LINEARIZING FLOWS AND A COHOMOLOGICAL INTERPRETATION OF LAX EQUATIONS

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In recent years a number of flows given by completely integrable Hamiltonian systems have been shown to be linearizable on the (real points of) the Jacobian variety $J(C)$ of an algebraic curve C associated to the problem. Adler and van Moerbeke [2], [3] have proved that, with the exception of Kowaleska's top, each of the then known finite dimensional completely integrable systems has an associated Lax equation

$$\text{(i)} \qquad \frac{dA(\xi, t)}{dt} = [B(\xi, t), A(\xi, t)]$$

containing a rational parameter ξ.

In this paper we shall consider an arbitrary such Lax equation given by (i) above, and associate to it an algebraic curve C (its spectral curve) together with a dynamical system $\{L_t\}$ on its Jacobian $J(C)$. Our main results give necessary and sufficient conditions on the B in the Lax pair (A, B) that the flow $t \mapsto L_t$ be linear on $J(C)$. Using this we may then easily verify the linearity of the classical flows in [2] as well as the recent one studied by Hitchin [10].

Our main philosophical point is the advantage gained by not specifying anything about B other than there be a Lax equation (i). In fact, B is not unique and so we may suspect that it should be naturally considered as a cohomology class somewhere. This happens as follows: The eigenvectors of the isospectral deformation give a family of holomorphic mappings

$$\text{(ii)} \qquad f_t : C \to \mathbf{P}V$$

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from the fixed curve C to a projective space; essentially f_t gives the time evolution of $A(\xi, t)$. Suppose that the image curve has degree d and set

$$L_t = f_t^*(\mathcal{O}_{\mathbf{P}V}(1)) \in \mathrm{Pic}^d(C) \cong J(C).$$

Applying more or less standard cohomological techniques from deformation theory, we may easily give necessary and sufficient conditions that the map

$$\text{(iii)} \qquad t \mapsto L_t$$

be linear for *any* family of holomorphic maps (ii). These conditions are cohomological, and the miracle is that in the situation arising from an isospectral deformation *the Lax equations turn out to have a very natural and elegant cohomological interpretation.* In carrying this out the crucial technical concept is the *residue* $\rho(B)$ associated to the $B(\xi)$ in the Lax equation (i) (cf. Section 7); $\rho(B)$ is a collection of "Laurent tails" lying over $\xi = 0, \infty$ on the spectral curve, and our main results give necessary and sufficient conditions for the flow (iii) to be linear expressed in terms of $\rho(B)$ in a way that is easily checked in examples.

In this paper sections 1 and 2 are preliminary giving a little background material and listing the examples to be discussed.

In section 3 we pose and informally discuss the main problem to be addressed. Then in section 4 we study the spectral curve and compute its genus, as well as some special features of its structure, in the examples.

Sections 5, 6, 7 are the main part of this work. A standard algebro-geometric principle is that the tangent space to any deformation lies in a suitable cohomology group (cf. [4], which contains an elementary account of deformation theory), and in section 5 we recall how this goes for the eigenvector maps (ii). Then in section 6 we give our first main theorem expressing the tangent vector to (iii) in terms of the B in the Lax equation. The aforementioned miracle is given by the simple computation (6.7)–(6.9). Another standard algebro-geometric principle is that on an algebraic curve "higher cohomology" may always be eliminated using duality theory (cf. the proof of (7.2) below). In section 7 this principle is applied to the problem at hand, and in this way we are led to the residue $\rho(B)$ of the Lax operator and our second main theorem.

Finally, in section 8 we show how our theorems apply to yield the results of Adler-van Moerbeke and Hitchin in the examples. It will be noted that in each case the verification of the conditions on $\rho(B)$ that the flow (iii) be linear is simple and natural.

(1) (cf. [1] and [5]). A symplectic manifold (M, ω) is given by a $2n$-dimensional manifold having a closed 2-form ω of maximal rank

$$d\omega = 0, \qquad \omega^n \neq 0.$$

Standard examples include the cotangent bundle T^*X of an n-dimensional manifold X and coadjoint orbits $\mathcal{O}_\mu \subset \mathfrak{g}^*$, where $\mathfrak{g}$ is the Lie algebra of a Lie group G and $\mathcal{O}_\mu = \{\mathrm{Ad}_g^*\mu : g \in G\}$ is the orbit of $\mu \in \mathfrak{g}^*$ under the coadjoint representation. Given a function H on M the Hamiltonian vector field X_H is defined by

$$\omega(X_H, v) = \langle dH, v\rangle$$

for all vectors $v \in TM$. The Poisson bracket of two functions G, H is defined by

$$\{G, H\} = \omega(X_G, X_H).$$

Under Poisson bracket the C^∞ functions $\mathfrak{F}(M)$ form a Lie algebra, and the mapping

$$\begin{array}{ccc} \mathfrak{F}(M) & \to & V(M) \\ \Downarrow & & \Downarrow \\ H & \mapsto & X_H \end{array}$$

to the vector fields $V(M)$ is a Lie algebra homorphism.

A Hamiltonian system is the dynamical system or flow given by a Hamiltonian vector field on a symplectic manifold. Such a Hamiltonian system (M, ω, H) is said to be *completely integrable* in case there are n functions $H = H_1, H_2, \ldots, H_n$ satisfying

$$\begin{cases} \{H_i, H_j\} = 0 & \text{(i.e., the } H_i \text{ are in involution)} \\ dH_1 \wedge \cdots \wedge dH_n \neq 0 & \text{on a dense open set of } M. \end{cases}$$

For generic $c = (c_1, \ldots, c_n)$ the level set

$$M_c = \{H_1 = c_1, \ldots, H_n = c_n\}$$

will be an n-manifold, and since

$$X_{H_i} H_j = \{H_i, H_j\} = 0$$

the integral curves of each X_{H_i} will lie in M_c. Since, moreover, $[X_{H_i}, X_{h_j}] = 0$ the vector fields X_{H_i} will be tangent to M_c and will Poisson commute there. In fact, in case M_c is compact and connected it follows that it will be an n-torus $\mathbf{R}^n/\mathbf{Z}^n = T$ and each $X_{H_i} = \Sigma_j a_{ij}\partial/\partial x^j$ will define a *linear* flow there. Thus, completely integrable Hamiltonian systems have an especially nice structure.

In this paper we are interested in realizing the linear flows given by some completely integrable systems.[*1] This shall mean that the torus T should be given in a more or less explicit manner and similarly for the vector field $\Sigma_j a_{ij}\partial/\partial x^j$. For us "more or less explicit" means that T consists of the real points on the Jacobian variety of an explicitly given algebraic curve C whose invariants, such as its genus g and space $H^0(\Omega_C)$ of holomorphic differentials, are readily computable in examples. Similarly, $\Sigma_j a_{ij}\partial/\partial x^j$ should be easily determined in practice.

One of the most remarkable developments in recent mathematics is the interplay between completely integrable systems and algebraic curves (cf. [3], [7], [16], and [18] and the references cited there). Our work may be viewed as one unifying observation, from an algebro-geometric viewpoint, of a particularly beautiful aspect of this theory that we shall recall briefly in the next section.

(2) In recent years a number of extremely interesting completely integrable Hamiltonian systems have been found (a few of these are classical) and extensively investigated. Among these we mention:

[*1] Instead of "realizing the linear flow" one frequently simply says "linearizing the flow." The difference between proving that a Hamiltonian system is completely integrable and realizing it is roughly this: Showing that (M, ω, H) is completely integrable gives qualitative information, such as quasi-periodicity and ergodicity of the flow on a general M_c (cf. [5]). On the other hand, realizing it will mean that the system is actually solved in the sense that the integral curves will be expressible in terms of abelian functions defined on an explicitly given T.

Example 1. The Euler equations of a free rigid body moving about a fixed point ([14], [21]). In this case, M is a coadjoint orbit in $so(n)^*$ (which is isomorphic to $so(n)$ via the Killing form).

Example 2. The Euler-Poisson equations for a symmetric heavy rigid body ([15], [22], [23]). In this case M is a coadjoint orbit in the semi-direct product of $SO(n)$ with its Lie algebra.

Example 3. The Toda lattice and its generalizations ([2], [24]). Here the motion takes place on coadjoint orbits in a Kac-Moody Lie algebra.

Example 4. Nahm's equations ([10], [20]) in which M is essentially an adjoint orbit in $u(n)$.

Example 5. The geodesics on an ellipsoid E ([12], [17]) in which $M = T^*E$, and the Neumann's problem [13], [17] of Newtonian motion on a sphere S^n with a quadratic potential in which again $M = T^*S^n$.

A common feature of all these systems is that they can be written in Lax form with a parameter ξ. More precisely, we consider matrix functions

$$A(\xi, t) = \sum_{k=-p}^{q} A_k(t)\xi^k$$

which are finite Laurent series in a variable ξ and whose coefficients lie in a linear Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(n)$.

