L-Indistinguishability for SL(2)*

J.-P. Labesse and R.P. Langlands

1. Introduction. The notion of L-indistinguishability, like many others current in the study of L-functions, has yet to be completely defined, but it is in our opinion important for the study of automorphic forms and of representations of algebraic groups. In this paper we study it for the simplest class of groups, basically forms of $SL(2)$. Although the definition we use is applicable to very few groups, there is every reason to believe that the results will have general analogues [**12**].

The phenomena which the notion is intended to express have been met — and exploited — by others (Hecke [**5**] §13, Shimura [**17**]). Their source seems to lie in the distinction between conjugacy and stable conjugacy. If F is a field, G a reductive algebraic group over F, and $\bar{F}$ the algebraic closure of F, then two elements of $G(F)$ may be conjugate in $G(\bar{F})$ without being conjugate in $G(F)$. In addition, if F is a local field then in many cases there is a rough duality between conjugacy classes in $G(F)$ and equivalence classes of irreducible representations of $G(F)$, and one might expect the coarse classification of stable conjugacy to lead to a grouping of these equivalence classes. One of the groups is now called an L*-packet* and the elements in it are said to be L*-indistinguishable* because in the cases that are understood they have the same L-functions.

It was the L-packets with which we started. If G is $GL(2)$, or even $GL(n)$, then stable conjugacy is the same as conjugacy and the L-packets will consist of a single element, and there is no need to introduce them. They do not appear in [**6**]. The group $GL(2, F)$ acts on $SL(2, F)$ by $g : h \to h^g = g^{-1}hg$ and, if F is a local field, on the irreducible representations of $SL(2, F)$ by $\pi \to \pi^g$ with $\pi^g(h^g) = \pi(h)$. Two elements of $SL(2)$ are *stably conjugate* if and only if they lie in the same orbit under $GL(2, F)$ and it is expedient to define two irreducible representations of $SL(2, F)$ to be L*-indistinguishable* if they lie in the same orbit under $GL(2, F)$, or more precisely, if the induced representations of the Hecke algebra lie in the same orbit. This definition can only be provisional but it will serve our purpose, which is to explore the notion for $SL(2)$ and some related groups thoroughly, attempting to formulate and verify theorems which are likely to be of general validity.

Our original purpose was more specific. Suppose F is a global field and $\pi = \otimes\pi_v$, the product being taken over all places of F, is an automorphic representation of $SL(2, A_F)$.

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If for each v we choose a π'_v which is L-indistinguishable from π_v and equivalent to it for almost all v then $\pi' = \otimes\pi'_v$ might or might not be an automorphic representation. We wished to show that it is, except for a very special class of π, those associated to characters of the group of idèles of norm one in a quadratic extension, and this we could do without too much difficulty. The problem was posed and solved in the spring of 1971 while we were together at the Mathematical Institute in Bonn in the Sonderforschungsbereich Theoretische Mathematik, and the paper could have been written then, except that we could not formulate the results in a satisfying fashion. For this the groups H of [**12**] are needed, for which an adequate general definition was found only after many conversations with Shelstad, as well as the groups S and S^0, whose introduction was suggested by the work of Knapp-Zuckerman [**7**].

Because we had some specific applications in mind we have considered groups slightly more general than twisted forms of $SL(2)$, but they can be left to the body of the paper. If G is $SL(2)$ or a twisted form then the L-group ${}^LG$ can, for the present purpose, be taken to be $PGL(2, \mathbf{C})$. If F is a local field and φ a homomorphism of the Weil group W_F into ${}^LG$ there is in general (cf. [**18**]) an associated L-packet $\Pi(\varphi)$, and according to the results of §3 it will contain only finitely many equivalence classes. If S_φ is the centralizer of $\varphi(W_F)$ in ${}^LG$ and S^0_φ the connected component of the identity in S_φ then, as will be seen in §6 and §7, there is a pairing $\langle s, \pi\rangle$ between $S^0_\varphi\backslash S_\varphi$ and $\Pi(\varphi)$ which is often but not always a duality.

This local pairing is of interest in itself, and is also of some significance in global multiplicity questions. To form a global L-packet Π one chooses local L-packets Π_v, such that Π_v contains the unramified representation π^0_v for almost all v, and takes Π to be the collection

$$\Pi = \{\pi = \otimes\pi_v \mid \pi_v \in \Pi_v \text{ for all } v \text{ and } \pi_v = \pi^0_v \text{ for almost all } v\}.$$

Some of the π may be automorphic and others not. It is shown in §6 and §7 that they are automorphic simultaneously unless Π is the L-packet $\Pi(\varphi)$ associated to a homomorphism $\varphi : W_F \to {}^LG = PGL(2, \mathbf{C})$ obtained from an irreducible induced two-dimensional representation of W_F. If φ_v is the restriction of φ to the decomposition group at v and $\pi = \otimes\pi_v$ lies in $\Pi(\varphi)$ then π_v lies in $\Pi(\varphi_v)$ and $S_\varphi \subseteq S\varphi_v, S^0_\varphi \subseteq S^0\varphi_v$. We may define $\langle s, \pi\rangle$ to be $\Pi_v\langle s, \pi_v\rangle$. One of the principal conclusions of this paper is that the multiplicity with which π occurs in the space of cusp forms is

$$\frac{1}{[S_\varphi : S^0_\varphi]} \sum_{s \in S^0_\varphi\backslash S_\varphi} \langle s, \pi\rangle.$$

One hopes that a similar result is valid for every L-packet containing an automorphic representation, but even its formulation would demand the introduction of the problematical group $G_{\Pi(F)}$ of §2 of [**15**].

Although the main results of the paper are in §6 and §7, the technical burden is carried by §5, in which the analysis of the trace formula suggested in **[12]** is carried out at length. The trace formula seldom functions without some local harmonic analysis, but usually with much less than appears necessary at first sight, and once it is primed it will start to pump out many local results. Since we are dealing with an easy group for which we could establish many of the local results directly, we have done so. For other groups, where local information is harder to come by, it will be necessary to bring the trace formula into play sooner, and so the reader who has his eye on generalizations should not spend too much time on the details of §2, §3, and §4. The critical observations are that the function $\Phi^{T'}(f) : \gamma \to \Phi^{T'}(\gamma, f)$ is smooth and that the map on distributions dual to $f \to \Phi^{T'}(f)$ sends a character to a difference of characters.

Finally we observe that **[8]** and **[16]** serve to some extent as introductions to this paper and that to avoid technical complications we have confined ourselves to fields of characteristic zero.

2. Local theory. Let F be a local field of characteristic zero and G the group $SL(2)$. Let T be a Cartan subgroup of G defined over F. Since G is simply-connected and

$$H^1(F, G) = \{1\}$$

the two sets $\mathfrak{D}(T)$ and $\mathfrak{E}(T)$ introduced in **[12]** are equal to each other and to

$$H^1(F, T).$$

Let $\widetilde{G}$ be the group $GL(2)$. Then the centralizer $\widetilde{T}$ of T in $\widetilde{G}$ is a Cartan subgroup of $\widetilde{G}$. Since

$$H^1(F, \widetilde{T}) = 1$$

any g in $\mathfrak{A}(T)$ (**[12]**) may be written as a product sh with $s \in \widetilde{T}(\bar{F})$ and h in $\widetilde{G}(F)$. Conversely any h in $\widetilde{G}(F)$ is a product $s^{-1}g$ with $g \in G(\bar{F})$ and $s \in \widetilde{T}(\bar{F})$. The element g must lie in $\mathfrak{A}(T)$ for

$$h^{-1}th = g^{-1}tg \quad t \in T(\bar{F}).$$

If L is the centralizer of $T(F)$ in the algebra of 2×2 matrices over F then

$$\{\det\ t | t \in \widetilde{T}(F)\} = \{\mathrm{Nm}_{L/F} x | x \in L^\times\}$$

and

$$g \to \det\ h\ (\mathrm{mod} \mathrm{Nm}_{L/F} L^\times)$$

yields an isomorphism

$$\mathfrak{D}(T) = \mathfrak{E}(T) \simeq F^\times / \mathrm{Nm}_{L/F} L^\times.$$

More generally we could suppose that F was an extension of some field E and then consider a group G' over E with

$$\mathrm{Res}_{L/E}G \subseteq G' \subseteq \mathrm{Res}_{L/E}\widetilde{G}.$$

Thus G' is defined by a subgroup A of $\mathrm{Res}_{L/E}G_m$ and

$$G'(F) = \{g \in \widetilde{G}(F) | \det\, g \in A(E)\}.$$

If T' is the centralizer of $\mathrm{Res}_{F/E}\ T$ in G' then one shows, just as above, that

$$\mathfrak{D}(T') = \mathfrak{D}(T'/E) \simeq F^\times / A(E)\mathrm{Nm}_{L/F}L^\times.$$

For our purposes it is best simply to take a closed subgroup A of $F^\times$ and to let

$$G' = \{g \in \widetilde{G}(F) | \det\, g \in A\},$$

so that G' may no longer be the set of points on an algebraic group rational over some field. T' will be the intersection of G' with $\widetilde{T}(F)$ and we set

$$\mathfrak{D}(T') = \widetilde{T}(F)\backslash\widetilde{G}(F)/G' \simeq F^\times / A\ \mathrm{Nm}_{L/F}L^\times.$$

It is a group, and is either trivial or of order two.

We return for a moment to G. Suppose, as in **[12]**, that κ is a homomorphism of $X_*(T)$ into $\mathbf{C}^\times$ that is invariant under the Galois group. There are two possibilities.

a) T is split and the Galois group acts trivially. Then κ is any homomorphism of $X_*(T)$ into $\mathbf{C}^\times$. On the other hand $X_*(T)$ has no elements of norm $0, \mathfrak{D}(T)$ is trivial, and so κ restricted to $\mathfrak{D}(T)$ is also trivial.

b) T is not split. Then the action of the Galois group factors through $\mathfrak{G}(L/F) = \{1, \sigma\}$ and σ acts as -1. Thus κ is of order 2, every element is of norm 0, and

$$\mathfrak{D}(T) = X_*(T)/2X_*(T)$$

is of order 2. Since neither root $\alpha^\vee$ lies in $2X_*(T), \kappa(\alpha^\vee) \neq -1$ if and only if κ is not trivial.

The group H associated to the pair T, κ (**[12]**) is either G or T, and we shall only be interested in the case that it is T. Yet in the following discussion it is the restriction of κ to $\mathfrak{D}(T)$ which plays a role, and this is not enough to determine H. What we do is introduce a character κ' of $\mathfrak{D}(T')$ and assume that if T is not split then κ' is not trivial.

We fix Haar measures on G' and T' and let γ be a regular element in T'. If $h \in \widetilde{G}(F)$ we may transfer the measure from T' to $h^{-1}T'h$. If f is a smooth function on G' with compact support and δ is the image of h in $\mathfrak{D}(T')$ we set

$$\Phi^{\delta}(\gamma, f) = \int_{h^{-1}T'h\backslash G'} f(g^{-1}h^{-1}\gamma hg)dg.$$

We are going to introduce a function $d(\gamma)$ on the set of regular elements of T' and will set

$$\Phi^{T'\kappa'}(\gamma, f) = \Phi^{T'}(\gamma, f) = d(\gamma) \sum\nolimits_{\mathfrak{D}(T')} \kappa'(\delta)\Phi^{\delta}(\gamma, f).$$

Let γ_1 and γ_2 be the eigenvalues of γ. If T is split

$$d(\gamma) = |(\gamma_1 - \gamma_2)^2|^{1/2}/|\gamma_1\gamma_2|^{1/2}.$$

If d is not split the definition is more complicated, and requires several choices to be made. κ' may now be regarded as the non-trivial character of $F^{\times}/\mathrm{Nm}\, L^{\times}$. Let γ^0 be a fixed regular element in $\widetilde{T}(F)$ and let ψ be a fixed non-trivial additive character of F. The factor

$$\lambda(L/F, \psi)$$

has been introduced in [**13**]. Moreover, an order on the eigenvalues γ_1^0, γ_2^0 of γ^0 determines an order γ_1, γ_2 on those of γ. Set

$$d(\gamma) = \lambda(L/F, \psi)\kappa'\left(\frac{\gamma_1 - \gamma_2}{\gamma_1^0 - \gamma_2^0}\right) \frac{|(\gamma_1 - \gamma_2)^2|^{1/2}}{|\gamma_1\gamma_2|^{1/2}}.$$

Different choices of ψ and γ^0 lead either to $d(\gamma)$ once again or to $-d(\gamma)$. The change of sign is not important.

Lemma 2.1 *We may extend*

$$\gamma \to \Phi^{T'}(\gamma, f)$$

to a smooth function on T' with compact support.

What we must do is define $\Phi^{T'}(\gamma, f)$ when γ is a scalar matrix in G' and show that the resultant function is smooth in the neighbourhood of such a γ. This is not difficult but some care must be taken with the normalization of measures. The coset space $T'\backslash G'$ is open in $\widetilde{T}(F)\backslash\widetilde{G}(F)$ and the given measure on $T'\backslash G'$ defines one on $\widetilde{T}(F)\backslash\widetilde{G}(F)$. We may write

$$\Phi^{T'}(\gamma, f) = d(\gamma) \int_{\widetilde{T}(F)/\widetilde{G}(F)} f(g^{-1}\gamma g)\kappa'(\det\, g)dg.$$

It is enough to prove the lemma for one choice of the measure on $\widetilde{T}(F)\backslash\widetilde{G}(F)$. For a given regular γ, $\widetilde{T}(F)\backslash\widetilde{G}(F)$ may be identified with the orbit $O(\gamma)$ of γ on $\widetilde{G}(F)$ under conjugacy. We may assume that the measure on $\widetilde{T}(F)\backslash\widetilde{G}(F)$ is

$$|\omega_\gamma|/|\gamma_1-\gamma_2|$$

if ω is defined as in Lemma 6.1 of **[9]** and ω_γ as on p. 77 of the same paper. Then

$$\Phi^{T'}(\gamma,f)=\lambda(L/F,\psi)\kappa'\left(\frac{\gamma_1-\gamma_2}{\gamma_1^0-\gamma_2^0}\right)\frac{1}{|\gamma_1\gamma_2|^{1/2}}\int_{O(\gamma)}\epsilon(h)f(h)|\omega_\gamma|$$

if

$$\epsilon(g^{-1}\gamma g)=\kappa'(\det\ g).$$

It is understood that

$$\lambda(L/F,\psi)\kappa'\left(\frac{\gamma_1-\gamma_2}{\gamma_1^0-\gamma_2^0}\right)=1$$

if $\widetilde{T}$ is split.

If $a\in F^\times$ let

$$\gamma(a)=a\begin{pmatrix}1&1\\0&1\end{pmatrix}.$$

The form $\omega_{\gamma(a)}$ is still defined on $O(\gamma(a))$.

Definition 2.2. If $\widetilde{T}$ is split and a lies in the centre of T' set

$$\Phi^{T'}(a,f)=\frac{1}{|a|}\int_{O(\gamma(a))}f(h)dh.$$

If T' is defined by a quadratic extension L, we may regard $\widetilde{G}(F)$ as the group of invertible linear transformations of L. Then $\widetilde{T}(F)=L^\times$, the elements acting by multiplication. Choose a basis $\{1,\tau\}$ for L over F and let

$$\tau^2=u\tau+v.$$

Let $\gamma=a+b\tau$ lie in $\widetilde{T}(F)$ or $L^\times$. Its eigenvalues are then $\gamma_1=a+b\tau$, $\gamma_2=a+b\bar{\tau}$, and

$$\gamma_1-\gamma_2=b(\tau-\bar{\tau}).$$

Moreover, γ corresponds to the matrix

$$\begin{pmatrix}a&bv\\b&a+bu\end{pmatrix}.$$

If

$$g = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}$$

then

$$h = g^{-1}\gamma g = \begin{pmatrix} * & -b\mathrm{Nm}_{L/F}(b_1 + d_1 v)/\det\ g \\ b\,\mathrm{Nm}_{L/F}(a_1 + c_1\tau)/\det g & * \end{pmatrix}.$$

If

$$h = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

then

$$\kappa'\left(\frac{\gamma_1 - \gamma_2}{\gamma_1^0 - \gamma_2^0}\right) = \kappa'\left(\frac{\tau - \bar{\tau}}{\gamma_1^0 - \gamma_2^0}\right)\kappa'(c_2) = \kappa'\left(\frac{\tau - \bar{\tau}}{\gamma_1^0 - \gamma_2^0}\right)\kappa'(-b_2).$$

However, an element on $O(\gamma(a))$ has the form

$$g^{-1}\gamma(a)g = a\begin{pmatrix} * & \frac{d_1^2}{\det\ g} \\ \frac{-c_1^2}{\det\ g} & * \end{pmatrix} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

If both b_2 and c_2 are not zero then their quotient is a square. Moreover, one of them is always different from zero; so

$$\epsilon(h) = \kappa'(c_2) = \kappa'(-b_2)$$

is a well-defined function on $O(\gamma(a))$.

Definition 2.3. If $\widetilde{T}$ is not split and a lies in the centre of T' set

$$\Phi^{T'}(a, f) = \lambda(L/F, \psi)\kappa'\left(\frac{\tau - \bar{\tau}}{\gamma_1^0 - \gamma_2^0}\right)\frac{1}{|a|}\int_{O(\gamma(a))} \epsilon(h)f(h)|\omega_{\gamma(a)}|.$$

We have not mentioned it before but we identify $\widetilde{G}(F)$ with a set of linear transformations of L by choosing a vector x in F^2 and identifying L with F^2 by means of $\gamma \to \gamma x$. The standard basis of F^2 then yields a basis of L. It is understood in the above discussion that $\{1, \tau\}$ can be obtained from this basis by an element of G'.

With these definitions the function $\Phi^{T'}(\gamma, f)$ is certainly smooth when the support of f does not meet the set of scalar matrices. To prove it in general we have only to show that there exists a function $c(a)$ on $F^\times$ such that $\Phi^{T'}(\cdot, f)$ extends to a smooth function on T' which equals

$$c(a)\int_{O(\gamma(a))} \epsilon(h)f(h)|\omega_{\gamma(a)}|$$

on $F^\times$. Here $\epsilon(h)$ is to be identically 1 if $\widetilde{T}$ is split.

For split $\widetilde{T}$ this is a well-known and basic fact about orbital integrals. If F is $\mathbf{R}$ but $\widetilde{T}$ is not split it follows readily from Harish-Chandra's study of orbital integrals for real groups (cf. [**4**]). For non-archimedean fields and non-split $\widetilde{T}$ we carry out the necessary calculation.

Again we regard $\widetilde{G}(F)$ as the group of invertible linear transformations of L. At a cost of no more than a change of sign for $\Phi^{T'}(\cdot, f)$ we may suppose that $\{1, \tau\}$ is a basis over O_F of the ring of integers O_L in L. Let $\widetilde{G}(O_F)$ be the stabilizer of O_L in $\widetilde{G}(F)$. Replacing f by

$$g \to \int_{G(O_F)} f(k^{-1}gk)\kappa'(\det\ k)dk$$

if necessary, we may assume that

$$f(k^{-1}gk) = \kappa'(\det\ k)f(g), \quad k \in \widetilde{G}(O_F).$$

The calculation now proceeds along the lines of the proof of Lemma 7.3.2 of [**6**]. If ϖ is a generator of the maximal ideal of O_F then every double coset in $\widetilde{T}(F)\backslash\widetilde{G}(F)/\widetilde{G}(O_F)$ contains a g such that

$$gO_L = O_F + \varpi^m O_F \tau \quad m \geqq 0.$$

In other words it contains a representative

$$\begin{pmatrix} 1 & 0 \\ 0 & \varpi^m \end{pmatrix}, \ m \geqq 0.$$

It is clear that m is uniquely determined.

If m is unramified the index

$$\delta_m = \left[\widetilde{T}(F)\begin{pmatrix} 1 & 0 \\ 0 & \varpi^m \end{pmatrix}\widetilde{G}(O_F) : F^\times \widetilde{G}(O_F)\right]$$

is given by

$$\delta_0 = 1,$$
$$\delta_m = (q+1)q^{m-1}, m > 0.$$

If L is ramified

$$\delta_m = 2q^m.$$

Here q is the number of elements in the residue field. Moreover, apart from a constant that does not depend on f or on γ, the function $\Phi^{T'}(\gamma, f)$ is given by

$$\sum_{m=0}^{\infty} \kappa'(b\varpi^{-m})|b|\delta_m f\left(\begin{pmatrix} a & bv\varpi^m \\ b\varpi^{-m} & a+bu \end{pmatrix}\right) \tag{2.1}$$

if $\gamma = a + b\tau$.

If L is unramified then apart from a factor 2 the index δ_m is $|\varpi^{-m}|$. If $|b| = |\varpi|^N$ the above sum is twice

$$\int_{|x| \geqq |\varpi|^N} \kappa'(x) f\left(\begin{pmatrix} a & b^2 v/x \\ x & a+bu \end{pmatrix}\right) dx.$$

If γ is close to a scalar a_0 and N therefore very large then

$$|b^2 v/x| \leqq \varpi|^N |v|$$

and

$$f\left(\begin{pmatrix} a & b^2 v/x \\ x & a+bu \end{pmatrix}\right) = f\left(\begin{pmatrix} a_0 & 0 \\ x & a_0 \end{pmatrix}\right).$$

Since

(2.2) $$\int_{|x| < |\varpi|^N} \kappa'(x) dx = 0,$$

the above integral equals

$$\int \kappa'(x) f\left(\begin{pmatrix} a_0 & 0 \\ x & a_0 \end{pmatrix}\right) dx.$$

The desired assertion now follows from simple, standard integration formulae.

