A LETTER TO HERVÉ JACQUET

Contents

1. Gaussian sums

In the second paragraph I shall discuss the representations of the group of 2×2 non-singular matrices over a non-archimedean field. In the discussion a number of identities for Gaussian sums will be required. In this paragraph the necessary identities, trivial or not, are stated and proved.

Let k be a non-archimedean local field, let o be the ring of integers in k, let $\mathfrak{p}$ be the maximal ideal of o, and let π be a generator of $\mathfrak{p}$. Let $k^\times$ be the multiplicative group of k and let $o^\times$ be the group of units. If $n \geqslant 0$ then $o_n^\times = \left\{ \alpha \in o^\times \;\middle|\; \alpha - 1 \in \mathfrak{p}^n \right\}$. Fix a character ξ_0 of k with the property that o is the largest ideal of k on which ξ_0 is trivial.

If μ is a character of $o^\times$ and x belongs to k set

$$\Delta(\mu, x) = \int_{o^\times} \xi_0(\alpha x)\mu(\alpha)\, d\alpha.$$

It is clear that if β belongs to $o^\times$

$$\Delta(\mu, \beta x) = \mu^{-1}(\beta)\Delta(\mu, x).$$

Lemma 1.1. *Let $\mathfrak{p}^n$ be the conductor of μ.*

(i) If $n = 0$ then $\Delta(\mu, \pi^m) = 1$ if $m \geqslant 0$, $\Delta(\mu, \pi^{-1}) = \frac{|\pi|}{|\pi|-1}$, and $\Delta(\mu, \pi^m) = 0$ if $m < -1$.

(ii) If $n > 0$ then $\Delta(\mu, \pi^m) = 0$ if $m \neq -n$ but

$$\left|\Delta(\mu, \pi^{-n})\right| = \frac{|\pi|^{n/2}}{1-|\pi|}.$$

If $n = 0$ then μ is trivial and it is clear that $\Delta(\mu, \pi^m) = 1$ for $m \geqslant 0$. It is also clear that if $m < 0$

$$1 + \sum_{k=m}^{-1} \frac{1-|\pi|}{|\pi|^k}\Delta(\mu, \pi^k) = 0.$$

The first part of the lemma follows immediately.

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Certainly

$$\int_{o^\times} \xi_0(\alpha\pi^m)\mu(\alpha)\,d\alpha = \int_{o^\times/o_n^\times} \mu(\alpha)\left\{\int_{o_n^\times} \xi_0(\alpha\beta\pi^m)d\beta\right\}d\alpha.$$

If $n > 0$ the inner integral is equal to

$$\frac{\xi_0(\alpha\pi^m)}{1-|\pi|}\int_{\mathfrak{p}^n} \xi_0(\alpha\pi^m y)\,dy.$$

This is zero if the character $y \to \xi_0(\alpha\pi^m y)$ is not trivial on $\mathfrak{p}^n$, that is, if $m < -n$. On the other hand if $m > -n$ so that for some ℓ, with $0 \leqslant \ell < -n$, $m + \ell \geqslant 0$ then

$$\int_{o^\times} \xi_0(\alpha\pi^m)\mu(\alpha)\,d\alpha = \int_{o^\times/o_\ell^\times} \xi_0(\alpha\pi^m)\mu(\alpha)\left\{\int_{o_\ell^\times} \mu(\beta)d\beta\right\}d\alpha.$$

The inner integral on the right is zero.

Finally

$$\begin{aligned}\bigl|\Delta(\mu,\pi^{-n})\bigr|^2 &= \int_{o^\times} d\alpha \int_{o^\times} d\beta\,\xi_0\bigl((\alpha-\beta)\pi^{-n}\bigr)\mu\Bigl(\frac{\alpha}{\beta}\Bigr)\\ &= \int_{o^\times} d\alpha \int_{o^\times} d\beta\,\xi_0\bigl(\beta(\alpha-1)\pi^{-n}\bigr)\mu(\alpha).\end{aligned}$$

By part (i) of the lemma the integral with respect to β is 1 if $\alpha \in o_n^\times$, $\frac{|\pi|}{|\pi|-1}$ if $\alpha \in o_{n-1}^\times - o_n^\times$, and zero otherwise. Since

$$\frac{|\pi|}{|\pi|-1}\int_{o_{n-1}^\times - o_n^\times} \mu(\alpha)\,d\alpha = \frac{|\pi|}{1-|\pi|}\int_{o_n^\times} \mu(\alpha)\,d\alpha,$$

we have

$$\bigl|\Delta(\mu,\pi^{-n})\bigr|^2 = \frac{1}{1-|\pi|}(\text{measure}\, o_n^\times) = \frac{|\pi|^n}{\bigl(1-|\pi|\bigr)^2}.$$

If the conductor of μ is $\mathfrak{p}^n$ we shall refer to n as the order of μ.

Lemma 1.2. *Suppose μ and ν are characters of $o^\times$. Let the order of $\mu\nu$ be r. If $r \geqslant 1$ then*

$$\frac{\Delta(\mu,\pi^m)\Delta(\nu,\pi^n)}{\Delta(\mu\nu,\pi^{-r})} = \int_{\{\,\alpha\in o^\times \,\mid\, \pi^{r+m}\alpha+\pi^{r+m}\in o^\times\,\}} \mu(\alpha)(\mu\nu)^{-1}(\pi^{r+m}\alpha+\pi^{r+m})\,d\alpha.$$

If $r = 0$ then $\Delta(\mu,\pi^m)\Delta(\nu,\pi^n)$ is equal to

$$\int_{\{\,\alpha\in o^\times \,\mid\, \pi^m\alpha+\pi^n\in o\,\}} \mu(\alpha)\,d\alpha + \frac{|\pi|}{|\pi|-1}\int_{\{\,\alpha\in o^\times \,\mid\, \pi^{m+1}\alpha+\pi^{n+1}\in o^\times\,\}} \mu(\alpha)\,d\alpha.$$

The product $\Delta(\mu,\pi^m)\Delta(\nu,\pi^n)$ is equal to

$$\int_{o^\times}\int_{o^\times} \xi_0(\pi^m\alpha+\pi^n\beta)\mu(\alpha)\nu(\beta)\,d\alpha\,d\beta = \int_{o^\times}\int_{o^\times} \xi_0\bigl(\beta(\pi^m\alpha+1)\bigr)\,d\beta\,d\alpha.$$

If $r \geqslant 1$ the right side is equal to

$$\Delta(\mu\nu,\pi^{-r})\int_{\{\,\alpha\in o^\times \,\mid\, \pi^{r+m}\alpha+\pi^{r+m}\in o^\times\,\}} \mu(\alpha)(\mu\nu)^{-1}(\pi^{r+m}\alpha+\pi^{r+m})\,d\alpha.$$

If $r = 0$ the right side equals

$$\int_{\{\,\alpha \mid \pi^m\alpha+\pi^n \in o\,\}} \mu(\alpha)\,d\alpha + \frac{|\pi|}{|\pi|-1}\int_{\{\,\alpha \,\mid\, \pi^{m+1}\alpha+\pi^{n+1}\in o^\times\,\}} \mu(\alpha)\,d\alpha.$$

Now let K be a two-dimensional commutative algebra over k with a non-degenerate trace. There are two possibilities for K. Either it is the direct sum of k with itself or it is a separable quadratic extension of k. In both cases k has exactly one non-trivial automorphism over k. We will denote this automorphism by s. If $x \in K$ then $Sx = x + x^s$ and $Nx = xx^s$. Let O be the elements of k integral over o and let O^* be the group of units of O. If $K = k \oplus k$ set $\Pi = \pi \otimes \pi$ and if n_1 and n_2 are any two integers set $\pi^{n_1,n_2} = \pi^{n_1} \oplus \pi^{n_2}$. If K is an unramified extension of k set $\Pi = \pi$ and if $n_1 = n_2$ set $\pi^{n_1,n_2} = \pi^{n_1}$. If K is a ramified extension choose π and Π so that $N\Pi = \pi$, if $n_2 = 0$ set $\pi^{n_1,n_2} = \Pi^{n_1}$. Thus the symbol π^{n_1,n_2} has a meaning only for certain values of n_1 and n_2. We shall adhere to the convention that any expression in which the symbol π^{n_1,n_2} occurs with values of n_1 and n_2 for which it has no meaning is equal to zero. If $n_1 \geqslant 0$, $n_2 \geqslant 0$ and π^{n_1,n_2} is defined set $O^\times_{n_1,n_2} = \{\, \alpha \in O^\times \bigm| \alpha - 1 \in \pi^{n_1,n_2}O \,\}$. If M is a character of $O^\times$ then amongst all groups of this type on which M is trivial there is a maximal one $O^\times_{m_1,m_2}$. (m_1, m_2) will be called the order of M.