Definition. By a *Lax equation with a parameter* we shall mean an equation

$$\boxed{(1) \qquad \dot{A}(\xi) = [B(\xi), A(\xi)] \qquad (\dot{} = d/dt).}$$

Here, $B(\xi)$ is to be also a finite Laurent series in ξ whose coefficients are in $\mathfrak{g}$. In practice these coefficients will be functions of the A_k, but for the moment it is clearer *not* to assume this. In a very beautiful series of papers, built upon a great deal of previous work by themselves and other mathematicians, Adler and van Moerbeke [2], [3] showed that all of the above examples (except for (4) which did not yet exist) were completely integrable Hamiltonian systems (cf. also [8], [23]). Moreover, they showed that each could be written as a Lax equation with a parameter and that the linearized flow could be realized on the Jacobian of an algebraic curve associated to (1). More precisely, we define the *spectral curve* C associated to (1) to be given by the characteristic polynomial

$$Q(\xi, \eta) = \det \| \eta I - A(\xi, t) \| = 0 \tag{2}$$

(this will be explained in more detail below). We note that by (1) the polynomial $Q(\xi, \eta)$ is independent of t; i.e., the flow

$$t \mapsto A(\xi, t)$$

is *isospectral.* Then it was proved in [2], [3] that for each of examples 1-3, 5 the corresponding Hamiltonian flow may be realized as a linear flow on the real part of the Jacobian $J(C)$ of the spectral curve (or on an abelian variety closely related to the Jacobian). For example 4 this was proved in [10].

We shall now discuss the form of the Lax equations with a parameter associated to each of the examples above.

Example 1. For the free rigid body we let $\Omega \in so(n)$ be variable, $J = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ $(\lambda_i > 0)$, and $M = \Omega J + J\Omega$. Then the Euler equations are

$$\dot{M} = [M, \Omega]. \tag{3}$$

The basic observation, due to Manakov [14], is that (3) is equivalent to

$$(M + J^2\xi) = [M + J^2\xi, \Omega + J\xi], \tag{4}$$

which is a Lax equation with a parameter of the form (1) with $B = -(\Omega + J\xi)$. We note that since $M + {}^tM = 0$,

$$Q(\xi, \eta) = (-1)^n Q(-\xi, -\eta).$$

Thus there is an involution of the spectral curve

$$j : C \to C$$

given by

$$j(\xi, \eta) = (-\xi, -\eta).$$

We note that Ω moves on an adjoint orbit $\mathcal{O}_\mu \subset so(n)$, and for general μ

$$\dim \mathcal{O}_\mu = n(n-1)/2 - \left[\frac{n}{2}\right]. \tag{5}$$

To linearize the flow given by (3) we need ½ dim $\mathcal{O}_\mu$ integrals of motion that are in involution. A nice count of how (4) gives this many integrals appears in [21], where their involutivity is also proved.

Example 2. The Euler-Poisson equations are

$$\begin{cases} \dot{\Gamma} = [\Gamma, \Omega] \\ \dot{M} = [M, \Omega] + [\Gamma, \chi] \end{cases} \tag{6}$$

where $\Gamma, \Omega \in so(n)$, $M = \Omega J + J\Omega$ with $J = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ as in Example 1, and $\chi \in so(n)$ is fixed. It is proved in [22] and [23] that these may be written in the particular form

$$(\Gamma + M\xi + C\xi^2)^{\cdot} = [\Gamma + M\xi + C\xi^2, \Omega + \chi\xi] \tag{7}$$

of a Lax equation with a parameter if, and only if,

$$\begin{cases} \lambda_1 = \lambda_2 = \alpha, \lambda_3 = \ldots = \lambda_n = \beta \\ \text{and } \chi_{12} \neq 0, \text{ all other } \chi_{ij} = 0\ (i < j) \end{cases} \qquad \text{(Lagrange top)}$$

or

$$\lambda_1 = \cdots = \lambda_n = \alpha, \ \chi \text{ arbitrary} \qquad \text{(heavy symmetric top)}$$

in which case

$$C = (\alpha + \beta)\chi$$

The Equations (6) are physically meaningful when $n = 3$ but seem to be a definition when $n \geq 4$; accordingly, we shall concentrate on the Lagrange top in the case $n = 3$. A very nice treatment of this case, which may serve as a general introduction to the work of Adler-van Moerbeke, may be found in [25].

Example 3. On $\mathbf{R}^{2n}$ with coordinates $(x^1, \ldots, x^n, \xi_1, \ldots, \xi_n)$ and symplectic form $\omega = \Sigma_i\, dx^i \wedge d\xi_i$ we consider the Hamiltonian function

$$H(x, \xi) = \frac{1}{2} \sum_{i=1}^{n} \xi_i^2 + \sum_{i=1}^{n} e^{x_i - x_{i+1}}, \qquad x_{n+1} = x_1$$

whose corresponding flow is given by

$$\begin{cases} \dot{x}^k = \dfrac{\partial H}{\partial \xi_k} \\ \dot{\xi}_k = -\dfrac{\partial H}{\partial x^k} \end{cases} \tag{8}$$

This is the famous Toda lattice. By the Flaschka transformation

$$a_k = \frac{1}{2} e^{(x_k - x_{k+1})}, \qquad b_k = -\frac{\xi_k}{2}$$

the Equations (8) are transformed into

$$\begin{cases} \dot{b}_k = 2(a_k^2 - a_{k-1}^2), & a_0 = a_n \\ \dot{a}_k = a_k(b_{k+1} - b_k), & b_{n+1} = b_1 \end{cases} \tag{9}$$

We set

$$A = \begin{pmatrix} b_1 & a_1 & & & a_n\xi^{-1} \\ a_1 & b_2 & & & \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ a_n\xi & & & a_{n-1} & b_n \end{pmatrix}$$

$$B = -\begin{pmatrix} 0 & a_1 & & & -a_n\xi^{-1} \\ -a_1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & 0 & a_{n-1} \\ a_n\xi & & & -a_{n-1} & 0 \end{pmatrix} \tag{10}$$

From the first equation in (9) we have

$$\sum_k \dot{b}_k = 0,$$

and if we normalize by requiring that

$$\sum_k b_k = 0$$

then (9) are equivalent to the Lax equations with a parameter

$$\dot{A} = [B, A] \tag{11}$$

where A, B are given above. We note that

$$a_k(0) \neq 0 \Rightarrow a_k(t) \neq 0 \quad \text{for all } t,$$

and we make the usual requirement that all $a_k \neq 0$.

Observe that since ${}^tA(\xi) = A(\xi^{-1})$

$$Q(\xi, \eta) = Q(\xi^{-1}, \eta).$$

Thus there is again an involution of the spectral curve

$$j : C \to C$$

given by

$$j(\xi, \eta) = (\xi^{-1}, \eta). \tag{12}$$

We also note that in addition to the integral $\Sigma_k b_k = 0$, we need $n - 1$ further integrals in involution to show that the Toda lattice is a completely integrable Hamiltonian system.

The literature on the Toda lattice is enormous. In addition to [2] and [3], two important papers for its study via algebraic curves are [16] and [24].

Example 4. Nahm's equations are

$$\dot{T}_i = \frac{1}{2} \sum_{j,k} \epsilon_{ijk} [T_j, T_k] \tag{13}$$

where

$$\begin{cases} i, j, k = 1, 2, 3 \\ \epsilon_{ijk} = \operatorname{sgn}(i, j, k) = \epsilon^{ijk}_{123} \\ T_i \in u(n). \end{cases}$$

In contrast to our other examples, they did not (as far as this author knows) arise from a (generalized) classical mechanical system, but rather in the construction of monopoles as explained in [20] and [10].

If we set

$$\text{(14)} \qquad \begin{cases} A(\xi) = A_0 + \xi A_1 + \xi^2 A_2 \quad \text{and} \\ B(\xi) = -\dfrac{1}{2}\dfrac{dA(\xi)}{d\xi} = -\dfrac{1}{2}A_1 - \xi A_2 \end{cases}$$

where

$$\text{(15)} \qquad \begin{cases} A_0 = T_1 + iT_2 \\ A_1 = -2iT_3 \\ A_2 = T_1 - iT_2, \end{cases}$$

then it is straightforward to verify that Nahm's equations (13) are equivalent to the Lax equations with a parameter

$$\dot{A} = [B, A]$$

(what is not straightforward is to guess the substitution (15)—cf. [10]).