If L is unramified the sum (2.1) is equal to

$$\kappa'(b)|b| f\left(\begin{pmatrix} a & bv \\ b & a+bu \end{pmatrix}\right) + \left(1 + \frac{1}{q}\right) \times \sum_{m=1}^{\infty} \kappa'(b\varpi^{-m})|b\varpi^{-m}| f\left(\begin{pmatrix} a & b\varpi^m v \\ b\varpi^{-m} & a+bu \end{pmatrix}\right).$$

The second term is equal to

$$\left(1 + \frac{1}{q}\right) \int_{|x| \geq |\varpi^{-1}|} \kappa'(x) f\left(\begin{pmatrix} a & b^2 v/x \\ x & a+bu \end{pmatrix}\right) dx.$$

We argue as before except that (2.2) is replaced by

$$\int_{|x| < |b\varpi^{-1}|} \kappa'(x) dx = \sum_{m=0}^{\infty} \kappa'(b\varpi^m)|b\varpi^m| = \kappa'(b)|b|/(1 + q^{-1}).$$

There is a supplement to the lemma which we will have to take account. Suppose f is the restriction to G' of the characteristic function of $\widetilde{G}(O_F)$ divided by its measure. It is clear that

$$\phi^{T'}(\gamma, f) = 0$$

if L is ramified. If L is unramified and $\widetilde{T}(F)$ intersected with $\widetilde{G}(O_F)$, which we denote $\widetilde{T}(O_F)$, corresponds to the units of O_L then $\phi^{T'}(\gamma, f)$ is 0 unless γ is a unit, but then, if $|b| = |\varpi|^n$, it is given by

$$(\text{meas } \widetilde{T}(O_F))^{-1}\{(-1)^n q^{-n} + (1+q^{-1}\sum_{m=1}^{n}(-1)^{n-m}q^{m-n}\} = (\text{meas } \widetilde{T}(O_F))^{-1}.$$

The map

$$f \to \phi^{T'}(\cdot, f) = \phi^{T'}(f)$$

induces an adjoint map on distributions. We want to examine its effect on characters. If $\widetilde{T}$ is split then every character θ' of T' defines a principal series representation $\pi(\theta')$ of G' and it is easily seen that

$$\theta' \to \chi_{\pi(\theta')}$$

if $\chi_{\pi(\theta')}$ is the character of $\pi(\theta')$. Before considering $\widetilde{T}$ that are not split we state and prove some simple lemmas.

Lemma 2.4. *If $\tilde{\pi}$ is an irreducible admissible representation of $\widetilde{G}(F)$ then the restriction of $\tilde{\pi}$ to G' is the direct sum of finitely many irreducible representations.*

Lemma 2.5. *If π' is an irreducible admissible representation of G' then there exists an irreducible admissible representation $\tilde{\pi}$ of $\widetilde{G}(F)$ which contains π'.*

More general forms of the first lemma are known, but we are dealing with a very simple situation. We may replace G' by $F^\times G'$ and hence suppose that

$$\widetilde{G}(F)/G' \simeq \mathbf{Z}_2^n$$

with some integer n, for we have assumed that F is of characteristic 0. If the obvious induction is applied, it is enough to prove the two lemmas with $\widetilde{G}(F)$ replaced by G'', where $G' \subseteq G'' \subseteq \widetilde{G}(F)$ and

$$G''/G' \simeq \mathbf{Z}_2.$$

$\tilde{\pi}$ is replaced by π''. Suppose π'' acts on V. The restriction of π'' to G' is admissible. If it is irreducible the first lemma is valid for π''. Otherwise V contains a non-trivial invariant subspace W. If $g \in G'' - G'$ then

$$V = W + \pi''(g)W$$

and

$$W \cap \pi''(g)W$$

is invariant under G''. Thus it must be 0 and

$$V = W \oplus \pi''(g)W.$$

It is clear that G' must act irreducibly on the two summands.

To prove the second lemma we start from π' and consider

$$\sigma = \mathrm{Ind}(G'', G', \pi').$$

If $g \in G'' - G'$ and

$$h \to \pi'(g^{-1}hg), \quad h \in G',$$

is not equivalent to π' then σ is irreducible and contains π'. Otherwise if π' acts on W then

$$\pi'(g^{-1}hg) = A^{-1}\pi'(h)A, \quad h \in G'.$$

We may assume that

$$A^2 = \pi'(g^2)$$

and then extend π' to a representation π'' of G'' on W satisfying

$$\pi''(g) = A.$$

The map from W to the space of σ which takes w to

$$g \to \pi''(g)w$$

yields π'' as a component of σ. Indeed if ω is the non-trivial character of G''/G' it is clear that

$$\sigma \simeq \pi'' \oplus (\pi'' \otimes \omega).$$

Lemma 2.6. *The restriction of $\tilde{\pi}$ to G' contains no representation π' with multiplicity greater than one.*

This lemma is known for archimedean fields. We verify it only for non-archimedean. We may certainly suppose that $\tilde{\pi}$ is not one-dimensional, and hence that it is infinite-dimensional and possesses a Whittaker model [**6**].

We also assume once again that $G' \supseteq F^\times$. We start from a given infinite-dimensional π' and consider

$$\sigma = \mathrm{Ind}(\widetilde{G}(F), G', \pi').$$

By the Frobenius reciprocity law, π' is contained in a representation $\tilde{\pi}$ with the same multiplicity that $\tilde{\pi}$ is contained in σ.

Suppose ψ is a given non-trivial additive character of F and thus of

$$N(F) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in F \right\}.$$

To say that $\tilde{\pi}$ has a Whittaker model is to say that it is contained in

$$\text{Ind}(\widetilde{G}(F), N(F), \psi).$$

However

$$N(F) \subseteq G' \subseteq \widetilde{G}(F)$$

and every coset of G' in $\widetilde{G}(F)$ is represented by a matrix of the form

$$\begin{pmatrix} a & 0 \\ 0 & \beta \end{pmatrix}.$$

Since these matrices normalize $N(F)$ we infer from Lemma 2.5 that for some non-trivial character ψ' of F the representation π' is a constituent of

$$\text{Ind}(G', N(F), \psi').$$

The transitivity of induction and the uniqueness of the Whittaker model imply that $\tilde{\pi}$ is contained at most once in σ.

We associate to π' the group $G(\pi')$ of all $g \in \widetilde{G}(F)$ for which

$$h \to \pi'(g^{-1}hg) \quad h \in G'$$

is equivalent to π' and we associate to $\tilde{\pi}$ the set $X(\tilde{\pi})$ of all characters ω of $\widetilde{G}(F)/G'$ for which

$$\tilde{\pi} \otimes \omega \simeq \tilde{\pi}.$$

Any ω in $X(\tilde{\pi})$ is trivial on squares and hence of order two.

Corollary 2.7. *Suppose π' is a component of Ind$(G', N(F), \psi')$. Then it is also a component of Ind$(G', N(F), \psi'_1)$ if and only if $\psi'_1(x) \equiv \psi'(\beta x)$ for some β in $\{\det\, g | g \in G(\pi')\}$.*

If $\psi'_1(x) = \psi'(\beta x)$ and π' is realized on a space V of functions φ satisfying

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi'(x)\varphi(g)$$

and if

$$h = \begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$$

lies in $G(\pi')$ then it is also realized on

$$\{\varphi' | \varphi'(g) = \varphi(hgh^{-1}), \varphi \in V\}$$

and

$$\varphi'\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) = \psi'_1(x)\varphi'(g). \tag{2.3}$$

Conversely suppose π' is realized on V and on a space of functions satisfying (2.3). Then π' and

$$\pi'_1 : g \to \pi'(h^{-1}gh)$$

are both contained in

$$\mathrm{Ind}(G', N(F), \psi').$$

The uniqueness of the Whittaker model for $\widetilde{G}(F)$ implies that $\pi' \sim \pi'_1$ and that $h \in G(\pi')$.

Lemma 2.8. *Suppose π' is a component of $\tilde{\pi}$. The character ω belongs to $X(\tilde{\pi})$ if and only if it is trivial on $G(\pi')$. Moreover, the number of components of the restriction of $\tilde{\pi}$ to G' is $|X(\tilde{\pi})|$.*

Let the restriction of $\tilde{\pi}$ to G' be a direct sum

$$\pi'_1 \oplus \cdots \oplus \pi'_r$$

of irreducible representations with $\pi'_1 = \pi'$. If $\tilde{\pi}$ acts on V and

$$V = V_1 \oplus \cdots \oplus V_r$$

then V_1 is invariant under $G(\pi')$. If π_1 is the representation of $G(\pi')$ on V_1 then

$$\tilde{\pi} = \operatorname{Ind}(\widetilde{G}(F), G(\pi'), \pi_1).$$

Consequently

$$r = [\widetilde{G}(F) : G(\pi')]$$

and every character of $\widetilde{G}(F)/G(\pi')$ belongs to $X(\tilde{\pi})$. If ω belongs to $X(\tilde{\pi})$ and A intertwines $\tilde{\pi}$ and $\tilde{\pi} \otimes \omega$ then $A : V_i \to V_i$ and it acts as a scalar on V_i. Therefore π_1 and $\pi_1 \otimes \omega$ are not merely equivalent but in fact equal. Hence ω is trivial on $G(\pi')$.

Suppose $\tilde{\pi}$ and $\tilde{\sigma}$ are two irreducible representations of $\widetilde{G}(F)$ whose restrictions to G' contain π'. We may also decompose the restriction of $\tilde{\sigma}$ to G' into a direct sum

$$\sigma_1' \oplus \cdots \oplus \sigma_r'$$

with $r = [\widetilde{G}(F) : G(\pi')]$ and with $\sigma_1' = \pi'$. If $\tilde{\sigma}$ acts on W and σ_1' on W_1 then W_1 is invariant under $G(\pi')$ and if σ_1 is the representation of $G(\pi')$ on W_1 then

$$\sigma_1 = \omega_1 \otimes \pi_1$$

where ω_1 is a character of $G' \backslash G(\pi')$. Thus

$$\tilde{\sigma} = \operatorname{Ind}(\widetilde{G}(F), G(\pi'), \sigma_1)$$

is equivalent to $\omega \otimes \tilde{\pi}$ where ω is any extension of ω_1 from $G(\pi')$ to $\widetilde{G}(F)$. In particular if $\tilde{\pi}$ and $\omega \otimes \tilde{\pi}$ have the same restriction to G' then $\omega = \omega_1\omega_2$ with ω_1 trivial on A and $\omega_2 \otimes \tilde{\pi} \simeq \tilde{\pi}$.

Suppose ω is a character of $F^\times$ of order two and L the corresponding quadratic extension. One knows **[6]** that to each character θ of $L^\times$, of absolute value one or not, there is associated an irreducible admissible representation $\pi(\theta)$ of $\widetilde{G}(F)$. Moreover, by Lemma 5.16 of **[11]**, if ω lies in $X(\tilde{\pi})$ there is a character θ such that $\tilde{\pi} = \pi(\theta)$. If $G(\omega)$ is the kernel of ω, regarded as a character of $\widetilde{G}(F)$, then $\tilde{\pi}$ restricted to $G(\omega)$ is the direct sum of two irreducible representations. Suppose $\widetilde{T}(F)$ is a Cartan subgroup corresponding to L, ψ a non-trivial character of F, and γ^0 a regular element of $\widetilde{T}(F)$. According to Lemma 5.18 of **[11]** we may label these two components as $\pi^+(\theta)$ and $\pi^-(\theta)$ in such a way that

$$\chi_{\pi^+(\theta)}(\gamma) - \chi_{\pi^-(\theta)}(\gamma) = \lambda(L/F, \psi)\omega\left(\frac{\gamma_1 - \gamma_2}{\gamma_1^0 - \gamma_2^0}\right)\frac{\theta(\gamma) + \theta(w\gamma w^{-1})}{\Delta(\gamma)}. \tag{2.4}$$

Here w lies in the normalizer of $\widetilde{T}(F)$ but not in $\widetilde{T}(F)$ and

$$\Delta(\gamma) = |(\gamma_1 - \gamma_2)^2|^{1/2}/|\gamma_1\gamma_2|^{1/2}.$$

Let

$$\bar{\theta}(\gamma) = \theta(w\gamma w^{-1}) = \theta(\bar{\gamma}).$$

Observe in particular that $\pi(\theta_1) \ncong \pi(\theta)$ unless $\theta_1 = \theta$ or $\theta_1 = \bar{\theta}$.

Suppose $\Pi^+(\theta)$ and $\Pi^-(\theta)$ are the components of the restrictions of $\pi^+(\theta)$ and $\pi^-(\theta)$ to G'; then a simple calculation shows that the map adjoint to

$$f \to \Phi^{T'}(f)$$

takes θ to

$$\sum\nolimits_{\pi' \in \Pi^+(\theta)} \chi_{\pi'} - \sum\nolimits_{\pi' \in \Pi^-(\theta)} \chi_{\pi'}. \tag{2.5}$$

If $X(\tilde{\pi})$ has r elements then there must be $r-1$ non-isomorphic L and to each L a θ such that $\tilde{\pi} = \pi(\theta)$. The span of the distributions (2.5) together with $\sum \chi_{\pi'}$ where the sum is taken over all components of the restriction of $\tilde{\pi}$ to G' is the same as the span of the $\chi_{\pi'}$.

We say that two irreducible admissible representations π_1', π_2' of G' are L-indistinguishable if they both occur in the restriction of an irreducible, admissible representation of $\widetilde{G}(F)$ to G' or, in other words, if π_2' is equivalent to $h \to \pi_1'(g^{-1}hg)$ for some g in $\widetilde{G}(F)$. This gives a partition of the equivalence classes of irreducible admissible representations of G' into finite sets. If there is no L and θ such that π' is a component of $\pi(\theta)$ or if the quadratic character corresponding to L is not trivial on G', then the L-indistinguishable class of π' consists of π' alone. Otherwise it consists of two representations, or more precisely two equivalence classes, unless there are two different L_1, L_2 as well as θ_1, θ_2 such that G' is contained in the kernel of ω_1 and of ω_2 and π' is a component of both $\pi(\theta_1)$ and $\pi(\theta_2)$.

Indeed applying the Weyl integration to the formula (2.4) we see that

$$\sum\nolimits_{T'} ([\Omega(T')] \text{ meas } (Z'\backslash T'))^{-1} \int_{Z'\backslash T'} |\chi_{\pi^+(\theta)}(\gamma) - \chi_{\pi^-(\theta)}|^2 \Delta^2(\gamma) d\gamma$$

is equal to 4 if

$$\theta(\gamma) = \theta(w\gamma w^{-1})$$

for all γ in T' and to 2 otherwise. Here $\Omega(T')$ is the Weyl group of T' in G' and the sum is over the conjugacy classes of elliptic Cartan subgroups. Since $w\gamma w^{-1}$ corresponds to the conjugation $\gamma \to \bar{\gamma}$ we conclude from the orthogonality relations for square-integrable representations that if $\theta(\gamma) \not\equiv \theta(\bar{\gamma})$ then the L-indistinguishable class of π' consists of two elements if $\theta(\gamma) \equiv \theta(\bar{\gamma})$ on T' and of four if $\theta(\gamma) \equiv \theta(\bar{\gamma})$ on T'.

In the latter case

$$\gamma \to \theta(\gamma/\bar{\gamma})$$

is not trivial but of order two. If $B \subseteq F^\times$ is

$$\{\mathrm{Nm}_{L/F}\gamma | \theta(\gamma/\bar{\gamma}) = 1\}$$

then $[F^\times : B] = 4$. If $L_1 = L, L_2, L_3$ are the three quadratic extensions with $B \subseteq \mathrm{Nm}\ L_i$ then it is easily seen that there are characters $\theta_1 = \theta, \theta_2, \theta_3$ of $L_i^\times$ such that the three representations

$$\rho_i = \ \mathrm{Ind}(W_{L_i/F}, W_{L_i/L_i}, \theta_i)$$

of the Weil group are equivalent after inflation. Thus

$$\pi(\theta_1) = \pi(\theta_2) = \pi(\theta_3).$$

Since $[F^\times : B] = 4$ and there are only four classes L-indistinguishable from $\pi(\theta)$,

$$\pi(\theta) \sim \omega \otimes \pi(\theta)$$

only if ω is trivial on B. Thus there cannot exist a further field L_4 different from L_1, L_2, and L_3 and a θ_4 such that $\pi(\theta_1) \simeq \pi(\theta_4)$.

If $\theta(\gamma) \equiv \theta(\bar{\gamma})$ then $\pi(\theta)$ lies in the principal series and the L-indistinguishable class of π' is easily seen to consist of two elements. In general, therefore, an L-indistinguishable class consists of $1, 2$, or 4 elements.

As we have described them the sets $\Pi^+(\theta)$ and $\Pi^-(\theta)$ seem to depend on four choices, that of an additive character, a Cartan subgroup T' corresponding to the field L, an isomorphism of T' with

$$\{x \in L^\times | \mathrm{Nm}\ x \in A\}$$

and a regular element γ^0. However, the last three choices can be dispensed with, and it can be arranged that $\Pi^+(\theta)$ and $\Pi^-(\theta)$ depend only on ψ. This may not be significant. Given L we choose τ to lie in L but not in F and as above let

$$\tau^2 = u\tau + v.$$

Take the imbedding of $L^\times$ in $\widetilde{G}(F)$ that assigns to $a + b\tau$ its matrix with respect to the basis $\{1, \tau\}$, viz.

$$\begin{pmatrix} a & bv \\ b & a + bu \end{pmatrix}.$$

This fixes T' and the isomorphism. We take γ^0 to be the image of the conjugate of τ. If we replace τ by $x + y\tau$ we replace T' by $g^{-1}Tg$ where $\det g = y$ and $(\gamma_1 - \gamma_2)/(\gamma_1^0 - \gamma_2^0)$ by $y^{-1}(\gamma_1 - \gamma_2)/(\gamma_1^0 - \gamma_2^0)$. Consequently the sets $\Pi^+(\theta)$ and $\Pi^-(\theta)$ remain the same. According to the proof of Lemma 5.18 of **[11]**, the elements of $\Pi^+(\theta)$ are the constituents of the representation $\pi(\theta, \psi)$ of Theorem 4.6 of **[6]**.

It follows from the linear independence of characters that $\Pi^+(\theta)$ and $\Pi^-(\theta)$ are determined by the restriction θ' of θ to T'; it is often convenient to write $\Pi^+(\theta')$ and $\Pi^-(\theta')$. We set

$$\Pi(\theta') = \Pi^+(\theta') \cup \Pi^-(\theta').$$

If θ_1' and θ_2' are two characters of T' and $\Pi(\theta_1')$ and $\Pi(\theta_2')$ are the same, then θ_1' and θ_2' extend to characters θ_1 and θ_2 of $\widetilde{T}(F)$ or $L^\times$ with $\pi(\theta_1) \simeq \omega \otimes \pi(\theta_2)$. Here ω is a character of $G' \backslash \widetilde{G}(F)$ or of $A \backslash F^\times$. If $\omega'(\gamma) = \omega(\mathrm{Nm}\, \gamma)$ then we may replace θ_2 by $\omega'\theta_2$ and suppose $\pi(\theta_1) \simeq \pi(\theta_2)$. Then $\theta_1 = \theta_2$ or $\bar{\theta}_1 = \theta_2$. Consequently $\Pi(\theta_1') = \Pi(\theta_2')$ if and only if $\theta_1' = \theta_2'$ or $\bar{\theta}_1' = \theta_2'$.

We have described the properties of the transform

$$f \to \Phi^{T'}(f)$$

for functions with compact support. If ${}_0Z'$ is a closed subgroup of the centre of G' the transform may also be defined for functions which transform according to a character of ${}_0Z'$ and have support which is compact modulo ${}_0Z'$. It has similar properties, and they can be easily deduced from what we have already done (cf. **[11]**).

3. Stably invariant distributions. Two possible ways to define the notion of a stably invariant distribution on G' present themselves. We could define a distribution to be stably invariant if it can be approximated by finite linear combinations of the distributions

$$f \to \sum\nolimits_{\delta \in \mathfrak{D}(T')} \Phi^\delta(\gamma, f)$$

with γ regular in T' or, more naively, if it is invariant under conjugation by elements of $\widetilde{G}(F)$. The two possibilities are likely to be equivalent, but we do not trouble ourselves about this, and simply choose the second, because it is easier to work with. In this paragraph we shall examine a number of distributions that arise in the trace formula to see how far they depart from stable invariance.

Let A' be the group of diagonal matrices in G', let

$$N(F) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \big| x \in F \right\},$$

and let η be a character of A'. We do not assume that η has absolute value 1. We introduce the representation

$$\rho(\eta) : g \to \rho(g, \eta)$$

of G' on the space of smooth functions φ on $N(F)\backslash G'$ satisfying

$$\varphi(ag) = \eta(a) \left|\frac{\alpha}{\beta}\right|^{1/2} \varphi(g), \;\; a = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in A'.$$

If f is a smooth function on G' with compact support set

$$\rho(f, \eta) = \int_{G'} f(g)\rho(g, \eta)dg.$$

Lemma 3.1. *The distribution*

$$f \to \text{ trace } \rho(f, \eta)$$

is stably invariant.