If $K = k \otimes k$ or K is an unramified extension we set $f = 0$. Otherwise (Π^{-f}) is the inverse different. The index of $NK^\times$ in $k^\times$ is either 1 or 2. If it is 1 let χ be the trivial character of $k^\times$; if it is 2 let χ be the unique non-trivial character of $k^\times$ whose restriction to $NK^\times$ is trivial. Let χ_0 be the restriction of χ to $o^\times$. The order of χ_0 is f.

Before going on I recall some facts whose proofs are either completely trivial or are to be found in the book “Corps Locaux” of Serre.

Lemma 1.3.

(*i*) *Let n_1 and n_2 be non-negative integers. If $K = k \oplus k$ the map $x \to Sx$ takes $\pi^{n_1,n_2}O$ onto $\mathfrak{p}^r$ with $r = \min\{n_1, n_2\}$. The map $x \to Nx$ maps $O^\times_{n_1,n_2}$ onto $o^\times_r$.*

(*ii*) *If K is an unramified extension of k the map $x \to Sx$ maps $\pi^{n,n}O$ onto $\mathfrak{p}^n$. The map $x \to Nx$ takes $O^\times_{n,n}$ onto $o^\times_n$.*

(*iii*) *If K is a ramified extension of k the map $x \to Sx$ maps $\pi^{n,0}O$ onto $\mathfrak{p}^r$ with $r = \left[\frac{n+f}{2}\right]$. If $n \geqslant f$ the smallest number m such that $N(O^\times_{m,0}) = o^\times_n$ is $2n - f$; the largest such number is $2n - f + 1$. If $n < f$ then $N(O^\times_{n,0})$ is contained in $o^\times_n$ and if $0 \leqslant m < n$ the map $N : O^\times_{m,0}/O^\times_{n,0} \to o^\times_m/o^\times_n$ is an isomorphism. If $m < f$ the kernel and the cokernel of the map $N : O^\times_{n,0}/O^\times_{f,0} \to o^\times_n/o^\times_f$ both have order two.*

If μ is a character of $o^\times$ let μ^{1+s} be the character of $O^\times$ defined by $\mu^{1+s}(\alpha) = \mu(\alpha\alpha^s)$. Let n be the order of μ. If $K = k \otimes k$ or K is unramified the order of μ^{1+s} is (n, n). If K is ramified the order of μ^{1+s} is $(2n - f, 0)$ if $n > f$; it is $(n, 0)$ if $n < f$, but if $n = f$ all one can say is that it is $(r, 0)$ with $r \leqslant f$.

If M_0 is a character of $O^\times$ set

$$\Delta(M_0, \pi^{n_1,n_2}) = \int_{O^\times} \xi_0\bigl(S(\alpha\pi^{n_1,n_2})\bigr) M_0(\alpha)\,d\alpha.$$

The following lemma is an immediate consequence of Lemma 1.2 but it is convenient to state it explicitly.

Lemma 1.4. *Suppose M_0 and H_0 are two characters of O^*. Let the order of M_0H_0 be (r_1, r_2). If $r_1 > 0$ and $r_2 + f > 0$ then*

$$\frac{\Delta(M_0, \pi^{m_1,m_2})\Delta(H_0, \pi^{n_1,n_2})}{\Delta(M_0H_0, \pi^{-r_1-f,-r_2})} = \int_{\left\{\, \alpha\in O^\times \;\middle|\; \pi^{r_1+m_1+f,r_2+m_2}\alpha+\pi^{r_1+n_1+f,r_2+n_2}\in O^* \,\right\}} M_0(\alpha) \cdot (M_0H_0)^{-1}(\pi^{r_1+m_1+f,r_2+m_2}\alpha + \pi^{r_1+m_1+f,r_2+n_2})\, d\alpha.$$

If $K = k \oplus k$ and $r_1 = 0$ and $r_2 = 0$ the left hand side is equal to the sum of

$$\int_{\left\{\, \alpha\in O^\times \;\middle|\; \pi^{m_1,m_2}\alpha+\pi^{n_1,n_2}\in O \,\right\}} M_0(\alpha)\, d\alpha$$

and $\frac{|\pi|}{|\pi|-1}$ times

$$\int_{\left\{\, \alpha\in O^\times \;\middle|\; \pi^{m_1+1,m_2}\alpha+\pi^{n_1+1,n_2}\in o^\times\oplus o \,\right\}} M_0(\alpha)\, d\alpha + \int_{\left\{\, \alpha\in O^\times \;\middle|\; \pi^{m_1,m_2+1}\alpha+\pi^{n_1,n_2+1}\in o\oplus o \,\right\}} M_0(\alpha)\, d\alpha$$

and $\left(\frac{|\pi|}{|\pi|-1}\right)^2$ times

$$\int_{\left\{\, \alpha\in O^\times \;\middle|\; \pi^{m_1+1,m_2+1}\alpha+\pi^{n_1+1,n_2+1}\in O^\times \,\right\}} M_0(\alpha)\, d\alpha.$$

If K is an unramified extension and $r_1 = r_2 = 0$ it is the sum of

$$\int_{\left\{\, \alpha\in O^* \;\mid\; \pi^{m_1,m_2}\alpha+\pi^{n_1,n_2}\in O \,\right\}} M_0(\alpha)\, d\alpha$$

and

$$\frac{|\pi|^2}{|\pi|^2-1}\int_{\left\{\, \alpha\in O^\times \;\middle|\; \pi^{m_1+1,m_2+1}\alpha+\pi^{n_1+1,n_2+1}\in O^\times \,\right\}} M_0(\alpha)\, d\alpha.$$

If K is a ramified extension and $r_1 = 0$ it is the sum of

$$\int_{\left\{\, \alpha\in O^\times \;\middle|\; \pi^{m_1+f,m_2}\alpha+\pi^{n_1+f,n_2}\in O \,\right\}} M_0(\alpha)\, d\alpha$$

and

$$\frac{|\pi|}{|\pi|-1}\int_{\left\{\, \alpha\in O^\times \;\middle|\; \pi^{m_1+1+f,m_2}\alpha+\pi^{n_2+1+f,n_2}\in O^\times \,\right\}} M_0(\alpha)\, d\alpha.$$

Lemma 1.5. *Let M_0 be a character of $O^\times$ of order (m_1, m_2) and let μ and ν be characters of $o^\times$ of orders n_1 and n_2 respectively. Suppose that $M_0 = \chi_0\mu\nu$ on $o^\times$ and that the order of $M_0^{-1}\nu^{1+s}$ is (ℓ_1, ℓ_2) with $n_1 \geqslant \ell_1 + \ell_2 + f$. If $\ell_1 \geqslant \ell_2$, $n_1 \geqslant n_2$, and $n_1 + n_2 = m_1 + m_2 + f$ then*

$$\frac{\Delta(M_0, \pi^{-m_1-f,-m_2})\Delta(\chi_0, \pi^{-f})}{\left|\Delta(M_0, \pi^{-n_1-f,-m_2})\Delta(\chi_0, \pi^{-f})\right|} = \frac{M_0(\Pi^{(f+m_1-n_2)+s(m_2-n_2)})}{\chi(\pi^{n_1})}\frac{\Delta(\mu, \pi^{-n_1})\Delta(\nu, \pi^{-n_2})}{\left|\Delta(\mu, \pi^{-n_1})\Delta(\nu, \pi^{-n_2})\right|}.$$