Example 5. The geodesics on an ellipsoid and Neumann's forced harmonic motion are discussed in many places (e.g. [17]). Here we shall simply follow [2], pages 275–277 and 302, and explain how they may be written as a Lax equation with a parameter. For this we let $W = \mathbf{R}^n$ or $\mathbf{C}^n$, make the identification $W \otimes W \cong \operatorname{Hom}(W, W)$, and define maps

$$\begin{cases} \Gamma_{xy} : W \times W \to \operatorname{Hom}(W, W) \\ \Lambda_{xx} : W \to \operatorname{Hom}(W, W) \\ \Delta_{xy} : W \times W \to \operatorname{Hom}(W, W) \end{cases}$$

by

$$\begin{cases} \Gamma_{xy} = x \otimes y - y \otimes x & \text{(i.e., } (\Gamma_{xy})_{ij} = x_i y_j - x_j y_i) \\ \Lambda_{xx} = x \otimes x & \text{(i.e., } (\Lambda_{xx})_{ij} = x_i x_j) \\ \Delta_{xy} = x \otimes y + y \otimes x & \text{(i.e., } (\Lambda_{xy})_{ij} = x_i y_j + x_j y_i) \end{cases}$$

Remark that

$$\begin{cases} \text{Image } \Gamma = \text{skew-symmetric matrices of rank 2} \\ \text{Image } \Lambda = \text{matrices of rank 1} \\ \text{Image } \Delta = \text{symmetric matrices of rank } \leq 2, \end{cases}$$

and that the rank of a matrix is invariant when it moves on an adjoint orbit.

We also set

$$\begin{cases} \alpha = \operatorname{diag}(\alpha_1, \ldots, \alpha_n), & \alpha_i > 0, \\ \beta = \operatorname{diag}(\beta_1, \ldots, \beta_n) \\ \Gamma = (\beta\alpha^{-1})\Gamma_{xy}(\beta\alpha^{-1})^{-1}. \end{cases}$$

Then the equations for geodesics on an ellipsoid and for Neumann's system are respectively

$$(16) \qquad (-\Lambda_{xx} + \xi\dot{\Gamma}_{xy} + \xi^2\alpha) = [-\Lambda_{xx} + \xi\Gamma_{xy} + \xi^2\alpha, \Gamma + \xi\beta]$$

$$(17) \quad (\Delta_{xy} - \alpha + \xi\dot{\Gamma}_{xy} + \xi^2\alpha) = [\Delta_{xy} - \alpha + \xi\Gamma_{xy} + \xi^2\alpha, \Gamma + \xi\beta]$$

Remark. Equation (16) is equivalent to

$$\begin{cases} \text{(i) } \dot{\Lambda}_{xx} = [\Lambda_{xx}, \Gamma] \\ \text{(ii) } \dot{\Gamma}_{xy} =]\Gamma_{xy}, \Gamma] - [\Lambda_{xx}, \beta] \end{cases}$$

Since ${}^t\Gamma = -\Gamma$, ${}^t\Lambda_{xx} = \Lambda_{xx}$ and so Λ_{xx} moves in the space of matrices of rank 1. In (ii)

$$\begin{cases} {}^t\dot{\Gamma}_{xy} = -\dot{\Gamma}_{xy} \\ (\dot{\Gamma}_{xy} v, w) = 0 \quad \text{if} \quad (v, x) = (v, y) = (w, x) = (w, y) = 0, \end{cases}$$

from which it follows that Γ_{xy} moves in the space of skew-symmetric matrices of rank 2.

In addition to [17], the relationship between the geodesics on an ellipsoid and Neumann's system to algebraic curves is discussed in [12] and [13].

(3) The purpose of this paper is to address the following

Problem. Given any Lax equation with parameter (2.1), determine necessary and sufficient conditions on B that the corresponding flow on $J(C)$ be linear.

We shall explain in a moment what the corresponding flow on $J(C)$ is. Our answer to this problem is expressed by Theorems (6.3) and (7.7) below (cf. also (7.8), (7.10) and (7.11)). Using these one may recover the results of Adler-van Moerbeke and Hitchin by verifying that the conditions of the general theorems are satisfied in the particular examples.

We shall express our results in the language of algebraic geometry, as it is in this setting that the problem is perhaps most naturally posed. Indeed, even though the eigen*values* of $A(\xi, t)$ are fixed as time evolves, the eigen*vectors* of $A(\xi, t)$ will change with t. This leads to the eigenvector mappings

$$f_t : C \to \mathbf{P}V,$$

and we set

$$L_t = f_t^*(\mathcal{O}_{\mathbf{P}V}(1)) \in \mathrm{Pic}^d(C),$$

where $\mathcal{O}_{\mathbf{P}V}(1)$ is the standard hyperplane bundle on $\mathbf{P}V$ and $\mathrm{Pic}^d(C)$ is the set of line bundles of degree d on C. Choosing a reference line bundle $L_0 \in \mathrm{Pic}^d(C)$, it is well known that the mapping

$$L \mapsto L \otimes L_0^{-1}$$

induces an isomorphism $\mathrm{Pic}^d(C) \cong J(C)$. Moreover, there are canonical identifications

$$T_{L_t}(\mathrm{Pic}^d(C)) \cong H^1(\mathcal{O}_C),$$

and so we may write

$$\frac{dL_t}{dt} \in H^1(\mathcal{O}_C).$$

Our problem then becomes to determine the conditions on B that the *acceleration vector* $d^2L_t/dt^2 =: d/dt(dL_t/dt)$ be a multiple of dL_t/dt; i.e., that we have

$$d^2L_t/dt^2 = \mu_t dL_t/dt. \tag{1}$$

(In our examples it will turn out that $\mu_t = 0$, so that t is a natural linear parameter.) We shall find that this question has a very simple and elegant answer, whose main point is to understand the cohomological meaning of the Lax equation (2.1). As a portent of this we note that the Lax equation is invariant under a substitution

$$B \mapsto B + P(A, \xi)$$

where $P(x, \xi) \in \mathbf{C}[x, \xi]$. Thus B lives naturally in a quotient space, and this suggests that it has invariant cohomological meaning.

In this paper we shall use the standard notations of algebraic geometry as in [4] and [9], and some of which we shall momentarily recall. Aside from a little deformation theory, which is explained in [4], everything is quite elementary once one grants the essential point that the computations may be done simply and naturally using sheaf cohomology, as this is where infinitesimal deformations live naturally.

For a smooth variety X we let

$$\begin{cases} \mathcal{O}_X = \text{structure sheaf} \\ \Theta_X = \text{tangent sheaf} \\ \mathcal{O}_X(1) = \text{hyperplane bundle in case } X \subset \mathbf{P}^N; \end{cases}$$

remark that we identify line bundles with invertible sheaves with linear equivalence classes of Cartier divisors. If C is a smooth curve and $D =$

$\Sigma\, n_i p_i$, $n_i \geq 0$, an effective divisor on C, then $\mathcal{O}_C(D)$ is the sheaf of meromorphic functions with poles no worse than D. Thus

$$H^0(\mathcal{O}_C(D)) = \{f \text{ meromorphic on } C\colon (f) + D \geq 0\};$$

this vector space is also frequently denoted by $\mathcal{L}(D)$. We shall denote by Ω_C the sheaf of holomorphic 1-forms on C; then the Jacobian variety

$$J(C) =: \mathrm{Pic}^0(C) = \text{line bundles of degree zero}$$

$$= H^1(\mathcal{O}_C)/H^1(C, \mathbf{Z})$$

via the exponential sheaf sequence

$$\cong H^0(\Omega_C)^*/H_1(C, \mathbf{Z})$$

via the duality given by Abel's theorem.

(4) We shall also use the following particular notations:

$$\begin{cases} V \text{ is a complex vector space of dimension } m; \\ P = \mathbf{P}^1 \text{ with homogeneous coordinates } [\xi_0, \xi_1] \\ \qquad \text{and affine coordinate } \xi = \xi_1/\xi_0; \\ \mathcal{O}_P(1) \text{ is the standard line bundle over } P; \end{cases}$$

we also set $V = V \otimes_{\mathbf{C}} \mathcal{O}_P$ and $V(k) = \mathbf{V} \otimes \mathcal{O}_P(k)$; the context should avoid confusing the sheaf V and vector space V.

Remark. In dynamical systems problems one is of course interested in real solutions to the equations (e.g., to the Lax equations (2.1)). However, it will be convenient to work complex analytically, and then at the end put in the real structure as given by the fixed point of an involution corresponding to complex conjugation.

We assume given the following data:

$$A(t, \xi) = \sum_{k=0}^{n} A_k \xi^k = \xi_0^n A_0 + \cdots + \xi_1^n A_n \in H^0(P, \mathrm{Hom}(V, V(n)));$$

$$B(t, \xi) = \sum_{\ell=0}^{N} B_\ell \xi^\ell \in H^0(P, \mathrm{Hom}(V, V(N))); \qquad \text{and a Lax equation}$$

$$(1) \qquad \dot{A} = [B, A] \qquad (\dot{\ } = d/dt).$$

Remarks. (i) The case where A and B are finite Laurent series may be handled by a completely straightforward extension of the method we shall use in the polynomial case. Our only Laurent series example is the Toda lattice, and the referee remarks that it may be reduced to the polynomial case by multiplying A by ξ and replacing B by $B + A$, since the latter does not contain ξ^{-1}.

(ii) If A is suitably generic, then for general (ξ, t) the matrix $B(\xi, t)$ is determined by the Lax equation (1) only up to polynomials in A and ξ. In any case it is certainly not unique.