If $g \in \widetilde{G}(F)$ we set

$${}^g f(h) = f(g^{-1}hg).$$

We have to show that

$$\text{trace } \rho({}^g f, \eta) = \text{ trace } \rho(f, \eta),$$

but it is enough to do this when g is diagonal because $\widetilde{G}(F) = A(F)G'$. Then

$$R_a : \varphi \to {}^a\varphi$$

maps the space on which $\rho(\eta)$ acts to itself. Since

$$\rho({}^a f, \eta)R_a = R_a \rho(f, \eta)$$

and

$$R_a^{-1} = R_{a^{-1}}$$

the lemma is clear.

If F is non-archimedean

$$K' = G' \cap \widetilde{G}(O_F)$$

where O_F is the ring of integers in F. Otherwise let K' be the group of orthogonal or unitary matrices in G'. Let A' be the group of diagonal matrices in G'. Every element of G' may be written as a product $g = nak$ with $k \in K'$,

$$a = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \in A',$$

and

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Set $\beta(g) = |\alpha/\beta|$ and $\lambda(g) = \beta(g) + \beta(wg)$ with

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then $\lambda(g) = \beta(wn)$. If γ lies in A' and has distinct eigenvalues α, β set

$$\Delta(\gamma) = |\alpha - \beta| / |\alpha\beta|^{1/2}$$

and introduce two distributions

$$F(\gamma, f) = \Delta(\gamma) \int_{A'\backslash G'} f(g^{-1}\gamma g) dg,$$

$$A_1(\gamma, f) = \Delta(\gamma) \int_{A'\backslash G'} f(g^{-1}\gamma g) \ln \lambda(g) dg.$$

In both cases the integral may be replaced by one over $\widetilde{A}(F)\backslash\widetilde{G}(F)$ because $A'\backslash G' = \widetilde{A}(F)\backslash\widetilde{G}(F)$. The first distribution is clearly stably invariant.

The second is not even invariant. However, it is invariant under $\widetilde{G}(O_F)$. Since $A'\backslash G'$ is equal to $\widetilde{A}(F)\backslash\widetilde{G}(F)$ we may regard the space of functions on which $\rho(\eta)$ acts as a space of functions on $\widetilde{G}(F)$. We extend η to a character $\tilde{\eta}$ of $\widetilde{A}(F)$ and treat $\rho(\eta)$ as the restriction of $\rho(\tilde{\eta})$. We may identify the space of $\rho(\tilde{\eta})$ with a space of functions on $\widetilde{G}(O_F)$, for the functions on which $\rho(\tilde{\eta})$ acts are determined by their values on $\widetilde{G}(O_F)$. The space of functions is the same for $\tilde{\eta}$ and for

$$\tilde{\eta}_s : a = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \to \tilde{\eta}(a) \left| \frac{\alpha}{\beta} \right|^s ;$$

and this will allow us to introduce derivatives with respect to s. If $g \in \widetilde{G}(F)$ let $kg = n_1 a_1 k_1$ and let $N(g)$ be the operator on the space of $\rho(\tilde{\eta})$ given by multiplication by $\ln \beta(a_1)$. We introduce dual measures on A' and on D^0, its Pontrjagin dual. The kernel of $\rho(f, \eta)$ is

$$K_\eta(k_1, k_2) = \int_{A'} \int_{N(F)} f(k_1^{-1} a n k_2) \lambda(a)^{1/2} da dn$$

provided the measure on K' is so chosen that

$$\int_{G'} f(g)dg = \int_{A'} \int_{N(F)} \int_{K'} f(ank)dadndk.$$

The Fourier transform of

$$\int_{K'} K_\eta(k,k)dk$$

is easily seen to be $F(\gamma, f)$. The trace of $\rho(f,\eta)N(g)$ is

$$\int_{K'} K_\eta(k,k) \ln \beta(a_1)dk.$$

Taking the Fourier transform with respect to η we obtain

$$\int_{N(F)} \int_{K'} f(k^{-1}ank)\beta(a)^{1/2} \ln \beta(a_1)da'dndk'.$$

Lemma 3.2. *The difference*

$$A_1(\gamma, {}^{g^{-1}}f) - A_1(\gamma, f)$$

is the sum of the Fourier transform of

$$\text{trace } \rho(f,\eta)N(g)$$

at γ and $\tilde{\gamma}$ where

$$\tilde{\gamma} = \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$$

if

$$\gamma = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

The difference is equal to the sum of

$$\Delta(\gamma) \int_{\widetilde{A}(F)\backslash\widetilde{G}(F)} f(h^{-1}\gamma h)\{\ln \beta(hg) - \ln \beta(h)\}dh$$

and

$$\Delta(\gamma) \int_{\widetilde{A}(F)\backslash\widetilde{G}(F)} f(h^{-1}\gamma h)\{\ln \beta(whg) - \ln \beta(wh)\}dh.$$

Replacing h by $w^{-1}h$ in the second integral, we see that it is enough to show that the first integral is the value of the Fourier transform of trace $\rho(f,\eta)N(g)$ at γ. However, if $h = nak$ then $hg = nan_1a_1k_1$ and ln $\beta(hg) -$ ln $\beta(n) =$ ln $\beta(a_1)$. Standard manipulations complete the proof.

There are still more distributions to be investigated. Let $L(1+s, 1_F)$ be the local zeta-function of F at $1+s$. If a lies in $\mathbf{Z}'$ the centre of G' and

$$n_a = a\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

set

$$\theta(a,s,f) = \frac{1}{L(1+s,1_F)}\int_{\widetilde{G}_n(F)\backslash\widetilde{G}(F)} f(g^{-1}n_a g)\beta(g)^{-s}dg.$$

We suppose that the measure on $\widetilde{G}_n(F)$, the centralizer of n, is that associated to the form

$$\frac{dz}{z}dx$$

if a typical element of $\widetilde{G}_n$ is

$$z\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

and that the measure on $\widetilde{A}(F)$ is that associated to

$$\frac{d\alpha}{\alpha}\frac{d\beta}{\beta}$$

when the typical element is

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Lemma 3.3. *The distribution*

$$\frac{d}{ds}\theta(a,0,{}^{g^{-1}}\!f) - \frac{d}{ds}\theta(a,0,f)$$

is -1 *times the Fourier transform of*

$$\text{trace }\ \rho(f,\eta)N(g)$$

at a divided by $L(1, 1_F)$.

When we take the derivative and then the difference, the term involving the derivative of the L-function drops out and we are left with

$$\frac{-1}{L(1,1_F)}\int_{\widetilde{G}_n(F)\backslash\widetilde{G}(F)} f(h^{-1}n_a h)\{\ln\beta(hg) - \ln\beta(h)\}dh$$

or

$$\frac{-1}{L(1,1_F)}\int_{\widetilde{G}_n(F)\backslash\widetilde{G}(F)} f(h^{-1}an_a h)\ln\beta(a_1)dh$$

if a_1 is defined as above. However, the integral may be written

$$\int_{\widetilde{Z}(F)\backslash\widetilde{A}(F)}\int_{\widetilde{G}(O_F)} f(k^{-1}at^{-1}n_a tk)\ln\beta(a_1)\lambda(t)dtdk.$$

Changing variables we see that this equals

$$\int_{N(F)}\int_{G(O_F)} f(k^{-1}ank)\ln\beta(a_1)dndk$$

which is, because of the normalization of measures, also

$$\int_{N(F)}\int_{K'} f(k^{-1}ank)\ln\beta(a_1)dndk.$$

The next distributions to be considered are defined by intertwining operators. We associate to η the character

$$\mu : \alpha \to \eta\left(\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\right)$$

of $F^\times$ and consider the normalized intertwining operators $R(\eta)$ on the space of $\rho(\eta)$ or $\rho(\tilde{\eta})$

$$R(\eta)\varphi(g) = \epsilon(0,\mu,\psi)\frac{L(1,\mu)}{L(0,\mu)}\int_{N(F)}\varphi(wng)dn.$$

Here ψ is some non-trivial additive character of F and $\epsilon(0,\mu,\psi)$ is the usual factor. The measure on $N(F)$ is that associated to the form dx and the character ψ. The integrals converge when $|\mu(\tilde{\omega})| < 1$ and the intertwining operator can be defined by analytic continuation for $|\mu(\tilde{\omega})| = 1$. Here $\tilde{\omega}$ is a uniformizing parameter for F. When $|\mu(\tilde{\omega})| = 1$ the operator $R(\eta)$ is invertible.

Lemma 3.4. *The difference*

$$\text{trace } R^{-1}(\eta)R'(\eta)\rho({}^{g^{-1}}f,\eta) - \text{ trace } R^{-1}(\eta)R'(\eta)\rho(f,\eta)$$

is equal to

$$\text{trace } \rho(f,\eta)N(g) + \text{ trace } \rho(f,\eta_1)N(g).$$

The prime denotes differentiation with respect to the parameter s. It will be enough to show that

$$\rho(g,\tilde{\eta}_1)R'(\eta)\rho(g,\tilde{\eta})^{-1} - R'(\eta) = R(\eta)N(g) + N(g)R(\eta).$$

Here $\tilde{\eta}_1$ is defined by

$$\tilde{\eta}_1\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right) = \tilde{\eta}\left(\begin{pmatrix}\beta & 0\\ 0 & \alpha\end{pmatrix}\right)$$

and η_1 has been defined in a similar fashion. Because we can invoke the principle of permanence of functional relations, it suffices to verify the desired relation when $|\mu(\tilde{\omega})| < 1$ and the intertwining operator is defined by the integral. In the difference the derivative of the L-functions disappear and it is therefore adequate to verify the analogous relation for the unnormalized intertwining operators

$$M(\eta)\varphi(g) = \int_{N(F)} \varphi(wng)dn.$$

Then

$$M'(\eta)\varphi(k) = \int_{N(F)} \varphi'(wnk)dn$$

where in the integrand it is understood that we start from a fixed function on $\widetilde{G}(O_F)$ and then extend it to a function in the space of $\rho(\eta)$ or $\rho(\tilde{\eta})$. The extended function will then depend on the parameter s locally, and the prime indicates that the derivative with respect to s has been taken.

If $wnk = \eta_1 a_1 k_1$ the integrand is equal to ln $\beta(a_1)\varphi(wnk)$. If $kg = n_2 a_2 k_2$ and $wnk_2 = n_1' a_1' k_1'$ then

$$\rho(g,\tilde{\eta}_1)M'(\eta)\rho(g^{-1},\tilde{\eta})\varphi(k) = \int \tilde{\eta}_1(a_2)\beta^{1/2}(a_2) \text{ ln } \beta(a_1')\varphi(wnk_2g^{-1})dn.$$

Since

$$k_2g^{-1} = a_2^{-1}n_2^{-1}k,$$

we may change variables in the integrand, replacing n by $a_2^{-1}nn_2a_2$ to obtain

$$\int \ln\beta(a_1')\varphi(wnk)dn$$

where a_1' is now defined by

$$wa_2^{-1}nn_2a_2k_2 = n_1'a_1'k_1'.$$

But the left side is $wa_2^{-1}w^{-1}wnkg$. If $k_1g = n_3a_3k_3$ then

$$a_1' = wa_2^{-1}w^{-1}a_1a_3$$

and

$$\ln\beta(a_1') = \ln\beta(a_2) + \ln\beta(a_1) + \ln\beta(a_3).$$

Since

$$M(\eta)N(g)\varphi(k) = \int_{N(F)} \ln\beta(a_3)\varphi(wnk)dn$$

and

$$N(g)M(\eta)\varphi(k) = \int_{N(F)} \ln\beta(a_2)\varphi(wnk)dn$$

the lemma follows.

Suppose χ is a given character of ${}_0Z'$, a closed subgroup of the centre of G'. If f is a function which is smooth and compactly supported modulo ${}_0Z'$ and satisfies

$$f(zg) = \chi(z)f(g), \quad z \in {}_0Z',$$

and if η is a character of A' whose restriction to ${}_0Z$ is χ^{-1} then we may define $\rho(f,\eta)$ to be

$$\int_{{}_0\mathbf{Z}'\backslash G'} f(g)\rho(g,\eta)dg.$$

We can also define distributions for the class of these functions and carry out a completely analogous discussion.

There are two more lemmas about intertwining operators that we will need for our treatment of the trace formula. If $\eta = \eta_1$ then $R(\eta)$ intertwines the representation $\rho(\eta)$ with itself.

Lemma 3.5. *If η can be extended to a character $\tilde\eta$ of $\widetilde{A}(F)$ which satisfies $\tilde\eta_1 = \tilde\eta$, then $R(\eta)$ is the identity.*

This is basically the first part* of Lemma 5.7 of **[11]**. If $\eta = \eta_1$ but η does not extend to an $\tilde\eta$ with $\tilde\eta = \tilde\eta_1$ then the character μ of $F^\times$ associated to η by

$$\mu(\alpha) = \eta\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right)$$

is quadratic but not trivial and hence defines a quadratic extension L. We associate to L a Cartan subgroup $\widetilde{T}$ of $\widetilde{G}$ with $\widetilde{T}(F) \simeq L^\times$. Its intersection with G' is then a Cartan subgroup of G' and

$$T' \simeq \{x \in L^\times \mid \mathrm{Nm} x \in A\}.$$

Define a character θ of T' by

$$\theta(x) = \eta\left(\begin{pmatrix} \mathrm{Nm} x & 0 \\ 0 & 1 \end{pmatrix}\right).$$

The elements of $\Pi(\theta)$ are the irreducible constituents of $\rho(\eta)$. Since we have chosen the additive character ψ we can speak of $\Pi^+(\theta)$ and $\Pi^-(\theta)$.

Lemma 3.6. *The trace*

$$\text{trace } R(\eta)\rho(f,\eta)$$

is equal to

$$\sum\nolimits_{\pi\in\Pi^+(\theta)} \chi_\pi(f) - \sum\nolimits_{\pi\in\Pi^-(\theta)} \chi_\pi(f).$$

Observe that the previous lemma could be formulated in the same way if we took T' to be the split Cartan subgroup, $\Pi^-(\theta)$ to be empty, and $\Pi^+(\theta)$ to be the constituents of $\rho(\eta)$. Lemma 3.6 will be proved in much the same way as Lemma 5.8 of **[11]**. Notice how important the choice of normalization is! According to the discussion of **[11]**, we replace $R(\eta)$ by $\mu(a)R(\eta)$ if we replace $\psi(x)$ by $\psi(ax)$. If $\mu(a) = -1$ this is compensated by the interchange of $\Pi^+(\theta)$ and $\Pi^-(\theta)$.

It is certainly enough to prove the lemma when

$$G' = G(\mu) = \{g \mid \mu(\det\, g) = 1\}.$$

Then $\Pi^+(\theta)$ consists of a single element π^+ and $\Pi^-(\theta)$ of π^- and

$$\rho(\tilde\eta) = \pi^+ \oplus \pi^-.$$

* Observe that Section 5 of the notes **[11]** became Section 7 when they were published.

Suppose first that F is non-archimedean. If we take the Kirillov model of $\rho(\eta)$ then its space V is the sum of the space V^+ of functions in it with support on Nm $L^\times$ and the space V^- of functions in it with support in $F^\times -$ Nm $L^\times$. π^+ acts on V^+ and π^- on V^-. Since $R(\eta)$ commutes with G' it is multiplication by a function $a + b\mu(\alpha)$. Here a and b are constants and α is a variable in $F^\times$.

If $\tilde{\eta}$ is an extension of η to $A(\widetilde{F})$ then $\tilde{\eta}$ has the form

$$\tilde{\eta}\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\right) = \nu(\alpha\beta)\mu(\beta)$$

and $R(\tilde{\eta})$ is equal to $R(\eta)$. We are going to apply formula (1) of §5 of [**11**], taking account of the differences in notation. If

$$\psi(\alpha) = W\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right)$$

is a function in V then

$$\psi(\alpha) \sim c\nu(\alpha)|\alpha|^{1/2}\{\varphi(1) + \mu(\alpha)R(\eta)\varphi(1)\}.$$

Here c is a constant, namely

$$\mu(-1)\epsilon(0, \mu, \psi).$$

As we observed the effect of applying $R(\eta)$ is to multiply by $a + b\mu(\alpha)$. On the other hand the effect on the asymptotic expression above is to change it to

$$c\nu(\alpha)|\alpha|^{1/2}\{R(\eta)\varphi(1) + \mu(\alpha)R^2(\eta)\varphi(1)\}.$$

Lemma 5.8 of [**11**] implies that $R^2(\eta)$ is the identity. We conclude that

$$\begin{aligned} a\varphi(1) + bR(\eta)\varphi(1) &= R(\eta)\varphi(1) \\ b\varphi(1) + aR(\eta)\varphi(1) &= \varphi(1) \end{aligned}$$

and hence that

$$aI + (b-1)R(\eta) = 0.$$

The operator $R(\eta)$ cannot be a scalar because $\tilde{\eta}_1 \neq \tilde{\eta}$. We conclude that $a = b - 1 = 0$. The lemma follows.

If F is archimedean then F is **R**; and the lemma is proved by evaluating the appropriate definite integrals. This is straightforward. Since it is possible, at the cost of a little additional effort, to manage without the lemma for archimedean fields, and indeed to deduce it from the global considerations, we omit the calculations.

We should remark at some point and we do it now that if θ is unramified and O_F is the largest ideal on which ψ is trivial then there is exactly one element of $\Pi(\theta)$ that contains the trivial representation of K' and it lies in $\Pi^+(\theta)$. This can be seen by examining the constructions of [**6**].

4. Quaternion algebras. We now let $\widetilde{G}(F)$ be the multiplicative group $D^\times$ of a quaternion algebra D over F and, choosing A as before with $A \subseteq \text{Nm}\ D^\times$, define G' to be

$$\{g \in \widetilde{G}(F) \mid \text{Nm} g \in A\}.$$

A Cartan subgroup of T' of G' is again of the form $G' \cap \widetilde{T}(F)$ where $\widetilde{T}$ is a Cartan subgroup of $\widetilde{G}$ over F. T' or $\widetilde{T}$ correspond to a quadratic extension L of F. Set

$$\mathfrak{E}(T') = F^\times / A\text{Nm} L^\times$$
$$\mathfrak{D}(T') = \text{Nm}\ D^\times / A\text{Nm} L^\times.$$

If F is real $\mathfrak{D}(T')$ is a proper subset of $\mathfrak{E}(T')$.

If γ in T' is regular and δ in $\mathfrak{D}(T')$ is represented by h in $\widetilde{G}(F)$ we set

$$\Phi^\delta(\gamma, \xi) = \int_{h^{-1}T'h\backslash G'} f(g^{-1}h^{-1}\gamma h g)dg$$

as before and define

$$\Phi^{T'/\kappa'}(\gamma, f) = d(\gamma, \kappa') \sum\nolimits_{\mathfrak{D}(T')} \kappa'(\delta)\Phi^\delta(\gamma, f).$$

κ' is a character of $\mathfrak{E}(T')$. If κ' is not trivial we define

$$d(\gamma) = d(\gamma, \kappa')$$

as in §2 and set

$$\Phi^{T'}(\gamma, f) = \Phi^{T'/\kappa'}(\gamma, f).$$

Let Z' be the centre of G' and extend $\Phi^{T'}(f)$ to all of T' by the equation

$$\Phi^{T'}(a, f) = 0 \quad a \in Z'.$$

Lemma 4.1. *The function $\Phi^{T'}(f)$ is smooth.*

Since $Z'\backslash G'$ is compact the function $\gamma \to \Phi^\delta(\gamma, f)$ is given by

$$\frac{1}{\text{meas }(Z'\backslash T')} \int_{Z'\backslash G'} f(g^{-1}h^{-1}\delta h g)dg$$

and is defined and smooth on all of T'. If F is $\mathbf{R}$ then $d(\gamma)$ is also smooth on all of T' and 0 on Z'. If F is non-archimedean then $\mathfrak{D}(T') = \mathfrak{E}(T')$ and

$$\sum\nolimits_{\mathfrak{D}(T')} \kappa'(\delta)\Phi^{\delta}(\gamma, f)$$

is 0 near Z'.

The dual map on distributions takes the character θ' of T' to a function Θ which satisfies

$$\Theta(h^{-1}gh) = \kappa'(\delta)\Theta(g),$$

if $h \in \widetilde{G}(F)$ represents δ, and which is supported on the union of the Cartan subgroups stably conjugate to T'. Let ω be the quadratic character of $F^{\times}$ associated to L, so that $\omega(\mathrm{Nm}h) = \kappa'(\delta)$ and let the standard involution in D be denoted by a bar. If w in $\widetilde{G}(F)$ satisfies $w^{-1}\gamma w = \tilde{\gamma}$ for γ in T' then $\omega(\mathrm{Nm}w) = -\omega(-1)$. Formal manipulations then establish that on T' the function Θ is given by

$$\Theta(\gamma) = \lambda(L/F, \psi)\omega\left(\frac{\gamma_1 - \gamma_2}{\gamma_1^0 - \gamma_2^0}\right)\frac{\theta'(\gamma) - \theta'(w^{-1}\gamma w)}{\Delta(\gamma)}.$$

In particular Θ is identically 0 if $\theta'(\gamma) \equiv \theta'(\tilde{\gamma})$, and when F is non-archimedean is 0 near Z'.