Since both sides of this identity have the same absolute value all we need do is show that

$$\Delta(M_0, \pi^{-m_1-f,-m_2})\overline{\Delta(\mu, \pi^{-n_1})}\ \overline{\Delta(\nu, \pi^{-n_2})}$$

is equal to the product of

$$\overline{\Delta(\chi_0, \pi^{-f})} M_0(\Pi^{(f+m_1-n_1)+s(m_2-n_2)})\chi^{-1}(\pi^{n_1})$$

and a positive constant. As a start observe that it is equal to

$$\int_{O^\times} d\alpha \int_{o^\times} d\beta \int_{o^\times} d\gamma\, \xi_0\Big(S(\pi^{-m_1-f,-m_2}\alpha) - \pi^{-n_2}\beta - \pi^{-n_1}\gamma\Big) M_0(\alpha)\nu^{-1}(\beta)\mu^{-1}(\gamma)$$

which equals

$$\text{(A)}\quad \int_{O^\times} d\alpha \int_{o^\times} d\beta \left\{ \int_{o^\times} \xi_0\left[\frac{\gamma}{\pi^{n_1}}\Big(\pi^{n_1} S(\pi^{-m_1-f,-m_2}\alpha) - \pi^{n_1-n_2}\beta - 1\Big)\right]\chi_0(\gamma)\, d\gamma \right\} \cdot M(\alpha)\nu^{-1}(\beta).$$

If $f > 0$ the integral with respect to γ is zero unless

$$\pi^{n_1} S(\pi^{-m_1-f,-m_2}\alpha) - \pi^{n_1-n_2}\beta - 1 \in \mathfrak{p}^{n_1-f} - \mathfrak{p}^{n_1-f+1}.$$

However if this last condition is satisfied it is equal to

$$\Delta(\chi_0, \pi^{-f})\chi_0\left(\frac{S(\Pi^{(n_1-m_1-f)+(n_1-m_2)s}\alpha) - \pi^{n_1-n_2}\beta - 1}{\pi^{n_1-f}}\right).$$

Changing variables we see that the integral is equal to the product of

$$\Delta(\chi_0, \pi^{-f}) M_0^{-1}(\Pi^{(n_1-m,-f)+(n_2-m_2)s})$$

and

$$\int_{\{\,(\alpha,\beta)\ |\ S(\Pi^{s(n_1-n_2)}\alpha)-\pi^{n_1-n_2}\beta-1\in\mathfrak{p}^{n_1-f}-\mathfrak{p}^{n_1-f+1}\,\}} M_0(\alpha) \cdot \nu^{-1}(\beta)\chi_0\left(\frac{S(\Pi^{s(n_1-n_2)}\alpha) - \pi^{n_1-n_2}\beta - 1}{\pi^{n_1-f}}\right) d\alpha\, d\beta.$$

If $n_1 > f$ and $n_1 > n-2$ then the restriction of M to $o^\times$ has order n_1. Thus $m_1 \geqslant 2n_1$ and $m_1 + m_2 + f > 2n_1 > n_1 + n_2$ contrary to assumption. Consequently we need only consider the case that $n_1 = f$ or $n_1 = n_2$. If $n_1 > f$ or $n_1 > n_2$ then $S(\Pi^{\sigma(n_1-n_2)}\alpha) - \pi^{n_1-n_2}\beta - 1$ can belong to $\mathfrak{p}^{n_1-f} - \mathfrak{p}^{n_1-f+1}$ only if $S(\Pi^{\sigma(n_1-n_2)}\alpha) - 1$ belongs to $o^\times$.

Suppose that $n_1 = n_2 = f$ and $S(\alpha) - 1 \in \mathfrak{p}$. Replacing β by $\frac{1}{\beta}$ in

$$\int_{\{\,\beta\ |\ S\alpha-\beta-1\in o^\times\,\}} M_0(\alpha)\nu^{-1}(\beta)\chi_0\big(S(\alpha) - \beta - 1\big)\, d\beta$$

we obtain

$$M_0(\alpha)\chi_0(-1)\int_{o^\times} \nu(\beta)\chi_0(\beta)\chi_0\Big(1 - \beta\big(S(\alpha) - 1\big)\Big)\, d\beta.$$

Since $n_1 = n_2 = f$, $\ell_1 = 0$ and $M_0 = \nu^{1+\sigma}$. Since $M = \chi_0\mu\nu$ on $o^\times$, $\nu\chi_0 = \mu$ and the order of $\nu\chi_0$ is f. If $\beta \in o^\times$ and $\nu \in o^\times_{f-1}$ then

$$1 - \beta\gamma(S\alpha - 1) \equiv 1 - \beta(S\alpha - 1) \pmod{\mathfrak{p}^f}.$$

Thus the above expression is equal to

$$M_0(\alpha)\chi_0(-1)\int_{o^\times/o^\times_{f-1}} \nu(\beta)\chi_0(\beta)\chi_0\big(1+\beta(S\alpha-1)\big)\left\{\int_{o^\times_{f-1}} \nu(\gamma)\chi_0(\gamma)\,d\gamma\right\}d\beta = 0.$$

In all cases we can take the integral over

$$\left\{\, (\alpha,\beta) \;\middle|\; S(\Pi^{\sigma(n_1-n_2)}\alpha)-1\notin\mathfrak{p}, S(\Pi^{\sigma(n_1-n_2)}\alpha)-\pi^{n_1-n_2}\beta-1\in\mathfrak{p}^{n_1-f}-\mathfrak{p}^{n_1-f+1}\right\}.$$

Replacing β by $\Big[S(\Pi^{\sigma(n_1-n_2)}\alpha)-1\Big]\beta$ we obtain

$$\left\{\int_{\left\{\,\alpha\;\middle|\;S(\Pi^{\sigma(n_1-n_2)}\alpha)-1\notin\mathfrak{p}\right\}} M_0(\alpha)\nu^{-1}\chi_0\Big(S(\Pi^{\sigma(n_1-n_2)}\alpha)-1\Big)\,d\alpha\right\}$$
$$\cdot\left\{\int_{\left\{\,\beta\;\middle|\;\pi^{n_1-n_2}\beta-1\in\mathfrak{p}^{n_1-f}-\mathfrak{p}^{n_1-f+1}\right\}} \nu^{-1}(\beta)\chi_0\left(\frac{1-\pi^{n_1-n_2}\beta}{\pi^{n_1-f}}\right)d\beta\right\},$$

an expression we label (B).