In the following we will generally not distinguish between ξ as a homogeneous coordinate $[\xi_0, \xi_1]$ and as an affine coordinate ξ_1/ξ_0; hopefully the context will make clear which we mean. If $Y \xrightarrow{\pi} P$ is the bundle space of the line bundle $\mathcal{O}_P(n)$, then there is defined over Y the tautological section

$$\eta \in H^0(Y, \pi^*\mathcal{O}_P(n))^{(*2)}$$

Thus the characteristic polynomial

$$Q(\xi, \eta) = \det \| \eta I - A(\xi, t) \| \in H^0(Y, \pi^*\mathcal{O}_P(mn)).$$

The divisor of Q will be a complete curve $C_0 \subset Y$ and

$$\pi : C_0 \to P$$

(*2) Points of Y are pairs (ξ, ν) where $\xi \in P$ and $\nu \in \mathcal{O}_P(n)_\xi$; then

$$\eta(\xi, \nu) = \nu.$$

Remark that Y compactifies naturally to the rational ruled surface

$$F_n = \mathbf{P}(\mathcal{O}_P(n) \oplus \mathcal{O}_P).$$

In many examples we will have $n = 2$ and then $Y = T(P)$ is the tangent bundle of $\mathbf{P}^1$.

is a branched covering with

$$\pi^{-1}(\xi) = \{(\xi, \eta_1), \ldots, (\xi, \eta_m)\}$$

where $\eta_1, \ldots, \eta_m$ are the roots of the characteristic equation. We assume that C_0 is irreducible and denote its normalization by $C \to C_0$.

Clearly $\pi^{-1}(P\backslash\{\infty\}) \cong \mathbf{C}^2$ with coordinates (ξ, η) and C_0 is a compactification of the affine curve $\det\|\eta I - A(\xi, t)\| = 0$ in $\mathbf{C}^2$.

Example. For the free rigid body in $\mathbf{R}^3$ we have

$$\begin{aligned} Q(\xi, \eta) &= \det\|\eta I - \xi J^2 - M\| \\ &= \eta^3 + \delta\xi^3 + \cdots \end{aligned}$$

where $\delta = -\det J^2$. Thus C_0 is the compactification of a cubic curve in $\mathbf{C}^2$; we will see below that in general C does in fact have genus one (in general means that $A(\xi, 0)$ is chosen generically).

On the other hand, several of our examples will not satisfy such a genericity condition; in many of these cases the genus of the spectral curve is most easily computed using automorphisms of C arising from special features of A, such as $A \in so(n) \subset \mathfrak{gl}(n)$.

Definition. We shall call C the *spectral curve* associated to the Lax equation (1).

Let $\xi \in P$ be a general point and write

$$Q(\xi, \eta) = \prod_{\nu=1}^{m} (\eta - \eta_\nu(\xi)).$$

Assuming that the curve $C_0 = C$ is smooth, we shall show that the genus g of the spectral curve is given by

$$g = \frac{mn(m-1)}{2} - m + 1, \tag{2}$$

thereby confirming our claim that the curve in the example just above has genus one.

To prove (2) we consider the discriminant

$$\Delta = \prod_{\mu<\nu} (\eta_\nu - \eta_\mu)^2.$$

This is a well-defined function of $\xi = (\xi_0, \xi_1) \in \mathbf{C}^2$, and clearly

$$\Delta(\lambda\xi) = \lambda^{2n\binom{m}{2}}\Delta(\xi)$$

Thus

$$\Delta \in H^0(P, \mathcal{O}_P(mn(m-1))).$$

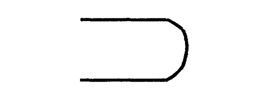
Suppose now that at some point $\xi \in P$, k of the eigenvalues, say $\eta_1, \ldots, \eta_k$, come together and cyclically permute as ξ turns around ξ. For a suitable local coordinate t centered at ξ we will have

$$\eta_\nu = \zeta^\nu t^{1/k}, \qquad \zeta = e^{2\pi i/k} \quad \text{and} \quad 1 \le \nu \le k.$$

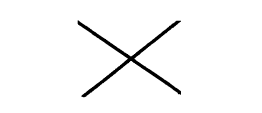
Since

$$\prod_{1 \le \mu < \nu \le k} (\zeta^\mu t^{1/k} - \zeta^\nu t^{1/k})^2 = ct^{k-1}, \qquad c \neq 0,$$

we see that the order of vanishing of Δ at ξ exactly gives the sum of the ramification indicies of the points of C lying over ξ. Moreover, by our assumption that $C = C_0$ is smooth the only way that we can have $\eta_\nu(\xi) = \eta_\mu(\xi)$ $(\nu \neq \mu)$ is for there to be branching as above; i.e., we have a picture



and not



It follows that $C \xrightarrow{\pi} P$ is an m-sheeted covering whose ramification divisor has degree given by

$$r = \deg(\Delta) = mn(m-1).$$

The Riemain-Hurwitz formula (cf. [4] or [9])

$$2g - 2 = -2m + r$$

then gives (2).

Remark. In case C_0 is singular the formula will be

$$g = \frac{mn(m-1)}{2} - m + 1 - \delta$$

where δ measures the number and type of singularity of C_0. For instance we will encounter singular curves in Examples 2, 3, 5 below.

Example 1. For the free rigid body problem discussed in Example 1 in section 2 we find that the spectral curve has genus

$$g = (n-1)(n-2)/2 \tag{3}$$

The involution has the n distinct points over $\xi = \infty$ (since $-\infty = \infty$), together with the origin (0, 0) in case n is odd, as its fixed points. By the Riemann-Hurwitz formula, the quotient curve $C' = C/j$ has genus

$$g' = \begin{cases} \dfrac{(n-2)^2}{4} & n \equiv 0 \bmod 2 \\[2ex] \dfrac{(n-1)(n-3)}{4} & n \equiv 1 \bmod 2. \end{cases} \tag{4}$$

Associated to the double covering $C \to C'$ is the *Prym variety* Prym (C/C') (cf. [4] for a definition), which may be described as

$$H^0(\Omega_C)^{-*}/H_1(C, \mathbf{Z})^-$$

where $V^{\pm}$ are the ± 1 eigenspaces for a vector space on which j acts. From (3) and (4) it follows that (cf. [3])

$$\dim \operatorname{Prym}(C/C') = \begin{cases} \dfrac{n(n-2)}{4} & n \equiv 0\ (2) \\[2ex] \dfrac{(n-1)^2}{4} & n \equiv 1\ (2) \end{cases}$$

On the other hand, comparing with (2.5) we obtain

$$\text{dim Prym}\,(C/C') = \frac{1}{2}\dim \mathcal{O}_\mu \tag{5}$$

This suggests that the motion of the free rigid linearizes on Prym (C/C'), which in fact turns out to be the case (cf. [3] and section 8 below).

Example 2. Referring to the Lagrange top in the case $n = 3$ discussed in Example 2 in section 2 above we set

$$A(\xi) = \Gamma + M\xi + C\xi^2 \in so(3)[\xi]. \tag{6}$$

From

$$\det \begin{Vmatrix} \eta & -a_1 & -a_3 \\ a_1 & \eta & -a_2 \\ a_3 & a_2 & \eta \end{Vmatrix} = \eta(\eta^2 + (a_1^2 + a_2^2 + a_3^2))$$

we infer that

$$Q(\xi, \eta) = \det\|\eta I - A(\xi)\| = \eta(\eta^2 + |A|^2)$$

where $|A|^2$ is the sum of the squares of the entries of A. By (6)

$$|A|^2 = \gamma_0 + \gamma_1\xi + \gamma_2\xi^2 + \gamma_3\xi^3 + \gamma_4\xi^4 \tag{7}$$

where $\gamma_0 = |\Gamma|^2$, $\gamma_4 = |C|^2$. It follows that the spectral curve is reducible with one component ($\eta = 0$) corresponding to the zero eigenvalue of any matrix in $so(3)$. The other component

$$\eta^2 + |A(\xi)|^2 = 0$$

is by (7) an elliptic curve, generally smooth, and realized by $(\xi, \eta) \to \xi$ as a 2-sheeted branched covering of $\mathbf{P}^1$ with sheet interchange given by $j(\xi, \eta) = (\xi, -\eta)$.

Example 3. Referring to the Toda lattice given by Example 3 in section 2 above, we have

$$A(\xi) = A_{-1}\xi^{-1} + A_0 + A_1\xi$$

where

$$A_{-1} = \begin{bmatrix} 0 & \cdots & a_n \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}, \qquad A_1 = {}^tA_{-1}.$$

$$A_0 = \begin{bmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & & & \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ & & & a_{n-1} & b_n \end{bmatrix}$$

It follows that

$$\begin{aligned} Q(\xi, \eta) &= \det \| \eta I - A(\xi) \| \\ &= a_1 \cdots a_{n-1}(\xi + \xi^{-1}) + \tilde{P}(\eta) \end{aligned}$$

where $\tilde{P}(\eta) = \eta^n + c_1\eta^{n-1} + \cdots + c_n$ is a polynomial in η. Multiplying by $a_1 \cdots a_{n-1} \neq 0$ we are led to the affine curve in $\mathbf{C}^* \times \mathbf{C}$ given by

(8) $$R(\xi, \eta) = \xi + \xi^{-1} + P(\eta) = 0$$

In general this affine curve will be smooth, but its completion C_0 in $\mathbf{P}^2$ will be singular as soon as $n \geq 4$.

