\operatorname{sgn}u\left(\frac{2}{|u|}\right)^{-z} & -2\beta_1\operatorname{sgn}u\left(\frac{2}{|u|}\right)^{z} & \ell=0,\ m=0,\ m_1=1,\ n=2. \\
\beta_0\left(\frac{2}{|u|}\right)^{-z} & \beta_0\left(\frac{2}{|u|}\right)^{z} & \ell=1,\ m=0,\ m_2=1,\ n=0. \\
\gamma_0\operatorname{sgn}u\left(\frac{2}{|u|}\right)^{-z} & i\gamma_0\left(\frac{2}{|u|}\right)^{z} & \ell=0,\ |m|=1,\ m_2=0,\ n=1. \\
\gamma_0\left(\frac{2}{|u|}\right)^{-z} & i\gamma_0\operatorname{sgn}u\left(\frac{2}{|u|}\right)^{z} & \ell=1,\ |m|=1,\ m_2=0,\ n=1. \\
\gamma_1\operatorname{sgn}u\left(\frac{2}{|u|}\right)^{-z} & i\gamma_1\left(\frac{2}{|u|}\right)^{z} & \ell=0,\ |m|=1,\ m_2=1,\ n=1. \\
\gamma_1\left(\frac{2}{|u|}\right)^{-z} & i\gamma_1\operatorname{sgn}u\left(\frac{2}{|u|}\right)^{z} & \ell=1,\ |m|=1,\ m_2=1,\ n=1.
\end{array}$$

If $s-m$ is an odd integer the two values are

$$\begin{array}{lll}
\delta_0\left(\frac{1}{|u|}\right)^{-z+n/2}(\operatorname{sgn}u)^{\ell} & (i)^{|s|+1}\delta_0\left(\frac{1}{|u|}\right)^{z+n/2}(\operatorname{sgn}u)^{\ell} & n=|s|+1,\ m=0. \\
\delta_0\left(\frac{1}{|u|}\right)^{-z+n/2}(\operatorname{sgn}u)^{\ell-1} & (i)^{|s|+1}\delta_0\left(\frac{1}{|u|}\right)^{z+n/2}(\operatorname{sgn}u)^{\ell} & n=|s|+1,\ |m|=1.
\end{array}$$

4. Representations of $\mathrm{GL}(2,\mathbf{C})$

In this paragraph and the next $G_{\mathbf{C}}$ will be $\mathrm{GL}(2,\mathbf{C})$ and $G^0_{\mathbf{C}}$ will be $\mathrm{SL}(2,\mathbf{C})$. U will be the group of unitary matrices in $G_{\mathbf{C}}$ and U^0 will be $U\cap G^0_{\mathbf{C}}$. $G_{\mathbf{C}}$ and $G^0_{\mathbf{C}}$ will be considered as real Lie groups. The Lie algebra of $G_{\mathbf{C}}$ is

$$\mathfrak{g}=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\oplus\begin{pmatrix}\overline{a}&\overline{b}\\\overline{c}&\overline{d}\end{pmatrix}\right\};$$

its complexification is

$$\mathfrak{g}_{\mathbf{C}}=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\oplus\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\right\}.$$

The Lie algebra of $G^0_{\mathbf{C}}$ is

$$\mathfrak{g}^0=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\oplus\begin{pmatrix}\overline{a}&\overline{b}\\\overline{c}&\overline{d}\end{pmatrix}\ \middle|\ a+d=0\right\};$$

its complexification is

$$\mathfrak{g}^0_{\mathbf{C}}=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}+\begin{pmatrix}a'&b'\\c'&d'\end{pmatrix}\ \middle|\ a+d=a'+d'=0\right\}.$$

The Lie algebra of U is

$$\mathfrak{u} = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & d \end{pmatrix} \oplus \begin{pmatrix} \bar{a} & \bar{b} \\ -b & \bar{d} \end{pmatrix} \,\middle|\, a = -\bar{a},\ d = -\bar{d} \right\};$$

its complexification is

$$\mathfrak{u}_{\mathbf{C}} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus \begin{pmatrix} -a & -c \\ -b & -d \end{pmatrix} \right\}.$$

Finally $\mathfrak{u}^0 = \mathfrak{u} \cap \mathfrak{g}^0$ and $\mathfrak{u}^0_{\mathbf{C}} = \mathfrak{u}_{\mathbf{C}} \cap \mathfrak{g}^0_{\mathbf{C}}$. When there is no risk of confusion an element of $\mathfrak{u}_{\mathbf{C}}$ will be identified by giving its first component.

Let V_n be the space of binary forms of degree n and let $\widetilde{V}_n$ be its dual. We write the elements of V_n as

$$\psi(x, y) = \sum_{\substack{|k| \leqslant n \\ \frac{n}{2} - k \in \mathbf{Z}}} \psi^k x^{\frac{n}{2}+k} y^{\frac{n}{2}-k}.$$

ψ^k will be called the kth component of ψ. If $|k| > \frac{n}{2}$ let $\psi^k = 0$. Let σ_n be the representation of U^0 on V_n defined by

$$\sigma_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} \psi(x, y) = \psi(ax + cy, bx + dy).$$

Denote the corresponding representation of $\mathfrak{u}^0_{\mathbf{C}}$ by σ_n also. If $\psi_1 = \sigma_n\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\psi$ then $\psi_1^{k-1} = \left(\frac{n}{2} + k\right)\psi^k = c_k \psi^k$ where $c_k \neq 0$ for $-\frac{n}{2} < k \leqslant \frac{n}{2}$ and if $\psi_2 = \sigma_n\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\psi$ then $\psi_2^{k+1} = \left(\frac{n}{2} - k\right)\psi^k = d_k \psi^k$ where $d_k \neq 0$ for $-\frac{n}{2} \leqslant k < \frac{n}{2}$.

Let $\mathfrak{A}$ be the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$ and $\mathfrak{A}^0$ that of $\mathfrak{g}^0_{\mathbf{C}}$. If π is a representation of $\mathfrak{A}$ on a vector space W then π^0 will be the restriction of π to $\mathfrak{A}^0$. Let W_n be the set of all vectors in W which transform under $\mathfrak{u}^0_{\mathbf{C}}$ according to σ_n. π will be called quasi-simple[4] if

(i) $W = \bigoplus_n W_n$

(ii) If Z lies in the centre of $\mathfrak{A}$ then $\pi(Z)$ is a scalar. Suppose π_1 and π_2 are two representations of $\mathfrak{A}$ on W_1 and W_2 respectively. π_2 will be said to be deducible from π_1 if there are two invariant subspaces W_3 and W_4 of W_1 with $W_3 \supseteq W_4$ and π_2 is equivalent to the representation of $\mathfrak{A}$ on W_3/W_4. The same notions will be used for representations of $\mathfrak{A}^0$.

Set

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The centre of the universal enveloping algebra $\mathfrak{A}^0$ is generated by

$$\begin{aligned} D &= (X \oplus 0)(Y \oplus 0) + (Y \oplus 0)(X \oplus 0) + \frac{1}{2}(Z \oplus 0)^2 \\ &= 2(Y \oplus 0)(X \oplus 0) + Z \oplus 0 + \frac{1}{2}(Z \oplus 0)^2 \\ &= 2(X \oplus 0)(Y \oplus 0) - Z \oplus 0 + \frac{1}{2}(Z \oplus 0)^2 \end{aligned}$$

[4]I use the expression in a different way than Harish-Chandra.

and

$$
\begin{aligned}
D' &= (0 \oplus X)(0 \oplus Y) + (0 \oplus Y)(0 \oplus X) + \frac{1}{2}(0 \oplus Z)^2 \\
&= 2(0 \oplus Y)(0 \oplus X) + (0 \oplus Z) + \frac{1}{2}(0 \oplus Z)^2 \\
&= 2(0 \oplus X)(0 \oplus Y) - (0 \oplus Z) + \frac{1}{2}(0 \oplus Z)^2.
\end{aligned}
$$

The centre of $\mathfrak{A}$ is generated by D, D', $J = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \oplus 0$, and $J' = 0 \oplus \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

Let ω be a continuous homomorphism of the group $A_{\mathbf{C}}$ of diagonal matrices into $\mathbf{C}^\times$. If $N_{\mathbf{C}}$ is the group of matrices of the form $\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$ let $L(\omega)$ be the space of infinitely differentiable U-finite functions on $N_{\mathbf{C}} \backslash G_{\mathbf{C}}$ satisfying $\psi(ag) = \left|\frac{\alpha_1}{\alpha_2}\right| \omega(a)\varphi(g)$ if $a = \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{smallmatrix}\right)$ is in $A_{\mathbf{C}}$. The restriction of ρ to $L(\omega)$ defines a representation π_ω of $\mathfrak{A}$ on $L(\omega)$. Define ω_1 and ω_2 on $\mathbf{C}^\times$ by $\omega_1(t) = \omega\big(\left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)\big)$ and $\omega_2(t) = \omega\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & t \end{smallmatrix}\right)\big)$. Let $\omega_i(t) = |t|^{s_i}\left(\frac{t}{|t_i|}\right)^{m_i}$ and set $s = \frac{s_1 - s_2}{2}$, $m = \frac{m_1 - m_2}{2}$.

Lemma 4.1. *$L(\omega)_n \neq \{0\}$ if and only if $\frac{n}{2} - |m|$ is a non-negative integer and then $L(\omega)_n$ is irreducible under $\mathfrak{u}^0_{\mathbf{C}}$. Moreover*

$$
\pi_\omega(D) = \frac{(s+m)^2 - 1}{2} I, \qquad \pi_\omega(J) = \left\{\frac{s_1 + s_2}{2} - i\left(\frac{m_1 + m_2}{2}\right)\right\} I,
$$

$$
\pi_\omega(D') = \frac{(s-m)^2 - 1}{2} I, \qquad \pi_\omega(J') = \left\{\frac{s_1 + s_2}{2} + i\left(\frac{m_1 + m_2}{2}\right)\right\} I.
$$

The first assertion is an immediate consequence of the Iwasawa decomposition and the Frobenius reciprocity law. Set

$$
Z_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \oplus \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},
$$

$$
X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -i \\ 0 & 0 \end{pmatrix},
$$

$$
Y_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad Y_2 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ -i & 0 \end{pmatrix},
$$

$$
W_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad W_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.
$$

Then

$$
Z \oplus 0 = \frac{Z_1 - iZ_2}{2}, \qquad 0 \oplus Z = \frac{Z_1 + iZ_2}{2},
$$

$$
X \oplus 0 = \frac{X_1 - iX_2}{2}, \qquad 0 \oplus X = \frac{X_1 + iX_2}{2},
$$

$$
Y \oplus 0 = \frac{Y_1 - iY_2}{2}, \qquad 0 \oplus Y = \frac{Y_2 + iY_2}{2}.
$$

It is clear that $\rho(D) = \lambda(D)$, that $\rho(D') = \lambda(D')$ and that $\lambda(X_i)\varphi = 0$ if φ belongs to $L(\omega)$. Thus

$$\rho(D)\varphi = \lambda(Z \oplus 0)\varphi + \frac{1}{2}\lambda\big((Z \oplus 0)^2\big)\varphi$$
$$\rho(D')\varphi = \lambda(0 \oplus Z)\varphi + \frac{1}{2}\lambda\big((0 \oplus Z)^2\big)\varphi.$$

Combining this with the relations $\lambda(Z_1)\varphi = -2(s+1)\varphi$ and $\lambda(Z_2)\varphi = -2im\varphi$ one obtains the asserted values for $\pi_\omega(D)$ and $\pi_\omega(D')$. The other two relations of the lemma are very simple to verify.

Lemma 4.2. *If neither $-s-1-|m|$ nor $s-1-|m|$ is a non-negative integer then π_ω is irreducible. If $-s-1-|m| = \frac{n_0}{2} - |m|$ is a non-negative integer then*

$$\sum_{\substack{|m| \leqslant n \leqslant n_0 \\ \frac{n}{2} - |m| \in \mathbf{Z}}} L(\omega)_n = M(\omega)$$

is invariant and the representations of $\mathfrak{A}$ on $M(\omega)$ and $L(\omega)/M(\omega)$ are irreducible. If $s-1-|m| = \frac{n_0}{2} - |m|$ is a non-negative integer then

$$\sum_{\substack{n > n_0 \\ \frac{n}{2} - |m| \in \mathbf{Z}}} L(\omega)_n = M(\omega)$$

is invariant and the representations of $\mathfrak{A}$ on $M(\omega)$ and $L(\omega)/M(\omega)$ are irreducible.

Set

$$U^+ = X \oplus -Y, \qquad U = Z \oplus -Z, \qquad U^- = Y \oplus -X,$$
$$V^+ = X \oplus Y, \qquad V = Z \oplus Z, \qquad V^- = Y \oplus X.$$

These six elements form a basis of $\mathfrak{g}^0_{\mathbf{C}}$. U^+, U, and U^- form a basis of $\mathfrak{u}^0_{\mathbf{C}}$. The space $\mathfrak{p}_{\mathbf{C}}$ spanned by V^+, V, and V^- is invariant under the adjoint action of $\mathfrak{u}^0_{\mathbf{C}}$ and the map $V^+ \to x^2$, $V \to -2xy$, $V^- \to -y^2$ extends to a $\mathfrak{u}^0_{\mathbf{C}}$-invariant map of $\mathfrak{p}_{\mathbf{C}}$ to V_2. The map $W \otimes \varphi \to \pi_\omega(W)\varphi$, $W \in \mathfrak{p}_{\mathbf{C}}$, $\varphi \in L(\omega)_n$ extends to a $\mathfrak{u}^0_{\mathbf{C}}$ invariant map of $\mathfrak{p}_{\mathbf{C}} \otimes L(\omega)_n$ into $L(\omega)$. It follows from the existence of the Clebsch-Gordan series that the image lies in $L(\omega)_{n-2} + L(\omega)_n + L(\omega)_{n+2}$. To prove the lemma all we need do is show that the image contains a non-zero element in $L(\omega)_{n+2}$ if and only if $s \neq -\big(\frac{n}{2}+1\big)$ and that if $\frac{n}{2} > |m|$ it contains a non-zero element in $L(\omega)_{n-2}$ if and only if $s \neq \frac{n}{2}$.

Let $\frac{n}{2} - k \in \mathbf{Z}$. If $|k| \leqslant \frac{n}{2}$ let $\delta_k(x,y) = x^{\frac{n}{2}+k}y^{\frac{n}{2}-k}$ and if $|k| > \frac{n}{2}$ let $\delta_k(x,y) = 0$. If $|k| \leqslant \frac{n}{2}$ let γ_k be the element of $\widetilde{V}_n$ such that $\gamma_k\Big(\sum_j \psi^j x^{\frac{n}{2}+j}y^{\frac{n}{2}-j}\Big) = \psi^k$; if $|k| > \frac{n}{2}$ let $\gamma_k = 0$. If $g = \left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) au$ with $a = \left(\begin{smallmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{smallmatrix}\right)$ in $A_{\mathbf{C}}$ and u in U^0 set

$$\varphi_{n,k}(g) = \left|\frac{\alpha_1}{\alpha_2}\right| \omega(a)\gamma_m\sigma_n(u)\delta_k \qquad |k| \leqslant \frac{n}{2}.$$

The functions $\varphi_{n,k}$ form a basis of $L(\omega)_n$.

Using the method described on pp. 129–130 of Weyl's book on quantum mechanics to decompose $\mathfrak{p}_{\mathbf{C}} \otimes V_n$ or $V_2 \otimes V_n$ into a direct sum of irreducible subspaces one finds that

$$\left(\frac{n}{2}+k\right)\left(\frac{n}{2}+k+1\right)\rho(V^+)\varphi_{n,k-1} - \left(\frac{n}{2}+k+1\right)\left(\frac{n}{2}-k+1\right)\rho(V)\varphi_{n,k} - \left(\frac{n}{2}-k\right)\left(\frac{n}{2}-k+1\right)\rho(V^-)\varphi_{n,k+1}$$

is equal to

$$\left(\frac{n}{2}+k+1\right)!\left(\frac{n}{2}-k+1\right)!\,a(n,\omega)\varphi_{n+2,k}$$

and

$$\rho(V^+)\varphi_{n,k-1}+\rho(V)\varphi_{n,k}-\rho(V^-)\varphi_{n,k+1}=\left(\frac{n}{2}+k-1\right)!\left(\frac{n}{2}-k-1\right)!\,b(n,\omega)\varphi_{n-2,k}$$

if $|k| \leqslant \frac{n}{2}-1$. The image contains a non-zero element in $L(\omega)_{n+2}$ if and only if $a(n,\omega) \neq 0$ and a non-zero element in $L(\omega)_{n-2}$ if and only if $b(n,\omega) \neq 0$. Since $\varphi_{n+2,m}(1) = 1$ and $\varphi_{n-2,m}(1) = 1$ if $\frac{n}{2} > |m|$ all we need do to find $a(n,\omega)$ and $b(n,\omega)$ is to take $k = m$ and evaluate the left sides of the above expressions at 1.

Now $V = Z_1$, $V^+ = \left(X_1 + \frac{W_1}{2}\right) - i\left(X_2 - \frac{W_2}{2}\right)$, and $V^- = \left(X_1 + \frac{W_1}{2}\right) + i\left(X_2 - \frac{W_2}{2}\right)$. Since

$$\begin{aligned}
\rho(Z_1)\varphi_{n,k}(1) &= 2(s+1)\varphi_{n,k}(1),\\
\rho(X_1)\varphi_{n,k}(1) &= \rho(X_2)\varphi_{n,k}(1) = 0,\\
\rho(W_1)\varphi_{n,k}(1) &= \gamma_m\sigma_n\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\delta_k,\\
\rho(W_2)\varphi_{n,k}(1) &= \gamma_m\sigma_n\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\delta_k,
\end{aligned}$$

one has $\rho(V)\varphi_{n,k}(1) = 2(s+1)\gamma_m\gamma_k$ and

$$\rho(V^+)\varphi_{n,k}(1) = -\gamma_m\sigma_n\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\delta_k; \quad \rho(V^-)\varphi_{n,k}(1) = \gamma_m\sigma_n\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}\delta_k.$$

Thus

$$\left(\frac{n}{2}+m+1\right)!\left(\frac{n}{2}-m+1\right)\ a(n,\omega)$$

is equal to

$$-\left(\frac{n}{2}+m\right)\left(\frac{n}{2}+m+1\right)\left(\frac{n}{2}-m+1\right)\left(\frac{n}{2}+m+1\right)\left(\frac{n}{2}-m+1\right)(s+1) - \left(\frac{n}{2}-m\right)\left(\frac{n}{2}-m+1\right)\left(\frac{n}{2}+m+1\right)$$

which equals

$$-2\left[\left(\frac{n}{2}+1\right)^2 - m^2\right]\left[s+\frac{n}{2}+1\right],$$

and

$$\begin{aligned}\left(\frac{n}{2}+m-1\right)!\left(\frac{n}{2}-m-1\right)!\,b(n,\omega) &= -\left(\frac{n}{2}-m+1\right)+2(s+1)-\left(\frac{n}{2}+m+1\right)\\ &= 2\left(s-\frac{n}{2}\right).\end{aligned}$$

Lemma 4.3. *Suppose π is an irreducible quasi-simple representation of $\mathfrak{A}$ on the vector space H. There is at least one continuous homomorphism ω of $A_{\mathbf{C}}^{\times}$ into $\mathbf{C}^{\times}$ such that*

$$\pi(D) = \frac{(s+m)^2-1}{2}I, \qquad \pi(J) = \left\{\frac{s_1+s_2}{2} - i\frac{(m_1+m_2)}{2}\right\}I,$$

$$\pi(D') = \frac{(s-m)^2-1}{2}I, \qquad \pi(J') = \left\{\frac{s_1+s_2}{2} + i\frac{(m_1+m_2)}{2}\right\}I,$$

and such that $H_{n_0} \neq 0$ for at least one n_0 with $\frac{n_0}{2} - |m|$ a non-negative integer. If ω is any such homomorphism then π is deducible from π_ω.

The lemma is a special case of a theorem of Harish-Chandra (*Representations of semi-simple Lie groups,* II, T.A.M.S., v. 76, 1954). It implies that H_n is irreducible under $\mathfrak{u}^0_{\mathbf{C}}$. A similar assertion is valid for $\mathfrak{A}^0$.

Lemma 4.4. *Suppose $\lambda(D)$, $\lambda(D')$, $\lambda(J)$, and $\lambda(J')$ are four given numbers. Apart from equivalence there are at most two quasi-simple irreducible representations of $\mathfrak{A}$ satisfying*

$$\pi(D) = \lambda(D)I, \quad \pi(D') = \lambda(D')I, \quad \pi(J) = \lambda(J)I, \quad \pi(J') = \lambda(J')I.$$

If there are two, then one of them is finite-dimensional.

If there is one such representation there is an ω such that $\lambda(D) = \frac{(s+m)^2-1}{2}$, $\lambda(D') = \frac{(s-m)^2-1}{2}$, $\lambda(J) = \frac{s_1+s_2}{2} - i\frac{(m_1+m_2)}{2}$, $\lambda(J') = \frac{s_1+s_2}{2} + i\frac{(m_1+m_2)}{2}$. If ω' is such that these representations are satisfied by s_1', s_2', m_1', m_2', one must have $\frac{s_1+s_2}{2} = \frac{s_1'+s_2'}{2}$ and $\frac{m_1+m_2}{2} = \frac{m_1'+m_2'}{2}$. In particular $m - m' = \frac{m_1-m_1'}{2} - \frac{m_2-m_2'}{2} = m_1 - m_1'$ is integral. The relations $(s+m)^2 = (s'+m')^2$ and $(s-m)^2 = (s'-m')^2$ are satisfied if and only if one of the following holds.

(i)	$s = s'$	$m = m'$	(iii)	$s = m'$	$m = s'$
(ii)	$s = -s'$	$m = -m'$	(iv)	$s = -m'$	$m = -s'$.

If $s - m$ is not integral only the first two are possible. π_ω and $\pi_{\omega'}$ are irreducible by Lemma 4.2 and equivalent by Lemma 4.3. If $s - m$ is integral one can choose ω so that $s \geqslant |m|$. It follows from Lemma 4.3 that every quasi-simple irreducible representation deducible from $\pi_{\omega'}$ is deducible from π_ω. There are only two such representations deducible from π_ω and one of them is finite-dimensional. It is clear that Lemma 4.4 could also be formulated for $\mathfrak{A}^0$.

5. The local functional equation for $\mathrm{GL}(2,\mathbf{C})$

If η is a continuous homomorphism of $A_{\mathbf{C}}$ into $\mathbf{C}^\times$ let $L(\eta)$ be the space of all U-finite infinitely differentiable functions on G satisfying

$$\varphi(ag) = \eta(a)\varphi(g)$$

for all a in $A_{\mathbf{C}}$. If φ lies in $L(\eta)$ and X lies in $\mathfrak{A}$ then $\rho(X)\varphi$ lies in $L(\eta)$ so that we have a representation $\rho(\eta)$ of $\mathfrak{A}$ on $L(\eta)$.

Lemma 5.1. *No irreducible, quasi-simple representation is contained more than once in $\rho(\eta)$.*

Let π be an irreducible, quasi-simple representation. Suppose it is deducible from π_ω and suppose its restriction to $\mathfrak{u}$ contains σ_n. If π occurs in $L(\eta)$ then $\eta_1\eta_2 = \omega_1\omega_2$ and for the proof we may as well assume that this is the case. Let $L^0(\eta)$ be the space of infinitely differentiable U^0-finite functions on $G^0_{\mathbf{C}}$ satisfying $\varphi(ag) = \eta(a)\varphi(g)$ for a in $A^0_{\mathbf{C}}$ and let $\rho^0(\eta)$ be the representation of $\mathfrak{A}^0$ on $L^0(\eta)$. We have to show that π^0 is contained at most once in $\rho^0(\eta)$.