Suppose $n_1 > f$ and consider the first integral. Replacing α by $\alpha(1+v)$ with $v\in\Pi^{n_1}O$ does not change the value of the integral. The integrand becomes

$$M_0(\alpha)\nu^{-1}(S\alpha-1)\chi_0(S\alpha-1)M_0(1+v)\nu^{-1}\left(1+\frac{S(\alpha v)}{S\alpha-1}\right)\chi_0\left(1+\frac{S(\alpha v)}{S\alpha-1}\right).$$

Since $n_1\geqslant \ell_1$, $M_0(1+v)=v(1+Sv+Nv)=\nu(1+Sv)$. Moreover $\left[\frac{n_1+f}{2}\right]\geqslant\left[\frac{2f}{2}\right]=f$ so that $\chi_0\Big(1+\frac{S(\alpha\nu)}{S\alpha-1}\Big)=1$. Also $\left[\frac{n_1+f}{2}\right]\geqslant\left[\frac{n_1+1}{2}\right]\geqslant\frac{n_1}{2}$ so that

$$v^{-1}\left(1+\frac{S(\alpha v)}{S\alpha-1}\right)=\nu\left(1-\frac{S(\alpha v)}{S\alpha-1}\right)$$

and

$$v(1+Sv)\nu^{-1}\left(1+\frac{S(\alpha v)}{S\alpha-1}\right)=\nu\big(1+S(\delta v)\big)$$

if $\delta=1-\frac{\alpha}{S\alpha-1}$. Integrating over $\Pi^{n_1}O$ we obtain 0 unless $|\delta|=|\pi|^s$ and $\left[\frac{s+n_1+f}{2}\right]\geqslant n_1$, that is, $s+n_1+f\geqslant 2n_1$ or $s\geqslant n_1-f$ when we obtain $|\pi|^{n_1}$. Since $|\delta|=|\alpha-1|$ we can in all cases write the first integral of (B) as

$$\int_{\left\{\,\alpha\in O^\times_{n_1-f,0}\;\middle|\;S(\Pi^{\sigma(n_1-n_2)}\alpha)-1\notin\mathfrak{p}\right\}} M_0(\alpha)\nu^{-1}\chi_0\Big(S(\Pi^{\sigma(n_1-n_2)}\alpha)-1\Big)\,d\alpha.$$

Since $n_1-f\geqslant\ell_1$ and $\chi_0(N\alpha)=1$ this may be written as

$$\int_{\left\{\,\alpha\in O^\times_{n_1-f,0}\;\middle|\;S(\Pi^{\sigma(n_1-n_2)}\alpha)-1\notin\mathfrak{p}\right\}} \nu^{-1}\chi_0\left(\pi^{n_1-n_2}-N\left(\frac{\Pi^{\sigma(n_1-n_2)}\alpha-1}{\alpha}\right)\right)d\alpha.$$

Set $\Pi^{n_1-f}\gamma=\frac{\Pi^{\sigma(n_1-n_2)}\alpha-1}{\alpha}$ so that $\alpha=\frac{1}{\Pi^{\sigma(n_1-n_2)}-\Pi^{n_1-f}\gamma}$. The integral is the product of a positive constant and

$$\text{(C)}\qquad \sum_{\left\{\,\gamma\in O/\Pi^f O\;\middle|\;\pi^{n_1-n_2}-\pi^{n_1-f}N\gamma\notin\mathfrak{p}\right\}} \nu^{-1}\chi_0(\pi^{n_1-n_2}-\pi^{n-1-f}N\gamma).$$

If $n_1 > n_2$ every γ appearing in this sum is a unit and the sum is equal to

$$\sum_{\left\{\beta\in O^*/O^*_{n_1}\,\middle|\,\pi^{n_1-n_2}-\beta\in\mathfrak{p}^{n_1-f}-\mathfrak{p}^{n_1-f+1}\right\}} \nu^{-1}\chi_0(\beta)\left[1+\chi_0\left(\frac{\pi^{n_1-n_2}-\beta}{\pi^{n_1-f}}\right)\right].$$

Since this sum is taken over all of $o^\times$ it is equal to

$$\sum_{\left\{\beta\in o^\times/o^\times_{n_1}\,\middle|\,\pi^{n_1-n_2}-\beta\in\mathfrak{p}^{n_1-f}-\mathfrak{p}^{n_1-f+1}\right\}} \nu^{-1}\chi_0(\beta)\chi_0\left(\frac{\pi^{n_1-n_2}-\beta}{\pi^{n_1-f}}\right).$$

If $n_1 = n_2$ then (C) is the sum of

$$\sum_{\left\{\gamma\in O^\times/O^\times_{f,0}\,\middle|\,1-\pi^{n_1-f}N\gamma\notin\mathfrak{p}\right\}} \nu^{-1}\chi_0(1-\pi^{n_1-f}N\gamma)$$

and

$$\sum_{r=1}^{f}\sum_{\gamma\in O^\times/O^\times_{f-r,0}} \nu^{-1}\chi_0(1-\pi^{n_1+r-f}N\gamma).$$

Since the map $\gamma\to N\gamma$ defines an isomorphism of $O^\times/O^\times_{f-r,0}$ and $o^\times/o^\times_{f-r}$ the latter sum is equal to

$$\sum_{\mathfrak{p}/\mathfrak{p}^f}\nu^{-1}\chi_0(1-w).$$

Since

$$\sum_{o^\times/o^\times_f}\nu^{-1}\chi_0(\beta)=0$$

we can subtract it from (C) without changing (C). The result is

$$\text{(D)}\qquad \sum_{\left\{\beta\in o^\times/o^\times_{n_1}\,\middle|\,\pi^{n_1-n_2}-\beta\in\mathfrak{p}^{n_1-f}-\mathfrak{p}^{n_1-f+1}\right\}} \nu^{-1}\chi_0(\beta)\left(\frac{\pi^{n_1-n_2}-\beta}{\pi^{n_1-f}}\right).$$

Thus (C) and (D) are equal in all cases.

Replace β by $\frac{1}{\beta}$ in the second integral of (B) to see that it is equal to the product of a positive constant and

$$\chi_0(-1)\left\{\sum_{\left\{\beta\in o^\times/o^\times_{n_1}\,\middle|\,\pi^{n_1-n_2}-\beta\in\mathfrak{p}^{n_1-f}-\mathfrak{p}^{n_1-f+1}\right\}} \nu\chi_0^{-1}(\beta)\chi_0\left(\frac{\pi^{n_1-n_2}-\beta}{\pi^{n_1-f}}\right)\right\}.$$

This is the product of $\chi_0(-1)$ and the complex conjugates of (D). Since $\overline{\Delta(\chi_0,\pi^{-f})} = \chi_0(-1)\Delta(\chi_0,\pi^{-f})$ the lemma is proved for $f > 0$.

If $f = 0$ then in the integral (A) we may replace $\chi_0(\gamma)$ by 1. If $n_1 = 0$ then $n_2 = 0$ and $m_1 = m_2 = 0$ so that μ, ν and M_0 are all trivial. The lemma is also; so we suppose $n_1 > 0$. If $n_1 > n_2$ then $K = k\oplus k$. Let $M_0(\alpha\oplus\beta) = \mu_1(\alpha)\nu_1(\beta)$. Then m_1 is the order of μ_1 and m_2 is the order of ν_1. Since $\mu_1\nu_1 = \mu\nu$ either $m_1 \geqslant n_1$ or $m_2 \geqslant n_1$. If $m_1 \geqslant n_1$ then $\ell_1 = m_1$ so that $\ell_2 = 0$. Then $\nu_1 = \nu$ and $\mu_1 = \mu$. If $m_2 \geqslant n_1$ then $\ell_2 = m_2$ so that $\ell_1 = 0$ which is contrary to the assumption that $\ell_1 \geqslant \ell_2$. Thus the lemma is trivial if $n_1 > n_2$; so we suppose that $n_1 = n_2$. Then $m_1 = m_2 = n_1$.