If $\tilde{\theta}$ is an extension of θ' to $\widetilde{T}(F)$ and $\pi(\widetilde{\theta})$ the associated irreducible admissible representation of $\widetilde{G}(F)$, let $\Pi(\theta')$ be components of the restriction of $\pi(\tilde{\theta})$ to G'. If $\tilde{\theta}(\gamma) \equiv \tilde{\theta}(\tilde{\gamma})$ then $\pi(\tilde{\theta})$ does not exist, but it does otherwise, and if $F = \mathbf{R}$ the set $\Pi(\theta')$ consists of a single element π. It is easily seen that

$$\chi_{\pi} = \pm\Theta.$$

As in §2 we say that two irreducible admissible representations π_1, π_2 of G' are L-indistinguishable if $\pi_2 = \pi_1^g$ with $g \in \widetilde{G}(F)$. It is clear that π_1 and π_2 are L-indistinguishable if and only if there is a representation $\tilde{\pi}$ of $\widetilde{G}(F)$ whose restriction to G' contains both π_1 and π_2. In general

$$\tilde{\pi} \mid G' \simeq \oplus_{i=1}^{r} c\pi_i$$

where $c = c(\tilde{\pi}) = c(\pi_i)$ is a positive integer and the sum is over an L-indistinguishable class. The number c may not be 1. In order to discover something about it we compare representations of $\widetilde{G}(F)$ with representations of $GL(2, F)$. We now denote $GL(2)$ by H and change the notation of §2 accordingly. Thus

$$H' = \{h \in GL(2, F) \mid \det h \in A\}.$$

Let $\tilde{\tau}$ be the representation of $\widetilde{H}(F)$ associated to $\tilde{\pi}$ and let

$$\tilde{\tau} \mid H' = \oplus_{i=1}^{s} \tau_i.$$

With no loss of generality we may assume that the central character of $\tilde{\pi}$ has absolute value 1. The character $\chi_{\tilde{\tau}}$ of $\tilde{\tau}$ on the elliptic elements is the negative of $\chi_{\tilde{\pi}}$, and $\chi_{\tilde{\pi}}$ restricted to G' and $\chi_{\tilde{\tau}}$ restricted to H' are stable. The orthogonality relations on H' state that

$$\frac{1}{2}\sum_{U'} \frac{[\mathfrak{D}(u')]}{\text{meas } Z'\backslash U'} \int_{Z'\backslash U'} |\chi_{\tilde{\tau}}(\gamma)\;|^2\;\Delta^2(\gamma) d\gamma = s.$$

The sum is over a set of representations for the stable conjugacy classes of elliptic Cartan subgroups. If F is non-archimedean the orthogonality relations on G' state that

$$\frac{1}{2}\sum_{T'} \frac{[\mathfrak{D}(T')]}{\text{meas } Z'\backslash T'} \int_{Z'\backslash T'} |\chi_{\tilde{\pi}}(\gamma)|^2 \Delta^2(\gamma) d\gamma = rc^2.$$

The sum is over a set of representatives for the stable conjugacy classes of Cartan subgroups. If F is non-archimedean and T' and U' are Cartan subgroups corresponding to the same quadratic extension then

$$[\mathfrak{D}(T')] = [\mathfrak{D}(U')].$$

If F is $\mathbf{R}$ then $[\mathfrak{D}(T)]$ is always 1 while $[\mathfrak{D}(U')]$ is 1 or 2, but then $s = [\mathfrak{D}(U')]$. The conclusion is that $rc^2 = s$ when F is non-archimedean and that $rc^2 = 1$ when F is $\mathbf{R}$. Thus when F is $\mathbf{R}$ the integers r and c are both 1. When F is non-archimedean s is $1, 2$, or 4. If s is 1 or 2 then $r = s$ and $c = 1$. If s is 4 either $r = 4$ and $c = 1$ or $r = 1$ and $c = 2$. We shall see eventually that only the latter possibility occurs, but we will need the help of the trace formula.

In order to apply the trace formula we shall need to compare distributions on G' and on H'. We set

$$\Phi^{T'/1}(\gamma, f) = \sum\nolimits_{\mathfrak{D}(T')} \Phi^{\delta}(\gamma, f).$$

If we agreed that $\Phi^{\delta}(\gamma, f) = 0$ for δ in $\mathfrak{E}(T')$ but not in $\mathfrak{D}(T')$ we could also write the sum on the right as

$$\sum\nolimits_{\mathfrak{E}(T')} \Phi^{\delta}(\gamma, f).$$

It is a stably invariant distribution and does not depend on the choice of T' within a stable conjugacy class. If we replace T' by $h^{-1}T'h$ and γ by $h^{-1}\gamma h$ with h in $\widetilde{G}(F)$ we obtain the same distribution. T' determines a stable conjugacy class $\{U'\}$ of H' and an isomorphism $\psi : T' \xrightarrow{\sim} U'$, determined up to stable conjugation. If ϕ is a smooth function on H' with compact support we may also introduce

$$\Phi^{U'/1}(\zeta, \phi) = \sum\nolimits_{\mathcal{D}(U')} \Phi^{\delta}(\zeta, \phi).$$

It follows easily from Lemma 4.1 of [**11**] that if f is given there is a ϕ satisfying

$$\Phi^{U'/1}(\zeta,\phi)\equiv 0$$

if U' is split and

$$\Phi^{U'/1}(\zeta,\phi)=\Phi^{T'/1}(\gamma,f)$$

if T' and U' are corresponding tori and $\zeta=\psi(\gamma)$. The adjoint map is only defined on stably invariant distributions.

Suppose T is an L-indistinguishable class for H'. If the elements of T belong to the principal series then

$$\sum\nolimits_{\tau\in T}\ \chi_\tau\to 0.$$

However, if the elements of T belong to the discrete series and Π is the corresponding class for G' then

$$\sum\nolimits_{\tau\in T}\ \chi_\tau\to-\sum\nolimits_{\pi\in\Pi}\ c(\pi)\chi_\pi.$$

If τ is one-dimensional and π is the corresponding one-dimensional representation of G' then $\chi_\tau\to\chi_\pi$.

5. The trace formula. F will now be a global field. Let A be a closed subgroup of $I_F=GL(1,\mathbf{A}_F)$ of the form

$$A=\prod_v A_v.$$

The product is over all places of F and A_v is a closed subgroup of $GL(1,F_v)$. We suppose in addition that $F^\times A$ is closed. If B is an open subgroup of A and $A_F=A\cap F^\times$ we also demand that $[A:A^2A_FB]<\infty$. Set

$$G'=\{g\in\widetilde{G}(\mathbf{A}_F)\mid \det\, g\in A\}.$$

Let ${}_0Z'$ be a closed subgroup of the centre Z' of G', with ${}_0Z'F^\times$ closed,

$${}_0Z'=\prod_\nu\ {}_0Z'_\nu,$$

and

$${}_0Z'Z'_F\backslash Z',\quad Z'_F=Z'\cap F^\times$$

compact. Let χ be a character of ${}_0Z'$ trivial on ${}_0Z'_F={}_0Z'\cap F^\times$ and of absolute value one.

We want to apply the trace formula to the space of measurable functions φ on $G'_F\backslash G'$, with $G'_F = \widetilde{G}(F) \cap G'$, which satisfy

$$\text{(i)}\quad \varphi(zg) = \chi^{-1}(z)\varphi(g), \qquad z \in {}_0Z',$$
$$\text{(ii)}\quad \int_{{}_0Z'G'_F\backslash G'} |\varphi(g)|^2 dg < \infty.$$

This is not the exact context in which a detailed proof has been published, but that may be in the nature of things, for it is a principle that Selberg discovered, more than a formula, and principles when they are effective are also plastic, and do not admit a definitive form. To carry out the verification of the formula with the minor modifications now required would not however be very profitable, for it would amount to little more than a transcription of [**6**]. We content ourselves with stating the result.

The space of functions φ is a direct sum of two subspaces

$$L'(G'_F\backslash G', \chi) \oplus L''(G'_F\backslash G', \chi)$$

both invariant under the action of G'. The representation of G' on the first subspace is a continuous direct integral of irreducible representations and is constructed by the Eisenstein series, while $L''(G'_F\backslash G', \chi)$ is a direct sum of irreducible representations. Let r be the representation of G' on it. Suppose

$$f : g \to \Pi f_v(g_v)$$

is a function on G'. We suppose that

$$f(zg) = \chi(z)f(g), \quad z \in {}_0Z',$$

and that its support is compact modulo ${}_0Z'$. Moreover, each f_v is to be smooth and for almost all v, f_v is to be supported on ${}_0Z'_v(G' \cap \widetilde{G}(O_{F_v}))$ and the relation

$$\text{meas } ({}_0Z'_v\backslash {}_0Z'_v(G' \cap \widetilde{G}(O_{F_v}))f(zk) = \chi(z), z \in {}_0Z'_v, k \in G' \cap \widetilde{G}(O_{F_v}),$$

is to be satisfied. We define

$$r(f) = \int_{{}_0Z'\backslash G'} f(g)r(g)dg.$$

The operator $r(f)$ is of trace class and the trace formula provides a complicated but useful expression for it.

This expression is usually presented as the sum of several parts ([**1**], [**3**], [**6**], [**11**]). The authors of [**6**] do not seem to have been able to keep a firm grip on the constants that arose in their discussion of the trace formula. We timorously suggest the following corrections: p. 516, line 2*, replace c by $c/4$; p. 531, divide the second, third, and fourth displayed expressions by 2; p. 540, lines 7, 9, and 11, replace c_1^{2s} and c_1^{2t} by c_1^s and c_1^t. The last change has then to be made in the ensuing calculations as well. However, we are going to state the formula in a slightly different situation, and will have the opportunity to make new errors all of our own. The first part is

$$\sum\nolimits_{\gamma \in {}_0Z'_F \backslash Z_{F'}} f(\gamma). \tag{5.1}$$

The second is a sum over the elliptic conjugacy classes $\{\gamma\}$ in $({}_0Z' \cap F^\times)\backslash G'_F$ of

$$\delta(\gamma)^{-1} \text{ meas } ({}_0Z'G'_F(\gamma)\backslash G'(\gamma)) \int_{G'(\gamma)\backslash G'} f(g^{-1}\gamma g)dg. \tag{5.2}$$

Here $G'(\gamma)$ and $G'_F(\gamma)$ are the centralizers of γ in G' and G'_F and $\delta(\gamma)$ is the index of ${}_0Z'\backslash {}_0Z'G'_F(\gamma)$ in the centralizer of γ in ${}_0Z'\backslash {}_0Z'G'_F$. It is 1 or 2.

As before, let A' be the group of diagonal matrices in G' and let

$$K'_v = G' \cap \widetilde{G}(O_{F_v}).$$

It is understood that $\widetilde{G}(O_{F_v})$ is to be the group of orthogonal or unitary matrices when v is real or complex. Let

$$A^0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & \beta \end{pmatrix} \in A' \;\middle|\; |\alpha| = |\beta| \right\}.$$

The group $A^0\backslash A'$ is isomorphic to $\mathbf{R}$ by means of

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \to x = \ln \left|\frac{\alpha}{\beta}\right|.$$

We choose a measure on ${}_0Z'\backslash A'$ composed of the measure dx on $A^0\backslash A'$ and a measure on ${}_0Z'\backslash A^0$ for which

$$\text{meas } {}_0Z'A'_F\backslash A^0 = 1.$$

Here A'_F is $A' \cap A(F)$. We may also suppose that this measure is given as a product measure. The next term of the trace formula is then

$$-\frac{\lambda}{2} \sum_v \sideset{}{'}\sum_{\gamma \in {}_0Z'_F\backslash A'_F} \frac{A_1(\gamma, f_v)}{L(1, F_v)} \prod_{w \neq v} \frac{F(\gamma, f_w)}{L(1, f_w)} \tag{5.3}$$

with ${}_0Z'_F = {}_0Z' \cap F^\times$. Here λ is the residue of the global L-function $L(s, 1_F)$ at $s = 1$. The factor λ and the denominators appear because of the relation between local and global Tamagawa measures.

On

$$N(\mathbf{A}) = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbf{A} \right\}$$

we choose a product measure for which

$$\text{meas } (N(F)\backslash N(\mathbf{A})) = 1.$$

The next part of the trace formula appears at first in a garb that conceals the features of concern to us. It is the constant term of the Laurent expansion of

$$\sum_{a \in {}_0Z'_F\backslash Z'_F} \sum_{x \in F^{\times 2} A_F \backslash F^\times} \text{meas } ({}_0Z'Z'_F\backslash Z') \int_{Z'N(\mathbf{A})\backslash G'} f\left(g^{-1}a \begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix} g\right) \lambda(g)^{-s} dg \tag{5.4}$$

at $s = 0$. Here $A_F = A \cap F^X$ and if

$$g = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} k, \; k \in K' = \prod K'_v,$$

then

$$\beta(g) = |\alpha/\beta|.$$

Suppose D^0 is the set of all characters of $A'_F\backslash A'$ which equal χ^{-1} on ${}_0Z'$. Another contribution to the trace formula is the sum over all η in D^0 for which $\eta = \eta_1$ of

$$-\frac{1}{4} \text{ trace } (M(\eta)\rho(f, \eta)). \tag{5.5}$$

Recall that

$$\eta_1 \left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right) = \eta \left(\begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} \right).$$

There are two more contributions. D^0 is the union of connected components, the component containing η^0 consisting of those η of the form

$$\eta : a = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \to \eta^0(a) \left| \frac{\alpha}{\beta} \right|^s$$

with s purely imaginary. The dual measure on D^0 is $1/2\pi|ds|$. If

$$\mu = \mu_\eta : \alpha \to \eta\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right)$$

then μ is a character of the idèle class group. We set

$$m(\eta) = L(1, \mu^{-1})/L(1, \mu).$$

One contribution is

$$\frac{1}{4\pi}\int_{D^0} m(\eta) \text{ trace } \rho(f, \eta)|ds|. \tag{5.6}$$

Another is

$$\sum_v \frac{1}{4\pi}\int_{D^0} \{\text{trace } R^{-1}(\eta_v)R'(\eta_v)\rho(f_v, \eta_v)\}\left\{\prod_{w\neq v} \text{trace } \rho(f_w, \eta_w)\right\}|ds|.$$

The representations $\rho(\cdot, \eta_v)$ have been defined in the third paragraph.

The distribution

$$f \to \text{ trace } r(f)$$

will not be stable. Our purpose is to write it as the sum of a stable term and a term that can be analyzed by means of the stabilized trace formula for groups of lower dimension and the principle of functoriality. The group $G = SL(2)$ is very special, and for it the lower dimensional groups are just Cartan subgroups, but one of our purposes is to illuminate the definitions of [**12**], and so we begin with them in mind, with $G' = G(\mathbf{A})$, and with ${}_0Z' = 1$.

Two elements of $G(F)$ will be called *stably conjugate** if they are conjugate in $G(\bar{F})$ or, what is the same, in $\widetilde{G}(F)$. We take the expression (5.2) and sum over the conjugacy classes within the stable conjugacy class of γ. If we introduce the global form of the notation of the second paragraph, this sum may be written

$$\text{meas } (T(F)\backslash T(\mathbf{A}))\sum\nolimits_{\mathfrak{D}(T/F)} \Phi^\delta(\gamma, f) \tag{5.7}$$

with $T = G(\gamma)$. With the present assumptions $\delta(\gamma)$ is 1.

* Prudence must be exercised when transferring this notion to other groups.

If $\mathfrak{D}(T/\mathbf{A})$ is the direct sum of the $\mathfrak{D}(T/F_v)$ we have a map

$$\mathfrak{D}(T/F) \to \mathfrak{D}(T/\mathbf{A}).$$

There are a number of simplifying factors in the present circumstances. First of all

$$\mathfrak{D}(T/F) = \mathfrak{E}(T/F) = H^1(F, T)$$

and

$$\mathcal{D}(T/\mathbf{A}) = \mathfrak{G}(T/\mathbf{A}) = \oplus H^1(F_v, T).$$

Moreover

$$H^1(F, T) \to \oplus H^1(F_v, T)$$

is injective. If L is the quadratic extension defined by T and C_L the idèle class group of L then

$$H^1(F, T) \to \oplus H^1(F_v, T) \to H^1(\mathfrak{E}(L/F), X_*(T) \otimes C_L) \tag{5.8}$$

is exact. Since T is not split the Tate-Nakayama theory shows that the last group is

$$H^{-1}(\mathfrak{G}(L/F), X_*(T)) = X_*(T)/\{\Sigma\sigma\lambda - \lambda \mid \lambda \in X_*(T), \sigma \in \mathfrak{G}(L/F)\}.$$

There are analogues of these statements for general groups. A special feature of $G = SL(2)$ is that the second arrow of (5.8) is surjective. Thus the dual of $\mathfrak{E}(T/F)\backslash\mathfrak{E}(T/\mathbf{A})$ is isomorphic to the set of $\mathfrak{G}(L/F)$-invariant homomorphisms κ of $X_*(T)$ into $\mathbf{C}^\times$.

The sum (5.7) is equal to

$$\text{meas } (T(F)\backslash T(\mathbf{A}))[\mathfrak{E}(T/\mathbf{A}) : \mathfrak{E}(T/F)]^{-1} \sum\nolimits_\kappa \sum\nolimits_{\mathfrak{E}(T/\mathbf{A})} \kappa(\delta)\Phi^\delta(\gamma, f).$$

According to the definitions of **[12]**, a group H is associated to the pair T, κ. If κ is trivial this group is just the quasi-split form of G, namely G itself, and

$$f \to \sum\nolimits_{\mathfrak{E}(T/\mathbf{A})} \Phi^\delta(\gamma, f)$$

is a stable distribution.

If κ is the non-trivial homomorphism, and there is only one of them, the group H is T. If w is a divisor of v in L then by the Tate-Nakayama theory

$$H^1(F_v, T) = \{\lambda \in X_*(T) | \sum\nolimits_{\mathfrak{G}(L_w/F_v)} \sigma\lambda = 0\}/\{\Sigma\sigma\lambda - \lambda | \lambda \in X_*(T), \sigma \in \mathfrak{G}(L_w/F_v)\}$$

and the map

$$H^1(F_v, T) \to H^1(\mathfrak{G}(L/F), X_*(T) \otimes C_E)$$

is simply $\lambda \to \lambda$. In particular κ determines a homomorphism κ_v of $\mathfrak{D}(T/F_v)$ into $\mathbf{C}^\times$. It is trivial if v splits and non-trivial otherwise. Choose γ^0 in $\widetilde{T}(F)$ and a non-trivial character of $F\backslash\mathbf{A}$. Then

$$\prod_v \lambda(L_v/F_v, \psi_v)\kappa_v \left(\frac{\gamma_1 - \gamma_2}{\gamma_1^0 - \gamma_2^0}\right) \frac{|(\gamma_1 - \gamma_2)^2|_v^{1/2}}{|\gamma_1\gamma_2|_v^{1/2}} = 1$$

and

$$\sum\nolimits_{\mathfrak{G}(T/\mathbf{A})} \kappa(\delta)\Phi^\delta(\gamma, f) = \Pi_v \Phi^{T/\kappa_v}(\gamma, f_v)$$

and

$$\text{meas } (Z'T(F)\backslash T(\mathbf{A}))\Pi_v \Phi^{T/\kappa_v}(\gamma, f_v)$$

is one term of the trace formula for the abelian pair $Z'T(F)\backslash T(\mathbf{A})$ and the function

$$\gamma \to \Pi_v \Phi^{T/\kappa_v}(\gamma, f_v) = \Phi^{T/\kappa}(\gamma, f)$$

on $T(\mathbf{A})$.

For each isomorphism class of quadratic extension we choose a torus T, in other words we choose a set of representatives for the stable conjugacy classes of non-split tori. We may choose the representatives γ of the stable conjugacy classes arising from (5.2) to lie in one of these T. The stable conjugacy class determines T, but within T there are two possible representatives for the stable conjugacy classes, and so we must divide by 2 if we want to sum over all regular elements in $T(F)$. Since the index $[\mathfrak{E}(T/\mathbf{A}) : \mathfrak{E}(T/F)]$ is also 2, the total unstable contribution from (5.2) is $\frac{1}{4}$ times the sum over T of

(5.9) $$\text{meas } (T(F)\backslash T(\mathbf{A})) {\sum}' \Phi^{T/\kappa}(\gamma, f).$$

The prime indicates that we only sum over the regular elements γ of $T(F)$. We almost have what we were seeking. We must now hope that when we remove the stable part from the remaining terms of the trace formula we will be left with the summands missing from (5.9), namely

$$\frac{1}{4} \sum\nolimits_T \text{ meas } (T(F)\backslash T(\mathbf{A})) \sum\nolimits_{\gamma \in Z(F)} \Phi^{T/\kappa}(\gamma, f).$$

Here Z is the centre of G and $Z(F)$ consists of two elements.

We have preferred to treat not merely the group $G(\mathbf{A})$ but the groups G' as well, and we carry out the complete discussion for them. We examine the terms (5.2) once again. The group

$G'_F(\gamma)$ is the intersection T'_F of G'_F with a torus $\widetilde{T}$ in $\widetilde{G}$. If $\widetilde{T}$ is associated to the quadratic extension L we define

$$\mathfrak{E}(T'/F) = F^\times/A_F \mathrm{Nm}_{L/F} L^\times, \quad A_F = A \cap F^\times,$$

and

$$\mathfrak{E}(T'/\mathbf{A}) = I_F/A\mathrm{Nm}_{L/F} I_L,$$

where I_F and I_L are the idèles of F and L. It is also useful to introduce the subsets $\mathfrak{D}(T/F)$ and $\mathfrak{D}(T/\mathbf{A})$ whose elements can be realized as norms from $\widetilde{G}$, but only for groups defined by quaternion algebras is this necessary. The natural map from $\mathfrak{E}(T'/F)$ to $\mathfrak{E}(T'/\mathbf{A})$ has a cokernel of order 1 or 2. Let μ be the order of its kernel.