To compute the genus of the normalization C of C_0 we observe that the involution j given by (2.12) realizes C as a 2-sheeted covering

(9) $$C \to \mathbf{P}^1 = \eta\text{-sphere}.$$

The fixed points of j, which coincide with the branch points of (9), occur when $\xi^2 = 1$; i.e., when $\xi = \pm 1$. In general there are n of these for each value $\xi = +1$, $\xi = -1$. Thus there are $2n$ branch points in all, and by the Riemann-Hurwitz formula the genus of C is given by

(10) $$g = n - 1.$$

Referring to Example 3 in section 2 we note that this is the number of required additional integrals of motion.

Example 4. In general the spectral curve C associated to Nahm's Equations (2.13) will be smooth and have genus given by (2) as

$$g = (n - 1)^2$$

This is in agreement with [10], where a detailed discussion of the structure of C may be found. In particular, for this example the reality question is somewhat subtle due to (2.15).

Example 5. The spectral curves associated to geodesics on an ellipsoid and Neumann's problem are given by

$$\text{(11)} \qquad \begin{cases} Q_1(\xi, \eta) = \det \| \eta I - \xi^2 \alpha - \xi \Gamma_{xy} + \Gamma_{xx} \| = 0 \\ Q_2(\xi, \eta) = \det \| \eta I - \xi^2 \alpha - \xi \Gamma_{xy} - \Delta_{xy} + \alpha \| = 0. \end{cases}$$

Each of these is of the form

$$\text{(12)} \qquad \det \| \eta I - \xi^2 \alpha - P \| = 0$$

where P is a rank 2 matrix. Set $T = \eta I - \xi^2 \alpha$ so that (12) is

$$\begin{aligned} 0 = \det \| T - P \| &= \det T \det \| I - T^{-1} P \| \\ &= \det T (1 - \operatorname{Trace} T^{-1} P + \operatorname{Trace}(\Lambda^2 T^{-1} P)) \end{aligned}$$

since $T^{-1}P$ has all $k \times k$ minors equal to zero for $k \geq 3$. Using this observation, which is due to Moser in the problem at hand, it is straightforward to compute the $Q_i(\xi, \eta)$. Actually, following [2] for the computation of the genus, it is more convenient to reparametrize so that the spectral curves are given by ([2], page 303)

$$\begin{cases} Q_1(\xi, \eta) = \xi^2 + \langle (\eta - \alpha)^{-1} x, y \rangle = 0 \\ Q_2(\xi, \eta) = \xi^2 - 1 + (-2\langle (\eta - \alpha)^{-1} x, y \rangle + \langle \langle \eta - \alpha \rangle^{-1} x, x \rangle \\ \qquad\qquad \langle (\eta - \alpha)^{-1} y, y \rangle - \langle (\eta - \alpha)^{-1} x, y \rangle^2) = 0 \end{cases}$$

These are hyperelliptic curves of the form (in $\mathbf{C} \times \mathbf{C}^*$)

$$\xi^2 = R(\eta)$$

where (loc cit) $R(\eta)$ is a rational function of the form

$$R(\eta) = c\,\frac{(\eta^{n-1} + \cdots)}{(\eta^n + \cdots)}.^{(*3)}$$

It follows that in each case the genus is $n - 1$. Remark that in each of these two problems we need $n - 1$ commuting integrals of motions in addition to total energy in order to render the system completely integrable.

(5) We now assume that for a general point $p = (\xi, \eta) \in C$

$$\dim \ker \| \eta I - A(\xi, t) \| = 1$$

Then there is determined, uniquely up to nonzero scalars, a vector $v(p, t) \in V$ satisfying

(1) $$A(\xi, t)v(p, t) = \eta v(p, t)$$

The assignment

$$p \mapsto \mathbf{C}v(p, t) \subset V$$

consequently determines a family of holomorphic mappings, depending holomorphically on t,(*4)

(2) $$f_t : C \to \mathbf{P}V.$$

Definition. We shall call (2) the *eigenvector mappings* associated to the Lax equation (2.1).

We set

$$L_t = f_t^*(\mathcal{O}_{\mathbf{P}V}(1)) \in \operatorname{Pic}^d(C), \qquad L = L_0,$$

(*3) In [2] the c depends on x and y but it is a constant of motion.

(*4) Here we are using the following fact: Given a *smooth* algebraic curve C and a holomorphic mapping

(3) $$f : C \backslash \{p_1, \dots p_N\} \to \mathbf{P}^N, \qquad p_i \in C,$$

and will address the problem stated in section 3 above. For this some general remarks will be useful.

Given a smooth algebraic curve C, a complex manifold X, and a nonconstant holomorphic mapping

$$f: C \to X, \tag{4}$$

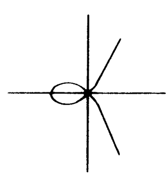
we define the *normal sheaf* N_f on C by the exact sequence

$$0 \to \Theta_C \xrightarrow{f_*} f^*\Theta_X \to N_f \to 0. \tag{5}$$

Here Θ_C, Θ_X are the respective tangent sheaves of C, X and f_* is the differential of f. It is important to remark that f_* is injective as a *sheaf* map but will fail to be injective as a *bundle* map at points of C that are ramified over their images. According to Horikawa's theory ([11], cf. also Chapter V of [4]) the Kodaira-Spencer tangent space to the moduli space of the situation (4) is given by $H^0(C, N_f)$.(*5)

---

such that $f^*(x_i/x_0)$ extends to meromorphic function on C where $[x_0, x_1, \ldots, x_N]$ are homogeneous coordinates on $\mathbf{P}^N$, then (3) extends uniquely to a *holomorphic* mapping $f: C \to \mathbf{P}^N$. This is false when C is singular; the meromorphic function x/y is not defined at the origin for the curve



and this is the reason that we have passed to the normalization of the spectral curve C_0. A more satisfactory theory would work directly with C_0 and its generalized Jacobian taking into due account the situation when the eigenvector mapping fails to give a Cartier divisor (cf. [19]).

(*5) For example, when $X = \mathbf{P}^1$ the normal sheaf N_f will be a skyscraper sheaf supported on the ramification divisor of $f: C \to \mathbf{P}^1$. The statement that $H^0(C, N_f) \cong H^0(\mathbf{P}^1, f_*N_f)$ gives the tangent space to the deformations of the map means geometrically that "we deform $f: C \to \mathbf{P}^1$ by moving the branch points." At the other extreme when f is an embedding, N_f is the usual normal bundle to $f(C) \subset X$ and $H^0(C, N_f)$ represents the infinitesimal displacements of $f(C)$ in X. Note that "moduli of the situation (4)" means families

$$f_t: C_t \to X$$

where the abstract curve together with the map to X both vary.

If

$$f_t : C_t \to X, \qquad f_0 = f,$$

is a deformation of (4), then we shall write

$$\dot{f} \in H^0(C, N_f) \tag{6}$$

for the corresponding infinitesimal deformation at $t = 0$. If, in local product coordinates (z, t) on $\bigcup_t C_t$ and $w = (w^1, \ldots, w^m)$ on X, f_t is given by

$$(z, t) \mapsto w(z, t),$$

then the section (6) is locally given by

$$\left.\frac{\partial w(z, t)}{\partial t}\right|_{t=0} \text{ modulo } \frac{\partial w(z, 0)}{\partial z}.$$

The exact cohomology sequence of (5) gives

$$H^0(\Theta_C) \to H^0(f^*\Theta_X) \to H^0(N_f) \xrightarrow{\tilde{\delta}} H^1(\Theta_C).$$

Recalling that $H^1(\Theta_C)$ is the tangent space to the moduli space of C as an abstract curve (cf. [4]), then it is easy to verify that

$$\tilde{\delta}(\dot{f}) =: \dot{C} \in H^1(\Theta_C)$$

is the tangent to the family of curves $\{C_t\}$. Thus, the tangent space to deformations of (4) *where the curve C remains fixed* is given by

$$H^0(f^*\Theta_X)/H^0(\Theta_C) \subset H^0(N_f).$$

It is this situation that we are interested in.