Let $H \subseteq L^0(\eta)$ be $\mathfrak{A}^0$-invariant and suppose that the restriction of $\rho^0(\eta)$ to H is equivalent to π^0. There is a map $\varphi \to \Phi$ of H_n to V_n and a function $\Psi(g)$ on G^0 with values in $\widetilde{V}_n$ such that $\varphi(g) = \Psi(g)\Phi$ and $\Psi(gk) = \Psi(g)\pi(k)$. Let $\omega_1(t) = |t|^{s_1}\left(\frac{t}{|t|}\right)^{m_1}$, $\omega_2(t) = |t|^{s_2}\left(\frac{t}{|t|}\right)^{m_2}$, $\eta_1(t) = |t|^{r_1}\left(\frac{t}{|t|}\right)^{\ell_1}$, $\eta_2(t) = |t|^{r_2}\left(\frac{t}{|t|}\right)^{\ell_2}$. If $z = x+iy$ let $\psi(z) = \Psi\left(\left(\begin{smallmatrix}1 & z\\ 0 & 1\end{smallmatrix}\right)\right)$. Ψ is uniquely determined by ψ. Let us rewrite the equations

$$\rho(D)\Psi = \frac{\left(\frac{(s_1-s_2)+(m_1-m_2)}{2}\right)^2 - 1}{2}\Psi = \frac{(s+m)^2-1}{2}\Psi,$$

$$\rho(D)\Psi = \frac{\left(\frac{(s_1-s_2)-(m_1-m_2)}{2}\right)^2 - 1}{2}\Psi = \frac{(s-m)^2-1}{2}\Psi$$

in terms of ψ. D may be written as

$$\begin{aligned}
&2\left(\frac{X_1-iX_2}{2}\right)\left(\frac{Y_1-iY_2}{2}\right) - \left(\frac{Z_1-iZ_2}{2}\right) + \frac{1}{2}\left(\frac{Z_1-iZ_2}{2}\right)^2\\
&\quad= \frac{(X_1-iX_2)(X_1+iX_2)}{2} + \frac{(X_1-iX_2)(W_1-iW_2)}{2} - \frac{(Z_1-iZ_2)}{2} + \frac{(Z_1-iZ_2)^2}{8}
\end{aligned}$$

and D' may be written as

$$\begin{aligned}
&2\frac{(X_1+iX_2)}{2}\frac{(Y_1+iY_2)}{2} - \frac{(Z_1+iZ_2)}{2} + \frac{1}{2}\left(\frac{Z+iZ_2}{2}\right)^2\\
&\quad= \frac{(X_1+iX_2)(X_1-iX_2)}{2} + \frac{(X_1+iX_2)(W_1+iW_2)}{2} - \frac{(Z_1+iZ_2)}{2} + \frac{(Z_1+iZ_2)^2}{8}.
\end{aligned}$$

It is easily seen that

$$\rho(X_1^2+X_2^2)\Psi\left(\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right) = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 4\frac{\partial^2\psi}{\partial z\partial\overline{z}}$$

if

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}+\frac{1}{i}\frac{\partial}{\partial y}\right)\qquad \frac{\partial}{\partial \overline{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}-\frac{1}{i}\frac{\partial}{\partial y}\right)$$

and that $r=\frac{r_1-r_2}{2}$ and $\ell=\frac{\ell_1-\ell_2}{2}$.

$$\rho\big((X_1-iX_2)(W_1-iW_2)\big)\Psi\left(\begin{pmatrix}1&z\\0&1\end{pmatrix}\right)=4\frac{\partial\psi}{\partial z}\sigma_n\begin{pmatrix}0&0\\1&0\end{pmatrix}$$

$$\rho\big((X_1+iX_2)(W_1+iW_2)\big)\Psi\left(\begin{pmatrix}1&z\\0&1\end{pmatrix}\right)=-4\frac{\partial\psi}{\partial z}\sigma_n\begin{pmatrix}0&1\\0&0\end{pmatrix}$$

$$\rho(Z_1)\Psi\left(\begin{pmatrix}1&z\\0&1\end{pmatrix}\right)=\left(2r\psi-2x\frac{\partial\psi}{\partial x}-2y\frac{\partial\psi}{\partial y}\right)$$

$$\rho(Z_2)\Psi\left(\begin{pmatrix}1&z\\0&1\end{pmatrix}\right)=\left(2i\ell\psi-2y\frac{\partial\psi}{\partial x}-2x\frac{\partial\psi}{\partial y}\right).$$

Putting everything together one obtains the equations

$$2\frac{\partial^2\psi}{\partial z\partial\overline{z}}+2\frac{\partial\psi}{\partial z}\sigma_n\begin{pmatrix}0&0\\1&0\end{pmatrix}+\frac{1}{2}\left[(r+\ell-1)+2z\frac{\partial}{\partial z}\right]^2\psi=\frac{(s+m)^2}{2}\psi$$

$$2\frac{\partial^2\psi}{\partial z\partial\overline{z}}-2\frac{\partial\psi}{\partial z}\sigma_n\begin{pmatrix}0&1\\0&0\end{pmatrix}+\frac{1}{2}\left[(r-\ell-1)+2\overline{z}\frac{\partial}{\partial \overline{z}}\right]^2\psi=\frac{(s-m)^2}{2}\psi.$$

There is an auxiliary equation corresponding to the relation

$$\psi(e^{2i\theta}z)\sigma_n\begin{pmatrix}e^{i\theta}&0\\0&e^{-i\theta}\end{pmatrix}=e^{2i\ell\theta}\psi(z).$$

It is

$$-2y\frac{\partial\psi}{\partial x}+2x\frac{\partial\psi}{\partial y}+\psi\sigma_n\begin{pmatrix}i&0\\0&-i\end{pmatrix}=2i\ell\psi$$

or

$$z\frac{\partial\psi}{\partial z}-\overline{z}\frac{\partial\psi}{\partial\overline{z}}=\psi\left\{\ell I-\frac{1}{2}\sigma_n\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\}.$$

Since $\psi(z)$ is an analytic function of x and y it can be expanded in a power series

$$\sum_{p,q=0}^{\infty}z^p\overline{z}^q\psi_{p,q}.$$

According to the auxiliary equation,

$$(p-q)\psi_{p,q}=\psi_{p,q}\left\{\ell I-\frac{1}{2}\sigma_n\begin{pmatrix}1&0\\0&-1\end{pmatrix}\right\}.$$

Thus $\psi^k_{p,q} = 0$ unless $p - q = \ell - k$. Substituting in the first two equations one obtains

$$2(p+q+2)\psi^k_{p+1,q+1} + 2c_k\psi^{k-1}_{p+1,q} + \frac{1}{2}\big[(r+\ell-1)+2p\big]^2\psi^k_{p,q} = \frac{(s+m)^2}{2}\psi^k_{p,q}$$

$$2(p+q+2)\psi^k_{p+1,q+1} - 2d_k\psi^{k+1}_{p,q+1} + \frac{1}{2}\big[(r-\ell-1)+2q\big]^2\psi^k_{p,q} = \frac{(s-m)^2}{2}\psi^k_{p,q}.$$

Here $\psi^j_{p,q} = 0$ if $|j| > \frac{n}{2}$ and $c_k \neq 0$ if $-\frac{n}{2} < k \leqslant \frac{n}{2}$. The second equation can be used to determine the numbers $\psi^{n/2}_{p,q}$ inductively. Then the first can be used to determine the numbers $\psi^k_{p,q}$ for $k < \frac{n}{2}$.

Let $\xi(z) = e^{i\,\mathrm{Re}(wz)}$ with $w \neq 0$ be a character of $\mathbf{C}$ and let $L(\xi)$ be the space of all infinitely differentiable U-finite functions on $G_{\mathbf{C}}$ satisfying

(i) $\varphi\left(\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}g\right) = \xi(z)\varphi(g)$.

(ii) If $X \in \mathfrak{A}$ and $g \in G$ there is a constant M such that

$$\left|\varphi\left(\begin{pmatrix}t_1 & 0\\ 0 & t_2\end{pmatrix}g\right)\right| \leqslant M\Big(|t_1|^M + |t_2|^M\Big)$$

if $|t_1| \geqslant |t_2|$.

Let $\rho(\xi)$ be the representation of $\mathfrak{A}$ on $L(\xi)$.

Lemma 5.2. *Every quasi-simple irreducible representation of $\mathfrak{A}$ is contained at most once in $L(\xi)$.*

Let π be such a representation. Suppose π is deducible from π_ω and the restriction of π to $\mathfrak{u}$ contains σ_n. Let $L^0(\xi)$ be the space of all infinitely differentiable U^0-finite functions on $G^0_{\mathbf{C}}$ such that

(i) $\varphi\big(\left(\begin{smallmatrix}1 & z\\ 0 & 1\end{smallmatrix}\right)g\big) = \xi(z)\varphi(g)$.

(ii) If $X \in \mathfrak{A}^0$ and $g \in G^0_{\mathbf{C}}$ there is a constant M such that

$$\left|\rho(X)\varphi\left(\begin{pmatrix}t^{1/2} & 0\\ 0 & t^{-1/2}\end{pmatrix}g\right)\right| \leqslant Mt^M \text{ for } t \geqslant 1.$$

Let $\rho^0(\xi)$ be the representation of $\mathfrak{A}^0$ on $L^0(\xi)$. It is enough to show that π^0 is contained at most once in $\rho^0(\xi)$.

Suppose $H \subseteq L^0(\xi)$ is invariant and the restriction of $\rho^0(\xi)$ to H is equivalent to π^0. There is a function $\Psi(g)$ on $G^0_{\mathbf{C}}$ with values in $\widetilde{V}_n$ such that H_n is the set of functions of the form $\Psi(g)\Phi$, $\Phi \in V_n$. $\Psi(gu) = \Psi(g)\sigma_n(u)$ if $u \in U^0$. Let $\psi(t) = \Psi\left(\left(\begin{smallmatrix}t^{1/2} & 0\\ 0 & t^{-1/2}\end{smallmatrix}\right)\right)$ for $t > 0$. Ψ is completely determined by ψ. It is necessary to write the equations $\rho(D)\Psi = \frac{(s+m)^2-1}{2}\Psi$

and $\rho(D')\Psi = \frac{(s-m)^2-1}{2}\Psi$ in terms of ψ. It is easy to verify that

$$\rho(X_1^2+X_1^2)\Psi\left(\begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2}\end{pmatrix}\right) = -t^2|w|^2\psi(t)$$

$$\rho\big((X_1-iX_2)(W_1-iW_2)\big)\Psi\left(\begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2}\end{pmatrix}\right) = 2tiw\psi(t)\sigma_n\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}$$

$$\rho\big((X_1+iX_2)(W_1+iW_2)\big)\Psi\left(\begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2}\end{pmatrix}\right) = -2ti\overline{w}\psi(t)\sigma_n\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$$

$$\rho(Z_1)\Psi\left(\begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2}\end{pmatrix}\right) = 2t\frac{\partial\psi}{\partial t}$$

$$\rho(Z_2)\Psi\left(\begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2}\end{pmatrix}\right) = \psi(t)\sigma_n\begin{pmatrix} i & 0 \\ 0 & -i\end{pmatrix}.$$

Thus

$$\frac{1}{2}t\frac{d}{dt}\left(t\frac{d\psi}{dt}\right) - t\frac{d\psi}{dt}\left\{I-\frac{1}{2}\sigma_n\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right\} + \frac{1}{2}\psi\left\{I-\frac{1}{2}\sigma_n\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right\}^2 - \frac{t^2|w|^2}{2}\psi + tiw\psi\sigma_n\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} = \frac{(s+m)^2}{2}\psi$$

$$\frac{1}{2}t\frac{d}{dt}\left(t\frac{d\psi}{dt}\right) - t\frac{d\psi}{dt}\left\{I+\frac{1}{2}\sigma_n\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right\} + \frac{1}{2}\psi\left\{I+\frac{1}{2}\sigma_n\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right\}^2 - \frac{t^2|w|^2}{2}\psi + ti\overline{w}\psi\sigma_n\begin{pmatrix} 0 & -1 \\ 0 & 0\end{pmatrix} = \frac{(s-m)^2}{2}\psi.$$

In terms of components these equations are

$$\text{(A)}\qquad \begin{aligned} &\frac{1}{2}\left[t\frac{d}{dt}+k-1\right]^2\psi^k - \frac{t^2|w|^2}{2}\psi^k + c_k tiw\psi^{k-1} = \frac{(s+m)^2}{2}\psi^k, \\ &\frac{1}{2}\left[t\frac{d}{dt}-k-1\right]^2\psi^k - \frac{t^2|w|^2}{2}\psi^k - d_k ti\overline{w}\psi^{k+1} = \frac{(s-m)^2}{2}\psi^k, \end{aligned}$$

where $\psi^j = 0$ if $|j| > \frac{n}{2}$. Since $c_k \neq 0$ for $-\frac{n}{2} < k \leqslant \frac{n}{2}$ and $d_k \neq 0$ for $-\frac{n}{2} \leqslant k < \frac{n}{2}$ these equations allow one to solve for all ψ^k in terms of $\psi^{n/2}$ or $\psi^{-n/2}$.

For $k = \frac{n}{2}$ the second equation is

$$\frac{1}{2}\left[t\frac{d}{dt}-\frac{n}{2}-1\right]^2\psi^{n/2} - \frac{t^2|w|^2}{2}\psi^{n/2} = \frac{(s-m)^2}{2}\psi^{n/2}$$

which may be written as

$$\frac{1}{2}\frac{d^2\psi^{n/2}}{dt^2}+\left(-\frac{1}{2}-\frac{n}{2}\right)\frac{1}{t}\frac{d\psi^{n/2}}{dt}+\left(-\frac{|w|^2}{2}+\frac{\left(\frac{n}{2}+1\right)^2}{2t^2}\right)\psi^{n/2}=\frac{(s-m)^2}{2t^2}\psi^{n/2}.$$

Dropping the terms in $\frac{1}{t}$ and $\frac{1}{t^2}$ one obtains an equation with the solutions $e^{\pm|w|t}$. Thus the given equation has one solution of the form $t^{-\mu}e^{-|w|t}\Big(1+O\big(\frac{1}{t}\big)\Big)$ and one of the form $t^{-\nu}e^{|w|t}\Big(1+O\big(\frac{1}{t}\big)\Big)$. Since $\psi^{n/2}(t)=O(t^M)$ as $t\to\infty$ it must be a multiple of the first solution. The lemma follows.

To find μ we examine the formal solution

$$\psi^{n/2}(t)=t^{-\mu}e^{-|w|t}\sum_{n=0}^{\infty}a_nt^{-n}.$$

If $a_{-1}=a_{-2}=0$ the first derivative is

$$e^{-|w|t}\left\{\sum_{n=0}^{\infty}-\big(|w|a_n+(\mu+n-1)a_{n-1}\big)t^{-n-\mu}\right\}$$

and the second derivative is

$$e^{-|w|t}\left\{\sum_{n=0}^{\infty}\big(|w|^2a_n+|w|(2\mu+2n-1)a_{n-1}+(\mu+n-1)(\mu+n-2)a_{n-2}\big)t^{-n-\mu}\right\}.$$

Substituting into the equation, dividing by $e^{-|w|t}$, and equating coefficients of $t^{\mu-1}$ we obtain $\mu+\frac{n}{2}=0$.

For $k=-\frac{n}{2}$ the first of the equations (A) is

$$\frac{1}{2}\left[t\frac{d\psi^{-n/2}}{dt}-\frac{n}{2}-1\right]^2\psi^{-n/2}-\frac{t^2|w|^2}{2}\psi^{-n/2}=\frac{(s+m)^2}{2}\psi^{-n/2}.$$

This is the equation just discussed except that $-m$ is replaced by m. Thus if $|k|=\frac{n}{2}$ then $\psi^k(t)=t^{|k|}e^{-|w|t}\Big(1+O\big(\frac{1}{t}\big)\Big)$ as $t\to\infty$.

During the preceding discussion we have assumed the existence of H and thus the existence of solutions of equations (A) which satisfy the required growth condition. We continue with our discussion of this assumption. Since 0 is a singular point of the first kind for the first equations of (A) there is an N such that, for all k, $\psi^k(t)=O\big(\frac{1}{t^N}\big)$ as $t\to 0$. Thus

$$\theta^k(u)=\int_0^\infty\psi^k(t)t^{u-1}\,dt$$

is defined for $\operatorname{Re}u$ sufficiently large. These functions satisfy the difference equations

$$|w|^2\theta^k(u+2)=\big[(u-k+1)^2-(s+m)^2\big]\theta^k(u)+2ic_kw\theta^{k-1}(u+1)$$
$$|w|^2\theta^k(u+2)=\big[(u+k+1)^2-(s-m)^2\big]\theta^k(u)-2id_k\overline{w}\theta^{k+1}(u+1).$$

Lemma 5.3. *If $|k|=\frac{n}{2}$ then $\theta^k(u)$ is a multiple of*

$$\left(\frac{2}{|w|}\right)^u\Gamma\left(\frac{u+1+s+|k-m|}{2}\right)\Gamma\left(\frac{u+1-s+|k+m|}{2}\right).$$

Since $\frac{n}{2} \geqslant |m|$ the second of the difference equations is, when $k = \frac{n}{2}$, just

$$|w|^2\theta^k(u+2) = \left(u+1+s+\left|\frac{n}{2}-m\right|\right)\left(u+1-s+\left|\frac{n}{2}+m\right|\right)\theta^k(u)$$

which is an equation satisfied by the function of the lemma. Thus the inverse Mellin transform of that function, which is bounded by a power of t, satisfies the differential equation determining $\psi^{n/2}$ and must be a multiple of $\psi^{n/2}$. A similar argument proves the lemma when $k = -\frac{n}{2}$.

Lemma 5.4. *If $|m| = \frac{n}{2}$ the functions*

$$\theta_0^k(u) = \frac{2^u}{|w|^u}\left(\frac{iw}{|w|}\right)^{k-\frac{n}{2}}\Gamma\left(\frac{u+1+s+|k-m|}{2}\right)\Gamma\left(\frac{u+1-s+|k+m|}{2}\right)$$

satisfy the difference equations. They are the only solutions of the equations for which

$$\theta_0^{n/2}(u) = \frac{2^u}{|w|^u}\Gamma\left(\frac{u+1+s+\left|\frac{n}{2}-m\right|}{2}\right)\Gamma\left(\frac{u+1-s+\left|\frac{n}{2}+m\right|}{2}\right).$$

The uniqueness is evident from the form of the equations. It is convenient to treat the cases $m = \frac{n}{2}$ and $m = -\frac{n}{2}$ separately when verifying that they satisfy the equations. If $m = \frac{n}{2}$ then $|w|^2\theta_0^k(u+2) - 2ic_k w\theta_0^{k-1}(u+1)$ is equal to

$$\begin{aligned}
&\frac{2^u}{|w|^u}\left(\frac{iw}{|w|}\right)^{k-\frac{n}{2}} \\
&\quad\times\left\{\left(u+1+s+\frac{n}{2}-k\right)\left(u+1-s+\frac{n}{2}+k\right)-2\left(\frac{n}{2}+k\right)\left(u+1+s+\frac{n}{2}-k\right)\right\} \\
&\qquad\qquad\times\Gamma\left(\frac{u+1+s+\frac{n}{2}-k}{2}\right)\Gamma\left(\frac{u+1-s+\frac{n}{2}+k}{2}\right) \\
&\qquad\qquad\qquad= \left(u+1+s+\frac{n}{2}-k\right)\left(u+1-s-\frac{n}{2}-k\right)\theta_0^k(u)
\end{aligned}$$

and $|w|^2\theta_0^k(u+2) + 2d_k i\overline{w}\theta_0^{k+1}(u+1)$ is equal to

$$\begin{aligned}
&\frac{2^u}{|w|^u}\left(\frac{iw}{|w|}\right)^{k-\frac{n}{2}} \\
&\quad\times\left\{\left(u+1+s+\frac{n}{2}-k\right)\left(u+1-s+\frac{n}{2}+k\right)-2\left(\frac{n}{2}-k\right)\left(u+1+s+\frac{n}{2}-k\right)\right\} \\
&\qquad\qquad\qquad\times\Gamma\left(\frac{u+1+s+\frac{n}{2}-k}{2}\right)\Gamma\left(\frac{u+1-s+\frac{n}{2}+k}{2}\right)
\end{aligned}$$

or

$$\left(u+1-s+\frac{n}{2}+k\right)\left(u+1+s-\frac{n}{2}+k\right)\theta_0^k(u).$$

It is not necessary to treat the case $m = -\frac{n}{2}$ because the equations are not changed if θ^k is replaced by θ^{-k}, m by $-m$, and w by $-\overline{w}$.

Corollary. *The quasi-simple irreducible representation π is contained in $\rho(\xi)$ if and only if π is infinite-dimensional.*

It is enough to show that π^0 is contained in $\rho^0(\xi)$ if and only if π^0 is infinite-dimensional. Suppose H is a finite-dimensional invariant subspace of $L^0(\xi)$. Let τ be the restriction of $\rho^0(\xi)$ to $L^0(\xi)$ and let $\widetilde{\tau}$ be the representation contragredient to τ. If $X_z = \left(\begin{smallmatrix} 0 & z \\ 0 & 0 \end{smallmatrix}\right)$ lies in g^0 all the eigenvalues of $\widetilde{\tau}(X_z)$ must be zero because $\widetilde{\tau}$ is finite-dimensional. On the other hand if $\widetilde{\varphi}$ is the element of $\widetilde{H}$, the dual space of H, defined by $\widetilde{\varphi}(\varphi) = \varphi(1)$, $\varphi \in H$, then

$$\big(\widetilde{\tau}(X_z)\widetilde{\varphi}\big)(\varphi) = -\widetilde{\varphi}\big(\tau(X_z)\varphi\big) = -\rho(X_z)\varphi(1) = -izw\varphi(1)$$

so that $-izw$ is an eigenvalue of X_z. This is a contradiction.

Suppose π is deducible from π_ω. Let W be the set of all functions in $L^0(\xi)$ satisfying $\rho(D)\varphi = \frac{(s+m)^2-1}{2}\varphi$, $\rho(D')\varphi = \frac{(s-m)^2-1}{2}\varphi$, and $\varphi\Big(g\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\Big) = (-1)^{2m}\varphi(g)$. If $\theta = \big\{\, n \bigm| W_n \neq \{0\} \,\big\}$ then $W = \sum_{n\in\theta} W_n$. Combining the results of the previous lemma with the arguments used to prove Lemma 5.3 one sees that when $\frac{n}{2} = |m|$ the equations (A) have a solution satisfying the desired growth conditions. Thus W_n is not zero for $n = \left|\frac{m}{2}\right|$. Although it is not important at present, I observe that if $s-m$ is integral then $W_{|2s|}$ is also not zero. The proof of Lemma 5.2 shows that W_n is irreducible under $\mathfrak{u}^0$, the Lie algebra of U^0. Consequently every invariant subspace is of the form $W(\sigma) = \sum_{n\in\sigma} W_n$ where σ is a subset of θ. Suppose $\sigma_0 \supsetneqq \sigma_3$ and $W(\sigma_0)$ and $W(\sigma_3)$ are invariant. Let $n_0 \in \sigma_0$, $n_0 \notin \sigma_3$. There is a minimal element in $\big\{\, \sigma \bigm| W(\sigma) \text{ is invariant, } \sigma_3 \subseteq \sigma \subseteq \sigma_0,\ n_0 \in \sigma_1 \,\big\}$; let it be σ_1. There is a maximal element in $\big\{\, \sigma \bigm| W(\sigma) \text{ is invariant, } \sigma_3 \subseteq \sigma \subseteq \sigma_1,\ n_0 \notin \sigma \,\big\}$; let it be σ_2. The representation of $\mathfrak{A}^0$ on $W(\sigma_1)/W(\sigma_2)$ is irreducible. Thus there is an irreducible representation deducible from the representation of $\mathfrak{A}^0$ on $W(\sigma_0)/W(\sigma_3)$. Suppose W itself is not irreducible and let $W(\sigma_1)$ be a proper invariant subspace. If $W(\sigma_1)$ were not irreducible there would be a proper invariant subspace $W(\sigma_2)$. No two of the irreducible representations deduced from the representations on $W/W(\sigma_1)$, $W(\sigma_1)/W(\sigma_2)$, $W(\sigma_2)$ could be equivalent because the restrictions to $\mathfrak{u}^0_{\mathbf{C}}$ would not be equivalent. This would contradict Lemma 4.4. For the same reason the representation on $W/W(\sigma_1)$ is irreducible. Thus either W is irreducible or W contains a proper invariant irreducible subspace W_1 such that W/W_1 is irreducible. Combining Lemma 4.4 with the earlier observations about finite-dimensional representations one sees that if π is infinite-dimensional the representation of $\mathfrak{A}^0$ on W is equivalent to π^0 if W is irreducible and that if W is not irreducible the representation of $\mathfrak{A}^0$ on W_1 is equivalent to π^0.

We return to the study of the functions $\psi^k(t)$.