If the cokernel is trivial and we sum over the stable conjugacy classes within the conjugacy class of γ we obtain

$$\mu\delta(\gamma)^{-1} \text{ meas } ({}_0Z'G'_F(\gamma)\backslash G'(\gamma)) \sum\nolimits_{\delta\in\mathfrak{D}(T'/\mathbf{A})} \Phi^\delta(\gamma, f),$$

which is a stable distribution. If it is not trivial the contribution of the stable conjugacy class of γ to the trace formula is $\frac{1}{2}$ the sum of this expression and

$$\mu\delta(\gamma)^{-1} \text{ meas } ({}_0Z'G'_F(\gamma)\backslash G'(\gamma)) \sum\nolimits_{\mathfrak{D}(T'/\mathbf{A})} \kappa'(\delta)\Phi^\delta(\gamma, f).$$

Here κ' is the non-trivial character of $\mathrm{Im}(T'/\mathbf{A})/\mathfrak{E}(T'/F)$. The stable conjugacy class of γ has two representatives within a given T' modulo ${}_0Z'$ if and only if $\delta(\gamma) = 1$. Thus if we choose a set of representatives T' for the stable conjugacy classes of Cartan subgroups with

$$[\mathfrak{E}(T'/\mathbf{A}) : \text{ Im } \mathfrak{E}(T'/F)] = 2$$

the total labile contribution to the trace formula of the sum over the terms (5.2) is

$$\frac{1}{4} \sum\nolimits_{T'} \mu \text{ meas } ({}_0Z'T'_F\backslash T') \sum\nolimits' \Phi^{T'}(\gamma, f). \tag{5.10}$$

The inner sum is over the regular elements of T'_F modulo ${}_0Z'_F$. For the T' occurring in this sum

$$I_F \neq F^\times A\mathrm{Nm}\, I_L.$$

Since $A = \prod A_v$ this is possible only if $A \subseteq \mathrm{Nm}\, I_L$ and then the formula given below shows that μ, which seems to depend on T, is in fact 1. But we prefer not to make use of this until it is necessary, and perhaps not at all.

The missing terms in (5.10) will be extracted from (5.4). It is convenient to perform a sequence of modifications of (5.4) and (5.10) before comparing them. We write

$$\text{meas}\ ({}_0Z'T'_F\backslash T') = \text{ meas}\ ({}_0Z'Z'_F\backslash Z')\ \text{meas}\ (Z'T'_F\backslash T').$$

Since meas $({}_0Z'Z'_F\backslash Z')$ is common to (5.4) and (5.10) it will be ignored. Suppose $B = AI_{F^2}$ and let

$$G'' = \{g \mid \det\ g \in B\}.$$

The various objects associated to G'' will be denoted in the same way as those associated to G', except that the prime will be doubled. Since $A''\backslash G'' = Z'\backslash G'$ we may take the measures on the two spaces to be the same and replace the space of integration in (5.4) by $Z''N(\mathbf{A})\backslash G''$. The sum over x appearing there may be replaced by $[B_F : A_F F^{X^2}]$ times a sum over $B_F\backslash F^\times$.

We want to replace G' by G'' and T' by T'' in (5.10). Since $T''\backslash G'' = T'\backslash G'$ and $Z''\backslash T'' = Z'\backslash T'$ this is certainly possible. However

$$\text{meas}\ (Z''T''_F\backslash T'') = [T''_F : T'_F F^{\times^2}]\ \text{meas}\ (Z''T''_F\backslash T'')$$

and if T' corresponds to the quadratic extension L,

$$\mu = \mu(T') = [F^\times \cap A\ \text{Nm}\ I_L : A_F\ \text{Nm}\ L^\times]$$

while

$$\mu(T'') = [F^\times \cap B\ \text{Nm}\ I_L : B_F\ \text{Nm}\ L^\times].$$

However

$$A\ \text{Nm}\ I_L = B\ \text{Nm}\ I_L;$$

so that

$$\mu(T')[T''_F : T'_F F^{\times^2}] = \mu(T'')[B_F\ \text{Nm}\ L^\times : A_F\text{Nm}L^\times][B_F \cap \text{Nm}L^\times : (A_F \cap \text{Nm}L^\times F^{\times^2}].$$

The middle factor equals

$$[B_F : B_F \cap A_F\ \text{Nm}\ L^\times].$$

Moreover, the map

$$B_F \cap\ \text{Nm}\ L^\times \to B_F \cap A_F\ \text{Nm}\ L^\times / A_F F^{\times^2}$$

is surjective with kernel

$$A_F F^{\times^2} \cap\ \text{Nm}\ L^\times = (A_F \cap\ \text{Nm}\ L^\times) F^{\times^2}.$$

Thus the product of the last two factors is $[B_F : A_F F^{\times^2}]$. If we disregard this integer, which is common to (5.4) and (5.10), we may replace G' by G''. In order not to burden the notation we now assume that A contains I_{F^2} and that G' is G''.

The symbols G'' and B are now free again and we set $B = AF^\times$ and define G'' in terms of B as before. The quotient $Z'\backslash G'$ is open in $Z''\backslash G''$ and we choose the invariant measure on $Z''\backslash G''$ so that it restricts to that on $Z'\backslash G'$. We may drop the sum over x from (5.4) provided we now integrate over $Z''N(\mathbf{A})\backslash G''$. The space $T'\backslash G'$ is the same as $T''\backslash T''G'$, which is open in $T''\backslash G''$, and we choose the measures to be compatible. Then $\mathfrak{D}(T''/F) = \{1\}$ and

$$\sum\nolimits_{\mathfrak{D}(T'/F)} \Phi^\delta(\gamma, f)$$

is equal to

$$\int_{T''\backslash G''} f(g^{-1}\gamma g)dg.$$

We now have no choice for the measure on $Z''\backslash T''$. It must be compatible with that on the open subset $Z'\backslash T'$.

Since

$$[T''_F \cap T' : T'_F] = [A_F F^\times \cap A : A_F] = 1,$$

we have

$$\text{meas } (Z''T''_F\backslash T'') = [T'' : T''_F T'] \text{ meas } (Z'T'_F\backslash T').$$

Moreover

$$[T'' : T''_F T'] = [AF^\times \cap \text{ Nm } I_L : (AF^\times \cap \text{ Nm } L^\times)(A \cap \text{ Nm } I_L)]$$

which may be simplified to

$$[AF^\times \cap \text{ Nm } I_L : \text{ Nm } L^\times(A \cap \text{ Nm } I_L)].$$

We claim that this index equals

$$\mu(T') = [F^\times \cap A \,\text{Nm } I_L : A_F \text{ Nm } L^\times].$$

Suppose u in Nm I_L equals xy with $x \in A, y \in F^\times$. Then y lies in $F^\times \cap A \,\text{Nm } I_L$. If $u = vw$ with $v \in \text{ Nm } L^\times, w \in A \cap \text{ Nm } I_L$, then $y = (x^{-1}w)v$ and $x^{-1}w = v^{-1}y$. Clearly $x^{-1}w$ lies in A_F. Conversely, if $y \in F^\times \cap A \,\text{Nm } I_L$ then we may find $u \in \text{ Nm } I_L, x \in A$ with $u = xy$. If $y = zv$ with $z \in A_F, v \in \text{ Nm } L^\times$ then $u = v(xz)$ and $v \in \text{ Nm } L^\times, xz \in A \cap \text{ Nm } I_L$. The conclusion is that we may suppose that A contains both $F^\times$ and I_{F^2}.

The quotient $G'\backslash\widetilde{G}(A)$ is now compact and abelian. We take its measure to be 1 and write the integral of (5.4) as a sum over its characters

$$\sum_{\kappa}\int_{I_F N(\mathbf{A})\backslash\widetilde{G}(\mathbf{A})}\kappa(\det\, g)f\left(g^{-1}a\begin{pmatrix}1&1\\0&1\end{pmatrix}g\right)\beta(g)^{-s}dg. \tag{5.11}$$

It should perhaps be stessed that for a given f only finitely many terms of this sum are not 0. By global class field theory each non-trivial κ occurring in this sum determines a quadratic extension of F and the quadratic extensions so obtained are precisely those for which

$$[\mathfrak{E}(T'/\mathbf{A}) : \text{ Im } \mathfrak{E}(T'/F)] = 2$$

when T' is the corresponding Cartan subgroup of G'. Since Z'_F is just the set of singular elements in T'_F and since there is in (5.4) a sum over ${}_0Z'_F\backslash Z'_F$, we can hope that the term of (5.11) corresponding to κ is just the missing term in (5.10) corresponding to a and the T' defined by κ. The expression

$$\frac{1}{4}\text{ meas }(Z'T'_F\backslash T')\sum\nolimits_{\mathfrak{D}(T'/\mathbf{A})}\kappa'(\delta)\Phi^{\delta}(\gamma,f) = \frac{1}{4}\text{ meas }(Z'T'_F\backslash T')\Phi^{T'}(\gamma,f)$$

may be written

$$\frac{1}{2}\text{ meas }(I_F\widetilde{T}(F)\backslash\widetilde{T}(\mathbf{A}))\int_{\widetilde{T}(\mathbf{A})\backslash\widetilde{G}(\mathbf{A})}f(g^{-1}\gamma g)\kappa(\det\, g)dg$$

if we so normalize measures that

$$\text{meas }(T'\backslash\widetilde{T}(\mathbf{A})) = 1.$$

We take κ to be non-trivial and write the corresponding integral in (5.11) as a product

$$\prod_{v}\int_{F_v^{\times}N(F_v)\backslash\widetilde{G}(F_v)}\kappa_v(\det\, g)f_v\left(g^{-1}a\begin{pmatrix}1&1\\0&1\end{pmatrix}g\right)\beta(g)^{-s}dg. \tag{5.12}$$

Suppose κ_v is unramified, and f_v is supported on ${}_0Z'_vK'_v$ and satisfies

$$f(zk) = \chi(z)f(k).$$

Then the local integral is equal to

$$\text{meas }(F_v^{\times}N(F_v)\cap\widetilde{G}(O_{F_v})\backslash\widetilde{G}(O_{F_v})f(a)\sum\nolimits_{n=0}^{\infty}\kappa_v(\varpi_v^n)|\varpi_v^n|^{1+s}$$

and the sum is equal to $L(1+s,\kappa_v)$. Since the global L-function

$$\Pi_v L(1+s,\kappa_v)=L(1+s,\kappa)$$

is regular at $s=0$ the constant term of the Laurent expansion of (5.12) at $s=0$ is

$$L(1,\kappa)\prod_v \frac{1}{L(1,\kappa_v)}\int_{F_v^\times N(F_v)\backslash\widetilde{G}(F_v)}\kappa_v(\det\, g)f_v\left(g^{-1}a\begin{pmatrix}1&1\\0&1\end{pmatrix}g\right)dg.$$

We have fixed the measure on $N(\mathbf{A})$ but not on $\widetilde{T}(\mathbf{A})$ or on $I_F\backslash\widetilde{T}(\mathbf{A})$. Since the results are independent of this choice we may pick any measure that is convenient for the completion of the computations. If $\widetilde{Z}$ is the centre of $\widetilde{G}$ then $I_F\backslash\widetilde{T}(\mathbf{A})$ is $\widetilde{Z}\backslash\widetilde{T}(\mathbf{A})$ and we take the unnormalized Tamagawa measure (**[9]**, §6). The measure on $N(\mathbf{A})$ may also be taken to be the unnormalized Tamagawa measure for it has the desired property that

$$\text{meas }(N(F)\backslash N(\mathbf{A}))=1.$$

The lattice of characters of $\widetilde{Z}\backslash\widetilde{T}$ is **Z** with the action of $\mathfrak{G}(L/F)=\{1,\sigma\}$ given by $\sigma z=-z$. With this action

$$[H^1(\mathfrak{G}(L/F),\mathbf{Z})]=2.$$

Moreover, the kernel of

$$H^1(F,\widetilde{Z}\backslash\widetilde{T})\to\Pi_v H^1(F_v,\widetilde{Z}\backslash\widetilde{T})$$

is trivial. Thus by a general theorem of Ono on the Tamagawa number of a torus

$$\frac{1}{2}\text{ meas }(I_F\widetilde{T}(F)\backslash\widetilde{T}(\mathbf{A}))=L(1,\kappa).$$

Finally

$$\int_{\widetilde{T}(\mathbf{A})\backslash\widetilde{G}(\mathbf{A})}f(g^{-1}\gamma g)\kappa(\det\, g)dg=\prod_v\int_{\widetilde{T}(F_v)\backslash\widetilde{G}(F_v)}f_v(g^{-1}\gamma g)\kappa_v(\det\, g)dg.$$

We have seen in §2 that if $a\in Z_v'$ then

$$\lim_{\gamma\to a}\int_{\widetilde{T}(F_v)\backslash\widetilde{G}(F_v)}f_v(g^{-1}\gamma g)\kappa_v(\det\, g)dg$$

is equal to a constant c_v times

$$\frac{1}{|a|_v L(1,\kappa_v)}\int_{F_v^\times N(F_v)\backslash\widetilde{G}(F_v)}\kappa_v(\det\, g)f_v\left(g^{-1}a\begin{pmatrix}1&1\\0&1\end{pmatrix}g\right)dg.$$

All we need do is show that c_v is 1 for almost all v and that

$$\prod_v c_v = 1.$$

We define the local Tamagawa measure on N by means of the form dx if

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

and on $\widetilde{Z}\backslash\widetilde{T}$ as the quotient of

$$\frac{1}{\gamma_1^0 - \gamma_2^0} \frac{d\gamma_1}{\gamma_1} \frac{d\gamma_2}{\gamma_2}$$

by

$$\frac{dz}{z}.$$

However, we might as well use a more natural Tamagawa measure locally, that obtained by suppressing the factor $L(1, \lambda_v)$ from the definition in [**9**]. We must then suppress the factor $L(1, \kappa_v)$ in the definition of c_v as well. We may assume that the form η on

$$A = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \right\}$$

appearing in Lemma 6.1 of [**9**] is

$$\frac{d\alpha}{\alpha} \frac{d\beta}{\beta}$$

and that the measure on $\widetilde{Z}$ is given by the form dz/z. Then Lemma 2.1 together with Definitions 2.2 and 2.3 and the discussion following the aforementioned Lemma 6.1 show that

$$c_v = |(\gamma_1^0 - \gamma_2^0)^2|^{1/2}.$$

We have still to consider the contribution to (5.4) from the trivial character as well as the remaining terms of the trace formula, but we first examine the contribution

$$\frac{1}{4} \; \sum\nolimits_{T'} \mu(T') \text{ meas } ({}_0Z'T'_F\backslash T')\Sigma \; \phi^{T'}(\gamma, f)$$

more closely. Applying the trace formula to the pairs T'_F, T' we see that this equals

$$\frac{1}{4} \; \sum\nolimits_{T'} \mu(T') \sum\nolimits_{\theta} \langle \theta, \Phi^{T'}(f) \rangle$$

where θ runs through all characters of $T'_F\backslash T'$ which equal $\kappa\chi^{-1}$ on ${}_0Z'$. Appealing to the discussion of §2 we transform this to

$$\frac{1}{4}\ \sum\nolimits_{T'} \mu(T') \sum\nolimits_{\theta} \Pi_v\big(\sum\nolimits_{\pi_v\in\Pi^+(\theta_v)} \chi_{\pi_v}(f_v) - \sum\nolimits_{\pi_v\in\Pi^-(\theta_v)} \chi_{\pi_v}(f_v)\big).$$

It should perhaps be stressed that for almost all v the trace $\chi_{\pi_v}(f_v)$ is 0 when $\pi_v \in \Pi^-(\theta_v)$. Moreover, when v splits, $\Pi^-(\theta_v)$ is empty and $\Pi^+(\theta_v)$ consists of the components of the principal series defined by θ_v.

We distinguish three types of θ. Let $\gamma \to \bar\gamma$ be the automorphism of T' or of $\widetilde{T}(\mathbf{A})$ corresponding to conjugation on the corresponding quadratic field. We first introduce a provisional classification.

a_1) On T'

$$\theta(\gamma/\bar\gamma) \not\equiv 1.$$

b_1) On T'

$$\theta(\gamma/\bar\gamma) \equiv 1$$

but θ cannot be extended to a character $\tilde\theta$ of $\widetilde{T}(F)\backslash\widetilde{T}(\mathbf{A})$ satisfying

$$\tilde\theta(\gamma) \equiv \tilde\theta(\bar\gamma).$$

c_1) θ can be extended to a character of $\widetilde{T}(F)\backslash\widetilde{T}(\mathbf{A})$ satisfying this identity.

Suppose there are two elliptic Cartan subgroups T'_1, T'_2 which are not stably conjugate and two characters θ^1 and θ^2 for which

$$\Pi(\theta^i) = \{\otimes\ \pi_v \mid \pi_v \in \Pi(\theta^i_v) = \Pi^+(\theta^i_v) \cup \Pi^-(\theta^i_v)\}$$

are the same. It is understood that in each product π_v contains the trivial representation of K'_v for almost all v. Suppose also that

$$[\mathfrak{E}(T'_i/\mathbf{A}) : \text{Im } \mathfrak{E}(T'_i/F)] = 2. \tag{5.13}$$

Extend θ^i to a character $\tilde\theta^i$ of $\widetilde{T}(F)\backslash\widetilde{T}(\mathbf{A})$. Then $\pi(\tilde\theta^1_v)$ and $\pi(\tilde\theta^2_v)$ determine the same L-indistinguishable class of $G(F_v)$ for all v. We may regard $\tilde\theta^i$ as a character of $L_i^\times\backslash I_{L_i}$ and consider

$$\rho^i = \text{ Ind } (W_{L_i/F}, W_{L_i/L_i}, \tilde\theta^i).$$

When we compose ρ^i with

$$GL(2,\mathbf{C}) \to PGL(2,\mathbf{C}) \to GL(3,\mathbf{C})$$

we obtain three-dimensional representations which are locally equivalent everywhere and hence, by Lemma 12.3 of [**6**], equivalent. It is easy to see that this is possible only if

$$\gamma \to \tilde{\theta}^i(\gamma/\bar{\gamma})$$

is of order 2 but not trivial. In addition $F^\times \mathrm{Nm} I_{L_2}$ must contain

$$F^\times\{\mathrm{Nm}_{L_1/F}\ x|\theta^1(x/\bar{x}) = 1\}.$$

Indeed this set must be

$$F^\times \mathrm{Nm} I_{L_1} \cap F^\times \mathrm{Nm} I_{L_2}.$$

It follows from (5.13) that T_1, θ^1 must be of type (b_1).

Conversely if the pair (T_1, θ^1) is of type (b_1) and θ^1 lifts to $\tilde{\theta}^1$ then

$$\gamma \to \theta^1(\gamma/\bar{\gamma})$$

is of order 2. There are exactly three different quadratic extensions L_1, L_2, L_3 of F for which

$$F^\times \mathrm{Nm} I_{L_i} \supseteq F^\times\{\mathrm{Nm}_{L_1/F}\ x|\theta^1(x/\bar{x}) = 1\}.$$

There are also three characters $\tilde{\theta}^1, \tilde{\theta}^2$, and $\tilde{\theta}^3$ such that the representations

$$\rho^i = \text{ Ind } (W_{L_i/F}, W_{L_i/L_i}, \tilde{\theta}^i)$$

become equivalent upon inflation. Thus

$$\Pi(\theta^1) = \Pi(\theta^2) = \Pi(\theta^3),$$

if θ^i is the restriction of $\tilde{\theta}^i$ to T^i. We say that (T_1, θ^1) is of type (b_1') if

$$A \subseteq \cap_i\ F^\times \mathrm{Nm} I_{L_i}$$

and of type (b_1'') otherwise.

The final classification is:

$$\begin{aligned}
\text{type (a)} &\Leftrightarrow \text{type } (a_1) \text{ or type } (b_1'');\\
\text{type (b)} &\Leftrightarrow \text{type } (b_1');\\
\text{type (c)} &\Leftrightarrow \text{type } (c_1).
\end{aligned}$$

If θ is of type (c) and extends to $\tilde{\theta}$ with $\tilde{\theta}(\gamma) \equiv \tilde{\theta}(\bar{\gamma})$ then

$$\tilde{\theta}(\gamma) = \omega(\mathrm{Nm}\,\gamma)$$

and the elements of $\Pi(\theta)$ are the components of the principal series associated to the character

$$\eta : \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \to \omega(\alpha\beta)\kappa(\beta)$$

of A'. Here κ is the quadratic character associated to the quadratic extension defined by T'. We do not expect to see them in the discrete spectrum or in the stable or labile part of the trace formula, and it turns out that the corresponding contribution to (5.14) is cancelled by the labile part of (5.5).