The integral (A) is equal to

$$\int_{O^\times} d\alpha \int_{o^\times} d\beta \left\{ \int_{o^\times} \xi_0 \left[\frac{\gamma}{\pi^{n_1}}(S\alpha - \beta - 1) \right] d\gamma \right\} M_0(\alpha)\nu^{-1}(\beta).$$

The inner integral is different from zero if and only if $S\alpha - \beta - 1 \in \mathfrak{p}^{n-1}$. If $n_1 > 1$ this implies that $S\alpha - 1 \notin \mathfrak{p}$. Set

$$M_0(\alpha)\nu^{-1}(\beta) \int_{o^\times} \xi_0 \left[\frac{\gamma}{\pi^{n_1}}(S\alpha - \beta - 1) \right] d\gamma = \psi(\alpha, \beta).$$

If $S\alpha - 1 \in \mathfrak{p}$ and $n_1 = 1$ then $\psi(\alpha, \beta) = \frac{|\pi|}{|\pi|-1} M_0(\alpha)\nu^{-1}(\beta)$. Since $n_1 = n_2$

$$\int_{o^\times} \psi(\alpha, \beta)\, d\beta = 0$$

if $S\alpha - 1 \in \mathfrak{p}$. Thus if

$$\varphi(x) = \int_{o^\times} \xi_0 \left(\frac{\gamma x}{\pi^{n_1}} \right) d\gamma$$

the integral (A) is equal to

$$\int_{\{\,\alpha \mid S\alpha - 1 \notin \mathfrak{p}\,\}} \int_{o^\times} M_0(\alpha)\nu^{-1}(S\alpha - 1)\nu^{-1}(\beta)\varphi\big((1-\beta)(S\alpha - 1)\big)\, d\beta\, d\alpha.$$

If $S\alpha - 1 \notin \mathfrak{p}$ then $\varphi\big((1-\beta)(S\alpha-1)\big) = \varphi(1-\beta)$. Moreover

$$\int_{o^\times} M_0(\alpha)\nu^{-1}(S\alpha - 1)\nu^{-1}(\beta)\varphi(1-\beta)\, d\beta$$

is equal to the product of $M_0(\alpha)\nu^{-1}(S\alpha - 1)$ and

$$\int_{o^\times_{n_1}} \nu^{-1}(\beta)\, d\beta + \frac{|\pi|}{|\pi| - 1} \int_{o^\times_{n_1 - 1} o^\times_{n_1}} \nu^{-1}(\beta)\, d\beta.$$

The first integral is equal to the measure of $o^\times_{n_1}$. The second is equal to

$$-\frac{|\pi|}{|\pi| - 1} \int_{o^\times_{n_1}} \nu^{-1}(\beta)\, d\beta = \frac{|\pi|}{1 - |\pi|} \text{ measure } o^\times_{n_1}.$$

Thus the integral (A) is the product of a positive constant and

$$\int_{\{\,\alpha \mid S\alpha - 1 \notin \mathfrak{p}\,\}} M_0(\alpha)\nu^{-1}(S\alpha - 1)\, d\alpha$$

$$= \int_{\{\,\alpha \mid S\alpha - 1 \notin \mathfrak{p}\,\}} M_0(\alpha)\nu^{-1}(N\alpha)\nu^{-1}\left(1 - N\left(\frac{\alpha - 1}{\alpha}\right)\right) d\alpha.$$

If $K = k \oplus k$ and $\ell_2 = 0$ the lemma is trivial. Suppose $K = k \oplus k$ and $\ell_2 > 0$. Let $\alpha = \alpha_1 \oplus \alpha_2$. If y is in $\mathfrak{p}^{\ell_2}$ then replacing α by $\alpha_1 \oplus \alpha_2(1+y)$ in the integrand does not change the value of the integral. The integrand becomes

$$M_0(\alpha)\nu^{-1}(N\alpha)\nu^{-1}\left(1 - N\left(\frac{\alpha - 1}{\alpha}\right) - \frac{\alpha_1 - 1}{\alpha_1}\ \frac{y}{\alpha_2(1+y)}\right).$$

The integral of this over $\mathfrak{p}^{\ell_2}$ is the measure of $\mathfrak{p}^{\ell_2}$ or zero according as $\alpha_1 - 1 \in \mathfrak{p}^{n_1-\ell_2}$ or not. The same observation applies to the first variable. Thus the integral is equal to

$$\int_{O^\times_{n_1-\ell_2, n_2-\ell_1}} M_0(\alpha)\nu^{-1}(\alpha)\nu^{-1}\left(1 - N\left(\frac{\alpha-1}{\alpha}\right)\right) d\alpha.$$

Since $n_1 - \ell_2 \geqslant \ell_1$, $n_2 - \ell_1 \geqslant \ell_2$ and $n_1 - \ell_2 + n_2 - \ell_1 \geqslant n_2$ the integrand is identically one. Thus the lemma is proved if $K = k \oplus k$.

If K is an unramified extension let $k_1 = k_2$ be the smallest integer greater than or equal to $\frac{n_1}{2}$. Let $y \in \pi^{k_1,k_2}O$. Replacing α by $\alpha(1+y)$ in the integrand does not change the value of the integral. Since $k_1 \geqslant \ell_1$ and $2k_1 \geqslant n_1$ the integrand becomes

$$M_0(\alpha)\nu^{-1}(N\alpha)\nu^{-1}\left(1 - N\left(\frac{\alpha-1}{\alpha}\right)\right)\nu^{-1}\left(1 + \gamma S\left(\frac{\alpha^s - 1}{1+y}\right)\right)$$

if $\gamma = -\frac{1}{N\alpha\left(1-N\left(\frac{\alpha-1}{\alpha}\right)\right)}$. The integral of this expression over $\pi^{k_1,k_2}O$ is the measure of $\pi^{k_1,k_2}O$ or zero according as $\alpha \in O^\times_{n_1-k_1,n_2-k_2}$ or not. Thus our integral is equal to

$$\int_{\left\{\alpha \in O^\times_{n_1-k_1,n_2-k_2} \,\middle|\, S\alpha - 1 \notin \mathfrak{p}\right\}} M_0(\alpha)\nu^{-1}(N\alpha)\nu^{-1}\left(1 - N\left(\frac{\alpha-1}{\alpha}\right)\right) d\alpha.$$

Since $n_1 - k_1 \geqslant \ell_1$ this is equal to

$$\int_{\left\{\alpha \in O^\times_{n_1-k_1,n_2-k_2} \,\middle|\, S\alpha - 1 \in \mathfrak{p}\right\}} \nu^{-1}\left(1 - N\left(\frac{\alpha-1}{\alpha}\right)\right) d\alpha.$$

If n_1 is even, $k_1 = \frac{n_2}{2}$ and the integrand is identically one. Thus the lemma is proved in this case. If n_1 is odd set $\frac{\alpha-1}{\alpha} = \pi^{n_1-k_2}\beta$ so that $\alpha = \frac{1}{1-\pi^{n_1-k_1}}\beta$. Since $2(n_1 - k_1) = n_1 - 1$ when n_1 is odd this integral is the product of a positive constant and

$$\sum_{\left\{\beta \in O/\Pi_O \,\middle|\, \pi^{n_1-1}N\beta \neq 1\right\}} \nu^{-1}(1 - \pi^{n_1-1}N\beta).$$

If $x \not\equiv \pmod{\mathfrak{p}}$ the equation $N\beta = x \pmod{\mathfrak{p}}$ has $\frac{|\pi|+1}{|\pi|}$ solutions modulo ΠO, otherwise it has just one. Thus if $n_1 > 1$ the sum equals

$$\frac{|\pi|+1}{|\pi|}\sum_{x \in o/\mathfrak{p}} \nu^{-1}(1 - \pi^{n_1-1}x) - \frac{1}{|\pi|}\nu^{-1}(1) = -\frac{1}{|\pi|}$$

and if $n_1 = 1$ it equals

$$\frac{|\pi|+1}{|\pi|}\sum_{o^\times/o_1^\times} \nu^{-1}(x) - \frac{1}{|\pi|}\nu^{-1}(1) = -\frac{1}{|\pi|}.$$

The lemma is completely proved.

If $K = k \oplus k$ we set $\epsilon = 1$; if K is an unramified extension of k we set $\epsilon = -1$, and if K is a ramified extension of k we set $\epsilon = 0$. If M is generalized a character of $K^\times$, if M_0 is its restriction to $O^\times$, and ν is a character of $o^\times$ set

$$T(M,\nu,n) = \big(1-|\pi|\big)\big(1-\epsilon|\pi|\big)\frac{\Delta(\chi_0,\pi^{-f})}{\big|\Delta(\chi_0,\pi^{-f})\big|}|\pi|^{\frac{n+f}{2}}\sum_{n_1+n_2=n} M\big((\pi^{n_1,n_2})^s\big)\Delta(M_0^{-1}\nu^{1+s},\pi^{n_1,n_2})$$

where the sum is taken over all n_1, n_2 for which π^{n_1,n_2} is defined.