Lemma 5.5. *If $s-m$ is not an integer then, near 0, $\psi^k(t)$ can be expanded in a series of the form*

$$\psi^k(t) = t^{|k-m|+1+s}\sum_{p=0}^{\infty} a_p^k t^{2p} + t^{|k+m|+1-s}\sum_{p=0}^{\infty} b_p^k t^{2p}.$$

If $s-m$ is an integer t and $|k+m|-s \geqslant |k-m|+s$ then

$$\psi^k(t) = t^{|k-m|+1+s}\sum_{p=0}^{\infty} a_p^k t^{2p} + t^{|k+m|+1-s}\log t\sum_{p=0}^{\infty} b_p^k t^{2p}$$

but if $|k+m|-s \leqslant |k-m|+s$ *then*

$$\psi^k(t) = t^{|k-m|+1+s} \log t \sum_{p=0}^{\infty} a_p^k t^{2p} + t^{|k+m|+1-s} \sum_{p=0}^{\infty} b_p^k t^{2p}.$$

As before when $k = \frac{n}{2}$ the second equation of (A) is

$$\frac{1}{2}\left[t\frac{d}{dt} - \frac{n}{2} - 1\right]^2 \psi^{n/2} - \frac{t^2|w|^2}{2} = \frac{(s-m)^2}{2}\psi^{n/2}.$$

The indicial equation $\frac{1}{2}\left[\lambda - \frac{n}{2} - 1\right]^2 = \frac{(s-m)^2}{2}$ has the roots $\lambda_1 = \frac{n}{2} + 1 - s + m$ and $\lambda_2 = \frac{n}{2} + 1 + s - m$ and $\lambda_1 - \lambda_2 = 2(m-s)$. The series $t^{\lambda_i} \sum_{p=0}^{\infty} a_p t^p$ will satisfy the equation if and only if

$$\left\{\left[\lambda_i + p - \frac{n}{2} - 1\right]^2 - (s-m)^2\right\} c_p = |w|^2 c_{p-2}.$$

Since $\left|\frac{n}{2} \pm m\right| = \frac{n}{2} \pm m$ the assertion of the lemma for $k = \frac{n}{2}$ follows from an application of the method of Frobenius.

To prove the lemma for general k we use induction and the equation

$$-c_k i w \psi^{k-1}(t) = \frac{1}{2t}\left\{\left[t\frac{d}{dt} + k - 1\right]^2 \psi^k - (s+m)^2\psi^k - t^2|w|^2\psi^k\right\}.$$

The symbol $A(t)$ will stand for a convergent series of the form $\sum_{p=0}^{\infty} a_p t^{2p}$ and $B(t)$ will stand for a convergent series of the form $\sum_{p=1}^{\infty} b_p t^{2p}$. The series represented by these symbols will vary but not within a given formula. One has

$$\begin{aligned}\frac{1}{2t}&\left\{\left[t\frac{d}{dt} + k - 1\right]^2 t^{|k\mp m|+1\pm s}A(t) - (s+m)^2 t^{|k\mp m|+1\pm s}A(t) - t^2|w|^2 A(t)\right\}\\ &= \frac{1}{2t}\left\{\left[\left(|k\mp m| + k \pm s\right)^2 - (s+m)^2\right] a_0 t^{|k\mp m|+1\pm s} + t^{|k\mp m|+1\pm s}B(t)\right\}.\end{aligned}$$

If $k > \pm m$ so that $|k-1\mp m| = |k\mp m| - 1$, this is of the form $t^{|k-1\mp m|+1\pm s}A(t)$. If $k \leqslant \pm m$ then $|k\mp m| = \pm m - k$ and $\left(|k\mp m| + k \pm s\right)^2 - (s+m)^2 = 0$ and it is of the form $t^{|k-1\mp m|+1\pm s}B(t)$ because $|k-1\mp m| = |k\mp m| + 1$. The first statement of the lemma follows immediately.

If $F(t)$ is any function

$$\left[t\frac{d}{dt} + k - 1\right] \log t F(t) = F(t) + \log t \left[t\frac{d}{dt} + k - 1\right] F(t)$$

and

$$\left[t\frac{d}{dt} + k - 1\right]^2 \log t F(t) = 2\left[t\frac{d}{dt} + k - 1\right] F(t) + \log t \left[t\frac{d}{dt} + k - 1\right]^2 F(t).$$

Thus

$$\frac{1}{2t}\left\{\left[t\frac{d}{dt}+k-1\right]^2 t^{|k\mp m|+1\pm s}\log tA(t)\right.$$
$$\left.-(s+m)^2 t^{|k\mp m|+1\pm s}\log tA(t)-t^2|w|^2 t^{|k\mp m|+1\pm s}\log tA(t)\right\}$$

is equal to the sum of a term of the form $t^{|k-1\mp m|+1\pm s}\log tA(t)$ and

$$\text{(B)}\qquad \frac{1}{t}\Big\{\big(|k\mp m|\pm s+k\big)t^{|k\mp m|+1\pm s}a_0+t^{|k\mp m|+1\pm s}B(t)\Big\}$$

Suppose $s-m$ is an integer and the assertions of the lemma are true for a given k. Let $|k\mp m|\pm s\geqslant |k\pm m|\mp s$ (Either all the top signs or all the bottom signs are taken). If $\big(|k\mp m|\pm s\big)-\big(|k\pm m|\mp s\big)$, which is an integer, is at least two then $|k\mp m|\pm s\geqslant |k-1\pm m|+1\mp s$ and the expression (B) is of the form $t^{|k-1\pm m|+1\mp s}$. Since $|k-1\mp m|\pm s$ will still be greater than or equal to $|k-1\pm m|\mp s$ the induction goes through.

The remaining possibility is $|k\mp m|\pm s=|k\pm m|\mp s$. If $k>\mp m$ then $|k-1\pm m|=|k\pm m|-1$ so that $|k-1\mp m|\pm s\geqslant |k-1\pm m|\mp s$ and the expression (B) is of the form $t^{|k-1\pm m|+1\mp s}A(t)$. If $k>\pm m$ then $|k-1\mp m|=|k\mp m|-1$ so that $|k-1\pm m|\mp s\geqslant |k-1\mp m|\pm s$ and the expression (B) is of the form $t^{|k-1\mp m|+1\pm s}A(t)$.

Thus we have only to treat the case that $k\leqslant \mp m$, $k\leqslant \pm m$ and $|k\mp m|\pm s=|k\pm m|\mp s$. Then $|k\mp m|=\pm m-k$ and $|k\pm m|=\mp m-k$ so $\pm m-k\pm s=\mp m-k\mp s$ or $m+s=0$ and $|k\mp m|\pm s+k=\pm(m+s)=0$. Thus $|k-1\mp m|\pm s=|k-1\pm m|\mp s$ and the expression (B) is of the form $t^{|k-1\pm m|+1\mp s}A(t)$.

Let $\psi(t)$ be the function with components $\psi^k(t)$. If 2ℓ is an integer and z is a fixed complex number set

$$\theta(u,\ell;z)=\int_0^\infty\left\{\frac{1}{4\pi}\int_0^{4\pi}e^{it\operatorname{Re}(e^{i\theta}z)}\psi(t)\sigma_n\begin{pmatrix}e^{\frac{i\theta}{2}} & 0\\ 0 & e^{-\frac{i\theta}{2}}\end{pmatrix}e^{-i\ell\theta}\,d\theta\right\}t^{u-1}\,dt.$$

The integral converges for $\operatorname{Re}u$ sufficiently large. The kth component of $\theta(u,\ell;z)$ is

$$\theta^k(u,\ell;z)=\int_0^\infty\left\{\frac{1}{4\pi}\int_0^{4\pi}e^{it\operatorname{Re}(e^{i\theta}z)}e^{i(k-\ell)}\,d\theta\right\}\psi^k(t)t^{u-1}\,dt.$$

Lemma 5.6. *For each ℓ and z the function*

$$\frac{\theta^k(u,\ell;z)}{\Gamma\Big(\frac{u+1+s+|\ell-m|}{2}\Big)\Gamma\Big(\frac{u+1-s+|\ell+m|}{2}\Big)}$$

is an entire function of u. Moreover $\theta^k(u,\ell;z)$ is bounded in any region of the form $|\operatorname{Re}u|\leqslant$ constant, $|\operatorname{Im}u|\geqslant$ constant $\gg 0$.

Let $m(t)$ be an infinitely differentiable function with compact support on the real line which is 1 in a neighbourhood of 0. Then $\theta^k(u,\ell;z)$ is the sum of

$$\widehat{\theta}^k(u,\ell;z)=\int_0^\infty\left\{\frac{1}{4\pi}\int_0^{4\pi}e^{it\operatorname{Re}(e^{i\theta}z)}e^{i(k-\ell)}\,d\theta\right\}\psi^{k(t)}t^{u-1}m(t)\,dt$$

and

$$\int_0^\infty \left\{ \frac{1}{4\pi} \int_0^{4\pi} e^{it\,\mathrm{Re}(e^{i\theta}z)} e^{i(k-\ell)}\, d\theta \right\} \psi^k(t) t^{u-1} \bigl(1-m(t)\bigr)\, dt.$$

The second integral defines an entire function of u which is bounded on vertical strips so it will be enough to prove the lemma with $\theta^k(u,\ell;z)$ replaced by $\widehat{\theta}^k(u,\ell;z)$.

The inner integral is equal to

$$\sum_{r=0}^\infty \frac{(it)^r}{2^r r!} \cdot \frac{1}{4\pi} \int_0^{4\pi} (e^{i\theta}z + e^{-i\theta}\overline{z})^r e^{i(k-\ell)\theta}\, d\theta.$$

It is zero if $k-\ell$ is not integral. If $k-\ell$ is integral let Δ be the set of integers r satisfying (i) $r \geqslant |k-\ell|$ and (ii) $\frac{r+\ell-k}{2}$ is integral. Then this expression equals

$$\sum_{r\in\Delta} \frac{(it)^r}{2^r} \frac{z^{r+\ell-k}}{\left(\frac{r+\ell-k}{2}\right)!} \frac{\overline{z}^{r+k-\ell}}{\left(\frac{r+k-\ell}{2}\right)!}.$$

If a real number c is given there is an R such that

$$\int_0^\infty \left\{ \sum_{\substack{r\in\Delta \\ r\geqslant R}} \frac{(it)^r}{2^r} \frac{z^{r+\ell-k}}{\left(\frac{r+\ell-k}{2}\right)!} \frac{\overline{z}^{r+k-\ell}}{\left(\frac{r+k-\ell}{2}\right)!} \right\} \psi^{k(t)} t^{u-1} m(t)\, dt$$

is analytic and bounded for $\mathrm{Re}\, u > c$. We need only study the analytic properties of

$$\int_0^\infty \psi^k(t) t^{r+u-1} m(t)\, dt \qquad r \in \Delta.$$

The same observation when combined with Lemma 5.5 shows that when $s-m$ is not an integer we need only study

$$\int_0^\infty t^{|k\pm m|\mp s+r+u+2p} m(t)\, dt \qquad r\in\Delta,\ p\in\mathbf{Z},\ p\geqslant 0$$

and that when $s-m$ is integral and $|k\mp m|\pm s \geqslant |k\pm m|\mp s$ we need only study

$$\int_0^\infty t^{|k\pm m|\mp s+r+u+2p} m(t)\, dt \qquad r\in\Delta,\ p\in\mathbf{Z},\ p\geqslant 0$$

and

$$\int_0^\infty t^{|k\mp m|\pm s+r+u+2p} m(t) \log t\, dt \qquad r\in\Delta,\ p\in\mathbf{Z},\ p\geqslant 0.$$

The second assertion of the lemma is going to be obvious and only the first will have to be dealt with explicitly. The first is going to follow from the observation that if $s-m$ is not an integer the denominator in the lemma has poles of order 1 at $-1\mp s-|\ell\mp m|-2$, $q\in\mathbf{Z}$, $q\geqslant 0$ and no zeros and that if $s-m$ is an integer and $|\ell\mp m|\pm s\geqslant|\ell\pm m|\mp s$ it has poles of order at least one at $-1\pm s-|\ell\pm m|-2q$ and poles of order two at $-1\mp s-|\ell\mp m|-2q$, $q\in\mathbf{Z}$, $q\geqslant 0$.

It has to be shown that these poles cancel the singularities of the numerator.

$$\int_0^\infty t^{a+u} m(t)\,dt = \frac{-1}{a+u+1}\int_0^\infty t^{a+u+1} m'(t)\,dt$$

$$\int_0^\infty t^{a+u}\log t\, m(t)\,dt = \frac{-1}{(a+u+1)^2}\int_0^\infty \big[(a+u+1)t^{a+u+1}\log t - t^{a+u+1}\big] m'(t)\,dt.$$

Since $m'(t)$ vanishes near 0 the first integral has at most a pole of order one at $-(a+1)$ and no other singularities while the second has at most a pole of order two at $-(a+1)$.

If $s-m$ is not an integer the lemma will follow if it is shown that, for $r\in\Delta$, $|k\pm m|+r = |\ell\pm m|+2q$, $q\in\mathbf{Z}$, $q\geqslant 0$. This is so because $r=|k-\ell|+2p$, $p\in\mathbf{Z}$, $p\geqslant 0$ and $|k\pm m|+|k-\ell|-|\ell\pm m|$ is a non-negative even integer. If $s-m$ is an integer one has to show in addition that if $|\ell\mp m|-|\ell\pm m|\pm 2s>0$ and $|k\pm m|-|k\mp m|\mp 2s>0$ (either all upper or all lower signs are taken so there are only two possibilities) then

$$|k\pm m|\mp s+|k-\ell| = |\ell\mp m|\pm s+2q \qquad q\in\mathbf{Z},\ q\geqslant 0.$$

The left side is

$$|k\mp m|\pm s+|k-\ell|+\big\{|k\pm m|-|k\mp m|\mp 2s\big\}.$$

The expression in brackets, which is a non-negative integer, is by assumption positive.

If π is an infinite-dimensional irreducible quasi-simple representation of $\mathfrak{A}$ let $L(\xi,\pi)$ be the unique subspace of $L(\xi)$ such that the restriction of $\rho(\xi)$ to $L(\xi,\pi)$ is equivalent to π. It follows from the proof of Lemma 4.4 that there is an ω such that π is equivalent to π_ω. As usual let $\omega_1(\alpha)=\omega\big(\left(\begin{smallmatrix}\alpha&0\\0&1\end{smallmatrix}\right)\big)$, $\omega_2(\alpha)=\omega\big(\left(\begin{smallmatrix}1&0\\0&\alpha\end{smallmatrix}\right)\big)$ for $\alpha\in\mathbf{C}^\times$ and let $\omega_i(te^{i\theta})=t^{s_i}e^{im_i\theta}$ for $t>0$.

If η is any character of $A_{\mathbf{C}}$ then $\widetilde{\eta}$ is the character defined by $\widetilde{\eta}\Big(\left(\begin{smallmatrix}\alpha_1&0\\0&\alpha_2\end{smallmatrix}\right)\Big)=\eta\Big(\left(\begin{smallmatrix}\alpha_2&0\\0&\alpha_1\end{smallmatrix}\right)\Big)$. If ζ is a character of $A_{\mathbf{C}}^\times$ such that $\zeta\big(\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)\big)\omega\big(\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)\big)\equiv 1$ and u and ℓ are defined by $\zeta\Bigg(\left(\begin{smallmatrix}t^{1/2}e^{i\theta/2}&0\\0&t^{-1/2}e^{-i\theta/2}\end{smallmatrix}\right)\Bigg)=t^u e^{i\ell\theta}$ then ζ is uniquely determined by u and ℓ and we shall occasionally write $\zeta=\zeta(u,\ell)$.

Lemma 5.7. *Suppose* $\zeta\big(\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)\big)\omega\big(\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)\big)\equiv 1$. *If* $\varphi\in L(\xi,\pi)$ *and* $\zeta=\zeta(u,\ell)$ *the function*

$$\Phi(g,\zeta,\varphi)=\int_0^\infty\left\{\frac{1}{2\pi}\int_0^{2\pi}\varphi\left(\begin{pmatrix}te^{i\theta}&0\\0&1\end{pmatrix}g\right)\zeta\left(\begin{pmatrix}te^{i\theta}&0\\0&1\end{pmatrix}\right)d\theta\right\}\frac{dt}{t}$$

is defined for $\operatorname{Re}u$ *sufficiently large. Set*

$$\Phi'(g,\zeta,\varphi)=\frac{\Phi(g,\zeta,\varphi)}{\Gamma\Big(\frac{u+1+s+|\ell+m|}{2}\Big)\Gamma\Big(\frac{u+1-s+|\ell-m|}{2}\Big)}.$$

Then $\Phi'\big(g,\zeta(u,\ell),\varphi\big)$ *is an entire function of* u *and* $\Phi\big(g,\zeta(u,\ell),\varphi\big)$ *is bounded in regions of the form* $|\operatorname{Re}u|\leqslant$ *constant*, $|\operatorname{Im}u|\geqslant$ *constant* $\gg 0$. *Moreover*

$$\left(\frac{2}{|w|}\right)^{-u}\Phi'\left(\begin{pmatrix}0&1\\-1&0\end{pmatrix}g,\zeta,\varphi\right)=\gamma(\ell,m)\left(\frac{iw}{|w|}\right)^{-2\ell}\left(\frac{2}{|w|}\right)^{u}\Phi'(g,\widetilde{\zeta},\varphi)$$

if $\gamma(\ell,m)=(-1)^{|\ell|+\ell}$ *for* $|\ell|\geqslant|m|$ *and* $\gamma(\ell,m)=(-1)^{|m|+\ell}$ *for* $|\ell|\leqslant|m|$.

It is enough to prove the lemma for φ in $L(\xi,\pi)_n$. There is a Φ in V_n such that if $g = a\left(\begin{smallmatrix}1 & z\\ 0 & 1\end{smallmatrix}\right)u$ with $a = \begin{pmatrix} t_1e^{i\theta_1} & 0\\ 0 & t_2e^{i\theta_2}\end{pmatrix}$ and u in U then $\varphi\Big(\left(\begin{smallmatrix}te^{i\theta} & 0\\ 0 & 1\end{smallmatrix}\right)g\Big)\zeta\Big(\left(\begin{smallmatrix}te^{i\theta} & 0\\ 0 & 1\end{smallmatrix}\right)\Big)$ is equal to the product of

$$e^{\frac{itt_1}{t_2}\,\mathrm{Re}(e^{i(\theta+\theta_1-\theta_2)}wz)}\omega\left(\begin{pmatrix}\sqrt{tt_1t_2}e^{i(\theta+\theta_1+\theta_2)} & 0\\ 0 & \sqrt{tt_1t_2}e^{i(\theta+\theta_1+\theta_2)}\end{pmatrix}\right)\zeta\left(\begin{pmatrix}te^{i\theta} & 0\\ 0 & 1\end{pmatrix}\right)$$

and

$$\psi\left(\frac{tt_1}{t_2}\right)\sigma_n\left(\begin{pmatrix}e^{\frac{i(\theta+\theta_1-\theta_2)}{2}} & 0\\ 0 & e^{\frac{i(\theta_2-\theta-\theta_1)}{2}}\end{pmatrix}u\right)\Phi$$

which equals the product of

$$\zeta^{-1}\left(\begin{pmatrix}t_1e^{i\theta_1} & 0\\ 0 & t_2e^{i\theta_2}\end{pmatrix}\right)e^{i\frac{tt_1}{t_2}\,\mathrm{Re}(e^{i(\theta+\theta_1+\theta_2)}wz)}\zeta\left(\begin{pmatrix}\sqrt{\frac{tt_1}{t_2}}e^{i\frac{(\theta+\theta_1-\theta_2)}{2}} & 0\\ 0 & \sqrt{\frac{t_2}{tt_1}}e^{i\frac{(\theta_2-\theta-\theta_2)}{2}}\end{pmatrix}\right)$$

and

$$\psi\left(\frac{tt_1}{t_2}\right)\sigma_n\left(\begin{pmatrix}e^{i\frac{(\theta+\theta_1-\theta_2)}{2}} & 0\\ 0 & e^{i\frac{(\theta_1-\theta-\theta_2)}{2}}\end{pmatrix}u\right)\Phi.$$

Consequently $\Phi\big(g,\zeta(u,\ell),\varphi\big)$ is equal to

$$\zeta^{-1}\left(\begin{pmatrix}t_1e^{i\theta_1} & 0\\ 0 & t_2e^{i\theta_2}\end{pmatrix}\right)\theta(u,-\ell,wz)\sigma_n(u)\Phi.$$

The first two assertions of the lemma follow immediately.

If $\eta = \widetilde{\zeta}^{-1}$ the maps

$$\varphi \longrightarrow \Phi'(g,\widetilde{\zeta},\varphi),$$
$$\varphi \longrightarrow \Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g,\zeta,\varphi\right)$$

are easily seen to be $\mathfrak{A}$-invariant maps of $L(\xi,\pi)$ into $L(\eta)$. It follows from Lemma 5.1 that one is a multiple of the other. To see what the multiple is choose $g=1$ and φ as above with $\Phi = \delta_\ell$. Then

$$\begin{aligned}\Phi'(1,\widetilde{\zeta},\varphi) &= \frac{\theta^\ell(-u)}{\Gamma\left(\frac{-u+1+s+|\ell-m|}{2}\right)\Gamma\left(\frac{-u+1-s+|\ell+m|}{2}\right)}\\ &= f_\ell(-u)\left(\frac{2}{|w|}\right)^{-u}\left(\frac{iw}{|w|}\right)^{\ell-\frac{n}{2}}\end{aligned}$$

if the functions $\theta^\ell(u)$ are normalized as in the appendix.

Since $\sigma_n\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\delta_\ell = (-1)^{\frac{n}{2}+\ell}\delta_{-\ell}$,

$$\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \zeta, \varphi\right) = \frac{(-1)^{\frac{n}{2}+\ell}\theta^{-\ell}(u)}{\Gamma\left(\frac{u+1+s+|\ell+m|}{2}\right)\Gamma\left(\frac{u+1-s+|\ell-m|}{2}\right)}$$
$$= (-1)^{\frac{n}{2}+\ell} f_{-\ell}(u)\left(\frac{2}{|w|}\right)^u\left(\frac{iw}{|w|}\right)^{-\ell-\frac{n}{2}}.$$

Taking $\frac{n}{2} = |m|$ if $|\ell| \leqslant |m|$ and $\frac{n}{2} = |\ell|$ if $|\ell| \geqslant m$ we see that

$$\left(\frac{2}{|w|}\right)^{-u}\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \zeta, \varphi\right) = \gamma(\ell, m)\left(\frac{iw}{|w|}\right)^{-2\ell}\left(\frac{2}{|w|}\right)^u\Phi'(1, \widetilde{\zeta}, \varphi)$$

because as is shown in the Appendix, $f_{\pm\frac{n}{2}}(u) = 1$ and, as is shown in Lemma 5.4, $f_k(u) = 1$ if $\frac{n}{2} = |m|$.

Appendix. Unfortunately the preliminary material of this paragraph was not sufficient to give the constant occurring in the functional equation. A little more information about the functions $\theta^k(u)$ is necessary. Normalize them by setting

$$\theta^{n/2}(u) = \left(\frac{2}{|w|}\right)^u\Gamma\left(\frac{u+1+s+\frac{n}{2}-m}{2}\right)\Gamma\left(\frac{u+1-s+\frac{n}{2}+m}{2}\right).$$

It is an immediate consequence of the difference equations that none of the functions $\theta^k(u)$, $|k| \leqslant \frac{n}{2}$, $\frac{n}{2} - k \in \mathbf{Z}$, can vanish identically.

Lemma A. *Let $\alpha_k = \min\left\{\frac{n}{2} - |k|, \frac{n}{2} - |m|\right\}$. Then $\theta^k(u)$ is of the form*

$$f_k(u)\left(\frac{2}{|w|}\right)^u\left(\frac{iw}{|w|}\right)^{k-\frac{n}{2}}\Gamma\left(\frac{u+1+s+|k-m|}{2}\right)\Gamma\left(\frac{u+1-s+|k+m|}{2}\right)$$

where $f_k(u)$ is a polynomial in u of degree α_k. Its coefficients are polynomials in s which do not depend on w.

We shall show that if $\theta^k(u)$ is of this form with a polynomial of degree β_k, the same is true of $\theta^{k-1}(u)$ with a polynomial of degree β_{k-1} where $\beta_{k-1} - \beta_k \leqslant \alpha_{k-1} - \alpha_k$. This is enough to prove the lemma because $\beta_{\frac{n}{2}} = \alpha_{\frac{n}{2}} = 0$ and if β_{k_0} were less than α_{k_0} for some k_0 then β_k would be less than α_k for all succeeding k. Since $\alpha_{-\frac{n}{2}} = 0$ this is impossible.

The first difference equations show that $2c_k\frac{|w|^{u+1}}{2^{u+1}}\left(\frac{iw}{|w|}\right)^{\frac{n}{2}-(k-1)}\theta^{k-1}(u+1)$ is the product of

$$\frac{1}{2}f_k(u+2)\left[u+1+s+|k-m|\right]\left[u+1-s+|k+m|\right]$$
$$-\frac{1}{2}f_k(u)[u-k+1+s+m][u-k+1-s-m]$$

and

$$\Gamma\left(\frac{u+1+s+|k-m|}{2}\right)\Gamma\left(\frac{u+1-s+|k+m|}{2}\right).$$

If $k > |m|$ then $\alpha_{k-1} = \alpha_k + 1$, $|k-1 \pm m| = |k \pm m| - 1$, the second factor is

$$\Gamma\left(\frac{(u+1)+s+|k-1-m|}{2}\right)\Gamma\left(\frac{(u+1)+1-s+|k-1+m|}{2}\right),$$

and the first factor is a polynomial in u and s of degree at most β_k+1 in u.