The operator $M(\eta)$ appearing in (5.5) is equal to

$$\frac{L(1,\mu)}{L(1,\mu^{-1})} \otimes_v R(\eta_v).$$

Here μ is of order 2. If it is trivial then the quotient $L(1,\mu)/L(1,\mu^{-1})$, which is defined as a limit, is equal to -1. Since each $R(\eta_v)$ is then the identity, the corresponding contribution to the trace is stable. If μ is not trivial, the quotient is 1 and by Lemmas 3.5 and 3.6

$$\prod_v \text{trace } R(\eta_v)\rho(f,\eta_v)$$

is equal to

$$\prod_v \Big(\sum\nolimits_{\pi_v \in \Pi^+(\theta_v)} \chi_{\pi_v}(f_v) \; - \sum\nolimits_{\pi_v \in \Pi^-(\theta_v)} \chi_{\pi_v}(f_v)\Big).$$

Observe that class field theory associates to μ a quadratic extension L. T' is the corresponding Cartan subgroup and θ is defined by

$$\theta(x) = \eta\left(\begin{pmatrix} \mathrm{Nm}\,x & 0 \\ 0 & 1 \end{pmatrix}\right).$$

For α in A,

$$\mu(\alpha) = \eta\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right)\eta_1\left(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}\right) = 1$$

because $\eta = \eta_1$. Thus $A \subseteq F^\times \mathrm{Nm} I_L$ and

$$[\mathfrak{E}(T'/\mathbf{A}) : \text{ Im } \mathfrak{G}(T'/F)] = 2.$$

In order to verify that the expected cancellation occurs we have to check that if θ extends to a character $\tilde{\theta}$ of $\widetilde{T}(F)\backslash\widetilde{T}(\mathbf{A})$ with $\tilde{\theta}(x) \equiv \tilde{\theta}(\bar{x})$ then there are exactly $\mu(T')$ characters η with $\eta = \eta_1$ yielding the pair (T, θ). If $\alpha \in \text{ Nm } I_L \cap A$ we must have

$$\eta\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) = \theta(\beta), \quad \alpha = \text{ Nm } \beta,$$

and we must have

$$\eta\left(\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}\right) = \kappa(\alpha)$$

if κ is the quadratic character associated to T'. Then

$$\eta(z) = \kappa(z)\theta(z), \quad z \in Z'.$$

In particular η is equal to χ^{-1} on ${}_0Z'$. The number of possibilities for η is therefore

$$[A : A_F(\text{Nm } I_L \cap A)].$$

However, A is contained in $F^\times \text{Nm} I_L$ and if we write $a = xy$ then $a \to x$ yields an injection of $A/A_F(\text{Nm}I_L \cap A)$ into $F^\times/A_F\text{Nm}L^\times$. The image is

$$F^\times \cap A\text{Nm}I_L/A_F\text{Nm}L^\times$$

and the order of this group is $\mu(T')$.

The terms (5.1) and (5.6) are clearly stably invariant. We shall show that the last term of the trace formula combines with (5.3) and the remaining part of (5.4) to give a stably invariant distribution. Denote these three terms by S_1, S_2, S_3. We take $g = \prod g_v$ in $\widetilde{G}(\mathbf{A})$ and show that

$$\sum_{i=1}^{3} (S_i^g - S_i) = 0.$$

According to Lemma 3.4 the distribution $S_1^{g^{-1}} - S_1$ is given by

$$\sum_v \frac{1}{2\pi} \int_{D^0} \{\text{trace } \rho(f_v, \eta_v) N(g_v)\} \left\{ \prod_{w \neq v} \text{trace } \rho(f_w, \eta_w) \right\} |ds|.$$

For brevity denote the function

$$\text{trace } \rho(f_v, \eta_v) N(g_v)$$

by $H_v(\eta_v)$ and the function trace $\rho(f_w, \eta_w)$ by $I_w(\eta_w)$. Then

$$H_v(\eta_v) \prod_w I_w(\eta_w) \tag{5.15}$$

is a function on $D^0_{\mathbf{A}}$, the set of all characters of A' which are of absolute value 1 and equal χ^{-1} on ${}_0Z'$. According to Lemma 3.2 the difference $S^g_2(f) - S_2(f)$ is equal to

$$-\lambda \sum_v \sum_{{}_0Z'_F \backslash \mathbf{A}_{F'}} \frac{H^\vee_v(\gamma)}{L(1, f_v)} \prod_{w \neq v} \frac{I^\vee_w(\gamma)}{L(1, F_w)}.$$

Here $H^\vee_v$ and $I^\vee_w$ are the local Fourier transforms of H_v and I_w. The global Fourier transform of (5.15) is

$$\gamma \to \lambda \frac{H^\vee_v(\gamma_v)}{L(1, F_v)} \prod_{w \neq v} \frac{I^\vee_w(\gamma_w)}{L(1, F_w)}.$$

If we can show that $S^g_3(f) - S_3(f)$ is equal to

$$-\lambda \sum_v \sum_{{}_0Z_{F'} \backslash Z_{F'}} \frac{H^\vee_v(\gamma)}{L(1, F_v)} \prod_{w = v} \frac{I^\vee_w(\gamma)}{L(1, F_w)}$$

when Z'_F is the group of scalar matrices in G', then an appeal to the Poisson summation formula will establish that the sum of the three differences is 0.

The value of S_3 at f is the constant term of the Laurent expansion at 0 of

$$\sum\nolimits_{{}_0Z_{F'} \backslash Z_{F'}} L(1 + s, 1_F) \prod_v \theta(a, s, f_v).$$

The distribution $\theta(a, 0, f_v)$ is stably invariant. Hence

$$S_3(f^{g^{-1}}) - S_3(f)$$

is equal to

$$\lambda \sum\nolimits_{{}_0Z_{F'} \backslash Z_{F'}} \sum\nolimits_v \{\theta'(a, 0, f_v^{g^{-1}}) - \theta'(a, 0, f_v)\} \ \{\prod_w \theta(a, 0, f_w)\}$$

if λ is the residue of $L(1 + s, 1_F)$ at $s = 0$. We now invoke Lemma 3.3.

6. Consequences. Let $m(\pi)$ be the multiplicity with which an irreducible admissible representation of G' occurs in the representation r. Then

$$\text{trace } r(f) = \sum m(\pi) \text{ trace } \pi(f).$$

If π belongs to no $\Pi(\theta)$ with

$$[\mathfrak{E}(T'/\mathbf{A}) : \text{Im}\mathfrak{E}(T'/F)] = 2$$

we set $n(\pi) = m(\pi)$. If π belongs to some such $\Pi(\theta)$ we say that π is of type (a) or type (b) according as θ is of type (a) or (b).

If π is of type (a) and θ^1 is a character of T_1' then π can belong to $\Pi(\theta^1)$ only if T_1' and T' are stably conjugate. If $T_1' = T'$ then

$$\Pi^+(\theta_v) = \Pi^+(\theta_v^1) \qquad \Pi^-(\theta_v) = \Pi^-(\theta_v^1)$$

for all v. Let $e(\pi)$ be the number of characters of $T_F'\backslash T'$ for which $\pi \in \Pi(\theta)$. Let $\langle \epsilon, \pi_v \rangle$ be 1 or -1 according as $\pi_v \in \Pi^+(\theta_v)$ or $\pi_v \in \Pi^-(\theta_v)$. Then $\langle \epsilon, \pi_v \rangle$ is 1 for almost all v, and we set

$$\langle \epsilon, \pi \rangle = \Pi_v \langle \epsilon, \pi_v \rangle.$$

The local factors depend upon a number of choices but $\langle \epsilon, \pi \rangle$ itself is well-defined when the local choices are made to depend in a consistent manner on global data, and this was done. Finally set

$$n(\pi) = m(\pi) - \frac{1}{4} \langle \epsilon, \pi \rangle e(\pi) \mu(T').$$

If π is of type (c) there are three distinct quadratic extensions L_1, L_2, L_3 with associated Cartan subgroups T_i' satisfying

$$[\mathfrak{E}(T_i'/\mathbf{A}),\ \text{Im}\ \mathfrak{E}(T_i'/F)] = 2$$

and characters θ^i of T_i' with $\pi \in \Pi(\theta^i)$. Let $e_i(\pi)$ be the number of characters of T_i' trivial on $T_i' \cap G_F'$ for which $\pi \in \Pi(\theta^i)$. For each i we may introduce $\langle \epsilon_i, \pi_v \rangle$ and set

$$\langle \epsilon_i, \pi \rangle = \Pi_v \langle \epsilon_i, \pi_v \rangle.$$

Then we introduce

$$n(\pi) = m(\pi) - \frac{1}{4} \sum_{i=1}^{3} \langle \epsilon_i, \pi \rangle e_i(\pi) \mu(T_i').$$

There is a curious property of the triple $\langle\epsilon_1,\pi\rangle$, $\langle\epsilon_2,\pi\rangle$, $\langle\epsilon_3,\pi\rangle$ which should be remarked. If we let $L_i = F(\tau_i)$ then, as at the end of §2, τ_i determines a particular T_i' and a particular $\gamma_i^0 \in T_i(\tilde{F})$. These we use to define $\Pi^+(\theta_v^i)$ and $\Pi^-(\theta_v^i)$. The set $\Pi(\theta_v^i)$ is the same for all i and a π_v in the set acts on the space of functions in the Kirillov model supported by

$$\{a \in F_v^\times | \kappa_v^i(a) = \langle\epsilon_i,\pi_v\rangle, i = 1,2,3\}.$$

Consequently

$$\langle\epsilon_1,\pi_v\rangle\langle\epsilon_2,\pi_v\rangle = \langle\epsilon_3,\pi_v\rangle$$

and

$$\langle\epsilon_1,\pi\rangle\ \langle\epsilon_2,\pi\rangle = \langle\epsilon_3,\pi\rangle.$$

The distribution

$$f \to \sum\nolimits_\pi n(\pi) \text{ trace } \pi(f)$$

is stable.

Lemma 6.1. *If* $g \in \widetilde{G}(\mathbf{A})$ *define* π^g *by*

$$\pi^g : h \to \pi(ghg^{-1}).$$

Then $n(\pi^g) = n(\pi)$.

Set

$$l(\pi) = n(\pi^g) - n(\pi)$$

and let B be the space of finite linear combinations of functions to which we have applied the trace formula. B is closed under $f \to f^*$ with

$$f^*(g) = \bar{f}(g^{-1})$$

and under the obvious convolution product. Moreover, for f in B

$$\sum l(\pi) \text{ trace } \pi(f)$$

is absolutely convergent and equal to 0. In particular

$$\sum\nolimits_{l(\pi)>0} l(\pi) \text{ trace } \pi(ff^*) = \sum\nolimits_{l(\pi)<0} -l(\pi) \text{ trace } \pi(ff*). \tag{6.1}$$

Let X^+ be the set of π with $l(\pi) > 0$ and X^- the set with $l(\pi) < 0$. All we need do is show that X^- is empty. If not, choose f so that the maximum eigenvalue $\lambda(\pi)$ of

$\pi(ff^*), \pi \in X^-$, is 1. Let $\lambda(\pi_0) = 1$ with π_0 in X^- and let x_0 be a unit vector in the space of π_0 with $\pi_0(ff^*)x_0 = x_0$.

Set

$$\delta = \max_{\pi \in X^+} \lambda(\pi).$$

δ is finite because the representations π are unitary. Moreover, it is positive. If f_1 lies in B let $\|\pi(f_1)\|$ be the Hilbert-Schmidt norm of $\pi(f_1)$. As on p. 498 of **[6]** we may choose f_1 so that

$$\sum\nolimits_{\pi \in X^+} l(\pi)\|\pi(f_1)\|^2 < -l(\pi_0)/2\delta$$

and so that

$$\|\pi_0(f_1)x_0\| = \|x_0\|.$$

In (6.1) we replace f by $f_1 f$. The left side is then less than $-l(\pi_0)/2$. The right side is at least

$$-l(\pi_0) \text{ trace } \pi_0(f_1 f f^* f_1^*) = -l(\pi_0) \text{ trace } \pi_0(f_1^* f_1 f f^*)$$

and this is greater than or equal to

$$-l(\pi_0)\|\pi_0(f_1)x_0\|^2 = -l(\pi_0).$$

The resulting contradiction proves the lemma.

Suppose $\tilde{\pi} = \otimes\, \tilde{\pi}_v$ is a representation of $\widetilde{G}(\mathbf{A})$ and let π be one of the irreducible components of its restriction to $G(\mathbf{A})$. Let $G(\pi)$ or $G(\tilde{\pi})$ be the set of all h in $\widetilde{G}(\mathbf{A})$ for which ${}^h\pi : g \to \pi(h^{-1}gh)$ is equivalent to π. Let $X(\pi) = X(\tilde{\pi})$ be the set of all characters of ω of I_F for which $\tilde{\pi} \simeq \omega \otimes\ \tilde{\pi}$. $X(\tilde{\pi})$ consists of the characters trivial on $\{\det\ h | h \in G(\tilde{\pi})\}$. Let Y be the set of characters of $F^\times I_F$ and $Y(\tilde{\pi})$ the set of all characters ω of I_F for which $\omega \otimes \tilde{\pi}$ is automorphic and cuspidal. Let

$$q(\pi) = q(\tilde{\pi}) = [Y(\tilde{\pi})/YX(\tilde{\pi})].$$

It will be seen in a moment that this index is finite; it is likely always to be 0 or 1 and to be 0 if $\tilde{\pi} = \pi(\mu, \nu)$ with two characters μ, ν in Y; but the only proof we can envisage at the moment is difficult and lies beyond the scope of this paper.

Lemma 6.2. *Suppose $G' = G(\mathbf{A})$ and π is infinite-dimensional with $\pi(z) = \chi^{-1}(z)I, z \in {}_0Z'$. Then*

$$q(\pi) = \sum\nolimits_{\widetilde{G}(F)G(\pi)\backslash\widetilde{G}(\mathbf{A})} m({}^h\pi).$$

It follows easily from this equation that $q(\pi')$ is finite. We may suppose that ${}_0Z' = Z'$. If $\widetilde{Z}$ is the centre of $\widetilde{G}$ let $G'' = \widetilde{Z}(\mathbf{A})G' = Z''G'$. From the relation

$$F^\times \cap I_F^2 = {F^\times}^2,$$

we conclude that

$$G' \cap Z''G''_F = Z'G'_F$$

and that

$$G'_F Z'\backslash G' \simeq G''_F Z''\backslash G''.$$

Thus if we extend χ to a character χ'' of Z'' and let π'' be the corresponding extension of π to G'', we need only show that

$$q(\pi) = \sum\nolimits_{\widetilde{G}(F)G(\pi)\backslash\widetilde{G}(\mathbf{A})} m({}^h\pi'').$$

Let L'', L_1, and $\tilde{L}$ be the spaces of cusp forms on $G''_F\backslash G''$, $\widetilde{G}(F)\backslash G''\widetilde{G}(F)$, and $\widetilde{G}(F)\backslash\widetilde{G}(\mathbf{A})$ transforming according to a given character of Z'', and let s'', s_1, and $\tilde{s}$ be the representations of the three groups G'', $G''\widetilde{G}(F)$, and $\widetilde{G}(\mathbf{A})$ on these three spaces. L'' and L are the same as spaces. Suppose π'' is a representation occurring in s'' with multiplicity $m(\pi'')$. Let π be its restriction to G' and let

$$G_1(\pi) = G''\widetilde{G}(F) \cap G(\pi).$$

It follows easily from Lemma 2.6 that π'' extends to a representation σ of $G(\pi)$ on the same space. Let σ_1 be the restriction of σ to $G_1(\pi)$.

The subspace V'' of L'' transforming according to π'' transforms under $G_1(\pi)$ according to

$$\oplus_{i=1}^{m(\pi'')} \omega_i \otimes \sigma_1$$

where ω_i is a character of $G''\backslash G_1(\pi)$. The smallest invariant subspace of L_1 containing V' transforms according to

$$\oplus_i \text{ Ind } (G''\widetilde{G}(F), G_1(\pi), \omega_i \otimes \sigma_1),$$

and each summand is irreducible.

Since

$$\tilde{s} = \text{ Ind } (\widetilde{G}(\mathbf{A}), G''\widetilde{G}(F), s_1)$$

we have

$$\tilde{s} = \oplus_{\{\pi\}} \oplus_i \text{ Ind } (\widetilde{G}(\mathbf{A}), G_1(\pi), \omega_i \otimes \sigma_1).$$

In this sum two representations π and π_1 are taken to be equivalent if $\pi_1 = {}^h\pi$ with $\det h \in F^\times$. The induction can be carried out in two steps, first from $G_1(\pi)$ to $G(\pi)$, and then from $G(\pi)$ to $\widetilde{G}(\mathbf{A})$. The quotient $G_1(\pi)\backslash G(\pi)$ is a subquotient of $F^\times I_F^2 \backslash I_F$ and hence compact; and

$$\text{Ind } (G(\pi), G_1(\pi), \omega_i \otimes \sigma_1) = \oplus_\omega \omega \otimes \sigma.$$

The sum is over all characters ω of $G(\pi)$ which agree with ω_i on $G_1(\pi)$. At the second step we obtain the summands

$$\text{Ind } (\widetilde{G}(\mathbf{A}), G(\pi), \omega \otimes \sigma),$$

which are irreducible. Since each of these representations contains π'', any one of them arises also from π_1'' if and only if $\pi_1'' = {}^h\pi''$ with $h \in \widetilde{G}(\mathbf{A})$. The lemma is therefore clear if $q(\pi) = 0$.

If $q(\pi) \neq 0$ we may take $\tilde{\pi}$ to be automorphic and cuspidal. Let $Y'(\pi)$ and Y' be the elements of order two in $Y(\pi)$ and Y. All elements of $X(\pi)$ are of order two and every element of $Y(\pi)$ is trivial on ${F^\times}^2$. Since

$$F^\times I_F^2/{F^\times}^2 \simeq F^\times/{F^\times}^2 \times I_F^2/{F^\times}^2$$

the set $Y(\pi)$ is equal to $Y'(\pi)Y$ and

$$q(\pi) = [Y'(\pi)/Y'X(\pi)].$$

The lemma now follows easily from multiplicity one for $\widetilde{G}(\mathbf{A})$ and the observation that a character lies in $Y'X(\pi)$ if and only if it is trivial on $G_1(\pi)$.

Now suppose G' is arbitrary and π' a cuspidal automorphic representation. We attempt to compute $m(\pi')$ in terms of the $m(\pi)$ for cuspidal automorphic forms for $G(\mathbf{A})$. We may suppose that ${}_0Z' = Z'$. We may also replace $G(\mathbf{A})$ by $G'' = Z'G(\mathbf{A})$ because

$$G(\mathbf{A}) \cap Z'G''_F = Z(\mathbf{A})G(F)$$

and

$$Z(\mathbf{A})G(F)\backslash G(\mathbf{A}) = Z'G''_F\backslash G''.$$

We proceed as in the proof of the lemma with G' replacing $\widetilde{G}(\mathbf{A})$. Set

$$G'(\pi) = G' \cap G(\pi), \qquad G'_1(\pi) = G''G'_F \cap G(\pi).$$

Let σ' and σ'_1 be the restriction of σ and σ_1 to $G'(\pi)$ and $G'_1(\pi)$. We obtain a direct sum decomposition of the representation of G' on the space of cusp forms transforming according to χ^{-1} under Z'. It is

$$\oplus_{\{\pi''\}} \oplus_{i=1}^{m(\pi'')} \oplus_{\omega'} \operatorname{Ind}\,(G', G'(\pi), \omega' \otimes \sigma').$$

For the purposes of this sum π'' and π''_1 are taken as equivalent if

$$\pi''_1 \simeq {}^h\pi'', \qquad \det\, h \in A_F.$$

For each π'' and each i there is a character ω'_i of $G''\backslash G'_1(\pi)$ and the inner sum is over all characters ω' of $G'(\pi)$ whose restriction to $G'_1(\pi)$ is ω'_i. The space of functions on

$$G''_F\backslash G'' = G'_F\backslash G''G'_F$$

transforming according to π'' transforms under $G'_1(\pi)$ according to

$$\oplus_{i=1}^{m(\pi'')} \omega_i \otimes \sigma'_1.$$

If the summands corresponding to π'', i, ω', and $\tilde{\pi}'', j, \bar{\omega}'$ are equivalent then

$$\tilde{\pi}'' \simeq {}^h\pi'', \quad h \in G'. \tag{6.2}$$

Suppose that (6.2) is satisfied and that $q(\pi)$ is 1. Then in addition $\tilde{\pi}'' \simeq {}^g\pi''$ with $g \in \widetilde{G}(F)$. Thus $h^{-1}g \in G(\pi)$. Also $m(\pi'') = m(\tilde{\pi}'') = 1$, and

$$\bar{\omega}'_1 \otimes \bar{\sigma}'_1 \simeq {}^g(\omega'_1 \otimes \sigma'_1).$$

Thus

$$\bar{\omega}'_1 \otimes \bar{\sigma}'_1 \simeq {}^h(\omega'_1 \otimes \sigma'_1)$$

and the summand indexed by $\pi'', 1$ and any ω' is equivalent to that indexed by $\tilde{\pi}'', 1$, and some ω'. We have established:

Corollary 6.3. *Suppose π' is a cuspidal automorphic representation of G' and π one of the irreducible components of the restriction of π' to $G(\mathbf{A})$. Let $A(\pi') = A(\pi) = \{\det g | g \in G(\pi)\}$. If $q(\pi)$ is 1 then the multiplicity with which π' occurs in the space of automorphic forms on $G'_F\backslash G'$ is*

$$[AA(\pi) \cap F^\times A(\pi) : A_F A(\pi))] = [F^\times \cap AA(\pi) : A_F A(\pi)_F].$$

Proposition 6.4. *Suppose π is a representation of G'. If $m(\pi^g)$ is not equal to $m(\pi)$ for all $g \in \widetilde{G}(\mathbf{A})$ there is a T' with $[\mathfrak{E}(T'/\mathbf{A}) :$ Im $\mathfrak{E}(T'/F)] = 2$ and a θ such that π belongs to $\Pi(\theta)$.*

If there were no such T' and θ then $m(\pi^g)$ would be $n(\pi^g)$ for all g.