Now suppose that

$$X = \mathbf{P}V$$

is a projective space and recall the *Euler sequence*

$$0 \to \mathcal{O}_{\mathbf{P}V} \to V \otimes \mathcal{O}_{\mathbf{P}V}(1) \to \Theta_{\mathbf{P}V} \to 0.$$

Pulling this back via f we obtain a diagram

$$\begin{tikzcd} & & 0 \arrow[d] & & \\ & & \mathcal{O}_C \arrow[d, "v"] & & \\ & & V \otimes L \arrow[d] & & \\ 0 \arrow[r] & \Theta_C \arrow[r, "f_*"] & f^*\Theta_{\mathbf{P}V} \arrow[r] \arrow[d] & N_f \arrow[r] & 0, \\ & & 0 & & \end{tikzcd} \qquad L = f^*\mathcal{O}_{\mathbf{P}V}(1),$$

a piece of whose cohomology diagram is

$$\begin{tikzcd} & H^0(V \otimes L) \arrow[d, "\tau"] & & \\ H^0(\Theta_C) \arrow[r] & H^0(f^*\Theta_{\mathbf{P}V}) \arrow[r, "j"] \arrow[d, "\delta"] & H^0(N_f) \arrow[r, "\bar{\delta}"] & H^1(\Theta_C) \\ & H^1(\mathcal{O}_C) & & \end{tikzcd} \tag{7}$$

Suppose that we have a family of holomorphic mappings

$$f_t : C \to \mathbf{P}V.$$

Locally we may choose a coordinate z on C and *position vector mapping*

$$(z, t) \mapsto v(z, t) \in V \backslash \{0\}$$

such that $f_t(z) = \mathbf{C} \cdot v(z, t) \subset V$[*6]. Set

$$\dot{v}(z) \equiv \left. \frac{\partial v(z, t)}{\partial t} \right|_{t=0} \text{ modulo } v(z, t).$$

[*6] By a position vector mapping we shall mean a local lifting of f_t to $V \backslash \{0\}$. Clearly, position vector mappings exist and any two differ by a nonvanishing holomorphic function. Note that the fibre

$$(f_t^* \mathcal{O}_{\mathbf{P}V}(-1))_z = \mathbf{C} \cdot v(z, t).$$

Another choice of position vector is given by

$$\tilde{v}(z, t) = \rho(z, t)v(z, t), \qquad \rho \neq 0,$$

and then

$$\dot{\tilde{v}} = \rho\dot{v} + \dot{\rho}v.$$

Recalling that the inclusion $\mathcal{O}_C \xhookrightarrow{v} V \otimes L$ is given locally by $\varphi \mapsto \varphi \cdot v$ ($\varphi \in \mathcal{O}_C$) it follows that

$$\dot{v} \in H^0(C, V \otimes L/\mathcal{O}_C) = H^0(C, f^*\Theta_{\mathbf{P}V}) \tag{8}$$

is well-defined. Clearly we have

$$j(\dot{v}) = \dot{f} \tag{9}$$

in (7).

As discussed above we are interested in the tangent vector

$$\dot{L} =: \left.\frac{dL_t}{dt}\right|_{t=0} \in H^1(\mathcal{O}_C).$$

It is a standard and easily verified fact that in (7)

$$\dot{L} = \delta(\dot{v}) \tag{10}$$

where $\dot{v}$ is the infinitesimal variation of the mappings $f_t : C \to \mathbf{P}V$ as described above. In particular,

$$\dot{L} = 0 \Leftrightarrow \dot{v} = \tau(w) \quad \textit{for some} \quad w \in H^0(V \otimes L) \quad \text{in} \quad (7). \tag{11}$$

(6) We now come to our first main result. Recalling that

$$B(\xi, t) \in H^0(C, \operatorname{Hom}(V, V(N))), \tag{1}$$

we let

$$D = (\xi_0^N)$$

be the divisor $N \cdot \pi^{-1}(\infty)$ on C where $C \xrightarrow{\pi} P$ is the branched covering of the spectral curve over the ξ-sphere. Then

$$\text{(2)} \qquad \begin{cases} B/\xi_0^N \in H^0(\mathrm{Hom}(V,\, V(D)), & \text{and} \\ v \in H^0(V \otimes L) \end{cases}$$

where $V(D) \cong V(N)$ are the sections of $V \otimes \mathcal{O}_C(D)^{(*7)}$. It follows that

$$\left(\frac{B}{\xi_0^N}\right) \cdot v \in H^0(V \otimes L(D)),$$

and our first cohomological interpretation of the Lax equation is given by the

THEOREM. *Referring to* (5.7) *we have*

$$\text{(3)} \qquad \boxed{\dot{v} = \tau\left(\frac{B}{\xi_0^N} \cdot v\right)}$$

Explanation. From the diagram

$$\text{(4)} \qquad \begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{O}_C \arrow[r] \arrow[d, "v"] & \mathcal{O}_C(D) \arrow[r] \arrow[d, "v"] & \mathcal{O}_D(D) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & V \otimes L \arrow[r] \arrow[d, "\tau"] & V \otimes L(D) \arrow[r] \arrow[d, "\tau"] & V \otimes L \otimes \mathcal{O}_D(D) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & f^*\Theta_{\mathbf{P}V} \arrow[r] \arrow[d] & f^*\Theta_{\mathbf{P}V}(D) \arrow[r] \arrow[d] & f^*\Theta_{\mathbf{P}V} \otimes \mathcal{O}_D(D) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

(*7) The difference here is that in (1) $B(\xi, t)$ is a holomorphic section of the bundle $\mathrm{Hom}(V, V) \otimes \mathcal{O}_C(N)$ where $\mathcal{O}_C(N) = \pi^*\mathcal{O}_P(N)$, whereas in (2) B/ξ_0^N is a matrix in $\mathrm{Hom}(V, V)$ with meromorphic functions in $H^0(\mathcal{O}_C(D))$ as entries. In other words, in (1) we are viewing $\xi = [\xi_0, \xi_1]$ as a homogeneous coordinate on $\mathbf{P}^1$ pulled up to C, whereas in (2) we are viewing $\xi = \xi_1/\xi_0$ as a function in $H^0(\mathcal{O}_C(D))$.

we obtain

$$
(5)\qquad
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] \\
 & H^0(\mathcal{O}_C(D)) \arrow[r] \arrow[d, "\nu"] & H^0(\mathcal{O}_D(D)) \arrow[d, "\sigma"] \\
H^0(V\otimes L) \arrow[r, "i"] \arrow[d, "\tau"] & H^0(V\otimes L(D)) \arrow[r, "j"] \arrow[d, "\tau"] & H^0(V\otimes L\otimes \mathcal{O}_D(D)) \arrow[d, "\tau"] \\
H^0(f^*\Theta_{\mathbf{P}V}) \arrow[r, "i"] \arrow[d, "\delta"] & H^0(f^*\Theta_{\mathbf{P}V}(D)) \arrow[r, "j"] & H^0(f^*\Theta_{\mathbf{P}V}\otimes \mathcal{O}_D(D)) \\
H^0(\mathcal{O}_D(D)) \xrightarrow{\delta_1} H^1(\mathcal{O}_C) & &
\end{tikzcd}
$$

and then the theorem means that

(i) $Bv/\xi_0^N \in H^0(V\otimes L(D))$
(ii) $\tau(Bv/\xi_0^N) = i(\dot{v})$

This may be compared with (5.11).

From the commutativity of the diagram (5) and (5.10) we obtain the

(6) COROLLARY. $\dot{L} = 0$ *if, and only if, there exists a meromorphic function* $\varphi \in H^0(\mathcal{O}_C(D))$ *such that*

$$\frac{Bv}{\xi_0^N} + \varphi v \in H^0(V\otimes L)$$

is holomorphic.

Indeed, the existence of φ is equivalent to the existence of $b \in H^0(V\otimes L)$ with $i(b) = Bv/\xi_0^N + \varphi v \in H^0(V\otimes L(D))$, and then by (5.10)

$$\dot{L} = \delta\dot{v} = \delta\tau(b) = 0. \qquad \text{Q.E.D.}$$

Proof of the theorem. The following simple computation is the whole point. Working in $\mathbf{C}^2$ with coordinates (ξ, η) where $A(\xi, t)$ and $B(\xi, t)$ are polynomials in $\xi \in \mathbf{C}$ whose coefficients are holomorphic functions of t, we have that

$$B(\xi, t) = \frac{B}{\xi_0^N}$$

where on the right hand side $B \in H^0(\mathrm{Hom}(V, V(N))$ is considered as a homogeneous polynomial in ξ_0, ξ_1. For a general point $p = (\xi, \eta) \in C$ we have by (5.1)

$$A(\xi, t)v(p, t) = \eta v(p, t)$$

where $p \mapsto v(p, t)$ is the position vector mapping. Letting $\quad$ = "d/dt at $t = 0$" this gives

$$\dot{A}v + A\dot{v} = \eta\dot{v} \tag{7}$$

Using the Lax equation (2.1) this becomes

$$A(\dot{v} - Bv) = \eta(\dot{v} - Bv). \tag{8}$$

By our assumption that the eigenspaces of A are generically 1-dimensional this implies that

$$Bv = \dot{v} + \lambda v \tag{9}$$

for some λ. But then clearly

$$\tau(Bv) = \dot{v} \in V \otimes L/\mathbf{C} \cdot v.$$ Q.E.D.

(7) Before giving our second main result concerning the problem posed in section 3, we shall discuss a few generalities concerning algebraic curves.