Suppose $|m| \geqslant k > -|m|$, such that $\alpha_k = \alpha_{k-1}$. Let $\pm m \geqslant 0$. The first factor is the product of $\frac{(u+1\pm s\pm m-k)}{2}$ and

$$f_k(u+2)\big[u+1\mp s+|k\mp m|\big]-f_k(u)[u-k+1\mp s\mp m]$$

which is a polynomial of degree at most β_k. Moreover $|k-1\pm m| = |k\pm m|-1$, $|k-1\mp m| = |k\mp m|+1$, and $|k\pm m| = \pm m-k$; so the product of $\frac{u+1\pm s\pm m-k}{2}$ and the second factor is

$$\Gamma\bigg(\frac{(u+1)+1+s+|k-1-m|}{2}\bigg)\Gamma\bigg(\frac{(u+1)+1-s|k-1+m|}{2}\bigg).$$

If $-|m| \geqslant k > \frac{n}{2}$ then $|k-m| = m-k$, $|m+k| = -m-k$, $|k-1-m| = |k-m|+1$, $|k-1+m| = |k+m|+1$, and $\alpha_{k-1} = \alpha_k - 1$. The first factor is the product of $\Big(\frac{u-k+1+s+m}{2}\Big)\Big(\frac{u-k+1-s-m}{2}\Big)$ and $2\big(f_k(u+2)-f_k(u)\big)$ which is either zero or a polynomial of degree at most β_k-1. Moreover the product of $\Big(\frac{u-k+1+s+m}{2}\Big)\Big(\frac{u-k+1-s-m}{2}\Big)$ and the second factor is

$$\Gamma\bigg(\frac{(u+1)+1+s+|k-1-m|}{2}\bigg)\Gamma\bigg(\frac{(u+1)+1-s+|k-1+m|}{2}\bigg).$$

It follows from the corollary to Lemma 5.4 that the equations (A) and thus the difference equations have a solution at least when $s-m$ is not an integer. We could have used the same ideas to show that they had a solution for all s and m. This also follows from the above lemma. To indicate explicitly the dependence of $f_k(u)$ on s and m we write $f_k(u) = f_k(u,s,m)$. The function $f_{-\frac{n}{2}}(u,s,m)$ is independent of u.

Lemma B.

$$f_{-\frac{n}{2}}(u,s,m)\equiv 1.$$

For the proof we observe that the functions

$$\widehat{\theta}^k(u) = f_{-k}(u,s,-m)\bigg(\frac{2}{|w|}\bigg)^n\bigg(-\frac{i\overline{w}}{|w|}\bigg)^{-k-\frac{n}{2}} \times\Gamma\bigg(\frac{u+1+s+|k-m|}{2}\bigg)\Gamma\bigg(\frac{u+1-s+|k+m|}{2}\bigg)$$

also satisfy the difference equations. From uniqueness and the relation

$$\widehat{\theta}^{n/2}(u) = f_{-\frac{n}{2}}(u,s,-m)\bigg(-\frac{i\overline{w}}{|w|}\bigg)^{-n}\theta^{n/2}(u)$$

we conclude that

$$f_{-k}(u,s,-m)\bigg(-\frac{i\overline{w}}{|w|}\bigg)^{-k-\frac{n}{2}} = f_{-\frac{n}{2}}(u,s,-m)f_k(u,s,m)\bigg(-\frac{i\overline{w}}{|w|}\bigg)^{-n}\bigg(\frac{iw}{|w|}\bigg)^{k-\frac{n}{2}}$$

or

$$f_{-k}(u,s,-m) = f_{-\frac{n}{2}}(u,s,-m)f_k(u,s,m).$$

Choosing $k=-\frac{n}{2}$ we see that

$$f_{-\frac{n}{2}}(u,s,-m)f_{-\frac{n}{2}}(u,s,m)\equiv 1.$$

Since both terms on the right are polynomials in s they must be independent of s and $f_{-\frac{n}{2}}(u,s,m)=\epsilon(m)$.

When $s=0$ the difference equations do not change when m is replaced by $-m$. Consequently $f_k(u,0,-m)=f_k(u,0,m)$ and

$$f_{-k}(u,0,m)=\epsilon(m)f_k(u,0,m).$$

If m is an integer we can take $k=0$ and conclude that $\epsilon(m)=1$. If m is a half-integer take $k=\frac{1}{2}$. Then $c_k=\frac{n+1}{2}$. We have just seen that if $\pm m\geqslant 0$,

$$\begin{aligned}(n+1)f_{-\frac{1}{2}}(u+1,0,m)&=f_{\frac{1}{2}}(u+2,0,m)\left[u+1\pm m+\frac{1}{2}\right]-f_{\frac{1}{2}}(u,0,m)\left[u+1-\frac{1}{2}\mp m\right]\\&=\left[f_{\frac{1}{2}}(u+2,0,m)-f_{\frac{1}{2}}(u,0,m)\right][u+1]+\left[f_{\frac{1}{2}}(u+2,0,m)+f_{\frac{1}{2}}(u,0,m)\right]\left[|m|+\frac{1}{2}\right].\end{aligned}$$

The degree of both sides is $\frac{n}{2}-|m|$. Let a be the coefficient of $u^{\frac{n}{2}-|m|}$ in $f_{\frac{1}{2}}(u,0,m)$. The coefficient of $u^{\frac{n}{2}-|m|}$ in the polynomial on the left is $(n+1)\epsilon(m)a$. The coefficient of $u^{\frac{n}{2}-|m|}$ in the polynomial on the right is $2\big(\frac{n}{2}-|m|\big)a+2\big(|m|+\frac{1}{2}\big)a=(n+1)a$. Thus $\epsilon(m)=1$.

6. The local functional equation at a non-archimedean prime

Let K be a non-archimedean local field, let O be the ring of integers in K, and let π be a generator of the prime ideal in O. Let $G_K=\mathrm{GL}(2,K)$ and let $G_O=\mathrm{GL}(2,O)$. If A is the group of diagonal matrices and N the group of matrices of the form

$$\begin{pmatrix}1&x\\0&1\end{pmatrix}$$

then the Haar measure on G_K may be so normalized that

$$\int_{G_K}f(g)\,dg=\int_{A_K/A_O}\left|\frac{\alpha_1}{\alpha_2}\right|^{-1}da\int_{N_K}dn\int_{G_O}dk\,f(nak)$$

if

$$a=\begin{pmatrix}\alpha_1&0\\0&\alpha_2\end{pmatrix}.$$

The Hecke algebra H is just the algebra, under convolution, of functions on G_K which have compact support and are bi-invariant under G_O. Let $\widetilde{H}$ be the algebra, under convolution, of functions with compact support on A_K/A_O which satisfy

$$f(a)=f(\overline{a}).$$

If

$$a=\begin{pmatrix}\alpha_1&0\\0&\alpha_2\end{pmatrix}$$

then

$$\overline{a}=\begin{pmatrix}\alpha_2&0\\0&\alpha_1\end{pmatrix}.$$

Lemma 6.1. *If $f \in H$ and $a \in A_K$ set*

$$\widetilde{f}(a) = \left|\frac{\alpha_1}{\alpha_2}\right|^{1/2} \int_{N_K} f(an)\,dn.$$

The map $f \to \widetilde{f}$ is an isomorphism of H with $\widetilde{H}$.

To show that $\widetilde{f}$ lies in $\widetilde{H}$ one has to show that $\widetilde{f}(a) = \widetilde{f}(\overline{a})$. This is clear if $a = \overline{a}$, so suppose $a \neq \overline{a}$. Since a is conjugate to $\overline{a}$ in G_K the integrals

$$\int_{A_K \backslash G_K} f(g^{-1}ag)\,dg$$

and

$$\int_{A_K \backslash G_K} f(g^{-1}\overline{a}g)\,dg$$

are equal if they exist. But

$$\begin{aligned}\int_{A_K \backslash G_K} f(g^{-1}ag)\,dg &= \int_{N_K} dn \int_{G_O} dk\big(f(k^{-1}n^{-1}ank)\big)\\ &= \int_{N_K} f\big(a(a^{-1}n^{-1}an)\big)\,dn.\end{aligned}$$

A simple change of variables shows that the last integral equals

$$\frac{\left|\frac{\alpha_2}{\alpha_1}\right|^{1/2}}{\left|1 - \frac{\alpha_2}{\alpha_2}\right|}\widetilde{f}(a).$$

Combining this with the relation

$$\frac{\left|\frac{\alpha_2}{\alpha_1}\right|^{1/2}}{\left|1 - \frac{\alpha_2}{\alpha_2}\right|} = \frac{\left|\frac{\alpha_1}{\alpha_2}\right|^{1/2}}{\left|1 - \frac{\alpha_1}{\alpha_1}\right|}$$

one sees that $\widetilde{f}(a) = \widetilde{f}(\overline{a})$.

If $f = f_1 * f_2$ then

$$\widetilde{f}(b) = \left|\frac{\beta_1}{\beta_2}\right|^{1/2} \int_{N_K} \left\{ \int_{G_K} f_1(bvg) f_2(g^{-1})\,dg \right\} dv.$$

The Haar measure has been so normalized that this equals

$$\left|\frac{\beta_1}{\beta_2}\right|^{1/2} \int_{A_K/A_O} da \int_{N_K} du \int_{N_K} dv \left\{ f_1(bvua) f_2(a^{-1}u^{-1}) \left|\frac{\alpha_1}{\alpha_2}\right|^{-1} \right\}.$$

Simple manipulation shows that this equals

$$\left|\frac{\beta_1}{\beta_2}\right|^{1/2} \int_{A_K/A_O} da \left\{ \int_{N_K} f_1(bav)\,dv \right\} \left\{ \int_{N_K} f_2(a^{-1}u)\,du \right\} = \widetilde{f}_1 * \widetilde{f}_2(b).$$

G_K is the disjoint union of the double cosets

$$G_O\begin{pmatrix}\pi^m & 0\\ 0 & \pi^n\end{pmatrix}G_O = G_O a(m,n) G_O \qquad m \leqslant n.$$

The characteristic function of such a double coset will be denoted by $f_{m,n}$. If $a(m',n')N_K$ meets $G_O a(m,n)G_O$ then $m+n = m'+n'$ and $m \leqslant m'$; moreover

$$a(m,n)N_K \cap G_O a(m,n) G_O = a(m,n)(G_O \cap N_K).$$

Thus $\widetilde{f}_{m,n}\big(a(m',n')\big) = 0$ unless $m+n = m'+n'$ and $m \leqslant m$! Moreover

$$\widetilde{f}_{m,n}\big(a(m,n)\big) = 1.$$

It follows readily that the map $f \to \widetilde{f}$ is an isomorphism. Consequently every homomorphism of H into $\mathbf{C}$ is of the form

$$\chi_\omega(f) = \int_{A_K/A_O} \widetilde{f}(a)\omega(a)\,da$$

where ω is a homomorphism of A_K/A_O into $\mathbf{C}^\times$.

If η is a homomorphism of A_K into $\mathbf{C}^\times$ let η_1 and η_2 be the functions on $K^\times$ defined by

$$\eta_1(\alpha) = \eta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right), \qquad \eta_2(\alpha) = \eta\left(\begin{pmatrix}1 & 0\\ 0 & \alpha\end{pmatrix}\right).$$

Lemma 6.2. *Let η be a homomorphism of A_K into $\mathbf{C}^\times$ and ω a homomorphism of A_K/A_O into $\mathbf{C}^\times$. There is up to a scalar factor at most one function φ on G_K satisfying*

(i) $\varphi(ag) = \eta(a)\varphi(g)$ for all a in A_K,

(ii)

$$\int_{G_K} \varphi(gh)f(h)\,dh = \chi_\omega(f)\varphi(g)$$

for all f in H.

If there is any non-zero solution of this equation then $\eta_1\eta_2 = \omega_1\omega_2$ so that η_1 and η_2 have the same conductor. Let it be (π^a). φ is determined by its restriction to N_K. If $y \in O^\times$ then

$$\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = \varphi\left(\begin{pmatrix}1 & x+y\\ 0 & 1\end{pmatrix}\right)$$

and if $\alpha \in O^\times$ then

$$\eta_1(\alpha)\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = \varphi\left(\begin{pmatrix}\alpha & \alpha x\\ 0 & 1\end{pmatrix}\right) = \varphi\left(\begin{pmatrix}1 & \alpha x\\ 0 & 1\end{pmatrix}\right).$$

If $x = \pi^{-b}$ and $b < a$ there is an α in $O^\times$ such that $\alpha \equiv 1 \pmod{\pi^b}$ and $\eta_1(\alpha) \neq 1$. Then

$$\eta_1(\alpha)\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) = \varphi\left(\begin{pmatrix}1 & x+(\alpha-1)x\\ 0 & 1\end{pmatrix}\right) = \varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)$$

so that

$$\varphi\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right) = 0.$$

To prove the lemma we need only show that if

$$\varphi\left(\begin{pmatrix}1 & \pi^{-a}\\ 0 & 1\end{pmatrix}\right)=0$$

then

$$\varphi\left(\begin{pmatrix}1 & \pi^{-b}\\ 0 & 1\end{pmatrix}\right)=0$$

for $b > a$. If O is the disjoint union $\bigcup_{i=1}^{q} x_i + (\pi)$ then $G_O a(0,1) G_O$ is the disjoint union

$$\bigcup_{i=1}^{q}\begin{pmatrix}\pi & x_i\\ 0 & 1\end{pmatrix}G_O \cup \begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}G_O.$$

Thus if $b \geqslant a$

$$\begin{aligned}
\chi_\omega(f_{0,1})\varphi\left(\begin{pmatrix}1 & \pi^{-b}\\ 0 & 1\end{pmatrix}\right) &= \sum_i \varphi\left(\begin{pmatrix}\pi & \pi^{-b}+x_i\\ 0 & 1\end{pmatrix}\right)+\varphi\left(\begin{pmatrix}1 & \pi^{-(b-1)}\\ 0 & \pi\end{pmatrix}\right)\\
&= \eta_1(\pi)\sum_i \varphi\left(\begin{pmatrix}1 & \pi^{-(b+1)}(1+\pi^b x_i)\\ 0 & 1\end{pmatrix}\right)\\
&\quad + \eta_2(\pi)\varphi\left(\begin{pmatrix}1 & \pi^{-(b-1)}\\ 0 & 1\end{pmatrix}\right)\\
&= q\eta_1(\pi)\varphi\left(\begin{pmatrix}1 & \pi^{-(b+1)}\\ 0 & 1\end{pmatrix}\right)+\eta_2(\pi)\varphi\left(\begin{pmatrix}1 & \pi^{-(b-1)}\\ 0 & 1\end{pmatrix}\right)
\end{aligned}$$

because $\eta_1(1+\pi^b x)=1$.

Lemma 6.3. *Let ξ be a non-trivial character of K and ω a homomorphism of A_K/A_O into $\mathbf{C}^\times$. Apart from a scalar factor there is exactly one function φ on G_K which satisfies*

(i)

$$\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)=\xi(x)\varphi(g)$$

(ii)

$$\int_G \varphi(gh)f(h)\,dh=\chi_\omega(f)\varphi(g)$$

for all f in H.

Suppose φ satisfies these relations. Take an A_K and set $\varphi'(g)=\varphi(ag)$. The function φ' satisfies (ii). Moreover

$$\varphi'\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right)=\varphi\left(\begin{pmatrix}1 & \alpha_1\times\alpha_2^{-1}\\ 0 & 1\end{pmatrix}ag\right)=\xi(\alpha_1\times\alpha_2^{-1})\varphi'(g);$$

thus if $\xi'(x) = \xi(\alpha_1 \times \alpha_2^{-1})$ it satisfies (i) with ξ replaced by ξ'. Assume then for simplicity that O is the largest ideal on which ξ is trivial.

If φ is to satisfy (i) it must be of the form

$$\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} ak\right) = \xi(x)\Phi(a)$$

with Φ a function on A_K/A_O. The function φ is well-defined if and only if $\xi(x)\Phi(a) = \Phi(a)$ when $\alpha_1^{-1} \times \alpha_2$ is in O. Thus $\Phi(a) = 0$ unless $\alpha_1\alpha_2^{-1}$ is in O. The relations (ii) will be satisfied if and only if

$$\text{(A)} \qquad \Phi\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix} a\right) = \omega_1(\alpha)\omega_2(\alpha)\Phi(a)$$

and

$$\text{(B)} \qquad \int_G \varphi(gh) f_{0,1}(h)\, dh = \chi_\omega(f_{0,1})\varphi(g).$$

We can satisfy (A) and the previous conditions while specifying in an arbitrary manner the value of Φ at $a = \left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)$, $\alpha \geqslant 0$. (B) will be satisfied for all g if it is satisfied for $g = \left(\begin{smallmatrix}\pi^\alpha & 0\\ 0 & 1\end{smallmatrix}\right)$ when it becomes

$$\sum_i \varphi\left(\begin{pmatrix}\pi^{\alpha+1} & \pi^\alpha x_i\\ 0 & 1\end{pmatrix}\right) + \varphi\left(\begin{pmatrix}\pi^{\alpha} & 0\\ 0 & \pi\end{pmatrix}\right) = q^{1/2}\big(\omega_1(\pi) + \omega_2(\pi)\big)\varphi\left(\begin{pmatrix}\pi^{\alpha} & 0\\ 0 & 1\end{pmatrix}\right).$$

If $\alpha < -1$ all terms on both sides are zero. If $\alpha = -1$ the right side is 0 and the left side is

$$\left\{\sum_i \xi\left(\frac{x_i}{\pi}\right)\right\}\varphi\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right) = 0.$$

If $\alpha \geqslant 0$ the left side is

$$q\varphi\left(\begin{pmatrix}\pi^{\alpha+1} & 0\\ 0 & 1\end{pmatrix}\right) + \omega_1(\pi)\omega_2(\pi)\varphi\left(\begin{pmatrix}\pi^{\alpha-1} & 0\\ 0 & 1\end{pmatrix}\right).$$

Some simple algebra then shows that (ii) will be satisfied if and only if

$$\text{(X)} \qquad \sum_{n=-\infty}^{\infty} x^n q^{n/2}\Phi\left(\begin{pmatrix}\pi^{n} & 0\\ 0 & 1\end{pmatrix}\right) = \frac{1}{\big(1-\omega_1(n)x\big)\big(1-\omega_2(n)x\big)}\Phi\left(\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right).$$

The lemma follows.

If $(\pi^{-\delta})$ is the largest ideal of K on which ξ is trivial let $\varphi(g, \omega, \xi)$ be that solution of (i) and (ii) which takes the value 1 at $\left(\begin{smallmatrix}\pi^\delta & 0\\ 0 & 1\end{smallmatrix}\right)$. If $\xi'(x) = \xi(\beta x)$ then

$$\varphi(g; \omega, \xi') = \varphi\left(\begin{pmatrix}\beta & 0\\ 0 & 1\end{pmatrix} g, \omega, \xi\right).$$

Let ζ be a character of A_K such that $\zeta_1\zeta_2 = \omega_1^{-1}\omega_2^{-1}$. Set $\zeta_1(\alpha) = \zeta_0(\alpha)|\alpha|^s$ where $\zeta_0(\pi) = 1$. ζ is uniquely determined by s and ζ_0 and we shall sometimes write $\zeta = \zeta(s, \zeta_0)$.

Lemma 6.4. *Let ζ be a homomorphism of A_K into $\mathbf{C}^\times$ such that $\zeta_1\zeta_2 = \omega_1^{-1}\omega_2^{-1}$. If $\zeta = \zeta(s,\zeta_0)$ the function*

$$\Phi(g,\zeta;\omega,\xi) = \int_{K^\times} \varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g;\omega,\xi\right)\zeta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) d^\times\alpha$$

is defined for $\operatorname{Re} s$ *sufficiently large. If $\zeta_0 = 1$ then*

$$\Phi'(g,\zeta;\omega,\xi) = \left(1 - \frac{\omega_1(\pi)\zeta_1(\pi)}{q^{1/2}}\right)\left(1 - \frac{\omega_2(\pi)\zeta_1(\pi)}{q^{1/2}}\right)\Phi(g,\zeta;\omega,\xi)$$

is, for each g, a polynomial in q^s and q^{-s} and if $(\pi^{-\delta})$ is the largest ideal on which ξ is trivial then

$$\zeta_1(\pi^\delta)\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} g,\zeta;\omega,\xi\right) = \widetilde{\zeta}_1(\pi^{+\delta})\Phi'(g,\widetilde{\zeta},\omega,\xi)$$

where $\widetilde{\zeta}\left(\left(\begin{smallmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{smallmatrix}\right)\right) = \zeta\left(\left(\begin{smallmatrix}\alpha_2 & 0\\ 0 & \alpha_1\end{smallmatrix}\right)\right)$. *If $\delta = 0$ then*

$$\Phi'(1,\zeta;\omega,\xi) = 1.$$

If the conductor of ζ_0 is (π^γ), $\gamma > 0$, and

$$g(\xi,\gamma) = \int_{O^\times} \xi\left(\frac{\alpha}{\pi^{\gamma+\delta}}\right)\zeta_1(\alpha)\, d^\times\alpha$$

then $\Phi(g,\zeta;\omega,\xi)$ is a polynomial in q^s and q^{-s} and

$$\frac{\zeta_1(\pi^{\gamma+\delta})}{g(\xi,\zeta)}\Phi\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} g,\zeta;\omega,\xi\right) = \zeta_1(-1)\frac{\widetilde{\zeta}_1(\pi^{\gamma+\delta})}{g(\xi,\widetilde{\gamma})}\Phi(g,\widetilde{\zeta},\omega,\xi).$$

If $\xi'(x) = \xi(\pi^\beta x)$ then

$$\text{(Y)}\qquad \begin{aligned}\Phi(g,\zeta;\omega,\xi') &= \int_{K^\times} \varphi\left(\begin{pmatrix}\pi^\beta\alpha & 0\\ 0 & 1\end{pmatrix} g;\omega,\xi\right)\zeta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) d^\times\alpha\\ &= \zeta_1^{-1}(\pi^\beta)\Phi(g,\zeta;\omega,\xi)\end{aligned}$$

and $g(\xi',\zeta) = g(\xi,\zeta)$; so it is enough to prove the lemma for $\delta = 0$.

If $g = a\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)k$ with $a = \left(\begin{smallmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{smallmatrix}\right)$ in A_K and k in G_O then

$$\int_{K^\times} \varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g;\omega,\xi\right)\zeta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) d^\times\alpha$$

is equal to

$$\zeta^{-1}\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)\int_{K^\times} \varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix};\omega,\xi\right)\zeta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) d^\times\alpha.$$

Because $\varphi\left(\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right);\omega,\xi\right) = \xi(\alpha x)\varphi\left(\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right);\omega,\xi\right)$ the function

$$\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix};\omega,\xi\right) - \varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix};\omega,\xi\right)$$

has, for a given x, compact support on $K^\times$. Since the integral

$$\zeta^{-1}\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)\int_{K^\times}\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix};\omega,\xi\right)\zeta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)d^\times\alpha$$

exists for $\operatorname{Re}s$ sufficiently large so does that of the lemma. Moreover the difference between $\Phi(g,\zeta;\omega,\xi)$ and this expression is a polynomial in q^s and q^{-s}. If $\zeta_0=1$ the expression equals

$$\zeta^{-1}\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)\sum_{n=-\infty}^{\infty}\zeta_1(\pi^n)\varphi\left(\begin{pmatrix}\pi^n & 0\\ 0 & 1\end{pmatrix}\right)=\frac{\zeta^{-1}\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)}{\left(1-\frac{\omega_1(\pi)\zeta_1(\pi)}{q^{1/2}}\right)\left(1-\frac{\omega_2(\pi)\zeta_1(\pi)}{q^{1/2}}\right)}$$

and if the conductor is (π^γ) and $\gamma>0$ it equals zero. All assertions of the lemma except the functional equations follow.

Let $\eta=\widetilde{\zeta}^{-1}$. If $\zeta_0=1$ then $\Phi'\left(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)g,\zeta;\omega,\xi\right)$ and $\Phi'(g,\widetilde{\zeta};\omega,\xi)$ both satisfy the assumptions of Lemma 6.2. Since they both take the value 1 at $g=1$ they are equal. If the conductor of ζ_0 is (π^γ) and $\gamma>0$ then

$$\Phi\left(\begin{pmatrix}1 & \pi^{-\gamma}\\ 0 & 1\end{pmatrix},\widetilde{\zeta};\omega,\xi\right)=\int_{K^\times}\xi(\alpha\pi^{-\gamma})\varphi\left(\begin{pmatrix}\alpha & 1\\ 0 & 1\end{pmatrix},\omega,\xi\right)\zeta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)d^\times\alpha.$$

The last integral is easily seen to equal $g(\xi,\widetilde{\zeta})$. Since

$$\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}1 & \pi^{-\gamma}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}-\pi^{-\gamma} & 0\\ 0 & \pi^{-\gamma}\end{pmatrix}\begin{pmatrix}1 & +\pi^{-\gamma}\\ 0 & 1\end{pmatrix}\begin{pmatrix}+1 & 0\\ -\pi^{\gamma} & -1\end{pmatrix}$$

the value of $\Phi\left(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right),\zeta;\omega,\xi\right)$ is $\zeta_1^{-1}(\pi^\gamma)\widetilde{\zeta}_1^{-1}(\pi^{-\gamma})\zeta_1(-1)g(\xi,\zeta)$. The functional equation again follows from Lemma 6.2.