Proposition 6.5. *Suppose ω is a non-trivial character of $F^\times\backslash I_F$ of order 2 and π is a constituent of the space of cusp forms of $\widetilde{G}(\mathbf{A})$. If $\pi \simeq \omega\otimes\pi$ then there is a character θ of $L^\times\backslash I_L$, where L is the quadratic extension of F defined by ω, for which $\pi = \pi(\theta)$.*

Let G' be the group defined by

$$A = \mathrm{Nm}_{L/F} I_L.$$

The restriction of π to $\widetilde{G}(F)G'$ is the direct sum of two irreducible representations π and π_1. Let π' and π_1' be irreducible components of the restrictions of π and π_1 to G'. One of π' and π_1' must occur in the space of cusp forms on G'. Suppose they both do. Taking the Fourier expansion with respect to the group $N(F)\backslash N(\mathbf{A})$ we see that there are two characters ψ and ψ_1 of $F\backslash\mathbf{A}$ such that π' is contained in Ind $(G', N(\mathbf{A}), \psi)$ and π_1' in Ind $(G', N(\mathbf{A}), \psi_1)$. Let

$$\psi(x) = \psi_1(\beta x) \qquad \beta\in F^\times.$$

There is a g in $\widetilde{G}(\mathbf{A})$ with $\omega(\alpha) = -1$ if $\alpha = \det\, g$ for which π is equivalent to

$$h \to \pi_1(ghg^{-1}).$$

Thus π is also contained in Ind $(G', N(\mathbf{A}), \psi')$ if $\psi'(x) = \psi_1(\alpha x)$. Since $G_v' = G(\pi_v')$ for all v it follows from Corollary 2.7 that $\omega(\beta) = \omega(\alpha)$. This is a contradiction, and so one of $m(\pi')$ and $m(\pi_1')$ is 0. The proof is completed by an appeal to Proposition 6.4.

Corollary 6.6. *Suppose L is a quadratic extension of F, θ a character of $L^\times\backslash I_L$ which does not factor through the norm, and $\tilde{\pi} = \pi(\theta)$. Then $q(\tilde{\pi}) = 1$.*

If ω is the quadratic character of $F^\times\backslash I_F$ defined by L then

$$\omega\otimes\tilde{\pi} \simeq \tilde{\pi}.$$

Suppose ν is a character of I_F and $\tilde{\pi}' = \nu\otimes\tilde{\pi}$ is also a constituent of the space of cusp forms. Since $\omega\otimes\tilde{\pi}' \simeq \tilde{\pi}'$ there is a θ' for which $\tilde{\pi}' = \pi(\theta')$. The two characters θ and θ' define representations of the Weil group $W_{L/F}$ in $GL(2,\mathbf{C})$ and because $\tilde{\pi}' = \nu\otimes\tilde{\pi}$ the three-dimensional representations obtained from

$$GL(2,\mathbf{C}) \to PGL(2,\mathbf{C}) \to GL(3,\mathbf{C})$$

have equivalent restrictions to the local Weil groups at every place. Glancing at Lemma 12.3 of [**6**], we conclude that the three-dimensional representations of $W_{L/F}$ itself are equivalent.

Writing the representations out explicitly and recalling that the first cohomology group of $L^\times\backslash I_L$ is trivial, we see that

$$\theta'(x) = \omega'(\mathrm{Nm}\, x)\theta$$

with some character of $F^\times\backslash I_F$. Thus

$$\tilde{\pi}' \sim \omega' \otimes \tilde{\pi}.$$

We suppose more generally that $\tilde{\pi} = \pi(\rho)$ where ρ is an irreducible two-dimensional representation of the Weil group. We assume that for every place v and every character ω_v of $F_v^\times$,

$$\pi(\rho_v) \simeq \omega_v \otimes \pi(\rho_v)$$

if and only if

$$\rho_v \simeq \omega_v \otimes \rho_v.$$

This is known in general, and in the case of primary concern to us that ρ is induced it is a consequence of the discussion of §2.

There are two notions of equivalence on the set of representations $\rho' = \omega\otimes\rho$, ω a character of $F^\times\backslash I_F$.

i) Global: $\rho' \sim \rho$ if and only if the representation ρ' is equivalent to $\omega' \otimes \rho$ with ω' trivial on $AF^\times$.

ii) Local: $\rho' \sim \rho$ if and only if for every place v the representation ρ'_v is equivalent to $\omega'_v \otimes \rho_v$ with ω'_v trivial on A_v.

Global equivalence means that

$$\omega = \omega'\omega''$$

with ω'' trivial on $A(\pi)F^\times$. Such a factorization is possible if and only if ω is trivial on $AF^\times \cap A(\pi)F^\times$. Local equivalence means that for every v

$$\omega_v = \omega'_v\omega''_v$$

with ω''_v trivial on $A(\pi)_v$. This factorization is possible if and only if ω is trivial on $F^\times(A\cap A(\pi))$. Thus the number of global equivalence classes within one local equivalence class is

$$[AF^\times \cap A(\pi)F^\times : F^\times(A \cap A(\pi))] = [A \cap A(\pi)F^\times : A_F(A \cap A(\pi))].$$

This is the index of Corollary 6.3. It depends only on the L-indistinguishability class to which π belongs, that is, it is the same for π and for ${}^g\pi, g \in \widetilde{G}(A)$. We denote it by $d(\pi)$. More generally if π' is a representation of G' and π a component of its restriction to $G(\tilde{A})$ we set $d(\pi') = d(\pi)$.

Suppose L is a quadratic extension and

$$\rho_i = \text{ Ind } (W_{L/F}, W_{L/L}, \theta_i).$$

The representations $\pi(\rho_i) = \pi(\theta_i)$ determine the same L-indistinguishability class of representations of G' if and only if

$$\rho_2 \sim \omega \otimes \rho_1$$

with ω trivial on $F^\times(A \cap A(\tilde{\pi})), \tilde{\pi} = \pi(\theta_1)$. However, $\rho_2 \sim \omega \otimes \rho_1$ if and only if

$$\theta_2(\gamma) \equiv \omega(\text{Nm}\,\gamma)\theta_1(\gamma)$$

or

$$\theta_2(\gamma) \equiv \omega(\text{Nm}\,\gamma)\theta_1(\bar{\gamma}).$$

The bar denotes conjugation on L. However, θ_1 and θ_2 have the same restrictions to T' if and only if

$$\theta_2(\gamma) = \omega(\text{Nm}\,\gamma)\theta_1(\gamma)$$

with ω trivial on $F^\times(A \cap \text{ Nm } I_L)$.

Thus the number of characters θ' of $T'_F\backslash T'$ which yield the same L-indistinguishability class $\Pi(\theta')$ is

$$2[F^\times(A \cap \text{ Nm } I_L) : F^\times(A \cap A(\pi))]$$

unless

$$\theta_1(x/\bar{x}) = 1$$

when it is simply

$$[F^\times\{A \cap \text{ Nm } I_L) : F^\times(A \cap A(\pi))].$$

We are going to multiply the latter number by $\mu(T')$. Any element a of $A \cap F^\times \text{ Nm } I_L$ is of course also equal to $xy, x \in F^\times, y \in \text{Nm } I_L$ and the map $a \to x$ yields an isomorphism

$$A \cap F^\times \text{ Nm } I_L/A_F(A \cap \text{ Nm } I_L) \simeq (F^\times \cap A \,\text{Nm } I_L)/A_F \text{ Nm } L^\times.$$

Thus

$$\mu(T') = [F^\times A \cap F^\times \text{Nm } I_L : F^\times(A \cap \text{Nm } I_L)]$$

and the product is

$$[F^\times A \cap F^\times \text{Nm } I_L : F^\times(A \cap A(\tilde{\pi}))].$$

If π' lies in $\Pi(\theta')$ this is $d(\pi')$ times

$$[F^\times A \cap F^X \mathrm{Nm}\, I_L : F^\times A \cap F^X A(\tilde{\pi})] = [A \cap F^\times \mathrm{Nm}\, I_L : A \cap F^\times A(\tilde{\pi})]. \tag{6.3}$$

Take a character θ of $L^\times$ and let $\tilde{\pi} = \pi(\theta)$. Every character of $I_F/A(\tilde{\pi})F^\times$ is of order two and $\tilde{\pi} \sim \omega \otimes \tilde{\pi}$. Thus $I_F/A(\tilde{\pi})F^\times$ is of order $1, 2$, or 4 and is of order 4 if and only if

$$x \to \theta(x/\bar{x})$$

is of order two but not identically 1. There are then three different quadratic extensions $L = L_1, L_2, L_3$ and characters $\theta_1, \theta_2, \theta_3$ such that $\tilde{\pi} = \pi(\theta_i)$. In addition

$$A(\tilde{\pi})F^\times = \bigcap_{i=1}^{3} F^\times\ \mathrm{Nm}\, I_{L_i}.$$

If $A \subseteq F^\times\ \mathrm{Nm}\, L$ then the index (6.3) is 1 unless

$$[I_F : A(\tilde{\pi})F^\times] = 4$$

and

$$A \not\subseteq F^\times\ \mathrm{Nm}\, I_{L_2} \qquad A \not\subseteq F^\times\ \mathrm{Nm}\, I_{L_3}$$

when it is 2.

Suppose θ does not factor through the norm. Since $q(\tilde{\pi})$ is 1 the discussion culminating in Corollary 6.3 shows in addition that for π' in the L-indistinguishable class $\Pi(\theta')$ defined by $\tilde{\pi}, m(\pi')$ is 0 or $d(\pi')$, that for one of these π',

$$m(\pi') = d(\pi'),$$

and that then

$$m({}^g\pi') = d(\pi')$$

if and only if det $g \in AA(\tilde{\pi})F^\times$. The results of the previous paragraph yield in combination with the considerations above a more precise statement.

Proposition 6.7. *Suppose θ is a character of $T'_F\backslash T'$ with*

$$[\mathfrak{E}(T'/\mathbf{A}) : \text{ Im } \mathfrak{E}(T'/F)] = 2$$

and π in $\Pi(\theta)$ is of type (a). *Then*

$$n(\pi) = d(\pi)/2$$

and

$$m(\pi) = \frac{d(\pi)}{2}(1 + \langle \epsilon, \pi \rangle).$$

One need only observe that when θ is of type (a)

$$e(\pi)\mu(T') = 2d(\pi).$$

Proposition 6.8. *Suppose π is of type* (b) *and lies in $\Pi(\theta^1), \Pi(\theta^2), \Pi(\theta^3)$, where θ^i is a character of T'_i trivial on $T'_i \cap G'_F$ and*

$$[\mathfrak{E}(T'_i/\mathbf{A}) : \text{ Im } \mathfrak{E}(T'_i/F)] = 2$$

Then

$$n(\pi) = d(\pi)/4$$

and

$$m(\pi) = \frac{d(\pi)}{4}\{1 + \langle \epsilon_1, \pi \rangle + \langle \epsilon_2, \pi \rangle + \langle \epsilon_3, \pi \rangle\}.$$

When θ is of type (b)

$$e(\pi)\mu(T') = d(\pi).$$

There is a more suggestive way to state the propositions. We first work locally. Suppose F is an extension of E such that

$$\widetilde{G}(F) = \text{ Res}_{F/E}\widetilde{G}(E)$$

and the determinant map is from

$$\text{Res}_{F/E}\widetilde{G}(F) \to \text{ Res}_{F/E}GL(1).$$

We replace A by $A(E)$ where A is a connected algebraic subgroup of $\text{Res}_{F/E}GL(1)$ defined over E. The group G' is now $G(E)$ where G is the inverse image of A. Let ${}^LG$ be the associate group of G. If $\tilde{\pi} = \pi(\tilde{\rho})$ is the irreducible, admissible representation associated to a two-dimensional representation of the Weil group W_F over F then the components of the restriction of $\tilde{\pi}$ to G' form the L-indistinguishable class $\Pi(\rho)$ associated to the corresponding homomorphism ρ of W_E to ${}^LG$, a quotient of the associate group of $\mathrm{Res}_{F/E}\tilde{G}$ by a central torus. Let S be the centralizer of $\rho(W_E)$ in ${}^LG^0$ and S^0 the product of its connected component and its intersection with the centre of ${}^LG^0$. We shall show that the quotient $S^0\backslash S$ is abelian and that the set $\Pi(\rho)$ may be mapped in a natural and bijective manner to its dual.

We first remind ourselves of the definitions of ρ and ${}^LG$ [**10**]. If K is a large Galois extension of E containing F then

$$X^* = X^*(\mathrm{Res}_{F/E}GL(1)) = \mathrm{Ind}\;(\mathfrak{G}(K/E), \mathfrak{G}(K/F), 1) = \oplus_{\mathfrak{G}(K/F)\backslash\mathfrak{G}(K/E)}\mathbf{Z}$$

and

$$X^*(A) = X^*/Y^*.$$

Let Y_* be the orthogonal complement of Y^* in the dual module. The group ${}^LG$ is the quotient of the semi-direct product

$$\prod\nolimits_{\mathfrak{G}(K/F)\backslash\mathfrak{G}(K/E)} GL(2,\mathbf{C}) \times \mathfrak{G}(K/F)$$

by

$${}^LZ^0 = \left\{ \prod \begin{pmatrix} z_\sigma & 0 \\ 0 & z_\sigma \end{pmatrix} \Big| \prod_{\mathfrak{G}(K/F)\backslash\mathfrak{G}(K/E)} \lambda_\sigma(z_\sigma) = 1 \text{ for all } \lambda = (\lambda_\sigma) \text{ in } Y_* \right\}.$$

Choose a set of representatives v for $W_{K/F}\backslash W_{K/E}$. If $w \in W_{K/E}$ let

$$vw = d_v(w)v', \qquad d_v(w) \in W_{K/F}.$$

If $w \to \sigma$ in $\mathfrak{G}(K/F)$ then $\rho(w)$ is the image in ${}^LG$ of

$$(\prod \bar{\rho}(d_v(w))) \times \sigma.$$

Notice that the cosets in $\mathfrak{G}(K/F)\backslash\mathfrak{G}(K/E)$ may also be labelled by the v.

We may suppose that one of the v is 1. Suppose $\prod a_v$ commutes with $\rho(W_{K/F})$. Then

$$a_v\tilde{\rho}(d_v(w))a_{v'}^{-1} = z_v\tilde{\rho}(d_v(w))$$

with

$$\prod_v \lambda_v(z_v) = 1$$

for all $\lambda = (\lambda_v)$ in ${}^LZ^0$. If $\tilde{\rho}$ is a representation by scalar matrices then S is the product of its intersection with the centre and the image of $GL(2, \mathbf{C})$ in ${}^LG^0$ under the diagonal map. Thus $S^0\backslash S = 1$. In this case $\Pi(\rho)$ consists, as we have seen, of a single element. If $\tilde{\rho}$ is not a representation by scalar matrices and is not induced from a one-dimensional representation of a Weil group over a quadratic extension then the associated projective representation is irreducible and S is contained in the centre of ${}^LG^0$. Again $S^0\backslash S = 1$ and $\Pi(\rho)$ consists of a single element.

Suppose

$$\tilde{\rho} = \operatorname{Ind}(W_{K/F}, W_{K/L}, \tilde{\theta})$$

where $\tilde{\theta}$ is a one-dimensional representation of $W_{K/L}$, that is, a character of $L^\times$. We suppose $\tilde{\theta}$ does not factor through the norm. Suppose $\tilde{T}$ is a Cartan subgroup of $\widetilde{G}$ associated to L and $T = G \cap \operatorname{Res}_{F/E}\tilde{T}$. The associate group ${}^L\tilde{T}$ is a semi-direct product

$$(\mathbf{C}^\times \times \mathbf{C}^\times)\mathfrak{G}(K/F).$$

$\mathfrak{G}(L/F)$ acts on $\mathbf{C}^\times \times \mathbf{C}^\times$ by permuting the two factors and $\mathfrak{G}(K/F)$ acts through its projection on $\mathfrak{G}(L/F)$. $\tilde{\theta}$ may be regarded as a character of $\tilde{T}(F)$ and is associated to a homomorphism $\tilde{\varphi} : W_{K/F} \to {}^L\tilde{T}$. There is a homomorphism

$$\tilde{\psi} : {}^L\tilde{T} \to {}^L\tilde{G}$$

given by

$$\tilde{\psi} : (a, b) \times \sigma \to \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \times \sigma$$

if σ acts trivially on L and by

$$\tilde{\psi} : (a, b) \times \sigma \to \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \times \sigma$$

if σ does not act trivially on L. Then $\tilde{\rho} = \tilde{\psi} \circ \tilde{\varphi}$ and $\pi(\tilde{\rho})$ is the image of $\tilde{\theta}$ under the map $\tilde{\psi}_*$ associated to $\tilde{\psi}$ by the principle of functoriality. The formalism of [**10**] then yields

$$\psi : {}^LT \to {}^LG$$

and if θ is the restriction of $\tilde{\theta}$ to $T(E)$ then ψ_* takes $\{\theta\}$ to the set $\Pi(\rho)$ and θ is associated to $\varphi : W_{K/E} \to {}^LT$.

Consider the set $\mathfrak{E}(T)$ introduced in **[12]** and suppose κ is a homomorphism of $\mathfrak{E}(T)$ into $\mathbf{C}^\times$, that is, a homomorphism of $X_*(T_{\mathrm{sc}})$ into $\mathbf{C}^\times$ which is trivial on

$$X_*(T_{\mathrm{sc}}) \cap (\textstyle\sum_{\mathfrak{G}(K/E)} (\sigma - 1) X_*(T)).$$

Then κ extends to a $\mathfrak{G}(K/F)$-invariant homomorphism κ' of $X_*(T)$ into $\mathbf{C}^\times$. Since

$${}^L T^0 = \mathrm{Hom}(X_*(T), \mathbf{C}^X)$$

the homomorphism κ' is an element of ${}^L T^0$ which commutes with $\mathfrak{G}(K/E)$ and hence with $\varphi(W_{K/E})$. Then $\epsilon = \psi(K')$ commutes with $\rho(W_{K/E})$ and lies in S. Since ϵ is uniquely determined modulo this centre by κ, its image in $S^0 \backslash S$ is uniquely determined.

If there is a non-trivial κ it is unique, and there exists a non-trivial character if and only if $[\mathfrak{G}(T)] = 2$. However, it is easy to see that if $T' = T(E) \subseteq \tilde{T}(F)$ and if $\mathfrak{E}(T')$ is defined as in §2 then

$$\mathfrak{E}(T) \simeq \mathfrak{E}(T').$$

If κ is non-trivial then ϵ is represented by an element which is congruent to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \times \ldots \times \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{6.4}$$

modulo scalars.

For each $L, \tilde{T}, \tilde{\theta}$ with $[\mathfrak{E}(T)] = 2$ and

$$\tilde{\rho} \sim \text{ Ind } (W_{K/F}, W_{K/L}, \tilde{\theta})$$

we obtain an ϵ, uniquely determined modulo the centre. When $\tilde{\rho}$ is fixed within its equivalence class then ϵ has the form (6.4) only after conjugation. If (a_v) lies in S then all the a_v have the same image in the projective group and lie in the projective centralizer of $\tilde{\rho}(W_{K/F})$. But the projective centralizer of $\tilde{\rho}(W_{K/F})$ consists of two or four elements, two when

$$\tilde{\theta}(\bar{x})^2 \not\equiv \tilde{\theta}(x)^2$$

and four otherwise. If G is $SL(2)$ so that $[\mathfrak{E}(T)]$ is always 2 it is easily seen that this centralizer consists of the identity and the ϵ obtained from non-isomorphic $L, \tilde{T}, \tilde{\theta}$.

To show that in general S is formed of S^0 and the $S^0\epsilon$, where ϵ is obtained from the $L, \tilde{T}, \tilde{\theta}$ with $[\mathfrak{E}(T)] = 2$ we show that if we start from $L, \tilde{T}, \tilde{\theta}$ with

$$\tilde{\rho} = \text{ Ind}(W_{K/F}, W_{K/L}, \tilde{\theta})$$

and form the corresponding $\bar{\epsilon}$ in the associate group of $\mathrm{Res}_{E/F}SL(2)$, which is

$$\prod\nolimits_{\mathfrak{G}(K/F)\backslash\mathfrak{G}(K/E)} PGL(2,\mathbf{C})\times\mathfrak{G}(K/E),$$

and $\bar{\epsilon}$ lifts to ϵ in ${}^LG^0$ then ϵ lies in S only if $[\mathfrak{E}(T)]=2$. But ϵ is still the image of κ' and if ϵ lies in S then κ' is invariant under $\mathfrak{G}(K/E)$ and its restriction to $X_*(T_{\mathrm{sc}})$ is non-trivial.