Lemma 1.6. *Let ω and M be homomorphisms of $k^\times$ and $K^\times$ respectively into $\mathbf{C}^\times$. Suppose that the restriction of M to $k^\times$ is $\omega\chi$. Let ν and η be characters of $o^\times$ and let ω_0 be the restriction of ω to $o^\times$. Suppose that the order n of $\nu\eta\omega_0^{-1}$ is positive. Then, for all integers k and ℓ,*

$$\omega(\pi^n)T(M,\eta;k-n)T(M,\nu,\ell-n) = \frac{\Delta(\nu\eta\omega_0^{-1},\pi^{-n})}{\big|\Delta(\nu\eta\omega_0^{-1},\pi^{-n})\big|^2}\sum_\rho \Delta(\eta\rho^{-1},\pi^k)\Delta(\nu\rho^{-1},\pi^\ell)T(M,\rho,k+\ell)$$

where the sum is over all characters of o^*.

The formula of the lemma will be referred to as formula (E). Notice that all but a finite number of terms in the sum on the right are zero. The sum on the right is the product of $\big(1-|\pi|\big)\big(1-\epsilon|\pi|\big)|\pi|^{\frac{k+\ell+f}{2}}\frac{\Delta(\chi_0,\pi^{-f})}{|\Delta(\chi_0,\pi^{-f})|}$ and

$$\sum_{n_1+n_2=k+\ell} M\big((\pi^{n_1,n_2})^s\big)\sum_\rho\int_{O^\times} d\alpha\int_{o^\times} d\beta\int_{o^*} d\alpha\,\xi_0\Big(S(\pi^{n_1,n_2}\alpha)+\pi^\ell\beta+\pi^k\gamma\Big) \cdot M_0^{-1}(\alpha)\rho\Big(\frac{N\alpha}{\beta\gamma}\Big)\nu(\beta)\eta(\gamma).$$

Given ν, η, M, k, and ℓ there is a number m such that this integral is zero if the order of ρ is greater than m. Thus we may restrict the sum to a sum over the characters of $o^\times/o_m^\times$. Replace α by $\beta\alpha$, γ by $\beta\gamma$, and take one of the summations under the integral sign to obtain, if μ is the restriction of M_0 to $o^\times$,

$$\sum_{n_1+n_2=k+\ell} M\big((\pi^{n_1,n_2})^s\big)\int_{O^\times} d\alpha\int_{o^\times} d\beta\int_{o^\times} d\gamma\,\xi_0\Big[\Big(S(\pi^{n_1,n_2}\alpha)+\pi^\ell+\pi^k\gamma\Big)\Big] \cdot M^{-1}(\alpha)\eta(N\alpha)\nu\eta\mu^{-1}(\beta)\Bigg\{\sum_\rho \rho\eta^{-1}\Big(\frac{N\alpha}{\gamma}\Big)\Bigg\}.$$

The summation over ρ is different from zero if and only if $\gamma\equiv N\alpha \pmod{\mathfrak{p}}^m$. If $K=k\oplus k$ set $\lambda(\alpha)=N(\pi^{n_1-\ell,n_2}\alpha+\pi^{0,\ell})$; if K is an unramified extension of k set $\lambda(\alpha)=\pi^k N\Big(\alpha+\pi^{\frac{\ell-k}{2},\frac{\ell-k}{2}}\Big)$; if K is a ramified extension set $\lambda(\alpha)=\pi^k N(\alpha+\Pi^{-k+s\ell})$. The above expression is equal to

$$\sum_{n_1+n_2=k+\ell} M\big((\pi^{n_1,n_2})^s\big)\int_{O^\times} d\alpha\int_{o^\times} d\beta\,\xi_0\big(\beta\lambda(\alpha)\big)M_0^{-1}(\alpha)\eta(N\alpha)\nu\eta\mu^{-1}(\beta).$$

If the order r of $\nu\eta\mu^{-1}$ is not zero this is equal to

$$\text{(F)}\quad \Delta(\nu\eta\mu^{-1},\pi^{-r}) \cdot \sum_{n_1+n_2=k+\ell} M\big((\pi^{n_1,n_2})^s\big)\int_{\{\,\alpha\in O^\times\,\mid\,\pi^r\lambda(\alpha)\in o^\times\,\}} M_0^{-1}(\alpha)\eta(N\alpha)\mu\nu^{-1}\eta^{-1}\big(\pi^r\lambda(\alpha)\big)\,d\alpha.$$

If the order of $\nu\eta\mu^{-1}$ is zero it is equal to

$$\sum_{n_1+n_2=k+\ell} M\big((\pi^{n_1,n_2})^s\big)\bigg\{\int_{\{\,\alpha\in O^\times\,\mid\,\lambda(\alpha)\in o\,\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha + \frac{|\pi|}{|\pi|-1}\int_{\{\,\alpha\in O^\times\,\mid\,\pi\lambda(\alpha)\in o^\times\,\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha\bigg\},$$

an expression that will be labelled (G).

If K is an unramified extension of k, $r=n>0$ and the expression (F) is zero unless $k-\ell$ and $k-n$ are even. There is only one term in (F) and the corresponding integral is

$$\int_{\left\{\,\alpha\in O^\times\,\middle|\,\pi^{r+\frac{k-r}{2},r+\frac{k-r}{2}}\alpha+\pi^{r+\frac{\ell-r}{2},\frac{\ell-r}{2}}\in O^\times\,\right\}} M_0^{-1}(\alpha)\eta(N\alpha)\mu\nu^{-1} \cdot\eta^{-1}\left(N\left(\pi^{r+\frac{k-r}{2},r+\frac{k-r}{2}}\alpha+\pi^{r+\frac{\ell-r}{2},r+\frac{\ell-r}{2}}\right)\right)d\alpha.$$

Set $M_0^s(\alpha)=M_0(\alpha^s)$. Since $M_0^{-1}\eta^{1+s}\cdot M_0^{-s}\nu^{1+s}=(\mu^{-1}\nu\eta)^{1+s}$ this integral is equal to

$$\frac{\Delta\Big(M_0^{-1}\eta^{1+s},\pi^{\frac{k-n}{2},\frac{k-n}{2}}\Big)\Delta\Big(M_0^{-1}\nu^{1+s},\pi^{\frac{\ell-n}{2},\frac{\ell-n}{2}}\Big)}{\Delta\big((\mu^{-1}\nu\eta)^{1+s},\pi^{-r,-r}\big)}.$$

Putting everything together and appealing to Lemmas 1.1 and 1.5, we see that the right side of (E) is equal to

$$\chi(\pi^n)M\left(\left(\pi^{\frac{k+\ell}{2},\frac{k+\ell}{2}}\right)^s\right)\big(1-|\pi|^2\big)^2|\pi|^{\frac{k+\ell}{2}-n} \cdot\Delta\Big(M_0^{-1}\eta^{1+s},\pi^{\frac{k-n}{2},\frac{k-n}{2}}\Big)\Delta\Big(M_0^{-1}\nu^{1+s},\pi^{\frac{\ell-n}{2},\frac{\ell-n}{2}}\Big).$$

Since $\Delta(M^{-\sigma},\pi^{m,m})=\Delta(M^{-1},\pi^{m,m})$ it is equal to the left side.