7. The Main Theorem

Let k be either the rational number field or the field of rational functions in one variable over a finite field and let K be a finite separable extension of k. Let S_∞ be the set of archimedean primes of K. Let $\mathbf{A}$ be the adele ring of K and let I be the group of ideles.

If R is any commutative ring with unit let G_R be the group of 2×2 matrices from R which have a determinant which is a unit of R. A_R will be the group of diagonal matrices in G_R. If $\mathfrak{p}$ is a non-archimedean prime let $U_{K_\mathfrak{p}}$ be $G_{O_\mathfrak{p}}$, where $O_\mathfrak{p}$ is the ring of integers in $K_\mathfrak{p}$, and if $\mathfrak{p}$ is an archimedean prime let $U_{K_\mathfrak{p}}$ be the group of unitary matrices which lie in $G_{K_\mathfrak{p}}$.

Lemma 7.1. [56] *There is a constant c_0 such that if g belongs to $G_\mathbf{A}$ there is a γ in $G_\mathbf{Q}$ such that* $\max\{|c|,|d|\}\leqslant c_0|\det g|^{1/2}$ *if* $\gamma_g=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$.

[5](1998) As observed in the comments this lemma is not what is needed. Indeed, neither it nor its proof make much sense. The correct lemma, which there is at this stage no need to state, would replace max $\{|c|,|d|\}$ by $\prod_\mathfrak{p}\max\{|c|_\mathfrak{p},|d|_\mathfrak{p}\}$. See Lemma 5.1 of the following letter.

[6](2023) Editorial comment: The letter to which the author refers in the previous footnote is this long one to Hervé Jacquet: `https://publications.ias.edu/rpl/paper/55`.

Fix a measure on $\mathbf{A}$. This determines a measure on $\mathbf{A} \oplus \mathbf{A}$. $K \oplus K$ is a discrete subgroup of $\mathbf{A} \oplus \mathbf{A}$ and the quotient $\mathbf{A} \oplus \mathbf{A}/K \oplus K$ has finite measure c_1. The lattice $Lg = (K \oplus K)g$, is discrete and the quotient $\mathbf{A} \oplus \mathbf{A}/Lg$ has measure $c_1|\det g|$. The non-zero elements of Lg are, for all practical purposes, the last rows of the matrices γ_g, $\gamma \in G_{\mathbf{Q}}$. There is a positive constant c_2 such that the measure of $\left\{ (x,y) \;\middle|\; \max\{|x|,|y|\} \leqslant d_0 \right\}$ is at least $c_2 d_0^2$.

Let c_0 be any number larger than $2\sqrt{\frac{c_1}{c_2}}$. If Lg contained no non-zero (c,d) with

$$\max\{|c|,|d|\} \leqslant c_0 |\det g|^{1/2}$$

the measure of the projection of $\left\{ (x,y) \;\middle|\; \max\{|x|,|y|\} \leqslant \frac{c_0}{2}|\det g|^{1/2} \right\}$ on $\mathbf{A} \oplus \mathbf{A}/Lg$ would be greater than $c_1|\det g|$.

$\mathfrak{L}$ will be the space of functions φ on $G_K\backslash G_{\mathbf{A}}$ satisfying conditions (i), (ii), and (iii) below

(i) If $U = \prod_{\mathfrak{p}} U_{K_{\mathfrak{p}}}$ then φ is U-finite on the right.

(ii) If $\mathfrak{p}$ is an archimedean prime the function $\varphi(hg)$, $g \in G_{K_{\mathfrak{p}}}$, is infinitely differentiable.

If $\mathfrak{p}$ is any such prime let $\mathfrak{A}_{\mathfrak{p}}$ be the universal enveloping of $G_{K_{\mathfrak{p}}}$. If, for each $\mathfrak{p}$, $X_{\mathfrak{p}}$ belongs to $\mathfrak{A}_{\mathfrak{p}}$ the function $\left\{\prod_{\mathfrak{p}} \rho(X_{\mathfrak{p}})\right\}\varphi$ is defined.

(iii) If c_1 is any constant there are constants M_1 and M_2 such that[7]

$$\left| \left\{ \prod_{\mathfrak{p}} \rho(X_{\mathfrak{p}}) \right\} \varphi(g) \right| \leqslant M_1 \left[\left\{ |\det g| + \frac{1}{|\det g|} \right\} \left\{ \frac{|\det g|^{1/2}}{\max\{|c|,|d|\}} \right\} \right]^{M_2}$$

on the set $\max\{|c|,|d|\} \leqslant c_1 |\det g|^{1/2}$.

If $\mathfrak{p}$ is a non-archimedean prime the group $G_{K_{\mathfrak{p}}}$ operates on $\mathfrak{L}$. If $\mathfrak{p}$ is a complex prime $\mathfrak{A}_{\mathfrak{p}}$ acts on $\mathfrak{L}$. If $\mathfrak{p}$ is a real prime let $\sigma_{\mathfrak{p}}$ be the element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ of $G_{K_{\mathfrak{p}}}$; the pair $\{\sigma_{\mathfrak{p}}, \mathfrak{A}_{\mathfrak{p}}\}$ acts on $\mathfrak{L}$.

If $\mathfrak{p}$ is a non-archimedean prime, a representation of $G_{K_{\mathfrak{p}}}$ on a vector space $H_{\mathfrak{p}}$ will be called quasi-simple if the isotropy group of every vector in $H_{\mathfrak{p}}$ is an open subgroup of $G_{K_{\mathfrak{p}}}$. It follows from Lemma 6.1 that the space of vectors whose isotropy group contains $U_{K_{\mathfrak{p}}}$ has dimension at most 1 if the presentation is irreducible.

Suppose that for every prime $\mathfrak{p}$ we are given a quasi-simple irreducible representation of either $G_{K_{\mathfrak{p}}}$, $\mathfrak{A}_{\mathfrak{p}}$, or $\{\sigma_{\mathfrak{p}}, \mathfrak{A}_{\mathfrak{p}}\}$, according to the nature of the prime, on a vector space $H_{\mathfrak{p}}$. Suppose there is a finite set S_0 of primes which contains S_∞ such that if $\mathfrak{p}$ is not in S_0 there is a non-zero vector in $H_{\mathfrak{p}}$ which is fixed by $U_{\mathfrak{p}}$. For each $\mathfrak{p}$ not in S_0 choose such a vector $X_{\mathfrak{p}}^0$. If S contains S_0 let $H_S = \bigotimes_{\mathfrak{p}\in S} H_{\mathfrak{p}}$. If $S_2 \supseteq S_1 \supseteq S_0$ let δ_{S_1,S_2} be the injection of H_{S_1} into H_{S_2} which sends $\bigotimes_{\mathfrak{p}\in S_1} X_{\mathfrak{p}}$ to

$$\left(\bigotimes_{\mathfrak{p}\in S_1} X_{\mathfrak{p}} \right) \otimes \left(\bigotimes_{\mathfrak{p}\in S_2 - S_1} X_{\mathfrak{p}}^0 \right)$$

and let H be the injective limit of the spaces H_S. Let $\mathfrak{A}$ be the system consisting of all the $G_{K_{\mathfrak{p}}}$, $\mathfrak{p}$ not in S_∞, $\mathfrak{A}_{\mathfrak{p}}$, $\mathfrak{p}$ complex, and $\{\sigma_{\mathfrak{p}}, \mathfrak{A}_{\mathfrak{p}}\}$, $\mathfrak{p}$ real. The system $\mathfrak{A}$ acts on H.

[7]See previous footnote.

For our purposes a divisor D is just a function $\mathfrak{p} \to m_\mathfrak{p}$ from the non-archimedean primes to the non-negative integers such that $m_\mathfrak{p} = 0$ for almost all $\mathfrak{p}$. $\mathfrak{p}|D$ means that $m_\mathfrak{p} > 0$ and $\mathfrak{p} \nmid D$ means that $m_\mathfrak{p} = 0$ or $\mathfrak{p} \in S_\infty$. If $\mathfrak{p}$ is not in S_∞ let $U^D_{K_\mathfrak{p}}$ be the set of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $U_{K_\mathfrak{p}}$ for which $c \equiv 0 \pmod{\mathfrak{p}^{m_\mathfrak{p}}}$ and let $U^D = \prod_{\mathfrak{p} \notin S_\infty} U^D_{K_\mathfrak{p}}$. Let $\widehat{U}^D_{K_\mathfrak{p}}$ be the set of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $U^D_{K_\mathfrak{p}}$ for which $a \equiv d \equiv 1 \pmod{\mathfrak{p}^{m_\mathfrak{p}}}$ and let $\widehat{U}^D = \prod_{\mathfrak{p} \notin S_\infty} \widehat{U}^D_{K_\mathfrak{p}}$. U^D is in the normalizer of $\widehat{U}^D$.

Lemma 7.2. *There is a D such that $H^D = \left\{ x \,\middle|\, \pi(u)x = x \text{ for all } u \text{ in } \widehat{U}^D \right\}$ contains a non-zero vector. Moreover H^D is the sum of one-dimensional subspaces invariant under U^D.*

Although we have not troubled to be explicit it is clear how $\prod_{\mathfrak{p} \notin S_\infty} G_{K_\mathfrak{p}}$ operates on H.

Given any divisor D let $\widetilde{U}^D_{K_\mathfrak{p}}$ be the set of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in $U_{K_\mathfrak{p}}$ which are congruent to I modulo $\mathfrak{p}^{m_\mathfrak{p}}$ and let $\widetilde{U}^D = \prod_{\mathfrak{p} \notin S_\infty} \widetilde{U}^D_{K_\mathfrak{p}}$. Given $x \neq 0$ in H there is a D' such that $\widetilde{U}^{D'}$ is contained in the isotropy group of x. Choose for each non-archimedean prime an $\alpha_\mathfrak{p}$ so that $(\alpha_\mathfrak{p}) = \mathfrak{p}^{m'_\mathfrak{p}}$ and set $g = \prod_{\mathfrak{p} \notin S_\infty} \left(\begin{smallmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then, if $m_\mathfrak{p} = 2m'_\mathfrak{p}$ and D is the divisor $\{m_\mathfrak{p}\}$, $g\widehat{U}^D g^{-1}$ is contained in $\widetilde{U}^{D'}$ and $\widehat{U}^D$ is contained in the isotropy group of $\pi(g^{-1})x$. The second assertion of the lemma is immediate because $U^D/\widehat{U}^D$ is a finite abelian group.

If ϵ is any homomorphism of U^D into $\mathbf{C}^\times$ which sends $\widehat{U}^D$ to 1, let

$$H^D_\epsilon = \left\{ x \,\middle|\, \pi(u)x = \epsilon(u)x \text{ for all } u \text{ in } U^D \right\}.$$

ϵ is determined by its restriction to the diagonal matrices. Let $\widetilde{\epsilon}$ be the homomorphism satisfying

$$\widetilde{\epsilon}\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = \epsilon\left(\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}\right).$$

If g is any matrix in $G_\mathbf{A}$ such that $g_\mathfrak{p} = I$, if $\mathfrak{p} \in S_\infty$ or $\mathfrak{p} \nmid D$ and $g_\mathfrak{p} = \left(\begin{smallmatrix} 0 & 1 \\ \alpha_\mathfrak{p} & 0 \end{smallmatrix}\right)$ with $(\alpha_\mathfrak{p}) = \mathfrak{p}^{m_\mathfrak{p}}$ if $\mathfrak{p}|D$ then $gU^Dg^{-1} = U^D$ and $\pi(g)H^D_\epsilon = H^D_{\widetilde{\epsilon}}$.

Let $\mathfrak{H}$ be a subspace of $\mathfrak{L}$ such that the representation of $\mathfrak{A}$ on $\mathfrak{H}$ is equivalent to that on H. We want to study some of the Dirichlet series associated to $\mathfrak{H}$. Let $\mathfrak{H}^D_\epsilon$ be the subspace of $\mathfrak{H}$ corresponding to H^D_ϵ. We suppose that H^D_ϵ is not $\{0\}$.

Choose a non-trivial character ξ of $\mathbf{A}$ which is trivial on K. If φ belongs to $\mathfrak{L}$ set

$$\varphi_0(g) = \frac{1}{\operatorname{measure}(K\backslash\mathbf{A})} \int_{K\backslash\mathbf{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx,$$

$$\varphi_1(g) = \frac{1}{\operatorname{measure}(K\backslash\mathbf{A})} \int_{K\backslash\mathbf{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\xi(x)}\, dx.$$

By the Fourier inversion formula

$$\varphi(g) = \varphi_0(g) + \sum_{\alpha \in K^\times} -\varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Let $G^D_\mathbf{A}$ be the set of all g in $G_\mathbf{A}$ such that $g_\mathfrak{p} \in U^D_{K_\mathfrak{p}}$ if $\mathfrak{p}|D$. Since $G_\mathbf{A} = G_K G^D_\mathbf{A}$, any function in $\mathfrak{L}$ is determined by its restriction to $G^D_\mathbf{A}$.

If $\mathfrak{p}$ is a non-archimedean prime which does not divide D and φ belongs to $\mathfrak{H}_\epsilon^D$ then φ must be an eigenfunction of the corresponding Hecke operators. Let it be an eigenfunction corresponding to the homomorphism $\omega_\mathfrak{p}$. Varying φ in $\mathfrak{H}_\epsilon^D$ does not change $\omega_\mathfrak{p}$. It follows from Lemmas 3.2, 5.2, and 6.3 that $\mathfrak{H}_\epsilon^D$ is spanned by functions φ for which

$$\varphi_1\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right) = a_\alpha\left\{\prod_{\mathfrak{p}\in S_\infty}\varphi_\mathfrak{p}\left(\begin{pmatrix}\alpha_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p}\right)\right\}\left\{\prod_{\substack{\mathfrak{p}\notin S_\infty\\ \mathfrak{p}\nmid D}}\varphi\left(\begin{pmatrix}\alpha_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p},\omega_\mathfrak{p},\xi_\mathfrak{p}\right)\right\}\epsilon(g_D)$$

for g in $G_\mathbf{A}^D$. a_α is a constant which depends on α and $\alpha_\mathfrak{p}$ is the image of α in $K_\mathfrak{p}$. g_D is the projection of G on $\prod_{\mathfrak{p}|D}G_{K_\mathfrak{p}}$. $\xi_\mathfrak{p}$ is the restriction of ξ to $K_\mathfrak{p}$ and $\varphi_\mathfrak{p}$, $\mathfrak{p}\in S_\infty$, is a function in $L(\xi_\mathfrak{p},\pi_\mathfrak{p})$ determined solely by φ. Let $I^D = \left\{\iota\in I \;\middle|\; |\iota_\mathfrak{p}| = 1 \text{ if } \mathfrak{p}|D\right\}$. If β lies in $K^\times\cap I^D$ then $a_{\alpha\beta} = \epsilon\left(\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)\right)a_\alpha$.

We shall only consider those φ for which the functions $\varphi_1\left(\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)g\right)$ are of the above form. $\varphi(g)$ is the sum of $\varphi_0(g)$ and

$$\sum_{\alpha\in K^\times/K^\times\cap I^D} a_\alpha \sum_{\beta\in K^\times\cap I^D}\left\{\prod_{\mathfrak{p}\in S_\infty}\varphi_\mathfrak{p}\left(\begin{pmatrix}\alpha_\mathfrak{p}\beta_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p}\right)\right\}\left\{\prod_{\substack{\mathfrak{p}\notin S_\infty\\ \mathfrak{p}\nmid D}}\varphi\left(\begin{pmatrix}\alpha_\mathfrak{p}\beta_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p},\omega_\mathfrak{p},\xi_\mathfrak{p}\right)\right\}$$
$$\times\epsilon\left(\begin{pmatrix}\beta_0 & 0\\ 0 & 1\end{pmatrix}g_D\right)$$

if β_D is the projection of β on $\prod_{\mathfrak{p}|D}K_\mathfrak{p}^\times$. In an appendix to this paragraph we shall discuss the form of the function ϕ_0. Lemma E of the appendix will eventually be used to show that φ is the sum of a cusp form and a function which is represented by an Eisenstein series. For the present we consider only the case that $\varphi_0(g)\equiv 0$.

Then $\varphi(g)$ is a cusp form. Let η be the homomorphism of $K^\times\backslash I$ into $\mathbf{C}^\times$ defined by

$$\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha\end{pmatrix}g\right) = \eta(\alpha)\varphi(g).$$

It is no real restriction to assume that $\left|\eta(\alpha)\right| = 1$ and we shall do so. It then follows from the general theory of automorphic forms that φ is bounded.

If $M_1 = \sup_{g\in G_\mathbf{A}}\left|\varphi(g)\right|$ and

$$M_2 = \sup_{g\in G_\mathbf{A}^D}\left|\prod_{\mathfrak{p}\in S_\infty}\varphi_\mathfrak{p}(g_\mathfrak{p})\right|\left|\prod_{\substack{\mathfrak{p}\notin S_\infty\\ \mathfrak{p}\nmid D}}\varphi(g_\mathfrak{p},\omega_\mathfrak{p},\xi_\mathfrak{p})\right|\left|\epsilon(g_D)\right|$$

then

$$|a_\alpha| \leqslant \frac{M_1}{M_2}.$$

If $\varphi\not\equiv 0$, as we certainly suppose, M_2 is not zero. Of course it is not ∞ either for then all the a_α would be zero. In any case a_α is a bounded function.

If the number M_2 is finite the function $\varphi(g_\mathfrak{p}, \omega_\mathfrak{p}, \xi_\mathfrak{p})$ is bounded. Appealing to the formula[8] at the top of p. 6.10 we see that the inequalities

$$\text{(A)} \qquad \left|\omega_{\mathfrak{p},1}(\pi)\right| \leqslant |\pi|^{-1/2} \qquad \left|\omega_{\mathfrak{p},2}(\pi)\right| \leqslant |\pi|^{-1/2}$$

must be satisfied.

If K is the real or complex field, π a quasi-simple irreducible representation of $\{\sigma, \mathfrak{A}\}$ or $\mathfrak{A}$ respectively, and ζ a homomorphism of A_K into $\mathbf{C}^\times$ satisfying the condition of Lemma 3.6 or 5.7, let $\Gamma(\zeta, \pi)$ be the function defined by

$$\Gamma(\zeta, \pi)\Phi'(g, \zeta, \varphi) = \Phi(g, \zeta, \varphi), \qquad \varphi \in L(\xi, \pi).$$

$\Phi(g, \zeta, \varphi)$ and $\Phi'(g, \zeta, \varphi)$ are the functions introduced in Lemmas 3.6 and 5.7.[9] $\Gamma(\zeta, \pi)$ also depends on ξ but we do not take this into account explicitly.

Let χ be a character of $K^\times \cap I^D \backslash I^D$. If s is a complex number define $\zeta = \zeta(s, \chi)$ by

$$\zeta\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right) = \eta^{-1}(\beta)|\alpha\beta^{-1}|^s \chi(\alpha\beta^{-1}).$$

Let $\zeta_\mathfrak{p}$ be the restriction of ζ to $A_{K_\mathfrak{p}}$.

Lemma 7.3. *The integral*

$$\int_{K^\times \cap I^D \backslash I^D} \varphi\left(\begin{pmatrix} \alpha & 1 \\ 0 & 1 \end{pmatrix} g\right) \zeta\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) d\alpha$$

converges absolutely for $\operatorname{Re} s$ *sufficiently large and* G *in* $G^D_\mathbf{A}$. *It is equal to zero if*

$$\zeta\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}\right) \epsilon\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is not identically 1 *in* $O_\mathfrak{p}^\times$. *There is a constant* $M > 0$ *such that* $a_\alpha = 0$ *if* $|\alpha_\mathfrak{p}| > M$ *for some* $\mathfrak{p}|D$. *Consequently the series*

$$\sum_{\alpha \in K^\times / K^\times \cap I^D} a_\alpha \prod_{\mathfrak{p}|D} \zeta_\mathfrak{p}\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}\right)$$

converges absolutely for $\operatorname{Re} s$ *sufficiently large. Let* R *be the set of non-archimedean primes which do not divide* D *for which* $\zeta_\mathfrak{p}$ *is not trivial on* $A_{O_\mathfrak{p}}$. *The product*

$$\prod_{\substack{\mathfrak{p} \notin S_\infty \cup R \\ \mathfrak{p} \nmid D}} \frac{1}{\left(1 - \omega_{\mathfrak{p},1}(\pi)\zeta_{\mathfrak{p},1}(\pi)|\pi|^{1/2}\right)\left(1 - \omega_{\mathfrak{p},2}(\pi)\zeta_{\mathfrak{p},2}(\pi)|\pi|^{1/2}\right)}$$

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[8](1998) labeled (X) in this version in which the pagination differs from that of the manuscript.

[9]In the digressions to establish notation we allow ourselves to use, in a new sense, symbols whose meaning has otherwise been fixed for the course of this paragraph.

also converges absolutely for $\operatorname{Re} s$ *sufficiently large. The integral is the product of the above two expressions with*

$$\left\{\prod_{\mathfrak{p}\in S_\infty}\Gamma(\zeta_\mathfrak{p},\pi_\mathfrak{p})\Phi'(g_\mathfrak{p},\zeta_\mathfrak{p},\varphi_\mathfrak{p})\right\}\left\{\prod_{\substack{\mathfrak{p}\notin S_\infty\cup R\\ \mathfrak{p}\nmid D}}\Phi'(g_\mathfrak{p},\zeta_\mathfrak{p},\omega_\mathfrak{p},\xi_\mathfrak{p})\right\}\left\{\prod_{\mathfrak{p}\in R}\Phi(g_\mathfrak{p},\zeta_\mathfrak{p},\omega_\mathfrak{p},\xi_\mathfrak{p})\right\}\epsilon(g_D).$$

According to Lemma 6.4 only a finite number of terms in the last product are different from 1. The absolute convergence of the other infinite product follows immediately from the inequalities (A). For each $\mathfrak{p}$ the character $\xi_\mathfrak{p}$ is non-trivial. If $\mathfrak{p}|D$, $x\in O_\mathfrak{p}$, $\alpha\in K_\mathfrak{p}^\times$, and $g\in G_\mathbf{A}^D$

$$\varphi_1\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right)=\varphi_1\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right).$$

If

$$g_\mathfrak{p}=\begin{pmatrix}a_\mathfrak{p} & b_\mathfrak{p}\\ c_\mathfrak{p} & d_\mathfrak{p}\end{pmatrix}$$

this equals

$$\varphi_1\left(\begin{pmatrix}1 & \frac{\alpha_\mathfrak{p}a_\mathfrak{p}}{d_\mathfrak{p}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right)=\xi_\mathfrak{p}\left(\frac{\alpha_\mathfrak{p}a_\mathfrak{p}}{d_\mathfrak{p}}x\right)\varphi_1\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right).$$

Thus if a_α is not zero, α must lie in the largest ideal of $K_\mathfrak{p}$ on which $\xi_\mathfrak{p}$ is trivial. The existence of the constant M follows immediately.

Recalling that, for almost all $\mathfrak{p}$, $\varphi(g_\mathfrak{p},\omega_\mathfrak{p},\xi_\mathfrak{p})$ equals 1 if $g_\mathfrak{p}$ lies in $U_{K_\mathfrak{p}}$, we see that

$$\int_{K^\times\cap I^D\backslash I^D}\left|\varphi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}g\right)\zeta\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\right)\right|d\gamma$$

is at most the sum over $K^\times/K^\times\cap I^D$ of the product of

$$|a_\alpha|\prod_{\mathfrak{p}\in S_\infty}\int_{K_\mathfrak{p}^\times}\left|\varphi_\mathfrak{p}\left(\begin{pmatrix}\alpha_\mathfrak{p}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p}\right)\zeta_\mathfrak{p}\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}\right)\right|d\gamma_\mathfrak{p}$$

and

$$\prod_{\substack{\mathfrak{p}\notin S_\infty\\ \mathfrak{p}\nmid D}}\int_{K_\mathfrak{p}^\times}\left|\varphi\left(\begin{pmatrix}\alpha_\mathfrak{p}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p},\omega_\mathfrak{p},\xi_\mathfrak{p}\right)\zeta_\mathfrak{p}\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}\right)\right|d\gamma_\mathfrak{p}.$$

Changing variables in the integral and recalling the product formula we see that the sum is the product of

$$\sum_{K^\times/K^\times\cap I^D}|a_\alpha|\prod_{\mathfrak{p}|D}\left|\zeta_\mathfrak{p}\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & 1\end{pmatrix}\right)\right|$$

and

$$\prod_{\mathfrak{p}\in S_\infty}\int_{K_\mathfrak{p}^\times}\left|\varphi_\mathfrak{p}\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p}\right)\zeta_\mathfrak{p}\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}\right)\right|d\gamma_\mathfrak{p}$$

and

$$\prod_{\substack{\mathfrak{p}\notin S_\infty\\ \mathfrak{p}\nmid D}}\int_{K_\mathfrak{p}^\times}\left|\varphi\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p},\omega_\mathfrak{p},\xi_\mathfrak{p}\right)\zeta_\mathfrak{p}\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}\right)\right|d\gamma_\mathfrak{p}.$$

The first term is certainly finite for $\operatorname{Re} s$ sufficiently large. The convergence of the integrals over $K_\mathfrak{p}^\times$, $\mathfrak{p}\in S_\infty$, was proved in Lemmas 3.6 and 5.7. It remains to show that if $\operatorname{Re} s$ is sufficiently large each of the integrals in the infinite product is finite and the product converges. It was proved in Lemma 6.4 that for a given $\mathfrak{p}$ the integral is finite if $\operatorname{Re} s$ is sufficiently large. Thus we can, in our considerations, drop any finite set of terms from the product.