If ϵ lies in S^0 we set $\langle\epsilon,\pi\rangle=1$. Otherwise ϵ in S is associated to a T' and a θ with $[\mathfrak{E}(T')]=2$ and $\pi\in\Pi(\theta)$. We set $\langle\epsilon,\pi\rangle$ equal to 1 or to -1 according as $\pi\in\Pi^+(\theta)$ or $\pi\in\Pi^-(\theta)$. We have already observed that $\langle\epsilon,\pi\rangle$ is then multiplicative in ϵ. If to each π in $\Pi(\theta)$ we associate the character $\epsilon\to\langle\epsilon,\pi\rangle$ we obtain a bijection from $\pi(\theta)$ to the dual of $S^0\backslash S$.

Now we treat a global field F. We again suppose that A is $A(E)$ where

$$A\subseteq\ \mathrm{Res}_{F/E}GL(1).$$

If w is a place of E then over E_w the group on the right becomes

$$\prod_{v|w}\mathrm{Res}_{F_v/E_w}GL(1).$$

In order to remain within the framework of the earlier paragraph we suppose that over E_w the group A is also a product $\prod A_v$ with

$$A_v\subseteq\ \mathrm{Res}_{F_v/E_w}GL(1),$$

but this is only a matter of convenience. The definition of S is now somewhat different. It is the set of all s such that for each place w there is an a_w in the centre of ${}^LG^0$ which is such that a_ws commutes with $\rho(W_{E_w})$. S^0 can therefore be taken to be the connected component of S. The analysis of $S^0\backslash S$ may be repeated, because $\mathfrak{E}(T'/\mathbf{A})/\mathfrak{E}(T'/F))$ may be identified by means of Tate-Nakayama duality with a quotient of $X_*(T_{\mathrm{sc}})$. If w is a place of E the local analogue of S is

$$S_w=\prod_{v|w}S_v$$

and the local analogue of S^0 is

$$S_w^0=\prod_{v|w}S_v^0.$$

We clearly have a map $\epsilon\to\epsilon_w=\Pi_v\epsilon_v$ of $S^0\backslash S$ to $S_w^0\backslash S_w$. If

$$\rho: W_{K/E}\to{}^LG$$

then $\Pi(\rho)$ consists of the representations $\pi = \otimes \pi_v$ where $\pi_v \in \Pi(\rho_v)$ for all v and π_v contains the trivial representation of K'_v for almost all v. We introduce the pairing of $S^0\backslash S$ with $\Pi(\rho)$ given by

$$\langle \epsilon, \pi \rangle = \Pi_v \langle \epsilon_v, \pi_v \rangle.$$

The product is over all places of F and yields a surjection from $\Pi(\rho)$ to the dual of $S^0\backslash S$. The numbers $m(\pi)$ appearing in Propositions 6.7 and 6.8 may be written as

$$\frac{d(\pi)}{[S:S^0]} \sum_{S^0\backslash S} \langle \epsilon, \pi \rangle.$$

We have also to find another interpretation for $d(\pi)$. π is $\pi(\rho)$ where

$$\rho : W_E \to {}^LG.$$

Suppose we have another

$$\sigma : W_E \to {}^LG.$$

ρ is defined by $\tilde\rho : W_F \to {}^L\tilde G$ and, by Lemma 3 of **[12]**, σ too is defined by $\tilde\sigma : W_F \to {}^L\tilde G$. It is easy enough to deduce from the results of **[14]** that σ_w and ρ_w are conjugate under ${}^LG^0$ for all places w of E if and only if $\tilde\sigma \simeq \omega \otimes \tilde\rho$ and $\tilde\sigma$ and $\tilde\rho$ are locally equivalent in the sense defined earlier. Thus local equivalence is more appropriately defined as the conjugacy of σ_w and ρ_w under ${}^LG^0$ for all w.

It also follows from **[14]** that ρ and σ themselves are conjugate under ${}^LG^0$ if and only if $\tilde\sigma \sim \omega \otimes \tilde\rho$ where ω is 1 on the group B of all $x \in I_F$ such that for some finite extension K of F there is a $y \in K^\times$ with $xy \in A(\mathbf{A}_K)$. Since B contains $F^\times A(\mathbf{A}_E)$ this is a stronger than global equivalence. $\tilde\rho$ and $\tilde\sigma$ are globally equivalent if and only if σ is conjugate under ${}^LG^0$ to

$$w \to \omega(w)\rho(w)$$

where $w \to \omega(w)$ is a continuous locally trivial 1-cocycle of W_E with values in ${}^LA^0$, the centre of ${}^LG^0$. In any event we can define local and global equivalence and the integer $d(\pi)$ entirely in terms of ρ and the associate group.

7. Quaternion algebras again. It is a straightforward matter to decompose the trace formula for a quaternion algebra into a stable and a labile part. Let $\tilde G$ be the group defined by a quaternion algebra over the global field F, let A be as in §5 with $A \subseteq \text{Nm}\, \tilde G(\mathbf{A})$, and let

$$G' = \{g \in \tilde G(\mathbf{A}) | \text{Nm}\, g \in A\}.$$

Otherwise our notation will be along the lines of §5. We want to evaluate the trace of the representation s on the space of functions φ which satisfy

$$\varphi(zg) = \chi^{-1}(z)\varphi(g) \qquad z \in {}_OZ'$$

and which are square integrable on ${}_0Z'G'_F\backslash G'$. If f is a function on $G(\mathbf{A})$ with

$$f(g) = \Pi_v f_v(g_v),$$

where the f_v satisfy the usual conditions, in particular

$$f_v(zg_v) = \chi(z)f_v(g_v), \qquad z \in {}_0Z'_v,$$

then

$$s(f)\varphi(h) = \int_{{}_0Z'\backslash G'} \varphi(hg)f(g)dg.$$

The results and definitions of §4 clearly have analogues for the functions f_v, and we shall employ them.

The first and trivial term of the trace formula is

$$\sum\nolimits_{\gamma\in {}_0Z_{F'}\backslash Z_{F'}} f(\gamma).$$

The second term will be broken up immediately into a stable and a labile part. If $\mu(T')$ is the order of the kernel of

$$\mathfrak{E}(T'/F) \to \mathfrak{E}(T'/\mathbf{A})$$

then the stable part is

$$\frac{1}{2}\sum_{T'} \frac{\mu(T')}{[\mathfrak{E}(T'/\mathbf{A}) :\ \mathrm{Im}\ \mathfrak{E}(T'/F)]} \text{ meas } ({}_0Z'T'_F\backslash T') \sum_{\gamma}{}' \Phi^{T'/1}(\gamma, f)$$

with

$$\Phi^{T'/1}(\gamma, f) = \Pi_v \Phi^{T'/1}(\gamma, f_v).$$

The outer sum is over a set of representatives for the stable conjugacy classes of Cartan subgroups. The inner sum is over the regular elements of T'_F modulo ${}_0Z'_F$. The labile part is a sum over representatives of those stable conjugacy classes for which

$$[\mathfrak{E}(T'/\mathbf{A}) :\ \mathrm{Im}\ \mathfrak{E}(T'/F)] = 2$$

of

$$(7.1) \qquad \frac{1}{4}\sum\nolimits_{\gamma}\mu(T')\text{ meas }({}_0Z'T'_F\backslash T')\Phi^{T'}(\gamma,f).$$

We note also that

$$\mathfrak{D}(T'/F) = \mathrm{Nm}D^\times/A_F\ \mathrm{Nm}L^\times$$

and that it is implicit in the above discussion that $\mathrm{Nm}D^\times$ consists of the elements of $F^\times$ that are positive at every real place where D does not split. The sum in (7.1) is at first only over the regular elements in T'_F modulo ${}_0Z'_F$ but we may extend it over all of T'_F for the additional terms will all be 0.

Let $\widetilde{H}$ be $GL(2)$ over F and define H' accordingly. For almost all v,

$$H'_v \simeq G'_v$$

and we may choose the isomorphism so that the maximal compact subgroups correspond. For these v let ϕ_v be the image of f_v by the isomorphism. For the v at which $\widetilde{H}$ does not split define ϕ_v as in §4. An easy comparison, as in §16 of [**6**], shows that the stable parts of trace $s(f)$ and trace $r(\phi)$ are equal. We write the first as

$$\sum\nolimits_{\pi} n(\pi)\text{ trace }\pi(f)$$

and the second as

$$\sum\nolimits_{\tau} n(\tau)\text{ trace }\tau(\phi).$$

The argument used to prove Lemma 6.1 and the remarks of §4 imply that if τ lies in the L-indistinguishable class corresponding to that of π then

$$n(\pi) = n(\tau)\Pi_v c(\pi_v).$$

Here $c(\pi_v)$ is 1 when $\widetilde{H}_v$ is split.

We first study the labile part for a particular G'. If F' is a given non-archimedean local field and L' a given quadratic extension of F' we choose a totally real field F and a totally imaginary quadratic extension of it so that for some place v of F the pair F_v, L_v is isomorphic to F', L'. Choose a quaternion algebra which splits at every finite place except v, and let

$$A = \text{ Nm } I_L.$$

There is only one stable conjugacy class of Cartan subgroups with

$$[\mathfrak{E}(T'/\mathbf{A}) :\ \text{Im }\mathfrak{E}(T'/F)] = 2$$

and the labile part of the trace formula is the sum over the characters of $T'_F\backslash T'$ which equal χ^{-1} on ${}_0Z'$ of

$$(7.2) \qquad \frac{1}{4}\,\Theta_v(f_v)\Pi_{w\neq v}(\sum\nolimits_{\pi_w\in\Pi^+(\theta_w)}\chi_{\pi_w}(f_w)-\sum\nolimits_{\pi_w\in\Pi^-(\theta_w)}\chi_{\pi_w}(f_w)).$$

We have fixed a non-trivial character ψ of $F\backslash\mathbf{A}$ and at a place where the quaternion algebra splits the sets $\Pi^+(\theta_w)$ and $\Pi^-(\theta_w)$ are defined with respect to ψ_w. Θ_v is defined with respect to ψ_v as in §4. At an archimedean place w where the algebra does not split $\Pi(\theta_w)$ consists of a single element π. We place it in $\Pi^+(\theta_w)$ or $\Pi^-(\theta_w)$ according as Θ_w equals $+\chi_\pi$ or $-\chi_\pi$. One of the two sets remains empty.

Define $\bar\theta$ by

$$\bar\theta(\bar\gamma)=\theta(\gamma)$$

where $\gamma\to\bar\gamma$ is the involution. If $\bar\theta_v=\theta_v$ then $\Theta_v=0$. Otherwise replacing θ by $\bar\theta$ changes the sign of an even number of factors in (7.2) but does not change the expression itself. We sum over pairs $\{\theta,\bar\theta\}$ with $\theta\neq\bar\theta$ and replace the $\frac{1}{4}$ by $\frac{1}{2}$.

It is clear that Θ_v is a finite linear combination of irreducible characters

$$\Theta_v=\sum a_i\chi_{\pi_v^i}.$$

Consider

$$\pi=\pi_v^i\otimes(\otimes_{w\neq v}\pi_w)$$

with π_w in $\Pi(\theta_w)$. If $m(\pi)$ is the multiplicity with which π occurs in s then

$$m(\pi)=n(\pi)\pm a_i/2.$$

Varying the π_w within $\Pi(\theta_w)$ does not change $n(\pi)$ but it does change the sign. We conclude that a_i is an integer. Moreover if $a_i\neq 0$ we may arrange that $m(\pi)$ be positive. Since π must lie in the restriction to G' of an automorphic representation of $\tilde G(\mathbf{A})$ we conclude from the strong form of the multiplicity one theorem that $\pi_v^i\in\Pi(\theta_v)$ when $a_i\neq 0$. We know that $\Pi(\theta_v)$ consists of two elements when $\bar\theta_v\neq\theta_v$.

The orthogonality relations for characters of G'_v show that

$$\frac{1}{2}\sum_{T'_v}\frac{[\mathfrak{D}(T'_v)]}{\text{meas } Z'_v\backslash T'_v}\int_{Z'_v\backslash T'_v}|\Theta_v(\gamma)|^2\Delta^2(\gamma)d\gamma=\sum a_i^2.$$

The sum is over a set of representatives for stable conjugacy classes of Cartan subgroups of G'_v. However, the formula for Θ_v shows immediately that the left side is 2 when $\bar\theta_v\neq\theta_v$. Thus

$a_i = \pm 1$. Since there are two a_i and Θ_v is not stable, they must have opposite signs, and with an appropriate choice of $\Pi^+(\theta_v)$ and $\Pi^-(\theta_v)$ we have

$$\Theta_v = \sum\nolimits_{\pi_v \in \Pi^+(\theta_v)} \chi_{\pi_v} - \sum\nolimits_{\pi_v \in \Pi^-(\theta_v)} \chi_{\pi_v}. \tag{7.3}$$

More generally if $\bar{G}'_v$ is defined by $\bar{A}_v$ and if $\bar{T}'_v$ belongs to the stable conjugacy class associated to L_v then

$$[\mathfrak{D}(\bar{T}'_v)] = 2$$

if and only if $\bar{A}_v \subseteq \mathrm{Nm} L_v^\times$ and $\bar{G}'_v \subseteq G'_v$. If π_v^+ and π_v^- are the restrictions of the elements of $\Pi^+(\theta_v)$ and $\Pi^-(\theta_v)$ then Θ_v is equal to

$$\chi_{\pi_v^+} - \chi_{\pi_v^-}$$

on $\bar{G}'_v$. If $\bar{\theta}_v$ and θ_v are not equal on $\bar{T}'_v$ then π_v^+ and π_v^- are not equivalent. Otherwise they are and $c(\pi_v^+) \geqq 2$. We know already that $c(\pi_v^+) \leqq 2$.

We can summarize the local results.

Lemma 7.1. *Suppose F is local field. The L-indistinguishable class of π consists of 1 or 2 elements. It consists of two elements if and only if π lies in $\Pi(\theta)$, where θ is a character of T', $[\mathfrak{D}(T')] = 2$, and $\theta(\gamma) \not\equiv \theta(\bar{\gamma})$. Moreover $c(\pi)$ is 1 unless π lies in $\Pi(\theta)$, where $[\mathfrak{D}(T')] = 2$, and $\theta(\gamma) \equiv \theta(\bar{\gamma})$ when $c(\pi) = 2$.*

When $\Pi(\theta)$ consists of two elements we may decompose it into $\Pi^+(\theta)$ and $\Pi^-(\theta)$ in such a way that

$$\Theta = \sum\nolimits_{\pi \in \Pi^+(\theta)} \chi_\pi - \sum\nolimits_{\pi \in \Pi^-(\theta)} \chi_\pi.$$

If θ is replaced by $\bar{\theta}$ then Θ is replaced by $-\Theta$, and consequently $\Pi^+(\bar{\theta}) = \Pi^-(\theta)$.

If F is a global field and θ a character of $T'_F \backslash T'$ we let $\Pi(\theta)$ be the set of tensor products $\otimes\, \pi_v$ with π_v in $\Pi(\theta_v)$ for all v. It is understood that in such a tensor product π_v contains the trivial representation of the maximal compact for almost all v.

Proposition 7.2. *Suppose $\pi = \otimes\, \pi_v$ is contained in no $\Pi(\theta)$ where θ is a character of $T'_F \backslash T'$ and*

$$[\mathfrak{D}(T'/\mathbf{A}) : \ \mathrm{Im}\, \mathfrak{D}(T'/F)] = 2.$$

Let τ be in the L-indistinguishable class of H' corresponding to that of π. For all $g \in \bar{G}(\mathbf{A})$

$$m(\pi^g) = m(\pi)$$

and

$$m(\pi) = n(\tau)\Pi_v c(\pi_v).$$

Now suppose π belongs to $\Pi(\theta)$ and $[\mathfrak{D}(T'/\mathbf{A}) : \operatorname{Im}\mathfrak{D}(T'/F)] = 2$. The type of π is again defined to be that of θ, either (a) or (b). Let τ be a representation of H' whose L-indistinguishable class is that of π. It is reasonable to set

$$d(\pi) = d(\tau).$$

Suppose π is of type (a). We introduce the group consisting of two elements $1, \epsilon$. At a place v where the quaternion algebra splits we define $\langle 1, \pi_v\rangle$ to be 1 and $\langle \epsilon, \pi_v\rangle$ to be as before. At a place v where the algebra does not split but $\theta_v \neq \bar{\theta}_v$ set $\langle 1, \pi_v\rangle = 1$ and $\langle \epsilon, \pi_v\rangle$ equal to $+1$ or -1 according as $\pi_v \in \Pi^+(\theta_v)$ or $\pi_v \in \Pi^-(\theta_v)$. Observe that $\langle \epsilon, \pi_v\rangle$ depends on θ_v. At a place where the quaternion algebra does not split and $\theta_v = \bar{\theta}_v$ we set $\langle 1, \pi_v\rangle = 2$ and $\langle \epsilon, \pi_v\rangle = 0$. Let

$$\langle 1, \pi\rangle = \Pi_v\langle 1, \pi_v\rangle$$

and

$$\langle \epsilon, \pi\rangle = \Pi_v\langle \epsilon, \pi_v\rangle.$$

Proposition 7.3. *If π is of type* (a) *then*

$$m(\pi) = \frac{d(\pi)}{2}(\langle 1, \pi\rangle + \langle \epsilon, \pi\rangle).$$

If $\theta_v = \bar{\theta}_v$ for some v then

$$m(\pi) = n(\pi) = n(\tau)\prod_v c(\pi_v) = \frac{d(\pi)}{2}\prod_v c(\pi_v) = \frac{d(\pi)}{2}\langle 1, \pi\rangle$$

and $\langle \epsilon, \pi\rangle = 0$. If $\theta_v \neq \bar{\theta}_v$ for all v then

$$m(\pi) = n(\pi) + \frac{d(\pi)}{2}\langle \epsilon, \pi\rangle = \frac{d(\pi)}{2}(\langle 1, \pi\rangle + \langle \epsilon, \pi\rangle).$$

If π is of type (b) we introduce a group consisting of four elements $1, \epsilon_1, \epsilon_2, \epsilon_3$ with $\epsilon_i^2 = 1$. The numbers $\langle \epsilon_i, \pi_v\rangle$ are defined just as those $\langle \epsilon, \pi_v\rangle$ of the previous lemma were and

$$\langle \epsilon_i, \pi\rangle = \Pi_v\langle \epsilon_i, \pi_v\rangle.$$

Since $\theta^i = \bar{\theta}^i, i = 1, 2, 3$ and $\pi \in \Pi(\theta)$, the algebra is split at the infinite places and not split at some finite places. Consequently

$$\langle \epsilon_i, \pi \rangle = 0 \qquad i = 1, 2, 3.$$

Proposition 7.4. *If* π *is of type* (b) *then*

$$m(\pi) = \frac{d(\pi)}{4}(\langle 1, \pi \rangle + \langle \epsilon_1, \pi \rangle + \langle \epsilon_2, \pi \rangle + \langle \epsilon_3, \pi \rangle).$$

For such a π,

$$m(\pi) = n(\pi) = n(\tau) \prod_v c(\pi_v) = \frac{d(\pi)}{4} \langle 1, \pi \rangle.$$

When G' is defined by a connected subgroup of $\mathrm{Res}_{F/E} GL(1)$, we may interpret the groups appearing in these two propositions as $S^0 \backslash S$, just as in the previous paragraph.

8. Afterword. There is a condition implicit when we take the group G' to be $G(\mathbf{A}_E)$ where G is a subgroup of $\mathrm{Res}_{F/E} \tilde{G}$ defined as the inverse image of the subgroup A of $G_1 = \mathrm{Res}_{F/E} GL(1)$. A is of course taken to be connected and

$$A(\mathbf{A}_E) = \prod_w A(E_w),$$

the product being restricted, and

$$A(E_w) \subseteq G_1(E_w) = \prod_{v|w} F_v^\times.$$

However, $A(E_w)$ need not be a product

$$\prod_{v|w} A_v$$

with $A_v \subseteq F_v^\times$, and we may not be free to apply the results of the early paragraphs.

We might have developed the local theory of L-indistinguishability for the groups $G(E_w)$. However $\mathfrak{E}(T/E_w)$ may no longer be of order 2. All of its characters would have to be considered, and some of them give rise to groups H which are not abelian. The local theory would provide a linear transformation from stable distributions on H to distributions on G. Although not difficult it would have been elaborate, and unnecessary for the global theory.

In the global theory it is the quotient

$$\mathfrak{E}(T/\mathbf{A}_E)/\text{Im } \mathfrak{E}(T/E)$$

which is central. If T is a Cartan subgroup of G associated to the quadratic extension L it is again

$$I_F/A(\mathbf{A}_E)F^\times \text{ Nm } I_L$$

and of order 1 or 2. The local theory need only be developed for the character of $\mathfrak{E}(T/E_w)$ obtained by pulling back the non-trivial character of the quotient via

$$\mathfrak{E}(T/E_w) \to \mathfrak{E}(T/\mathbf{A}_E) \to \mathfrak{E}(T/\mathbf{A}_E)/\text{Im } \mathfrak{E}(T/E).$$

The necessary results are easily deduced from §2 - §4.

The principal results of §5 - §7 remain valid, and the proofs are the same. We remark only that S_w is no longer a product $\Pi_{v|w} S_v$.

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Faculté des Sciences,
Dijon, France;
Institute for Advanced Study,
Princeton, New Jersey