If K is a ramified extension of k and $r>0$ there is only one term in the sum (F) and the integral appearing in that term is

$$\int_{\{\,\alpha\in O^\times\,\mid\,\Pi^{r+k}\alpha+\Pi^{r+s\ell}\in O^\times\,\}} M_0^{-1}(\alpha)\eta(N\alpha)\mu\nu^{-1}\eta^{-1}\Big(N(\Pi^{r+k}\alpha+\Pi^{r+s\ell})\Big)d\alpha.$$

Replace α by $\Pi^{-\ell+s\ell}\alpha$ to obtain

$$M(\Pi^{\ell-s\ell})\int_{\{\,\alpha\in O^\times\,\mid\,\Pi^{r+k}\alpha+\Pi^{r+\ell}\in O^\times\,\}} M_0^{-1}(\alpha)\eta(N\alpha)\mu\nu^{-1}\eta^{-1}\Big(N(\Pi^{r+k}\alpha+\Pi^{r+\ell})\Big)d\alpha.$$

If $r>f$ then $r=n$, the order of $(\mu\nu^{-1}\eta^{-1})^{1+s}$ is $2n-f$, 0, $r+k=2n-f+(k-n+f)$, $r+\ell=2n-f+(k-n+f)$. If $r<f$ then $n=f$, the order of $(\mu\nu^{-1}\eta^{-1})^{1+s}$ is $(r,0)$, and $r+k=r+(k-n+f)$, $r+\ell=r+(\ell-n+f)$. If $r=f$ then $n\leqslant f$, the order

of $(\mu\nu^{-1}\eta^{-1})^{1+s}$ is $(n,0)$, $r+k=n+(k-n+f)$, $r+\ell=n+(k-\ell+f)$. According to Lemma 1.4 the above expression is equal to

$$M(\Pi^{\ell-s\ell})\frac{\Delta(M_0^{-1}\eta^{1+s},\pi^{k-n,0})\Delta(M_0^{-s}\nu^{1+s},\pi^{\ell-n,0})}{\Delta\big((\mu^{-1}\nu\eta)^{1+s},\pi^{-r_1,-r_2}\big)}$$

if (r_1,r_2) is the order of $(\mu^{-1}\nu\eta)^{1+s}$. Observe that

$$\Delta(M_0^{-1}\nu^{1+s},\pi^{\ell-n,0})=M(\Pi^{(n-\ell)(1-s)})\Delta(M_0^{-1}\nu^{1+s},\pi^{(\ell-n,0)}).$$

Appealing to Lemmas 1.1 and 1.5, we see that, if $r>0$, the right side of (E) is equal to

$$M(\Pi^{n(1+s)})M(\Pi^{(k+\ell-2n)s})\big(1-|\pi|\big)^2|\pi|^a\left\{\frac{\Delta(\chi_0,\pi^{-f})}{\big|\Delta(\chi_0\pi^{-f})\big|}\right\}^2 \\ \Delta(M_0^{-1}\eta^{1+s},\pi^{k-n,0})\Delta(M_0^{-1}\nu^{1+s},\pi^{\ell-n,0})$$

with $a=\frac{k+\ell+f}{2}-\frac{n}{2}+\frac{r}{2}-\frac{r_1}{2}=\frac{k-n+f}{2}+\frac{\ell-n+f}{2}$. This is obviously equal to the left side.

If $r=0$ the expression (G) is equal to the product of $M(\Pi^{(k+\ell)s})$ and

$$\int_{\{\alpha\in O^\times\ |\ \Pi^k\alpha+\Pi^{s\ell}\in O\}}M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha \\ +\frac{|\pi|}{|\pi|-1}\int_{\{\alpha\in O^\times\ |\ \Pi^{k+1}\alpha+\Pi^{1+s\ell}\in O^\times\}}M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha.$$

After a change of variables this becomes

$$M(\Pi^{\ell-s\ell})\left\{\int_{\{\alpha\in O^\times\ |\ \Pi^k\alpha+\Pi^{\ell}\alpha\in O\}}M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha \right.\\ \left. +\int_{\{\alpha\in O^\times\ |\ \Pi^{k+1}\alpha+\Pi^{\ell+1}\in O^\times\}}M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha\frac{|\pi|}{|\pi|-1}\right\}.$$

Since $(\mu^{-1}\nu\eta)^{1+s}$ will also be trivial this is equal to

$$M(\Pi^{\ell-2s})\Delta(M_0^{-1}\eta^{1+s},\pi^{k-n,0})\Delta(M_0^{-s}\eta^{1+s},\pi^{\ell-n,0})$$

because $n=f$ in this case. Thus the right side of (E) is equal to

$$M(\Pi^{n(1+s)})M(\Pi^{(k+\ell-2n)s})\big(1-|\pi|\big)^2|\pi|^{\frac{k+\ell}{2}}\left\{\frac{\Delta(\chi_0,\pi^{-f})}{\big|\Delta(\chi_0,\pi^{-f})\big|}\right\}^2 \\ \cdot\Delta(M_0^{-1}\eta^{1+s},\pi^{k-n,0})\Delta(M_0^{-1}\nu^{1+s},\pi^{\ell-n,0}).$$

Since $\chi(\Pi^{n(1+s)})=1$ and $\Pi^{n(1+s)}=\pi^n$, it is equal to the left side.

It remains to consider the case that $K=k\oplus k$. Then $r-n$ is not zero and (F) is equal to the product of $\Delta(\nu\eta\mu^{-1},\pi^n)$ and

$$\sum_{(n_1+n_2=k+\ell)}\sum_{(m_1+m_2=r)}M(\pi^{n_1,n_2})\int_{\{\alpha\in O^\times\ |\ \pi^{n_1+m_1-\ell,n_2+m_2}\alpha+\pi^{m_1,m_2+\ell}\in O^\times\}}M_0^{-1}(\alpha)\eta(N\alpha) \\ \cdot\mu\nu^{-1}\eta^{-1}\Big(N(\pi^{n_1+m_1-\ell,n_2+m_2}\alpha+\pi^{m_1,m_2+\ell})\Big)\,d\alpha.$$

This is equal to

$$\sum_{(n_1+n_2=k+\ell)}\sum_{(m_1+m_2=r)} M(\pi^{n_2,n_1})\Delta(M_0^{-1}\eta^{1+s},\pi^{n_1+m_1-n-\ell,n_2+m_2-n})$$
$$\Delta(M_0^{-1}\nu^{1+s},\pi^{m_2+\ell-n,m_1-n})$$

divided by $\Delta\big((\mu^{-1}\nu\eta)^{1+s},\pi^{-n,-n}\big)$. Replace m_1 by m_1+n, m_2 by $m_2-\ell+n$, interchange the order of summation and replace n_1 by $n_1-m_1+\ell$, n_2 by $n_2-m_2+\ell$ to see that the sum is equal to

$$\sum_{(m_1+m_2=\ell-n)}\sum_{(n_2+n_1=k-n)} M(\pi^{n_1,n_2})M(\pi^{m_1,m_2})M(\pi^{n,n})$$
$$\Delta(M_0^{-1}\eta^{1+s},\pi^{n_1,n_2})\Delta(M_0^{-1}\eta^{1+s},\pi^{m_2,m_1}).$$

Appealing to Lemmas 1.1 and 1.5, we see that the right side of (E) is equal to

$$\omega(\pi^n)\big(1-|\pi|\big)^4|\pi|^{\frac{k+\ell}{2}+n}$$
$$\cdot\sum_{(n_1+n_2=k-n)}\sum_{(m_1+m_2=\ell-n)} M(\pi^{n_2,n_1})M(\pi^{m_2,m_1})\Delta(M_0^{-1}\eta^{1+s},\pi^{n_1,n_2})\Delta(M_0^{-1}\nu^{1+s},\pi^{m_1,m_2}).$$

This is of course just the left side.