The first formula[10] on the top of p. 6.10 shows that if $g_\mathfrak{p}$ is a unit and $O_\mathfrak{p}$ is the largest ideal on which $\xi_\mathfrak{p}$ is trivial then

$$\int_{K_\mathfrak{p}^\times}\left|\varphi\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p},\omega_\mathfrak{p},\xi_\mathfrak{p}\right)\zeta_\mathfrak{p}\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}\right)\right|d\gamma_\mathfrak{p}$$

is at most

$$\frac{1}{\left(1-|\pi|^s\right)\left(1-|\pi|^s\right)}$$

if $\operatorname{Re} s>0$. The infinite product converges if $\operatorname{Re} s>1$.

Thus the integral is finite. A simple formal manipulation which is now justified shows that it is the product of

$$\sum_{\alpha\in K^\times/K^\times\cap I^D} a_\alpha\prod_{\mathfrak{p}\mid D}\zeta_\mathfrak{p}\left(\begin{pmatrix}\alpha_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}\right)$$

and

$$\prod_{\mathfrak{p}\mid D}\int_{O_\mathfrak{p}^\times}\epsilon\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p}\right)\zeta_\mathfrak{p}\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}\right)d\gamma_\mathfrak{p}$$

and

$$\prod_{\mathfrak{p}\in S_\infty}\int_{K_\mathfrak{p}^\times}\varphi_\mathfrak{p}\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p}\right)\zeta_\mathfrak{p}\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}\right)d\gamma_\mathfrak{p}$$

and

$$\prod_{\substack{\mathfrak{p}\notin S_\infty\\ \mathfrak{p}\nmid D}}\int_{K_\mathfrak{p}^\times}\varphi\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}g_\mathfrak{p},\omega_\mathfrak{p},\xi_\mathfrak{p}\right)\zeta_\mathfrak{p}\left(\begin{pmatrix}\gamma_\mathfrak{p} & 0\\ 0 & 1\end{pmatrix}\right)d\gamma_\mathfrak{p}.$$

The remaining statements of the lemma are now just a matter of definition.

We shall be able to state the next lemma more succinctly if we first introduce some notation. First let K be the real or complex field and let π be a quasi-simple irreducible representation of $\{\sigma,\mathfrak{A}\}$ or $\mathfrak{A}$ respectively. Let ξ be a character of K and ζ a continuous

[10](1998) Labeled (X) for convenience.

homomorphism of A_K into $\mathbf{C}^\times$ which satisfies the condition of Lemma 3.6 or 5.7. Define $\epsilon(\zeta,\xi,\pi)$ by the relation

$$\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, \widetilde{\zeta}, \varphi\right) = \epsilon(\zeta,\xi,\pi)\Phi'(g,\zeta,\varphi).$$

The exact form of the factor is given in Lemmas 3.6 and 5.7. If K is a non-archimedean field, ω a homomorphism of A_K/A_O into $\mathbf{C}^\times$, ζ a continuous homomorphism of A_K into $\mathbf{C}^\times$ which satisfies the condition of Lemma 6.4, and ξ a character of K define $\epsilon(\zeta,\xi,\pi)$ by the relation

$$\Phi'\left(\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} g, \widetilde{\zeta}, \omega, \xi\right) = \epsilon(\zeta,\xi,\omega)\Phi'(g,\zeta,\omega,\xi)$$

if ζ is trivial on A_O and by the relation

$$\Phi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, \widetilde{\zeta}, \omega, \xi\right) = \epsilon(\zeta,\xi,\omega)\Phi(g,\zeta,\omega,\xi)$$

if it is not. The form of this factor is given in Lemma 6.4.

Choose A in $K^\times$ so that $(A_\mathfrak{p}) = \mathfrak{p}^{m_\mathfrak{p}}$ if $\mathfrak{p}|D$ and set

$$\widehat{\varphi}(g) = \varphi\left(g\prod_{\mathfrak{p}|D}\begin{pmatrix} 0 & 1 \\ A_\mathfrak{p} & 0 \end{pmatrix}\right).$$

$\widehat{\varphi}$ lies in $\mathfrak{H}^D_{\widetilde{\epsilon}}$. If ζ is the homomorphism of $A_\mathbf{A}$ into $\mathbf{C}^\times$ introduced in Lemma 7.3, let $\Xi(x,\chi)$ be the product of

$$\left\{\sum_{\alpha\in K^\times/K^\times\cap I^D} a_\alpha \prod_{\mathfrak{p}|D}\zeta_\mathfrak{p}\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}\right)\right\}\prod_{\mathfrak{p}\in S_\infty}\Gamma(\zeta_\mathfrak{p},\pi_\mathfrak{p})$$

and

$$\prod_{\substack{\mathfrak{p}\notin S_\infty\cup R \\ \mathfrak{p}\nmid D}} \frac{1}{\left(1-\alpha_{\mathfrak{p},1}(\pi)\zeta_{\mathfrak{p},1}(\pi)|\pi|^{1/2}\right)\left(1-w_{\mathfrak{p},2}(\pi)\zeta_{\mathfrak{p},2}(\pi)|\pi|^{1/2}\right)}.$$

Given φ the functions $\varphi_\mathfrak{p}$ are determined only up to a scalar factor. Thus there is an undetermined constant in the numbers a_α and hence in the function $\Xi(s,\chi)$. However we can certainly suppose that, for $\mathfrak{p}$ archimedean, $\widehat{\varphi}_\mathfrak{p}$, the function associated to $\widehat{\varphi}$, is the same as $\varphi_\mathfrak{p}$. This assumption is implicit in the statement and proof of the following lemma.

Lemma 7.4. $\Xi(s,\chi)$ *is an entire function of* s. *It satisfies the functional equation*[11]

$$\Xi(s,\chi) = \left\{\prod_{\mathfrak{p}|D}\zeta_\mathfrak{p}\left(\begin{pmatrix} -A_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}\right)\prod_{\mathfrak{p}\in S_\infty}\epsilon(\zeta_\mathfrak{p},\xi_\mathfrak{p},\pi_\mathfrak{p})\prod_{\mathfrak{p}\nmid D}\epsilon(\zeta_\mathfrak{p},\xi_\mathfrak{p},\omega_\mathfrak{p})\right\}\widehat{\Xi}\left(-s,(\chi\eta)^{-1}\right).$$

The integral in Lemma 7.3 is the sum of

$$\text{(B)}\qquad \int_{|\alpha|\geqslant 1}\varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)\zeta\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right)d\alpha$$

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[11] $\widehat{\Xi}\left(-s,(\chi\eta)^{-1}\right)$ is the function obtained on replacing φ by $\widehat{\varphi}$.

and

$$\int_{|\alpha|\leqslant 1} \varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix} g\right)\zeta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) d\alpha.$$

The latter integral is equal to

$$\int_{|\alpha|\geqslant 1} \varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ A & 0\end{pmatrix} g\right)\eta^{-1}(\alpha)\zeta^{-1}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) d\alpha$$

or

$$\text{(C)} \qquad \int_{|\alpha|\geqslant 1} \widehat{\varphi}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ A & 0\end{pmatrix} g \prod_{\mathfrak{p}|D}\begin{pmatrix}0 & A_{\mathfrak{p}}^{-1}\\ 1 & 0\end{pmatrix}\right)\widetilde{\zeta}\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) d\alpha.$$

If the first integral converges for $\operatorname{Re} s$ sufficiently large, as it does, it must converge for all s. The resulting function of s is entire. Since the substitution of $-s$ for s, $(\chi\eta)^{-1}$ for χ, $\widehat{\varphi}$ for φ, and $\left(\begin{smallmatrix}0 & 1\\ A & 0\end{smallmatrix}\right) g \prod_{m_{\mathfrak{p}}>0}\left(\begin{smallmatrix}0 & A_{\mathfrak{p}}^{-1}\\ 1 & 0\end{smallmatrix}\right)$ for g interchanges the integrals (B) and (C), the latter integral is also an entire function.

We conclude that the product of $\Xi(s,\chi)$ and

$$\text{(D)} \qquad \left\{\prod_{\mathfrak{p}\in S_\infty}\Phi'(g_{\mathfrak{p}},\zeta_{\mathfrak{p}},\varphi_{\mathfrak{p}})\right\}\left\{\prod_{\substack{\mathfrak{p}\notin S_\infty\cup R\\ \mathfrak{p}\nmid D}}\Phi'(g_{\mathfrak{p}},\zeta_{\mathfrak{p}},\omega_{\mathfrak{p}},\xi_{\mathfrak{p}})\right\}\left\{\prod_{\mathfrak{p}\in R}\Phi(g_{\mathfrak{p}},\zeta_{\mathfrak{p}},\omega_{\mathfrak{p}},\xi_{\mathfrak{p}})\right\}\epsilon(g_D)$$

is an entire function of s.

It is clear that if $\mathfrak{p}$ is an archimedean prime the function $\Xi(s,\chi)$ is not changed if φ is replaced by a non-zero linear combination of functions obtained from φ by operations of $\{\sigma_{\mathfrak{p}},\mathfrak{A}_{\mathfrak{p}}\}$ or $\mathfrak{A}_{\mathfrak{p}}$, according to the nature of the prime. Thus to prove the lemma we can choose the functions $\varphi_{\mathfrak{p}}$ in any way convenient. I claim that these functions and g in $G_{\mathbf{A}}^D$ can be so chosen that almost all of the factors in (D) are 1 and the rest are of the form ae^{bs} with $a\neq 0$. It will follow that $\Xi(s,\chi)$ is entire.

g_D may as well be taken to be I. If we take $g_{\mathfrak{p}} = I$ for $\mathfrak{p}\notin R\cup S_\infty$, $\mathfrak{p}\nmid D$ then according to Lemma 6.4 and the formula[12] at the top of p. 6.12 each of the functions $\Phi'(g_{\mathfrak{p}},\zeta_{\mathfrak{p}},\omega_{\mathfrak{p}},\xi_{\mathfrak{p}})$ is of this form and all but a finite number are identically 1. If $\mathfrak{p}\in R$ then, according to the formulae at the top of p. 6.12 and the bottom of p. 6.13[13]

$$\Phi\left(\begin{pmatrix}1 & \pi^{-\gamma_{\mathfrak{p}}}\\ 0 & 1\end{pmatrix},\zeta_{\mathfrak{p}},\omega_{\mathfrak{p}},\xi_{\mathfrak{p}}\right)$$

[12](1998) Labeled (Y).

[13](2023) Editorial comment: This is just before the start of section 7.

will be of this form for a suitable choice of $\gamma_\mathfrak{p}$. For a real prime choose $g_\mathfrak{p} = I$ and $\varphi_\mathfrak{p}$ so that the formulae on p. 3.37[14][15] can be applied. For a complex prime choose $g_\mathfrak{p} = I$ and $\varphi_\mathfrak{p}$ as on pp. 5.28 and 5.29.[16][17]

Now let us see what happens to the expression (D) when the substitution mentioned above is performed. The substitution replaces ζ by $\widetilde{\zeta}$ and ϵ by $\widetilde{\epsilon}$. The factor $\epsilon(g_D)$ is not changed. The functions occurring in the other factors are not changed but some of the variables are. $g_\mathfrak{p}$ is replaced by $\left(\begin{smallmatrix} 0 & 1 \\ A_\mathfrak{p} & 0 \end{smallmatrix}\right) g_\mathfrak{p}$ and $\zeta_\mathfrak{p}$ is replaced by $\widetilde{\zeta}_\mathfrak{p}$. Thus the expression (D) is multiplied by

$$\left\{\prod_{\mathfrak{p}\nmid D} \widetilde{\zeta}_\mathfrak{p}^{-1}\left(\begin{pmatrix} 1 & 0 \\ 0 & -A_\mathfrak{p} \end{pmatrix}\right)\right\}\left\{\prod_{\mathfrak{p}\in S_\infty} \epsilon(\zeta_\mathfrak{p}, \xi_\mathfrak{p}, \pi_\mathfrak{p})\right\}\left\{\prod_{\substack{\mathfrak{p}\notin S_\infty \\ \mathfrak{p}\nmid D}} \epsilon(\zeta_\mathfrak{p}, \xi_\mathfrak{p}, \omega_\mathfrak{p})\right\}$$

which equals

$$\left\{\prod_{\mathfrak{p}\mid D} \zeta_\mathfrak{p}\left(\begin{pmatrix} -A_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix}\right)\right\}\left\{\prod_{\mathfrak{p}\in S_\infty} \epsilon(\zeta_\mathfrak{p}, \xi_\mathfrak{p}, \pi_\mathfrak{p})\right\}\left\{\prod_{\substack{\mathfrak{p}\notin S_\infty \\ \mathfrak{p}\nmid D}} \epsilon(\zeta_\mathfrak{p}, \xi_\mathfrak{p}, \omega_\mathfrak{p})\right\}.$$

The lemma follows.

We want to prove a converse to this lemma. Suppose we are given the divisor D and hence U^D, a homomorphism ϵ of $U^D/\widehat{U}^D$ into $\mathbf{C}^\times$, and a non-trivial character ξ of $\mathbf{A}/K$. Suppose that we are given bounded functions a_α and $\widehat{a}_\alpha$ on $K^\times$ such that

$$a_{\alpha\beta} = \epsilon\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) a_\alpha \qquad \widehat{a}_{\alpha\beta} = \widetilde{\epsilon}\left(\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}\right) \widehat{a}_\alpha$$

if β lies in $K^\times \cap I^D$. Suppose moreover that $a_\alpha = 0$ if, for some $\mathfrak{p}$ dividing D, $\alpha_\mathfrak{p}$ does not lie in the largest ideal of $K_\mathfrak{p}$ on which $\xi_\mathfrak{p}$ is trivial. We will also have to be given, for each archimedean prime $\mathfrak{p}$, an irreducible quasi-simple representation of $\{\sigma_\mathfrak{p}, \mathfrak{A}_\mathfrak{p}\}$ or $\mathfrak{A}_\mathfrak{p}$ according to the nature of the prime and, for each non-archimedean prime which does not divide D, a character $\omega_\mathfrak{p}$ of $A_{K_\mathfrak{p}}/A_{O_\mathfrak{p}}$ which satisfies

$$\left|\omega_{\mathfrak{p},1}(\pi)\right| \leqslant |\pi|^{-1/2} \qquad \left|\omega_{\mathfrak{p},2}(\pi)\right| \leqslant |\pi|^{1/2}.$$

If $\mathfrak{p}$ is archimedean let $\pi_\mathfrak{p}$ be deducible from $\pi_{\omega_\mathfrak{p}}$. We shall also suppose that the homomorphism

$$\eta(\alpha) = \prod_{\mathfrak{p}\in S_\infty} \omega_\mathfrak{p}\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & \alpha_\mathfrak{p} \end{pmatrix}\right) \prod_{\substack{\mathfrak{p}\notin S_\infty \\ \mathfrak{p}\nmid D}} \omega_\mathfrak{p}\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & \alpha_\mathfrak{p} \end{pmatrix}\right) \prod_{\mathfrak{p}\mid D} \epsilon\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & \alpha_\mathfrak{p} \end{pmatrix}\right)$$

of I^D into $\mathbf{C}^\times$ is trivial on $K^\times \cap I^D$.

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[14](1998) Now p. 33.

[15](2023) Editorial comment: The formulae in the table on p. 27 after "In all but the last line $s - m$ is not an odd integer."

[16](1998) At the very end of the chapter, just before the appendix.

[17](2023) Editorial comment: P. 44. Page 5.28 starts at "If $\eta = \widetilde{\zeta}^{-1}$ the maps...".

Lemma 7.5. *Choose for each archimedean prime a function $\varphi_\mathfrak{p}$ in $L(\xi_\mathfrak{p}, \pi_\mathfrak{p})$. If $g \in G^D_\mathbf{A}$ the series*

$$\sum_{\alpha \in K^\times} a_\alpha \left\{ \prod_{\mathfrak{p} \in S_\infty} \varphi_\mathfrak{p}\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} g_\mathfrak{p}\right) \right\} \left\{ \prod_{\substack{\mathfrak{p} \notin S_\infty \\ \mathfrak{p} \nmid D}} \varphi\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} g_\mathfrak{p}, \omega_\mathfrak{p}, \xi_\mathfrak{p}\right) \right\} \epsilon(g_D)$$

converges absolutely. Moreover the convergence is uniform on compact subsets of $G^D_\mathbf{A}$. Let $\varphi(g)$ be its sum. If x belongs to K and $x_\mathfrak{p}$ lies in $O_\mathfrak{p}$ for $\mathfrak{p}|D$ then $\varphi\big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\big) = \varphi(g)$ and, if α, β lie in $K^\times \cap I^D$, $\varphi\Big(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \beta \end{smallmatrix}\right) g\Big) = \varphi(g)$.

Choose a compact subset C of $G^D_\mathbf{A}$. According to the discussion[18] on p. 6.8 there is for each non-archimedean prime $\mathfrak{p}$ which does not divide D a number $M_\mathfrak{p}$ such that if $g \in O$

$$\varphi\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} g_\mathfrak{p}, \omega_\mathfrak{p}, \xi_\mathfrak{p}\right) = 0$$

if $|\alpha_\mathfrak{p}| > M_\mathfrak{p}$. Moreover almost all of the numbers $M_\mathfrak{p}$ can be taken to be 1. Because of the assumption on the function $\{a_\alpha\}$ the sum in the lemma can be replaced by a sum over a finite set if K is a function field and by a sum over a lattice in K if K is a number field. If K is a function field the first two assertions of the lemma are immediate. Suppose K is a number field.

Combining the formula[19][20] at the top of p. 6.10 with our assumption on the magnitude of the numbers $\omega_{\mathfrak{p},1}(\pi)$ and $\omega_{\mathfrak{p},2}(\pi)$ we see that there is a positive constant b and for each non-archimedean prime $\mathfrak{p}$ which do not divide D a constant $C_\mathfrak{p}$ such that

$$\left|\varphi\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} g_\mathfrak{p}, \omega_\mathfrak{p}, \xi_\mathfrak{p}\right)\right| \leqslant C_\mathfrak{p} |\alpha_\mathfrak{p}|^{-b}$$

if g is in C. For all but a finite number of primes $C_\mathfrak{p}$ can be taken to be 1.

Because of the product formula we are reduced to considering the sum

$$\sum \prod_{\mathfrak{p} \in S_\infty} |\alpha_\mathfrak{p}|^b \left|\varphi'_\mathfrak{p}\left(\begin{pmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & 1 \end{pmatrix} g_\mathfrak{p}\right)\right|$$

over the non-zero points of a lattice in K. On pages[21] 3.9 and 5.9 we have discussed the behaviour of the functions $\psi(t)$ and $\psi^{n/2}(g)$ as $|t| \to \infty$. The first of the equations (A) on p. 5.8 can be used to determine the asymptotic behaviour of all the functions $\psi^k(t)$. In Lemmas 3.4 and 5.4 we have discussed the behaviour of these functions as $|t| \to 0$. Putting all the information together we see that these are positive constants c and d and a

[18](1998) Now following Lemma 6.3.

[19](1998) See previous footnotes.

[20](2023) Editorial comment: The formula labeled (X).

[21]Between Lemma 3.2 and its corollary and just before Lemma 5.3.

constant Q such that

$$\left|\varphi_{\mathfrak{p}}\left(\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix} g_{\mathfrak{p}}\right)\right| \leqslant Q|\alpha_{\mathfrak{p}}|^{-c}e^{-d|\alpha_{\mathfrak{p}}|}$$

if g is archimedean and g lies in C. The absolute and uniform convergence of the sum follows.

The last two statements of the lemma can be proved for both types of field simultaneously. If $x \in K$ and $x_{\mathfrak{p}} \in O_{\mathfrak{p}}$ for $\mathfrak{p}|D$ and $a_\alpha \neq 0$ then, by assumption, $\xi_{\mathfrak{p}}(\alpha_{\mathfrak{p}} x_{\mathfrak{p}}) = 1$ if $\mathfrak{p} \nmid D$. Thus

$$\prod_{\mathfrak{p}\nmid D} \xi_{\mathfrak{p}}(\alpha_{\mathfrak{p}} x_{\mathfrak{p}}) = 1.$$

The product

$$\left\{\prod_{\mathfrak{p}\in S_\infty} \varphi_{\mathfrak{p}}\left(\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x_{\mathfrak{p}}\\ 0 & 1\end{pmatrix} g_{\mathfrak{p}}\right)\right\}$$
$$\times\left\{\prod_{\substack{\mathfrak{p}\notin S_\infty\\ \mathfrak{p}\nmid D}} \varphi\left(\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x_{\mathfrak{p}}\\ 0 & 1\end{pmatrix} g_{\mathfrak{p}}, \omega_{\mathfrak{p}}, \xi_{\mathfrak{p}}\right)\right\}\epsilon\left(\begin{pmatrix}1 & x_D\\ 0 & 1\end{pmatrix} g_D\right)$$

is equal to

$$\prod_{\mathfrak{p}\nmid D}\xi_{\mathfrak{p}}(\alpha_{\mathfrak{p}} x_{\mathfrak{p}})\left\{\prod_{\mathfrak{p}\in S_\infty} \varphi_{\mathfrak{p}}\left(\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix} g_{\mathfrak{p}}\right)\right\}\left\{\prod_{\substack{\mathfrak{p}\notin S_\infty\\ \mathfrak{p}\nmid D}} \varphi\left(\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix} g_{\mathfrak{p}}, \omega_{\mathfrak{p}}, \xi_{\mathfrak{p}}\right)\right\}\epsilon\left(\begin{pmatrix}1 & x_D\\ 0 & 1\end{pmatrix} g_D\right).$$

Since $\epsilon\left(\left(\begin{smallmatrix}1 & x_D\\ 0 & 1\end{smallmatrix}\right) g_D\right) = \epsilon(g_D)$ the relation $\varphi\left(\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right) g\right) = \varphi(g)$ follows.

The relation $\varphi\left(\left(\begin{smallmatrix}\beta & 0\\ 0 & \beta\end{smallmatrix}\right) g\right) = \varphi(g)$ for $\beta \in K^\times \cap I^D$ is, essentially, one of the assumptions. To complete the proof of the lemma we need only show that $\varphi\left(\left(\begin{smallmatrix}\beta & 0\\ 0 & 1\end{smallmatrix}\right) g\right) = \varphi(g)$ when β lies in $K^\times \cap I^D$. After replacing g by $\left(\begin{smallmatrix}\beta & 0\\ 0 & 1\end{smallmatrix}\right) g$ in the sum defining φ we can change variables in the summation, replacing α by $\alpha\beta^{-1}$. The sum becomes

$$\sum_{\alpha\in K^\times} a_{\alpha,\beta^{-1}}\left\{\prod_{\mathfrak{p}\in S_\infty} \varphi_{\mathfrak{p}}\left(\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix} g_{\mathfrak{p}}\right)\right\}\left\{\prod_{\substack{\mathfrak{p}\notin S_\infty\\ \mathfrak{p}\nmid D}} \varphi\left(\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & 1\end{pmatrix} g_{\mathfrak{p}}, \omega_{\mathfrak{p}}, \xi_{\mathfrak{p}}\right)\right\}\epsilon\left(\begin{pmatrix}\beta_D & 0\\ 0 & 1\end{pmatrix} g_D\right).$$

The relation $\varphi\left(\left(\begin{smallmatrix}\beta & 0\\ 0 & 1\end{smallmatrix}\right) g\right) = \varphi(g)$ is thus a consequence of the assumption

$$a_{\alpha\beta^{-1}}\epsilon\left(\begin{pmatrix}\beta_D & 0\\ 0 & 1\end{pmatrix}\right) = a_\alpha.$$

With the same choice of functions $\varphi_{\mathfrak{p}}$ the function $\{\widehat{a}_\alpha\}$ determines a function $\widehat{\varphi}$. Of course ϵ must be replaced by $\widetilde{\epsilon}$.