Let C be a smooth curve of genus g and

$$D = \sum_i n_i p_i, \qquad n_i \geq 0,$$

an effective divisor on C. If z_i is a local coordinate centered at p_i, then by a *Laurent tail* we shall mean an expression

$$\varphi_i = \frac{a_{i,n_i}}{z_i^{n_i}} + \cdots + \frac{a_{i,1}}{z_i}$$

The Mittag-Leffler problem is this:

Given Laurent tails φ_i, when does there exist a meromorphic function $\varphi \in H^0(\mathcal{O}_C(D))$ (i.e., $(\varphi) + D \geq 0$) *such that $\varphi - \varphi_i$ is holomorphic near p_i?*

If as usual we denote by $H^0(\Omega_C)$ the space of holomorphic 1-forms on C, then for $\varphi \in H^0(\mathcal{O}_C(D))$ and $\omega \in H^0(\Omega_C)$ we have the residue theorem

$$\sum_i \operatorname{Res}_{p_i}(\varphi \cdot \omega) = 0. \tag{1}$$

We note that

$$\operatorname{Res}_{p_i}(\varphi \cdot \omega) = \operatorname{Res}_{p_i}(\varphi_i \cdot \omega)$$

depends only on the Laurent tail of φ at p_i. The main classical result concerning the Mittag-Leffler problem is this (cf. [4]):

(2) *Given Laurent tails φ_i, the necessary and sufficient condition that there exist $\varphi \in H^0(\mathcal{O}_C(D))$ such that $\varphi - \varphi_i$ is holomorphic near p_i is that*

$$\sum_i \operatorname{Res}_{p_i}(\varphi_i \cdot \omega) = 0 \tag{3}$$

for all $\omega \in H^0(\Omega_C)$.(*8)

We shall sketch the proof of this result as it is quite pertinent to the problem posed in section 3. In the exact sheaf sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \mathcal{O}_D(D) \to 0 \tag{4}$$

the last term $\mathcal{O}_D(D)$ is a skyscraper sheaf that may be identified with the collections $\{\varphi_i\}$ of Laurent tails. The cohomology sequence of (4) together with its dual are

$$\begin{cases} \text{(i) } H^0(\mathcal{O}_C(D)) \to H^0(\mathcal{O}_D(D)) \xrightarrow{\partial} H^1(\mathcal{O}_C) \to H^1(\mathcal{O}_C(D)) \to 0 \\ \text{(ii) } H^1(\mathcal{O}_C(-D)) \leftarrow H^0(\mathcal{O}_D(D))^* \xleftarrow{\partial^*} H^0(\Omega_C) \leftarrow H^0(\Omega_C(-D)) \leftarrow \end{cases} \tag{5}$$

(*8) Informally we may say that "the only constraints on Laurent tails to be the principal parts of a meromorphic function are those imposed by the residue theorem." The number of independent equations (3) is $g - i(D)$ where $i(D) = \dim H^0(\Omega_C(-D))$ is the number of linearly differentials $\omega \in H^0(\Omega_C)$ vanishing on D. It follows that (here we set $h^0(\mathcal{O}_C(D)) = \dim H^0(\mathcal{O}_C(D))$

$$h^0(\mathcal{O}_C(D)) = \deg D - g + 1 + i(D),$$

which is just the Riemann-Roch theorem. We may view (3) as a quantitative form of this theorem.

(here, dual vector spaces are aligned vertically). The basic fact is that for $\omega \in H^0(\Omega_C)$

$$\langle \partial^* \omega, \{\varphi_i\} \rangle = \sum_i \operatorname{Res}_{p_i}(\varphi_i \cdot \omega). \tag{6}$$

The assertion (2) is an immediate consequence of (6) together with the exactness of the above sequences.

Referring to (6.9) and noting that ν and $\dot{\nu}$ are holomorphic around D, we infer that λ induces a well-defined section of $\mathcal{O}_D(D)$. A fundamental invariant of the Lax equation (2.1) is given by the

Definition. The *residue* of B, denoted by $\rho(B)$, is the section of $\mathcal{O}_D(D)$ induced by λ in (6.9).

Our second main result is the

THEOREM. *Referring to* (i) *in* (5), *we have*

$$\boxed{\dot{L} = \partial(\rho(B))} \tag{7}$$

To state the first corollary we let

$$\mathcal{L} \subset H^0(\mathcal{O}_D(D))$$

be the Laurent tails of functions in $H^0(\mathcal{O}_C(D))$. Recalling that $B(\xi, t)$, and therefore also $\rho(B)$, depends on t, from (7) we may deduce the following

COROLLARY. *The necessary and sufficient condition that* $\{L_t\} \subset \operatorname{Pic}^d(C)$ *be linear is that*

$$\rho(\dot{B}) \equiv 0 \bmod \operatorname{span}\{\mathcal{L}, \rho(B)\} \tag{8}$$

Here, this equation takes place in the fixed vector space $H^0(\mathcal{O}_D(D)) \cong \mathbf{C}^k$ where $k = \deg D$.

For the second corollary we let $\rho_i(B)$ be the residue of B at p_i. We also recall that the Jacobian of C is

$$J(C) = H^0(\Omega_C)^*/H_1(C, \mathbf{Z}),$$

so that a linear flow on $J(C)$ is given, up to a fixed translation, by a bilinear map

(9) $$(t, \omega) \mapsto t\langle \lambda, \omega \rangle$$

where $\lambda \in H^0(\Omega_C)^*$. Recalling (3) we have our main conclusion concerning the problem posed in section 3:

COROLLARY. *Condition* (8) *is equivalent to*

(10) $$\sum_i \operatorname{Res}_{p_i}(\dot{\rho}_i(B)\omega) = \mu \sum_i \operatorname{Res}_{p_i}(\rho_i(B)\omega)$$

for all $\omega \in H^0(\Omega_C)$. *If this is satisfied then the linear flow is given in the form* (9) *by*

(11) $$(t, \omega) \mapsto t \sum_i \operatorname{Res}_{p_i}(\rho_i(B)\omega).$$

Proof of Theorem (7). Referring to the commutative diagram (6.5), we let

$$E \in H^0(V \otimes L(D))$$

satisfy

$$\tau(E) = i(w)$$

for some $w \in H^0(f^*\Theta_{\mathbf{P}V})$ (in practice we will have $E = Bv/\xi_0^N$ and $w = \dot{v}$ as in Theorem (6.3)). Then by commutativity

$$\begin{aligned} \tau j(E) &= j\tau(E) \\ &= ji(w) \\ &= 0, \end{aligned}$$

and so there exists $\lambda \in H^0(\mathcal{O}_D(D))$ with

$$\sigma(\lambda) = j(E).$$

Now the $H^0(\mathcal{O}_D(D))$ in the upper right hand corner of (6.5) is the same as the one in the lower left, and by commutativity

$$\delta(w) = \delta_1(\lambda).$$

This implies the result. Q.E.D.

Remark. We have commented several times on the nonuniqueness of the B in the Lax Equation (2.1). In particular, the condition (10) should be invariant under a substitution

$$B \mapsto B + P(\xi, A)$$

where $P(\xi, \eta)$ is a polynomial in two variables. To see why this should be so we remark that if D' is any divisor with $D' \geq D$ then we have an inclusion

$$\mathcal{O}_D(D) \subseteq \mathcal{O}_{D'}(D');$$

in particular the residue $\rho(B) \in H^0(\mathcal{O}_{D'}(D'))$ is defined. Moreover, the images $\mathcal{L} \subset H^0(\mathcal{O}_D(D))$ of $H^0(\mathcal{O}_C(D)) \to H^0(\mathcal{O}_D(D))$ and $\mathcal{L}' \subset H^0(\mathcal{O}_{D'}(D'))$ of $H^0(\mathcal{O}_C(D')) \to H^0(\mathcal{O}_{D'}(D'))$ are related by

$$\mathcal{L} = \mathcal{L}' \cap H^0(\mathcal{O}_D(D)).$$

Let now $D' \geq D$ correspond to $B' = B + P(\xi, A)$. At $p = (\xi, \eta) \in C$ we have

$$B'v = Bv + P(\xi, \eta)v,$$

from which it follows that

$$\text{(12)} \qquad \rho(B') = \rho(B) + \rho(P(\xi, \eta)), \qquad \Rightarrow \rho(B') \equiv \rho(B) \bmod \mathcal{L}'$$

since, by the definition (6.9), $\rho(P(\xi, \eta))$ is given by the Laurent tails of the rational function $P(\xi, \eta) \in H^0(\mathcal{O}_C(D'))$. In particular, (10) and (12) are consistent with (3).

(8) We now give some examples.

