Linearizing Flows and a Cohomology Interpretation of Lax Equations.

A talk given by Phillip Griffiths in Chern's Differential Equations Seminar on May 9, 1983

§1. By a Lax equation with a parameter we shall mean an equation

$$\dot{A}(\xi) = [B(\xi), A(\xi)] \tag{1}$$

where

$$\begin{cases} A(\xi) = \sum_{-p}^{q} A_k(t)\xi^k \\ B(\xi) = \sum_{-p}^{q} B_k(t)\xi^k \end{cases}$$

are finite Laurent series in a variable ξ whose coefficients are matrices depending on a parameter. It is known ([2], [3]) that, with the exception of Kowaleski's top, all of the known completely integrable Hamiltonian systems may be represented in the form (1). We give three cases.

Example 1. The Euler equations of a free rigid body in $\mathbb{R}^n$ are ([14], [21])

$$\begin{cases} \dot{M} = [M, \Omega] \quad \text{where} \\ \Omega \in so(n), \quad M = \Omega J + J\Omega \in so(n) \\ J = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \ \lambda_i > 0 \end{cases} \tag{2}$$

By Manakov's trick these are equivalent to (1) where

S. S. Chern (ed.), *Seminar on Nonlinear Partial Differential Equations*

$$\begin{cases} A = M + J^2\xi \\ B = -(\Omega + J\xi) \end{cases}$$

Example 2. The Toda lattice is the Hamiltonian system on $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$ corresponding to Hamiltonian function $H(x,y) = \frac{1}{2}\Sigma y_i^2 + \sum_{i=1}^{n} e^{x_i - x_{i+1}}$, $x_{n+1} = x_1$. By Flaschka's substitution

$$\begin{cases} a_k = \frac{1}{2} e^{(x_k - x_{k+1})} \\ b_k = -y_k/2 \end{cases}$$

the Hamiltonian equations are

$$(3) \qquad \begin{cases} \dot{b}_k = 2(a_k^2 - a_{k+1}^2), & a_{n+1} = a_1 \\ \dot{a}_k = a_k(b_{k+1} - b_k), & b_{n+1} = b_1. \end{cases}$$

Noting that $\sum_k b_k$ = constant, we normalize by requiring that $\sum_k b_k = 0$. Then by [24] the equations (3) are of the form (1) where

$$A(\xi) = \begin{bmatrix} b_1 & a_1 & a_n\xi^{-1} \\ a_1 & & \\ & & a_{n-1} \\ a_n & a_{n-1} & b_n \end{bmatrix}$$

$$B(\xi) = \begin{bmatrix} 0 & a_1 & -a_n\xi \\ -a_1 & & \\ & & a_{n-1} \\ a_n\xi & -a_{n-1} & 0 \end{bmatrix}$$

Example 3. Nahm's equations [20], which arise in the study of monopoles, are

$$\begin{cases} \dot{T}_i = \frac{1}{2} \epsilon_{ijk} [T_j, T_k] \\ T_i \in u(n). \end{cases}$$

By [10] these are equivalent to (1) where

$$\begin{cases} A(\xi) = (T_1 + iT_2) + (-2iT_3)\xi + (T_1 - iT_2)\xi^2 \\ B(\xi) = -A_\xi = -\frac{A_1}{2} - \xi A_2. \end{cases}$$

§2. Given (1) we define its spectral curve C to be the normalization of the complete algebraic curve whose affine equation is

$$Q(\xi,\eta) = \det\|\eta I - A(\xi,t)\| = 0.$$

Since the Lax equations give isospectral flows, C is independent of t. We assume that for general $p = (\xi,\eta)$ the corresponding eigenspace is 1-dimensional and is spanned by a vector $v(p,t) \in V \cong \mathbb{C}^n$. There is then a family of holomorphic mappings

(4) $$f_t: C \longrightarrow \mathbb{P}V$$

given by $p \longmapsto \mathbb{C}v(p,t)$; clearly these give the time evolution of $A(\xi,t)$. We set

$$L_t = f_t^*(\mathcal{O}_{\mathbb{P}V}(1)) \in J(C)$$

where $\mathcal{O}_{\mathbb{P}V}(1)$ is the hyperplane line bundle and

(5) $$J(C) = H^1(\mathcal{O}_C)/H^1(C,\mathbb{Z})$$

(6) $$\cong H^0(\Omega_C)^*/H_1(C,\mathbb{Z})$$

is a complex torus giving the Jacobian variety of C (c.f. [4] for definitions). Thus associated to (1) is the flow

(7) $$t \longrightarrow L_t \in J(C)$$

and motivated by [2], [3] we may consider the following

(8) **Problem**: Determine the necessary and sufficient conditions on $B(\xi)$ that the flow (7) be linear.