Let χ be a character of $K^\times \cap I^D \backslash I^D$ such that

$$\chi(\alpha_{\mathfrak{p}})\epsilon\left(\begin{pmatrix} \alpha_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\right) = 1$$

for $\mathfrak{p}|D$ and $\alpha_{\mathfrak{p}}$ in $O_{\mathfrak{p}}^\times$. If s is a complex number define $\zeta = \zeta(s,\chi)$ as before by

$$\zeta\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right) = \eta(\beta)|\alpha\beta^{-1}|^s \chi(\alpha\beta^{-1}).$$

The function $\Xi(s,\chi)$ given as the product of

$$\left\{\sum_{\alpha \in K^\times/K^\times \cap I^D} a_\alpha \prod_{\mathfrak{p}|D} \zeta_{\mathfrak{p}}\left(\begin{pmatrix} \alpha_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\right)\right\}\left\{\prod_{\mathfrak{p}\in S_\infty} \Gamma(\zeta_{\mathfrak{p}}, \pi_{\mathfrak{p}})\right\}$$

and

$$\prod_{\substack{\mathfrak{p}\notin S_\infty \cup R \\ \mathfrak{p}\nmid D}} \frac{1}{\left(1-\omega_{\mathfrak{p},1}(\pi)\zeta_{\mathfrak{p},1}(\pi)|\pi|^{1/2}\right)\left(1-\omega_{\mathfrak{p},2}(\pi)\zeta_{\mathfrak{p},2}(\pi)|\pi|^{1/2}\right)}$$

is defined for $\operatorname{Re} s$ sufficiently large. If $\{a_\alpha\}$ is replaced by $\{\widehat{a}_\alpha\}$ and χ by $\chi^{-1}\eta^{-1}$ we can define a similar function $\widehat{\Xi}(s,\chi^{-1}\eta^{-1})$.

Lemma 7.6. *If there is an A in $K^\times$ with $(A_{\mathfrak{p}}) = \mathfrak{p}^{m_{\mathfrak{p}}}$ for $\mathfrak{p}|D$ and if, for all possible choices of χ, $\Xi(s,\chi)$ is an entire function of s which is bounded in vertical strips and satisfies the functional equation*

$$\Xi(s,\chi) = \left\{\prod_{\mathfrak{p}|D} \zeta_{\mathfrak{p}}\left(\begin{pmatrix} -A_{\mathfrak{p}} & 0 \\ 0 & 1 \end{pmatrix}\right)\right\}\left\{\prod_{\mathfrak{p}\in S_\infty} \epsilon(\zeta_{\mathfrak{p}}, \xi_{\mathfrak{p}}, \pi_{\mathfrak{p}})\right\}\left\{\prod_{\substack{\mathfrak{p}\notin S_\infty \\ \mathfrak{p}\nmid D}} \epsilon(\zeta_{\mathfrak{p}}, \xi_{\mathfrak{p}}, \omega_{\mathfrak{p}})\right\}\widehat{\Xi}(-s,\chi^{-1}\eta^{-1})$$

then, for all g in $G_{\mathbf{A}}^D$,

$$\widehat{\varphi}\left(\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} g \prod_{\mathfrak{p}|D} \begin{pmatrix} 0 & A_{\mathfrak{p}}^{-1} \\ 1 & 0 \end{pmatrix}\right) = \varphi(g).$$

Let $\varphi_1(g)$ be the function on the left side of this equation and let I_0^D be the ideles of norm 1 in I^D. We have to show that for each g in $G_{\mathbf{A}}^D$

$$\varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) = \varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

for all α in I_0^D. Since both sides are continuous functions on $K^\times \cap I_0^D \backslash I_0^D$ which is compact, we just have to compare Fourier coefficients. Any character of $I_0^D \cap K^\times \backslash I_0^D$ is obtained by restricting a character χ of $K^\times \cap I^D$ to I_0^D. Set

$$\mu(\chi, g) = \int_{K^\times \cap I^D \backslash I_0^D} \varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(\alpha)\, d\alpha,$$

$$\mu_1(\chi, g) = \int_{K^\times \cap I^D \backslash I_0^D} \varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(\alpha)\, d\alpha.$$

$\mu(\chi, g)$ and $\mu_1(\chi, g)$ are both identically zero if $\chi(\alpha_\mathfrak{p})\epsilon\left(\left(\begin{smallmatrix} \alpha_\mathfrak{p} & 0 \\ 0 & 1 \end{smallmatrix}\right)\right) \neq 1$ for some $\mathfrak{p}|D$ and some $\alpha_\mathfrak{p}$ in $O_\mathfrak{p}^\times$. Thus we need only consider the χ satisfying the conditions of the lemma.

The functions

$$\chi(\alpha)\mu\left(\chi, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

$$\chi(\alpha)\mu_1\left(\chi, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

are continuous functions on $I_0^D \backslash I^D$ which is isomorphic to R^+ if K is a number field and to $\mathbf{Z}$ if K is a function field. As in the proof of Lemma 6.3, the Mellin transform

$$\int_{I_0^D \backslash I^D} \chi(\alpha)\mu\left(\chi, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) |\alpha|^s\, d\alpha = \int_{K^\times \cap I^D \backslash I^D} \varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \zeta\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) d\alpha$$

is defined for $\operatorname{Re} s$ sufficiently large and the Mellin transform

$$\int_{I_0^D \backslash I^D} \chi(\alpha)\mu_1\left(\chi, \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) |\alpha|^s\, d\alpha$$

which equals

$$\int_{K^\times \cap I^D \backslash I^D} \widehat{\varphi}\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} g \prod_{\mathfrak{p}|D} \begin{pmatrix} 0 & A_\mathfrak{p}^{-1} \\ 1 & 0 \end{pmatrix}\right) \widetilde{\zeta}\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) d\alpha$$

is defined for $\operatorname{Re} s$ sufficiently small.

To prove the lemma in the case of a function field we need only verify that both the Mellin transforms are entire functions of s and that they are equal. In the case of a number field we must show in addition that they are bounded in each vertical strip of finite width.[22]

As in Lemma 7.3 the first integral is the product of $\Xi(s, \chi)$ and

$$\text{(E)} \qquad \left\{\prod_{\mathfrak{p} \in S_\infty} \Phi'(g_\mathfrak{p}, \zeta_\mathfrak{p}, \varphi_\mathfrak{p},)\right\} \left\{\prod_{\substack{\mathfrak{p} \notin S_\infty \cup R \\ \mathfrak{p} \nmid D}} \Phi'(g_\mathfrak{p}, \zeta_\mathfrak{p}, \omega_\mathfrak{p}, \xi_\mathfrak{p})\right\} \left\{\prod_{\mathfrak{p} \in R} \Phi(g_\mathfrak{p}, \zeta_\mathfrak{p}, \omega_\mathfrak{p}, \xi_\mathfrak{p})\right\} \epsilon(g_D).$$

R is the set of non-archimedean primes which do not divide D such that ζ is not trivial on $A_{O_\mathfrak{p}}$. According to Lemmas 3.6, 5.7, and 6.4 each of the functions occurring in the product is an entire function of s and all but finitely many are identically 1. Thus the first

[22]This seems to be the simplest condition which allows the application of an inversion theorem to establish the identity of the original functions.

Mellin transform is an entire function of s. The second is the product of $\widehat{\Xi}(-s,\chi^{-1}\eta^{-1})$ and the factors

$$\prod_{\mathfrak{p}\in S_\infty}\widetilde{\zeta}_{\mathfrak{p}}^{-1}\left(\begin{pmatrix}1 & 0\\ 0 & -A_{\mathfrak{p}}\end{pmatrix}\right)\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g_{\mathfrak{p}},\widetilde{\zeta}_{\mathfrak{p}},\varphi_{\mathfrak{p}}\right)$$

$$\prod_{\substack{\mathfrak{p}\notin S_\infty\cup R\\ \mathfrak{p}\nmid D}}\widetilde{\zeta}_{\mathfrak{p}}^{-1}\left(\begin{pmatrix}1 & 0\\ 0 & -A_{\mathfrak{p}}\end{pmatrix}\right)\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g_{\mathfrak{p}},\widetilde{\zeta}_{\mathfrak{p}},\omega_{\mathfrak{p}},\xi_{\mathfrak{p}}\right)$$

$$\left\{\prod_{\mathfrak{p}\in R}\widetilde{\zeta}_{\mathfrak{p}}^{-1}\left(\begin{pmatrix}1 & 0\\ 0 & -A_{\mathfrak{p}}\end{pmatrix}\right)\Phi\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g_{\mathfrak{p}},\widetilde{\zeta}_{\mathfrak{p}},\omega_{\mathfrak{p}},\xi_{\mathfrak{p}}\right)\right\}\epsilon(g_D).$$

It is also an entire function of s and, by the definitions of the factors $\epsilon(\zeta_{\mathfrak{p}},\xi_{\mathfrak{p}},\pi_{\mathfrak{p}})$ and $\epsilon(\zeta_{\mathfrak{p}},\xi_{\mathfrak{p}},\omega_{\mathfrak{p}})$ together with the functional equation satisfied by the function $\Xi(s,\chi)$, equal to the first Mellin transform.

One of the Mellin transforms is bounded in vertical strips of a right half-plane, the other is bounded in vertical strips of a left half-plane. Thus to show they are bounded we can apply the Phragmén-Lindelöf theorem for strips. The function $\frac{1}{\Gamma(as+b)}$, a real, grows no faster than an exponential in vertical strips so it is enough to show that we can multiply the Mellin transforms by a product of functions of the form $\Gamma(as+b)$, a real, and obtain a function which is bounded in regions of the form $\operatorname{Re} s < \textit{constant}$, $|\operatorname{Im} s| \gg 0$. By assumption $\Phi(s,\chi)$ is bounded in such regions. The factors in the product (E) corresponding to the non-archimedean primes were shown in Lemma 6.4 to be bounded in vertical strips of finite width. If $\mathfrak{p}$ is an archimedean prime $\Gamma(\zeta_{\mathfrak{p}},\pi_{\mathfrak{p}})$ is a function of this form and

$$\Gamma(\zeta_{\mathfrak{p}},\pi_{\mathfrak{p}})\Phi'(g_{\mathfrak{p}},\zeta_{\mathfrak{p}},\varphi_{\mathfrak{p}})=\Phi(g_{\mathfrak{p}},\zeta_{\mathfrak{p}},\varphi_{\mathfrak{p}})$$

was shown in Lemmas 3.6 and 5.7 to be bounded in regions of the form $|\operatorname{Re} s|\leqslant \textit{constant}$, $|\operatorname{Im} s|\gg 0$.

Theorem 7.7. *If the assumptions of Lemma 7.6 are satisfied, the function φ is a function on $G_K\cap G_{\mathbf{A}}^D\backslash G_{\mathbf{A}}^D$.*

The set of all $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ in $G_K\cap G_{\mathbf{A}}^D$ which satisfy

$$\varphi\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}g\right)\equiv\varphi(g)$$

is a subgroup of $G_K\cap G_{\mathbf{A}}^D$. By Lemma 7.5 it contains all those matrices for which $c=0$. If $b=0$ then

$$\varphi\left(\begin{pmatrix}a & 0\\ c & d\end{pmatrix}g\right)=\widehat{\varphi}\left(\begin{pmatrix}0 & 1\\ A & 0\end{pmatrix}\begin{pmatrix}a & 0\\ c & d\end{pmatrix}g\prod_{m_{\mathfrak{p}}>0}\begin{pmatrix}0 & A_{\mathfrak{p}}^{-1}\\ 1 & 0\end{pmatrix}\right)$$

$$=\widehat{\varphi}\left(\begin{pmatrix}d & \frac{c}{A}\\ 0 & a\end{pmatrix}\begin{pmatrix}0 & 1\\ A & 0\end{pmatrix}g\prod_{m_{\mathfrak{p}}>0}\begin{pmatrix}1 & A_{\mathfrak{p}}^{-1}\\ 1 & 0\end{pmatrix}\right).$$

Applying Lemma 7.5 to $\widehat{\varphi}$ we see that the last expression is equal to

$$\widehat{\varphi}\left(\begin{pmatrix} 0 & 1 \\ A & 0 \end{pmatrix} g \prod_{m_{\mathfrak{p}}>0} \begin{pmatrix} 0 & A_{\mathfrak{p}}^{-1} \\ 1 & 0 \end{pmatrix}\right) = \varphi(g).$$

The theorem is a consequence of the following lemma.

Lemma 7.8. *$G_K \cap G_{\mathbf{A}}^D$ is generated by the matrices in it of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} a & 0 \\ c & d \end{smallmatrix}\right)$.*

Indeed

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & d - \frac{bc}{a} \end{pmatrix} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}.$$

If the matrix on the left is in $G_K \cap G_{\mathbf{A}}^D$ so are both the matrices on the right.

Appendix. Some preliminary remarks are necessary before the nature of the function $\varphi_0(g)$ can be determined. It is convenient to treat the various types of fields separately.

We consider the real field first and use the notation of paragraphs 2 and 3. Let L be the space of infinitely differentiable functions on $N_{\mathbf{R}}\backslash G_{\mathbf{R}}$ which are U-finite on the right.

Lemma A. *Let π be the infinite-dimensional irreducible quasi-simple representation of $\{\sigma, \mathfrak{A}\}$. Suppose π is deducible from π_ω. Let H be a subspace of L which transforms according to π.*

(*i*) *If $s - m$ is not an odd integer and $s \neq 0$ then $\omega \neq \widetilde{\omega}$ and H is contained in $L(\omega) + L(\widetilde{\omega})$.*

(*ii*) *If $s = 0$ and $m = 0$ let $L'(\omega)$ be the space spanned by the functions*

$$\varphi_n'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\right)$$

defined as

$$\left|\frac{\alpha_1}{\alpha_2}\right|^{1/2} \omega\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\right)\left\{\log\left|\frac{\alpha_1}{\alpha_2}\right| + \sum_{k=1}^{\left|\frac{n}{2}\right|} \frac{1}{2k-1}\right\} e^{in\theta},$$

$\frac{n}{2} \in \mathbf{Z}$. $L'(\omega)$ is an invariant irreducible subspace of L and the representation of $\{\sigma, \mathfrak{A}\}$ on $L'(\omega)$ is equivalent to π. H is contained in $L(\omega) + L'(\omega)$.

(*iii*) *If $s - m$ is an odd integer suppose, as we may, that $s \geqslant 0$. Define ω' by*

$$\omega'\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\right) = \operatorname{sgn}(\alpha_2\alpha_2)\omega\left(\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}\right).$$

Then H is contained in $L(\omega) + L(\omega')$.

ω' is of course defined for any ω. In Paragraph 2 we saw that if $s - m$ is not an odd integer and $s \neq 0$ then π is equivalent to the representation of $\{\sigma, \mathfrak{A}\}$ on $L(\omega)$ and $L(\widetilde{\omega})$ but is not contained in the representation of $\{\sigma, \mathfrak{A}\}$ on $L(\omega')$ or $L(\widetilde{\omega}')$. We also saw that if $s - m$ is an odd integer and $s \neq 0$ the representation π is contained once in the representation of $\{\sigma, \mathfrak{A}\}$ on $L(\omega)$ and $L(\omega')$ but is not contained in the representation of $\{\sigma, \mathfrak{A}\}$ on $L(\widetilde{\omega})$ or $L(\widetilde{\omega}')$. Thus if $s \neq 0$ we need only show that H is contained in $L(\omega) + L(\widetilde{\omega}) + L(\omega') + L(\widetilde{\omega}')$.

Suppose $s = 0$ and $m = 0$. It is clear that

$$\rho(\sigma)\varphi_n' = \omega\left(\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)\varphi_n', \quad \rho(J)\varphi_n' = (s_1+s_2)\varphi_n', \quad \rho(U)\varphi_n' = in\varphi_n'.$$

On the other hand taking $u = 0$ in the formulae[23][24] on pp. 3.7 and 3.8 we see that

$$\rho(V)\varphi_n'\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right)$$

is equal to

$$\left|\frac{\alpha_1}{\alpha_2}\right|^{1/2}\omega\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)\left\{(n+1)\log\left|\frac{\alpha_1}{\alpha_2}\right| + 1 + (n+1)\sum_{k=1}^{\left|\frac{n}{2}\right|}\frac{1}{2k-1}\right\}e^{i(n+2)\theta}$$

and that

$$\rho(W)\varphi_n'\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right)$$

is equal to

$$\left|\frac{\alpha_1}{\alpha_2}\right|^{1/2}\omega\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)\left\{(-n+1)\log\left|\frac{\alpha_1}{\alpha_2}\right| + 1 + (-n+1)\sum_{k=1}^{\left|\frac{n}{2}\right|}\frac{1}{2k-1}\right\}e^{i(n-2)\theta}.$$

Thus $\rho(V)\varphi_n' = (n+1)\varphi_{n+2}'$ and $\rho(W)\varphi_n' = (-n+1)\varphi_{n-2}'$. It follows from Lemma 2.1 that the representation on $L'(\omega)$ is equivalent to π_ω and hence to π. The representations of $\{\sigma, \mathfrak{A}\}$ on $L(\omega')$ and $L'(\omega')$ are not equivalent to π. Again we need only show that H is contained in $L(\omega) + L'(\omega) + L(\omega') + L'(\omega')$.

Suppose φ lies in H. There are functions $\varphi_n(\alpha_1, \alpha_2)$ on $\mathbf{R}^\times \times \mathbf{R}^+$, only a finite number of which do not vanish identically, such that for $\alpha_2 > 0$.

$$\varphi\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\right) = \sum_n \varphi_n(\alpha_1,\alpha_2)e^{in\theta}.$$

Moreover there are functions $\psi_n(t)$ on $\mathbf{R}^\times$ such that

$$\varphi_n(\alpha_1,\alpha_2) = \omega\left(\begin{pmatrix}|\alpha_1\alpha_2|^{1/2} & 0\\ 0 & |\alpha_1\alpha_2|^{1/2}\end{pmatrix}\right)\psi_n\left(\frac{\alpha_1}{\alpha_2}\right).$$

Since φ is in L, $\rho(D)\varphi = \lambda(Z)\varphi + \frac{1}{2}\lambda(Z^2)\varphi$ and the equation $\rho(D)\varphi = \frac{s^2-1}{2}\varphi$ reduces to the equations

$$-2t\frac{d\psi_n}{dt} + 2t\frac{d}{dt}\left(t\frac{d\psi_n}{dt}\right) = \frac{s^2-1}{2}\psi_n$$

[23](1998) See previous footnotes.

[24](2023) Editorial comment: ψ_i on p. 14.

or

$$4\left(t\frac{d}{dt}-\frac{1}{2}\right)^2\psi_n = s^2\psi_n.$$

If $s \neq 0$ four linearly independent solutions of this are $(\operatorname{sgn} t)^a|t|^{\frac{1\pm s}{2}}$, $a = 0$ or 1 and if $s = 0$ four linearly independent solutions of this are $(\operatorname{sgn} t)^a|t|^{1/2}$ and $(\operatorname{sgn} t)^a|t|^{1/2}\log|t|$, $a = 0$ or 1. The lemma follows for all representations except the one for which $s = 0$ and $|m| = 1$.

If $s = 0$ and $|m| = 1$ the space H_1 contains a non-zero vector. If φ lies in H_1 the function ψ_n is zero if $n \neq 1$. According to the first formula on p. 3.8[25] the equation $\rho(W)\varphi = 0$ is equivalent to

$$2t\frac{d\psi_1}{dt} - \psi_1(t) = 0.$$

Thus $\psi_1(t)$ is a linear combination of $|t|^{1/2}$ and $(\operatorname{sgn} t)|t|^{1/2}$. Thus H meets $L(\omega) + L(\omega')$. Since H is irreducible, H is contained in $L(\omega) + L(\omega')$.

For the complex field we use the notation of paragraphs 4 and 5. Let L be the space of infinitely differentiable functions on $N_{\mathbf{C}}\backslash G_{\mathbf{C}}$ which are U-finite on the right.

Lemma B. *Let π be an infinite-dimensional irreducible quasi-simple representation of $\mathfrak{A}$. Suppose π is deducible from π_ω. Let H be a subspace of L which transforms according to π.*

(i) If $s - m$ is not integral then $\omega \neq \widetilde{\omega}$ and H is contained in $L(\omega) + L(\widetilde{\omega})$. If $s - m$ is integral define ω' by

$$\omega'\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right) = |\alpha_1\alpha_2|^{\frac{s_1+s_2}{2}}\left|\frac{\alpha_1}{\alpha_2}\right|^m\left(\frac{\alpha_1}{|\alpha_1|}\right)^{\frac{m_1+m_2}{2}+s}\left(\frac{\alpha_2}{|\alpha_2|}\right)^{\frac{m_1+m_2}{2}-s}.$$

π is deducible from π_ω, $\pi_{\widetilde{\omega}}$, $\pi_{\omega'}$, and $\pi_{\widetilde{\omega}'}$.

(ii) If $|s| > |m|$ we can assume with no loss of generality that $s > |m|$. Then H is contained in $L(\omega) + L(\omega') + L(\widetilde{\omega}')$.

(iii) If $|s| = |m|$ and $s \neq 0$ either $\omega = \omega'$ or $\omega = \widetilde{\omega}'$. In this case H is contained in $L(\omega) + L(\widetilde{\omega})$.

(iv) If $s = 0$ and $m = 0$ define γ_0 and δ_k as on[26] p. 4.8 and let $c_n = \sum_{k=1}^{n/2}\frac{1}{k}$ if n is a non-negative even integer. If $t > 0$ set $\psi_n(t) = \log t + c_n$. Let $L'(\omega)$ be the space spanned by the functions

$$\widehat{\varphi}_{n,k}\left(\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}u\right) = \omega\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)\left|\frac{\alpha_1}{\alpha_2}\right|\psi_n\left(\left|\frac{\alpha_1}{\alpha_2}\right|\right)\gamma_0\sigma_n(u)\delta_k$$

with $\frac{n}{2} \in \mathbf{Z}$, $-k \in \mathbf{Z}$, and $|k| \leqslant \frac{n}{2}$. $L'(\omega)$ is an irreducible invariant subspace of L and the representation of $\mathfrak{A}$ on $L'(\omega)$ is equivalent to π. The space H must lie in $L(\omega) + L'(\omega)$.

The most complicated part of the lemma to verify is the assertion that $L'(\omega)$ is invariant and irreducible so we verify that first.

[25](2023) Editorial comment: The first formula on p. 3.8 is ψ_2 (not ψ_1) on page 14.

[26](1998) Just after Lemma 4.2.

For convenience set $\widehat{\varphi}_{n,k}(g) = 0$ if $k \in \mathbf{Z}$ and $|k| > \frac{n}{2}$. Just as[27][28] in Paragraph 4 the existence of the Clebsch-Gordan series allows us to assert that the function

$$\varphi^+_{n+2,k}\left(\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}u\right)$$

which equals

$$\left(\frac{n}{2}+k+1\right)\left(\frac{n}{2}+k\right)\rho(V^+)\widehat{\varphi}_{n,k-1} - \left(\frac{n}{2}+k+1\right)\left(\frac{n}{2}-k+1\right)\rho(V)\widehat{\varphi}_{n,k} - \left(\frac{n}{2}-k\right)\left(\frac{n}{2}-k+1\right)\rho(V^-)\widehat{\varphi}_{n,k+1}$$

is of the form

$$\left(\frac{n}{2}+k+1\right)!\left(\frac{n}{2}-k+1\right)!\,\omega\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)\left|\frac{\alpha_1}{\alpha_2}\right|\psi^+_{n+2}\left(\left|\frac{\alpha_1}{\alpha_2}\right|\right)\gamma_0\sigma_{n+2}(u)\delta_k,$$

that the function

$$\varphi^0_{n,k} = \left(\frac{n}{2}+k\right)\rho(V^+)\widehat{\varphi}_{n,k-1} + k\rho(V)\widehat{\varphi}_{n,k} + \left(\frac{n}{2}-k\right)\rho(V^-)\widehat{\varphi}_{n,k+1}$$

is of the form

$$\varphi^0_{n,k}\left(\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}u\right) = \left(\frac{n}{2}+k\right)!\left(\frac{n}{2}-k\right)!\,\omega\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)\left|\frac{\alpha_1}{\alpha_2}\right|\psi^0_{n}\left(\left|\frac{\alpha_1}{\alpha_2}\right|\right)\gamma_0\sigma_{n}(u)\delta_k,$$

and that the function

$$\varphi^-_{n-2,k} = \rho(V^+)\widehat{\varphi}_{n,k-1} + \rho(V)\widehat{\varphi}_{n,k} - \rho(V^-)\widehat{\varphi}_{n,k+1}$$

is of the form

$$\varphi^-_{n-2,k}\left(\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}u\right) = \left(\frac{n}{2}+k-1\right)!\left(\frac{n}{2}-k-1\right)!\,\omega\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)\left|\frac{\alpha_1}{\alpha_2}\right|\psi^-_{n-2}\left(\left|\frac{\alpha_1}{\alpha_2}\right|\right)\gamma_0\sigma_{n-2}(u)\delta_k.$$

In these three formulae δ_k is respectively $x^{\frac{n}{2}+1+k}y^{\frac{n}{2}+1-k}$, $x^{\frac{n}{2}+k}y^{\frac{n}{2}-k}$, and $x^{\frac{n}{2}+k-1}y^{\frac{n}{2}-k-1}$ and γ_0 lies in the dual of V_{n+2}, V_n, and V_{n-2} respectively.