Example 1. We consider the free rigid body as discussed in section

4. As noted in Example 1 in section 2, this is a Lax equation with a parameter where

$$B = -(\Omega + J\xi). \tag{1}$$

It follows that $D = \Sigma_{i=1}^{m} p_i$ where the p_i are the n distinct points lying over $\xi = \infty$. Moreover, if z_i is a local coordinate around p_i (e.g., we may take $z_i = \xi^{-1}$), it follows from (6.9) and (1) that the residue

$$\rho(B) = -\sum_i \frac{\lambda_i}{z_i}$$

where $J = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Clearly then $\rho(\dot{B}) = 0$ so that the flow is linearized on $J(C)$. Moreover, recalling the involution j of C, we have

$$j(B) = -(\Omega - J\xi)$$

and it follows that

$$j(\rho(B)) = -\rho(B).$$

From (7.11) we see that the linear flow on $J(C)$ is trivial on $H^0(\Omega_C)^+ = \{\omega \in H^0(\Omega_C): j^*\omega = \omega\}$, and hence the flow is actually linearized on $\mathrm{Prym}(C/C')$. According to the discussion in Example 1 of section 4 (cf. (4.5)), this is a torus of exactly the right dimension.

Example 2. We consider the Lagrange top discussed in Example 2 of section 4. The spectral curve is a 2-sheeted covering

$$\pi: C \to P = \mathbf{P}^1(\xi)$$

branched over 4 points ξ_ν with all $\xi_\nu \neq \infty$. Since

$$B(\xi) = \Omega + \chi \cdot \xi$$

it follows that

$$D = \pi^{-1}(\infty) = p + q$$

where

$$\begin{cases} \eta = \dfrac{\alpha + \beta}{\xi} + \cdots & \text{near } p \\ \eta = -\dfrac{(\alpha + \beta)}{\xi} + \cdots & \text{near } q \end{cases}$$

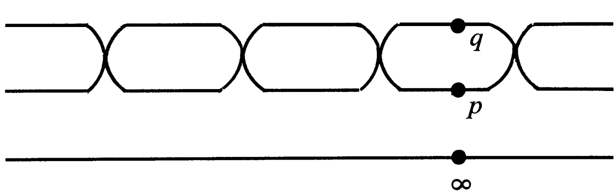
The residue $\rho(B) \in H^0(\mathcal{O}_D(D))$ is given by

$$\begin{cases} \rho(B) = -\dfrac{1}{\xi} + \cdots & \text{near } p \\ \rho(B) = -\dfrac{1}{\xi} + \cdots & \text{near } q \end{cases}$$

and thus (6.8) is satisfied.

We refer to [22] for further discussion of the Lagrange top and heavy symmetric top in n dimensions as well as $s\ell(n)$-analogue of the heavy symmetric top. The flow corresponding to the latter is linearized in the same way as Example 1 just above ([22], page 443), and presumably the flow corresponding to the former may also be linearized using our general results (although in this case the spectral curve $P(\xi, \eta) = 0$ will be quite singular).

Example 3. We consider the Toda lattice as discussed in Example 3 of section 2 and section 4. Since in this case the Lax equation (2.11) has a Laurent parameter our results of the two preceding sections must be modified slightly. We consider the hyperelliptic covering (4.9) and set $\eta^{-1}(\infty) = p + q$



From the affine equation (4.8) of C it follows that the divisor

$$(\xi) = np - nq$$

(the poles of ξ occur on one sheet lying over a neighborhood of ∞ and the zeroes on the other sheet, and each has multiplicity n). Consequently, setting

$$D = np + nq$$

it results from (2.10) that

$$B \in H^0(C, \operatorname{Hom}(V, V(D)).$$

Just as in section 7 above, we may define the residue

$$\rho(B) \in H^0(\mathcal{O}_D(D))$$

and then (6.3), (7.10) and (7.11) are valid.

Remark. It may be verified that the functions

$$1, \eta, \ldots, \eta^n, \xi \in H^0(\mathcal{O}_C(D))$$

give a basis. Consequently, the image $\mathcal{L} \subset H^0(\mathcal{O}_D(D))$ of $H^0(\mathcal{O}_C(D)) \to H^0(\mathcal{O}_D(D))$ has dimension $n + 1 = g + 2$ (by (4.10)). The mapping

$$H^0(\mathcal{O}_D(D))/\mathcal{L} \to H^1(\mathcal{O}_C)$$

is thus an isomorphism (this also follows from the exact cohomology sequence of $0 \to \mathcal{O}_C \to \mathcal{O}_C(D) \to \mathcal{O}_D(D) \to 0$ and $H^1(\mathcal{O}_C(D)) = (0)$).

Following [19] we may compute the eigenvector mapping as follows: Let $E = r_1 + \cdots + r_g$ be a general divisor of degree g satisfying

(2) $$h^0(\mathcal{O}_C(E + (k-1)p - kq) = 0$$

for all k. Since by the Riemann-Roch theorem

$$h^0(\mathcal{O}_C(E + kp - kq)) \geq 1$$

we find from (2) that

$$h^0(\mathcal{O}_C(E + kp - kq)) = 1$$

for all k. Let $f_k \in H^0(\mathcal{O}_C(E + kp - kq))$, $1 \le k \le n$, be a basis where $f_n = \xi$ and set

$$\tilde{v} = \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix}.$$

Then, as proved in [19], for suitable choice of E this $\tilde{v}$ will be an eigenvector for the spectral curve C

$$A(\xi)\tilde{v} = \eta\tilde{v} \quad \text{for} \quad (\xi, \eta) \in C \tag{3}$$

We will determine the residue of B at q; a similar discussion will hold at p. We will also restrict to the case $n = 3$, from which the general case will be clear. Recalling that $f_3 = \xi$, (3) is

$$\begin{cases} b_1 f_1 + a_2 f_2 + a_3 = \eta f_1 \\ a_1 f_1 + b_2 f_2 + a_2 \xi = \eta f_2 \\ a_3 \xi f_1 + a_2 f_2 + b_3 \xi = \eta \xi. \end{cases} \tag{4}$$

According to our general theory we must compute the residue of B using a *holomorphic* eigenvector v, and we take

$$v = \xi^{-1}\tilde{v}.$$

Multiplying (4) through by ξ^{-1}, everything becomes holomorphic except the last equation which gives

$$a_3 f_1 = \eta + \text{holomorphic terms.} \tag{5}$$

The residue of B is defined by (cf. (6.9))

$$Bv = \rho(B)v + \text{holomorphic terms.}$$

By (2.10) and (4) the left hand side of this equation is

$$\begin{pmatrix} a_1 f_2 \xi^{-1} - a_3 \xi^{-1} \\ -a_1 f_1 \xi^{-1} + a_2 \\ a_3 f_1 - a_2 f_2 \xi^{-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix} + \text{holomorphic terms},$$

where the equality follows from (5). Consequently

$$\rho(B) = \frac{\eta}{\xi}$$

and then obviously $\rho(\dot{B}) = 0$. By (7.10) the Toda lattice is linearized on $J(C)$; note that (cf. (4.10))

$$\dim J(C) = g = n - 1$$

is exactly the required number of integrals in the problem.

Remark. In this work we have taken the Lax equation (2.1) as given and from it deduced the spectral curve, eigenvector mappings $f_t : C \to \mathbf{P}V$, and flow $t \to \{L_t\}$. Conversely, in each of our examples, given the spectral curve plus some additional data such as L_t plus a suitable fixed divisor, we may reconstruct the eigenvector mappings. For instance, in the Toda lattice we have just said how to do this. This is just a part of a very beautiful "dictionary" that has been highly developed in recent years (cf. [6], [7], and [18]).

Example 4. We consider the spectral curve C associated to Nahm's equations as discussed in Example 2 of sections 2, 4. For the branched covering $C \to P$ given by ξ we set

$$D = \sum_{i=1}^{n} p_i = \xi^{-1}(\infty).$$

Let z_i be a local coordinate around p_i (e.g., $z_i = \xi^{-1}$ will do) and set

$$\eta_i = \frac{\lambda_i}{z_i^2} + \frac{1}{z_i} + \text{(holomorphic terms)} \tag{6}$$

near p_i. From (2.14) we have near p_i that

$$\begin{cases} A = A_0 + \dfrac{A_1}{z_i} + \dfrac{A_2}{z_i^2} \\ B = -\dfrac{1}{2}A_1 - \dfrac{A_2}{z_i} \end{cases}$$

The eigenvector v_i satisfies

$$Av_i = \eta_i v_i,$$

from which it follows first that

$$A_2 v_i(p_i) = \lambda_i v_i(p_i),$$

and secondly that the residue

$$\rho(B) = -\sum_i \frac{\lambda_i}{z_i}$$

where λ_i is the same as in (6). Consequently $\rho(\dot{B}) = 0$ and by (7.10) the flow on $J(C)$ corresponding to Nahm's equations is linear and given by (7.11).

Example 5. Referring to Example 5 in section 4, for the computation of the residue of $\rho(B)$ it is more convenient to take the equations (4.11), since even though the affine curves are quite singular they are "smooth over $\xi = \infty$" (this means that the map $C \to P$ from the normalization is unmodified over $\xi = \infty$). The exact same argument as in Example 4 above shows that (7.10) is satisfied and so the flow is linearized on $J(C)$.

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