Now, for an arbitrary family of homomorphic mappings (4), reasonably standard deformation theory ([4]) may be used to answer this problem. Moreover, since the tangent space to any algebro-geometric moduli space is computed cohomologically, the general answer to (8) is expressed in terms of an H^1 (by (5) this is reasonable).

Suppose now that (4) arises from (1). Note that B is not unique since any substitution

$$B \longmapsto B + P(\xi, A), \qquad P(\xi, \psi) \in \mathbb{C}[\xi, \psi],$$

leaves (1) invariant. This suggests that the B in a Lax pair (A,B) lives naturally in a cohomology group somewhere. By a very nice cohomological computation, this turns out to be the case and allows us to answer (8) in a way that is effective for the computation of examples.

To explain the result we assume for simplicity that

$$B(\xi) = \sum_{k=0}^{\ell} B_k(t)\xi^k$$

is a polynomial of degree ℓ. View ξ as a meromorphic function, set

$$D = \xi^{-1}(\infty) = \sum_i n_i p_i$$

and denote by $H^0(\mathcal{O}_D(P))$ the Laurent tails $\{\phi_i\}$ where

$$\phi_i = a_{i,n_i}/z_i + \cdots + a_{i,1}/z_i$$

and where z_i is a local coordinate around p_i. Near p_i we have

$$A\ v(p,t) = \eta\ v(p,t)$$

$$\Rightarrow \quad \dot{A}v + A\dot{v} = \eta\dot{v}$$

$$\Rightarrow \quad A(\dot{v} - Bv) = \eta(\dot{v} - Bv) \qquad \text{(by (1))}$$

(9) $$\Rightarrow \quad Bv = \dot{v} + \lambda_i v,$$

where λ_i is a Laurent tail as above.

Definition. We defined the <u>residue</u> $\rho(B) \in H^0(\mathcal{O}_D(D))$ to be the collection of Laurent tails $\{\lambda_i\}$ given by (9).

(10) **Theorem.** *Let $\mathcal{L} \subset H^0(\mathcal{O}_D(D))$ be the Laurent tails of meromorphic functions $g \in H^0(\mathcal{O}_C(D))$. Then the flow (7) is linear $\Leftrightarrow$*

$$\dot{\rho}(B) \equiv 0 \bmod \{\rho(B), \mathcal{L}\}. \tag{11}$$

If this is satisfied, then using (6) it is given by a translate of

$$(t,\omega) \longmapsto t \sum \mathrm{Res}_{p_i}(\lambda_i\omega), \qquad \omega \in H^0(\Omega_C). \tag{12}$$

§3. This result serves to unify the known linearizibility theorems given in [2], [3], [10], [17], [21] and [22]. We shall indicate how it applies in two of the above examples.

Example 1 reconsidered. In this case

$$D = \xi^{-1}(\infty) = \sum_i p_i$$

and setting $z_i = \xi^{-1}$ near p_i

$$\rho(B) = \sum_i \lambda_i/z_i$$

Thus $\rho(\dot{B}) = 0$ and so (11) and (12) apply.

Actually, in this case since $A = M + J^2\xi$ where ${}^tM = -M$ and ${}^tJ^2 = J^2$ we have that

$$Q(\xi,\eta) = (-1)^n Q(-\xi,-\eta).$$

Thus $j(\xi,\eta) = (-\xi,-\eta)$ gives an involution of C with quotient curve C' = C/j. If C and C' have respective genera g and g', then an easy computation using the Riemann-Hurwitz formula gives

$$g - g' = \frac{1}{2}\left[\frac{n(n-1)}{2} - [\tfrac{n}{2}]\right]. \tag{13}$$

Since clearly

$$j(\rho(B)) = -\rho(B)$$

the flow (7) actually occurs on the complex torus

$$\mathrm{Prym}(C/C') =: H^0(\Omega_C)^-/H_1(C,\mathbb{Z})^-$$

where $^-$ denotes the -1 eigenspace of j. By (13)

$$\dim \mathrm{Prym}(C/C') = \frac{1}{2}\left[\frac{n(n-1)}{2} - [\tfrac{n}{2}]\right]. \tag{14}$$

On the other hand, in the Euler equation (2), Ω moves on an adjoint orbit $\mathcal{O}_\mu \subset so(n)$ and in general

(15) $$\dim \mathcal{O}_\mu = \frac{n(n-1)}{2} - [\tfrac{n}{2}].$$

Comparing (14) and (15) we see that our linearization occurs on a torus of exactly the correct dimension.