[27] The right hand sides of the formula on p. 4.9 are not correct. They should be

$$\left(\frac{n}{2}+k+1\right)!\left(\frac{n}{2}-k+1\right)!\,a(n,\omega)\varphi_{n+2,k}$$

and

$$\left(\frac{n}{2}+k-1\right)!\left(\frac{n}{2}-k-1\right)!\,b(n,\omega)\varphi_{n-2,k}$$

[28] (2023) Editorial comment: p. 31 here.

To show that $L'(\omega)$ is invariant we need only verify that ψ^+_{n+2} is a multiple of ψ_{n+2}, that ψ^0_n is a multiple of ψ_n, and that ψ^-_{n-2} is a multiple of ψ_{n-2}. If $n=0$ only ψ^+_{n+2} is defined. Evaluating $\varphi^+_{n+2,0}$ at $\left(\begin{smallmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{smallmatrix}\right)$ we see that[29][30] $\left[\left(\frac{n}{2}+1\right)!\right]^2 t\psi^+_{n+1}(t)$ is equal to

$$\left(\frac{n}{2}+1\right)\left(\frac{n}{2}\right)t(\log t + c_n)\gamma_0\sigma_n\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}\delta_{-1} \\ - 2\left(\frac{n}{2}+1\right)^2 t\frac{d}{dt}(t\log t + c_n t) \\ - \left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)t(\log t + c_n)\gamma_0\sigma_n\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\delta_1,$$

which equals (cf. p. 4.2)

$$-2\left(\frac{n}{2}+1\right)^2\left(\frac{n}{2}+1\right)t\left(\log t + c_n + \frac{1}{\frac{n}{2}+1}\right) = -2\left(\frac{n}{2}+1\right)^3 t\psi_{n+2}(t).$$

In the same way we see that $\left(\frac{n}{2}!\right)^2 t\psi^0_n(t)$ is equal to

$$\left[-\left(\frac{n}{2}\right)\gamma_0\sigma_n\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\delta_{-1} + \left(\frac{n}{2}\right)\gamma_0\sigma_n\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\delta_1\right](t\log t + c_n t),$$

which equals

$$\left[-\left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right) + \left(\frac{n}{2}\right)\left(\frac{n}{2}+1\right)\right](t\log t + c_n t) = 0.$$

Finally $\left[\left(\frac{n}{2}-1\right)!\right]^2 t\psi^-_{n-2}(t)$ is equal to

$$\left[-\gamma_0\sigma_n\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\delta_{-1} - \gamma_0\sigma_n\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\delta_1\right](t\log t + c_n t) + 2t\frac{d}{dt}(t\log t + c_n t)$$

which equals

$$-n\left(t\log t + c_n t - \frac{t}{\frac{n}{2}}\right) = -nt\psi_{n-2}(t).$$

[29]The formula at the top of p. 4.10 is not correct. It should be

$$V^- = \left(X_1 + \frac{W_1}{2}\right) + i\left(X_2 - \frac{W_2}{2}\right).$$

[30](2023) Editorial comment: P. 4.10 is p. 31 here.

If the functions $\varphi_{n,k}$ are defined as on p. 4.9[31] then, as we have seen[32][33] when $s = 0$ and $m = 0$

$$\left(\frac{n}{2}+k\right)\left(\frac{n}{2}+k+1\right)\rho(V^+)\varphi_{n,k-1} - \left(\frac{n}{2}+k+1\right)\left(\frac{n}{2}-k+1\right)\rho(V)\varphi_{n,k} - \left(\frac{n}{2}-k\right)\left(\frac{n}{2}-k+1\right)\rho(V^-)\varphi_{n,k+1}$$

is equal to

$$-2\frac{\left(\frac{n}{2}+k+1\right)!}{\left(\frac{n}{2}+1\right)!}\frac{\left(\frac{n}{2}-k+1\right)!}{\left(\frac{n}{2}+1\right)!}\left(\frac{n}{2}+1\right)^3\varphi_{n+2,k}$$

and

$$\rho(V^+)\varphi_{n,k-1}+\rho(V)\varphi_{n,k}-\rho(V^-)\varphi_{n,k+1}$$

is equal to

$$\frac{\left(\frac{n}{2}+k-1\right)!}{\left(\frac{n}{2}-1\right)!}\frac{\left(\frac{n}{2}-k-1\right)!}{\left(\frac{n}{2}-1\right)!}(-n)\varphi_{n-2,k}.$$

Moreover one shows readily that

$$\left(\frac{n}{2}+k\right)\rho(V^+)\varphi_{n,k-1}+k\rho(V)\varphi_{n,k}+\left(\frac{n}{2}-k\right)\rho(V^-)\varphi_{n,k+1}$$

is equal to zero. It follows immediately that the representation on $L'(\omega)$ is equivalent to the representation on $L(\omega)$.

The remarks of the lemma can now be verified rather easily. Choose n so that $H_n \neq 0$. There is a function $\Psi(g)$ on $G_{\mathbf{C}}$ with values in $\widehat{V}_n$ such that H_n is the set of functions of the form $\Psi(g)\Phi$, $\Phi \in V_n$. Moreover $\Psi(gu) = \Phi(g)\sigma_n(u)$ if $u \in U^0$ and $\Psi\big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right)g\big) = \omega\big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right)\big)\Psi(g)$. Let $\psi(t) = \Psi\left(\left(\begin{smallmatrix}t^{1/2} & 0\\ 0 & t^{-1/2}\end{smallmatrix}\right)\right)$; Ψ is determined by ψ. According to the formulae[34] on p. 5.8 the equations $\rho(D)\Psi = \frac{(s+m)^2-1}{2}\Psi$ and $\rho(D')\Psi = \frac{(s-m)^2-1}{2}\Psi$ reduce to

$$\left[t\frac{d}{dt}+k-1\right]^2\psi^k = (s+m)^2\psi^k$$

$$\left[t\frac{d}{dt}-k-1\right]^2\psi^k = (s-m)^2\psi^k.$$

If either $(s+m) \neq 0$ or $(s-m) \neq 0$ these equations imply that each ψ^k is a power of t. Thus H_n, and hence H, is contained in a space of the form $\sum_{i=1}^r L(\omega_i)$ for some $\omega_1, \ldots, \omega_r$. Parts (i), (ii) and (iii) of the lemma follow from Lemma 4.2 and the proof of Lemma 4.4. If $s+m = 0$ and $s-m = 0$ then $s = m = 0$. Then $\psi^k \equiv 0$ if $k \neq 0$ and $\psi^0(t)$ is a linear combination of t and $t\log t$. Part (iv) of the lemma also follows.

[31](2023) Editorial comment: P. 30.

[32]According to a remark in a previous footnote the left hand sides of the equation on p. 4.10 should be $\left(\frac{n}{2}+m+1\right)!\left(\frac{n}{2}-m+1\right)!\,a(n,\omega)$ and $\left(\frac{n}{2}+m-1\right)!\left(\frac{n}{2}-m-1\right)!\,b(n,\omega)$.

[33](2023) Editorial comment: P. 31.

[34](1998) Between Lemmas 5.2 and 5.3.

For a non-archimedean local field we use the notation of paragraph 6. If ω is a homomorphism of A_K/A_O into $\mathbf{C}^\times$ define the function φ_ω by

$$\varphi_\omega\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}u\right) = \left|\frac{\alpha_1}{\alpha_2}\right|^{1/2}\omega\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right) \qquad u \in G_O.$$

If $\omega \neq \widetilde{\omega}$ then $\varphi_\omega \neq \varphi_{\widetilde{\omega}}$. If $\omega = \widetilde{\omega}$ define φ'_ω by

$$\varphi'_\omega\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}u\right) = \left|\frac{\alpha_1}{\alpha_2}\right|^{1/2}\omega\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right)\log\left|\frac{\alpha_1}{\alpha_2}\right|.$$

Lemma C. *Suppose φ is a function on $N_K\backslash G_K$ which satisfies $\varphi(gu) \equiv \varphi(g)$ for u in G_O and suppose that for all f in H*

$$\int_{G_K}\varphi(gh)f(h)\,dh = \chi_\omega(f)\varphi(g).$$

If $\omega \neq \widetilde{\omega}$, φ is a linear combination of φ_ω and $\varphi_{\widetilde{\omega}}$ and, if $\omega = \widetilde{\omega}$, φ is a linear combination of φ_ω and φ'_ω.

Choosing f to be the characteristic function of $a(1,1)G_O$ and $G_Oa(0,1)G_O$ we obtain the relations

$$\varphi\left(\begin{pmatrix}\pi\alpha_1 & 0\\ 0 & \pi\alpha_2\end{pmatrix}\right) = \omega(\pi)\varphi\left(\begin{pmatrix}\alpha_1 & \pi\\ w_1 & \alpha_2\end{pmatrix}\right)$$

$$q\varphi\left(\begin{pmatrix}\pi\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right) + \varphi\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \pi\alpha_2\end{pmatrix}\right) = q^{1/2}\left[w\left(\begin{pmatrix}\pi & 0\\ 0 & 1\end{pmatrix}\right) + \omega\left(\begin{pmatrix}1 & 0\\ 0 & \pi\end{pmatrix}\right)\right] \times \varphi\left(\begin{pmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{pmatrix}\right).$$

It is easy to see that these relations are satisfied by φ_ω, $\varphi_{\widetilde{\omega}}$ and, if $\omega = \widetilde{\omega}$, by φ'_ω. If $\omega \neq \widetilde{\omega}$ then $\varphi_\omega\big(\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)\big) = \varphi_{\widetilde{\omega}}\big(\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)\big) \neq 0$ but $\varphi_\omega\big(\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)\big) \neq \varphi_{\widetilde{\omega}}\big(\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)\big)$. Subtracting from φ a suitable linear combination of φ_ω and $\varphi_{\widetilde{\omega}}$ we obtain a function ψ which satisfies the relations and vanishes at $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)$. If $\omega = \omega'$ then $\varphi_\omega\big(\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)\big) \neq \omega$ but $\varphi'_\omega\big(\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)\big) = 0$ while $\varphi'_\omega\big(\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)\big) \neq 0$. We can again subtract from φ a suitable linear combination of φ_ω and φ'_ω and obtain a function ψ which satisfies these relations and vanishes at $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right)$. To prove that, in either case, ψ vanishes identically we need only show that it vanishes at the matrices $\left(\begin{smallmatrix}\pi^{m+n} & 0\\ 0 & \pi^n\end{smallmatrix}\right)$. The first relation implies this is so if $n = 0$ or 1. Taking $\left(\begin{smallmatrix}\alpha_1 & 0\\ 0 & \alpha_2\end{smallmatrix}\right) = \left(\begin{smallmatrix}\pi^{m+n} & 0\\ 0 & \pi^n\end{smallmatrix}\right)$ and substituting in the second relation we see that if this is so if for all m and $n = n_0$ and $n_0 + 1$ it is true for all m and $n = n_0 - 1$ and that if this is so for all m and $n = n_0$ and $n_0 - 1$ it is true for all m and $n = n_0 + 1$. The lemma follows by induction.

Let S be a finite set of primes containing the archimedean primes and the primes which divide D. Let $I_S = \left\{\iota \mid \iota_\mathfrak{p} \text{ is a unit if } \mathfrak{p} \notin S\right\}$. We suppose S is so large that $I^D(K^\times \cap I^D)I_S^D$ if $I_S^D = I_S \cap I^D$. Let $G_S = \prod_{\mathfrak{p}\in S} G_{K_\mathfrak{p}} \times \prod_{\mathfrak{p}\notin S} G_{O_\mathfrak{p}}$ and let $G_S^D = G_\mathbf{A}^D \cap G_S$. According to the previous three lemmas the restriction of φ_0 to G_S^D is a linear combination of functions of the form[35]

$$\psi\left(\left\{\prod_{\mathfrak{p}\in S}\begin{pmatrix}1 & x_{\mathfrak{p}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{pmatrix}u_{\mathfrak{p}}\right\}\left\{\prod_{\mathfrak{p}\notin S}u_{\mathfrak{p}}\right\}\right) =$$

$$\left\{\prod_{\mathfrak{p}\in S}\left|\frac{\alpha_{\mathfrak{p}}}{\beta_{\mathfrak{p}}}\right|^{1/2}\right\}\eta\left(\prod_{\mathfrak{p}\in S}\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{pmatrix}\right)\zeta\left(\prod_{\mathfrak{p}\in S}u_{\mathfrak{p}}\right)\prod_{\mathfrak{p}\in S_1}\log\left|\frac{\alpha_{\mathfrak{p}}}{\beta_{\mathfrak{p}}}\right|.$$

Here $\left(\begin{smallmatrix}1 & x_{\mathfrak{p}}\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{smallmatrix}\right)u_{\mathfrak{p}}$ lies in $U^D_{K_{\mathfrak{p}}}$ if $\mathfrak{p}|D$, η is a homomorphism of the group of diagonal matrices with entries from I^D_S into $\mathbf{C}^\times$ such that $\eta\left(\left(\begin{smallmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{smallmatrix}\right)\right) = 1$ if $\mathfrak{p}\notin S$ and $\alpha_{\mathfrak{p}}$, $\beta_{\mathfrak{p}}$ lie in $O^\times_{\mathfrak{p}}$, and S_1 is a subset of S. If γ and δ belong to $K^\times\cap I^D_S$ then $\varphi_0\left(\left(\begin{smallmatrix}\gamma & 0\\ 0 & \delta\end{smallmatrix}\right)g\right) = \varphi_0(g)$. Moreover $\sum_{\mathfrak{p}\in S}\log|\gamma_{\mathfrak{p}}| = 0$ is the only linear relation satisfied by all the vectors $\left\{\log|\gamma_{\mathfrak{p}}| \;\middle|\; \mathfrak{p}\in S\right\}$ as γ varies over $K^\times\cap I^D_S$. A simple argument then shows that the restriction of φ_0 to G^D_S is of the form

$$\varphi_0\left(\left\{\prod_{\mathfrak{p}\in S}\begin{pmatrix}1 & x_{\mathfrak{p}}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{pmatrix}u_{\mathfrak{p}}\right\}\right) =$$

$$\prod_{\mathfrak{p}\in S}\left|\frac{\alpha_{\mathfrak{p}}}{\beta_{\mathfrak{p}}}\right|^{1/2}\sum_{i=1}^{r}\eta^{(i)}\left(\prod_{\mathfrak{p}\in S}\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{pmatrix}\right)\left\{\zeta^{(i)}_1\left(\prod_{\mathfrak{p}\in S}u_{\mathfrak{p}}\right) + \zeta^{(i)}_2\left(\prod_{\mathfrak{p}\in S}u_{\mathfrak{p}}\right)\sum_{\mathfrak{p}\in S}\log\left|\frac{\alpha_{\mathfrak{p}}}{\beta_{\mathfrak{p}}}\right|\right\}.$$

The homomorphisms $\eta^{(1)},\dots,\eta^{(r)}$ are to be distinct and for each i either $\zeta^{(i)}_1$ or $\zeta^{(i)}_2$ is to be different from zero. If α and β lie in $K^\times\cap I^D_S$ then $\eta^{(i)}\left(\prod_{\mathfrak{p}\in S}\left(\begin{smallmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{smallmatrix}\right)\right) = 1$.

Each $\eta^{(i)}$ determines a homomorphism of the diagonal matrices with entries from I^D into $\mathbf{C}^\times$. This homomorphism, which will be 1 on the matrices with entries from $K^\times\cap I^D$ we again call $\eta^{(i)}$. The value of φ_0 at $\prod_{\mathfrak{p}}\left(\begin{smallmatrix}1 & x_{\mathfrak{p}}\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{smallmatrix}\right)u_{\mathfrak{p}}$ is the same as its value at $\left\{\prod_{\mathfrak{p}\in S}\left(\begin{smallmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{smallmatrix}\right)u_{\mathfrak{p}}\right\}\left\{\prod_{\mathfrak{p}\notin S}\left(\begin{smallmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{smallmatrix}\right)\right\}$ which is

$$\left\{\prod_{\mathfrak{p}}\left|\frac{\alpha_{\mathfrak{p}}}{\beta_{\mathfrak{p}}}\right|^{1/2}\right\}\sum_{i=1}^{r}\eta^{(i)}\left(\prod_{\mathfrak{p}}\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{pmatrix}\right)\left\{\zeta^{(i)}_1\left(\prod_{\mathfrak{p}\in S}u_{\mathfrak{p}}\right) + \zeta^{(i)}_2\left(\prod_{\mathfrak{p}\in S}u_{\mathfrak{p}}\right)\sum_{\mathfrak{p}}\log\left|\frac{\alpha_{\mathfrak{p}}}{\beta_{\mathfrak{p}}}\right|\right\}.$$

Define $\widetilde{\eta}^{(i)}$ by

$$\widetilde{\eta}^{(i)}\left(\prod_{\mathfrak{p}}\begin{pmatrix}\alpha_{\mathfrak{p}} & 0\\ 0 & \beta_{\mathfrak{p}}\end{pmatrix}\right) = \eta^{(i)}\left(\prod_{\mathfrak{p}}\begin{pmatrix}\beta_{\mathfrak{p}} & 0\\ 0 & \alpha_{\mathfrak{p}}\end{pmatrix}\right).$$

Lemma D. *If $i\neq j$ then $\eta^{(j)} = \widetilde{\eta}^{(i)}$.*

[35] In this formula and the similar ones following the absolute value at the complex primes is the square of the usual absolute value.

Let

$$\eta^{(i)^{-1}}\eta^{(j)}\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right) = |\alpha|^a|\beta|^b\chi\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right)$$
$$\widetilde{\eta}^{(i)^{-1}}\eta^{(j)}\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right) = |\alpha|^c|\beta|^d\chi'\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right)$$

for α, β in I^D. Here a, b, c, d are real numbers and χ and χ' are characters in the usual sense. Lemma C implies that if $\mathfrak{p} \notin S$ the restriction of either $\eta^{(i)^{-1}}\eta^{(j)}$ or $\widetilde{\eta}^{(i)^{-1}}\eta^{(j)}$ to $\left\{ \left(\begin{smallmatrix}\alpha_\mathfrak{p} & 0\\ 0 & \beta_\mathfrak{p}\end{smallmatrix}\right) \,\middle|\, \alpha_\mathfrak{p}, \beta_\mathfrak{p} \in K_\mathfrak{p}^\times \right\}$ is trivial. This can only happen if $a = b = 0$ or $c = d = 0$. Suppose that $a \neq 0$ or $b \neq 0$. Then $c = d = 0$ and $\widetilde{\eta}^{(i)^{-1}}\eta^{(j)}$ is an ordinary character. It is known that the values $\widetilde{\eta}^{(i)^{-1}}\eta^{(j)}$ takes on the matrices $\left(\begin{smallmatrix}\alpha_\mathfrak{p} & 0\\ 0 & \beta_\mathfrak{p}\end{smallmatrix}\right)$, $\alpha_\mathfrak{p}$, $\beta_\mathfrak{p} \in K_\mathfrak{p}^\times$, $\mathfrak{p} \notin S$ are dense in the set of values which $\widetilde{\eta}^{(i)^{-1}}\eta^{(j)}$ takes on. It follows that $\eta^{(j)} = \widetilde{\eta}^{(i)}$. In the same way we show that if $c \neq 0$ or $d \neq 0$ then $\eta^{(i)} = \eta^{(j)}$. This is of course excluded. It remains to treat the case $a = b = c = d = 0$. In this case the values taken by the vector-valued function $(\eta^{(i)^{-1}}\eta^{(j)}, \widetilde{\eta}^{(i)^{-1}}\eta^{(j)})$ on the matrices $\left(\begin{smallmatrix}\alpha_\mathfrak{p} & 0\\ 0 & \beta_\mathfrak{p}\end{smallmatrix}\right)$, $\alpha_\mathfrak{p}$, $\beta_\mathfrak{p} \in K_\mathfrak{p}^\times$, $\mathfrak{p} \notin S$ are dense in the set of all values it assumes. It follows from Lemma C that $(1 - \eta^{(i)^{-1}}\eta^{(j)})(1 - \widetilde{\eta}^{(i)^{-1}}\eta^{(j)})$ vanishes identically. If $\widetilde{\eta}^{(i)} \neq \eta^{(j)}$ there is an $\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)$ such that $\widetilde{\eta}^{(i)^{-1}}\eta^{(j)}\left(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)\right) \neq 1$. Then, necessarily $\eta^{(i)^{-1}}\eta^{(j)}\left(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)\right) = 1$. Since $\eta^{(i)} \neq \eta^{(j)}$ there is a $\left(\begin{smallmatrix}\gamma & 0\\ 0 & \delta\end{smallmatrix}\right)$ such that $\eta^{(i)^{-1}}\eta^{(j)}\left(\left(\begin{smallmatrix}\gamma & 0\\ 0 & \delta\end{smallmatrix}\right)\right) \neq 1$. Then $\widetilde{\eta}^{(i)^{-1}}\eta^{(j)}\left(\left(\begin{smallmatrix}\gamma & 0\\ 0 & \delta\end{smallmatrix}\right)\right) = 1$. One sees immediately that $(1 - \eta^{(i)^{-1}}\eta^{(j)})(1 - \widetilde{\eta}^{(i)^{-1}}\eta^{(j)})$ will not vanish at $\left(\begin{smallmatrix}\alpha\gamma & 0\\ 0 & \beta\delta\end{smallmatrix}\right)$. This is a contradiction.

Lemma E. *There are two possible forms for the function* φ_0.

(*i*) *There is a homomorphism* ω *of the diagonal matrices with entries from* I^D *into* $\mathbf{C}^\times$, *which is* 1 *on the matrices with entries from* $K^\times \cap I^D$, *such that* $\omega \neq \widetilde{\omega}$ *and two functions* ζ *and* ζ' *on* $\prod_{\mathfrak{p}\in S} U_{K_\mathfrak{p}}$ *such that if* $g = \prod_\mathfrak{p} \left(\begin{smallmatrix}1 & x_\mathfrak{p}\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha_\mathfrak{p} & 0\\ 0 & \beta_\mathfrak{p}\end{smallmatrix}\right) u_\mathfrak{p}$ *lies in* $G_\mathbf{A}^D$ *then* $\varphi_0(g)$ *equals*

$$\left\{\prod_\mathfrak{p}\left|\frac{\alpha_\mathfrak{p}}{\beta_\mathfrak{p}}\right|^{1/2}\right\}\left\{\omega\left(\prod_\mathfrak{p}\begin{pmatrix}\alpha_\mathfrak{p} & 0\\ 0 & \beta_\mathfrak{p}\end{pmatrix}\right)\zeta\left(\prod_{\mathfrak{p}\in S} u_\mathfrak{p}\right) + \widetilde{\omega}\left(\prod_\mathfrak{p}\begin{pmatrix}\alpha_\mathfrak{p} & 0\\ 0 & \beta_\mathfrak{p}\end{pmatrix}\right)\zeta'\left(\prod_{\mathfrak{p}\in S} u_\mathfrak{p}\right)\right\}.$$

(*ii*) *There is a homomorphism* ω *of the diagonal matrices with entries from* I^D *into* $\mathbf{C}^\times$, *which is* 1 *on the matrices with entries from* $K^\times \cap I^D$, *such that* $\omega = \widetilde{\omega}$ *and two functions* ζ *and* ζ' *on* $\prod_{\mathfrak{p}\in S} U_{K_\mathfrak{p}}$ *such that if* $g = \prod_\mathfrak{p} \left(\begin{smallmatrix}1 & x_\mathfrak{p}\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha_\mathfrak{p} & 0\\ 0 & \beta_\mathfrak{p}\end{smallmatrix}\right) u_\mathfrak{p}$ *lies in* $G_\mathbf{A}^D$ *then* $\varphi_0(g)$ *equals*

$$\left\{\prod_\mathfrak{p}\left|\frac{\alpha_\mathfrak{p}}{\beta_\mathfrak{p}}\right|^{1/2}\right\}\left\{\omega\left(\prod_\mathfrak{p}\begin{pmatrix}\alpha_\mathfrak{p} & 0\\ 0 & \beta_\mathfrak{p}\end{pmatrix}\right)\right\}\left\{\zeta\left(\prod_{\mathfrak{p}\in S} u_\mathfrak{p}\right) + \zeta'\left(\prod_{\mathfrak{p}\in S} u_\mathfrak{p}\right)\sum_{\mathfrak{p}\in S}\log\left|\frac{\alpha_\mathfrak{p}}{\beta_\mathfrak{p}}\right|\right\}.$$

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