<u>**Example 3 reconsidered**</u>. In this case also

$$D = \sum_i p_i.$$

Near p_i we have

$$\begin{cases} (A_2\xi^2)v_i = \eta_i v_i + O(\xi) \\ Bv_i = -A_2\xi v_i + O(1) \end{cases}$$

$\Longrightarrow \rho^{(B)} = \sum_i \lambda_i/z_i$

$$\text{where } \lambda_i = (\eta^2/\xi)(p_i).$$

Clearly then (11) is satisfied and (12) linearizes Nahm's equations (cf. [10] for an extensive discussion).

The remaining integrable systems, such as Toda lattice, heavy symmetric top ([22],[23]), geodesics on an ellipsoid ([17],[12]), and Neumann's mechanical problem ([17],[13]) may be treated in a similar way. The details may be found in an upcoming paper by the author.

[1] R. Abraham and J. Marsden, Foundations of mechanics, Benjamin/Cummings (1978).

[2] M. Adler and P. van Moerbeke, Completely integrable systems, Euclidean Lie algebras, and curves, Advances in Math., 38 (1980), 267-317.

[3] M. Adler and P. van Moerbeke, Linearization of Hamiltonian systems, Jacobi varieties, and representation theory, Advances in Math., 38 (1980), 318-379.

[4] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, Topics in the geometry of algebraic curves, Grundlehren Math., Springer-Verlag, (1984).

[5] V.I. Arnold, Mathematical methods of classical mechanics, Springer-Verlag (1978).

[6] B.A. Dubrovnin, V.B. Mateev, and S.P. Novikov, Uspehi Mat. Nauk., 31 (1976), and Russian Math Surveys, 1 (1976).

[7] B.A. Dubrovnin, Theta functions and non-linear equations, Usp. Mat. Nauk., 36 (1981), 11-80.

[8] I. Frankel, A. Reiman, and M. Senenov-Tian-Shansky, Graded Lie algebras and completely integrable dynamical systems, Soviet Math. Dokl., 20 (1979), 811-814.

[9] R. Hartshorne, Algebraic geometry, Springer-Verlag.

[10] N. Hitchin, On the construction of monopoles, preprint (1982).

[11] E. Horikawa, On deformations of quintic surfaces, Invent. Math., 31 (1975), 43-85.

[12] H. Knörrer, Geodesics on the ellipsoid, Invent. Math., 59 (1980), 119-144.

[13] H. Knörrer, Geodesics on the ellipsoid, Invent. Math., 59 (1980), 119-144.

[14] S.V. Manakov, Note on the integration of Euler's equations of the dynamics of an n-dimensional rigid body, Functional Anal. and

App., 11 (1976), 328-329.

[15] A.S. Mishchenko and A.T. Fomenko, Euler equations on finite dimensional Lie groups, Math. USSR Izvestija, 12 (1978), 371-389.

[16] H.P. McKean, Integrable systems and algebraic curves, Global Analysis, Lecture Notes in Math. number 755 (1979), 83-200.

[17] J. Moser, Various aspects of integrable Hamiltonian systems, CIME, Bressanone, Progress in Math., 8 (1980), Birkhäuser-Boston, 233-289.

[18] D. Mumford, An algebraic-geometrical construction of commuting operators and of solutions to the Toda lattice equation, Proc. Kyoto Conference in algebraic geometry, Publ. Math. Society, Japan (1977).

[19] P. van Moerbeke and D. Mumford, The spectrum of difference operators and algebraic curves, Acta Math., 143 (1979), 93-154.

[20] W. Nahn, All self-dual multimonopoles for all gauge groups, preprint CERN (1981).

[21] T. Ratiu, The motion of the free n-dimensional rigid body, Indiana Univ. Math. J., 24 (1980), 609-629.

[22] T. Ratiu, Euler-Poisson equations on Lie algebras and the N-dimensional heavy rigid body, Amer. J. Math., 104 (1982), 409-448.

[23] A. Reiman and M. Semernov-Tian-Shansky, Reduction of Hamiltonian systems, affine Lie algebras, and Lax equations, Invent. Math., 54 (1979), 81-101.

[24] P. van Moerbeke, The spectrum of Jacobi matrices, Invent. Math., 37 (1976), 45-81.

[25] J.L. Verdier, Algèbres de Lie, systèmes Hamiltoniens, courbes algèbriques, Sem. Bourbaki 566 (1980/81).