Lemma 1.7. *Let ω and M be homomorphisms of $k^\times$ and $K^\times$ respectively into $\mathbf{C}^\times$. Suppose that the restriction of M to $k^\times$ is $\omega\chi$. Let ν and η be characters of $o^\times$ and let ω_0 be the restriction of ω to $o^\times$. If $\nu\eta\omega_0^{-1}$ is trivial then for all integers k and ℓ*

$$\sum_{-\infty}^{-2}-\omega(\pi^{-m})T(M,\eta,k+m)T(M,\nu,\ell+m)$$
$$+\frac{1}{|\pi|-1}\omega(\pi)T(M,\eta,k-1)T(M,\nu,\ell-1)+\omega_0(-1)\delta_{\ell,k}\omega(\pi^\ell)$$
$$=\sum_\rho\Delta(\eta,\rho^{-1},\pi^k)\Delta(\nu\rho^{-1},\pi^\ell)T(M,\rho,k+\ell).$$

$\delta_{\ell,k}$ is of course Kronecker's delta. For brevity denote the left side by $L_{k,\ell}$ and the right side by $R_{k,\ell}$. Suppose at first that $k\ll 0$ and $\ell\ll 0$. Then $L_{k,\ell}=\omega_0(-1)\omega(\pi^\ell)\delta_{\ell,k}$. The only terms which contribute anything to the right hand side are those for which $\operatorname{order}(\rho)=-k$ and $\operatorname{order}(\rho)=-\ell$. Thus the right side is zero if $k\neq\ell$. Suppose $\operatorname{order}(\rho)=-\ell$ and $k=\ell$. The order, (r_1,r_2), of $M_0^{-1}\rho^{1+s}$ is $(-\ell,-\ell)$ if $K=k\oplus k$ or K is an unramified extension of k. It is $(-2\ell-f,0)$ if K is a ramified extension of k. Moreover if $n_1+n_2=k+\ell$

$$\Delta(M_0^{-1}\rho^{1+s},\pi^{n_1,n_2})=0$$

if $-n_1\neq r_1+f$. The orders of $\eta^{-1}\rho$ and $\nu^{-1}\rho$ are both $-\ell$. The orders of

$$(M_0^{-1}\rho^{1+s})^{-1}(\eta^{-1}\rho)^{1+s}=M_0\eta^{-1-s}$$

and

$$(M_0^{-1}\rho^{1+s})^{-1}(\nu^{-1}\rho)^{1+s}=M_0\nu^{-1-s}$$

are independent of ℓ. Moreover the restriction of $M_0^{-1}\rho^{1+s}$ to $o^\times$ is equal to $\chi_0(\eta^{-1}\rho)(\nu^{-1}\rho)$. According to Lemma 1.5

$$\Delta(M_0^{-1}\rho^{1+s},\pi^{-r_1-f_1-r_2}) \\ = \frac{M(\Pi^{\ell-s\ell})}{\chi(\pi^\ell)}\Delta(\eta^{-1}\rho,\pi^\ell)\Delta(\nu^{-1}\rho,\pi^\ell)\frac{\overline{\Delta(\chi_0,\pi^{-f})}}{\left|\Delta(\chi_0,\pi^{-f})\right|}\frac{1-|\pi|}{1-\epsilon|\pi|}|\pi|^{\frac{r_1+r_2+2\ell}{2}}.$$

Since $\Delta(\eta^{-1}\rho,\pi^\ell)=\eta\rho^{-1}(-1)\overline{\Delta(\eta\rho^{-1},\pi^\ell)}$ the term corresponding to a ρ with order $\rho=-\ell$ is

$$\frac{M(\Pi^{\ell-s\ell})M\big((\pi^{-r_1-f,-r_2})^s\big)}{\chi(\pi^\ell)}\nu\eta(-1)\frac{1}{|\pi|^\ell\big(1-|\pi|\big)^2}.$$

This is clearly equal to

$$\frac{M(\pi^\ell)}{\chi(\pi^\ell)}\omega_0(-1)\frac{1}{|\pi|^\ell\big(1-|\pi|\big)^2}=\frac{\omega(\pi^\ell)\omega_0(-1)}{|\pi|^\ell\big(1-|\pi|\big)^2}.$$

Since the number of such characters is $|\pi|^\ell\big(1-|\pi|\big)^2$ the lemma is valid if $k\ll 0$ and $\ell\ll 0$.

Thus to prove the lemma it is enough to show that

$$L_{k+1,\ell+1}-\omega(\pi)L_{k,\ell}=R_{k+1,\ell+1}=\omega(\pi)R_{k,\ell}$$

for all k and ℓ. The left-hand side is equal to

$$\frac{\omega(\pi)}{|\pi|-1}T(M,\eta,k)T(M,\nu,\ell)-\frac{|\pi|}{|\pi|-1}\omega^2(\pi)T(M,\eta,k-1)T(M,\nu,k-1).$$

Suppose K is an unramified extension. If $k-\ell$ is odd both of these terms are zero and so is the right side. We suppose then that $k-\ell$ is even. If k is even only the first of these two terms can be different from zero. If k is odd only the second can be. Remembering that $\chi(\pi)=-1$ so that $\omega(\pi)=-M(\pi^{1,1})$ we apply formula (G) to see that the right side is the product of $\big(1-|\pi|^2\big)|\pi|^{\frac{k+\ell}{2}}M\Big(\pi^{\frac{k+\ell}{2},\frac{k+\ell}{2}+1}\Big)$ and

$$|\pi|\int_{\left\{\alpha\in O^\times\,\middle|\,\pi^{k+1}N\left(\alpha+\pi^{\frac{\ell-k}{2},\frac{\ell-k}{2}}\right)\in o\right\}}M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha \\ +\frac{|\pi|^2}{|\pi|-1}\int_{\left\{\alpha\in O^\times\,\middle|\,\pi^{\frac{k+2}{2},\frac{k+2}{2}}\alpha+\pi^{\frac{\ell+2}{2},\frac{\ell+2}{2}}\in O^\times\right\}}M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha \\ +\int_{\left\{\alpha\in O^\times\,\middle|\,\pi^{k}N\left(\alpha+\pi^{\frac{\ell-k}{2},\frac{\ell-k}{2}}\right)\in o\right\}}M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha \\ +\frac{|\pi|}{|\pi|-1}\int_{\left\{\alpha\in O^\times\,\middle|\,\pi^{\frac{k+1}{2},\frac{k+1}{2}}\alpha+\pi^{\frac{\ell+1}{2},\frac{\ell+1}{2}}\in O^\times\right\}}M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha.$$

If k is even $\pi^{k+1}N\Big(\alpha+\pi^{\frac{\ell-k}{2},\frac{\ell-k}{2}}\Big)\in o$ if and only if $\pi^k N\Big(\alpha+\pi^{\frac{\ell-k}{2},\frac{\ell-k}{2}}\Big)\in o$ so that, if k is even, this expression equals

$$\big(|\pi|+1\big)\Bigg\{\int_{\left\{\alpha\in O^\times\;\middle|\;\pi^{\frac{k}{2},\frac{k}{2}}\alpha+\pi^{\frac{\ell}{2},\frac{\ell}{2}}\in O^\times\right\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha + \frac{|\pi|^2}{|\pi|^2-1}\int_{\left\{\alpha\in O^\times\;\middle|\;\pi^{\frac{k+2}{2},\frac{k+2}{2}}\alpha+\pi^{\frac{\ell+2}{2},\frac{\ell+2}{2}}\in O^\times\right\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha\Bigg\}$$

which equals

$$\big(|\pi|+1\big)\Delta\Big(M_0^{-1}\eta^{1+s},\pi^{\frac{k}{2},\frac{k}{2}}\Big)\Delta\Big(M_0^{-1}\nu^{1+s},\pi^{\frac{\ell}{2},\frac{\ell}{2}}\Big).$$

The identity, for even k, follows immediately. If k is odd the expression above simplifies to

$$\big(|\pi|+1\big)\Bigg\{\int_{\left\{\alpha\in O^\times\;\middle|\;\pi^{\frac{k-1}{2},\frac{k-1}{2}}\alpha+\pi^{\frac{\ell-1}{2},\frac{\ell-1}{2}}\in O\right\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha + \frac{|\pi|^2}{|\pi|^2-1}\int_{\left\{\alpha\in O^\times\;\middle|\;\pi^{\frac{k+1}{2},\frac{k+1}{2}}\alpha+\pi^{\frac{\ell+1}{2},\frac{\ell+1}{2}}\in O^\times\right\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha\Bigg\}$$

which equals

$$\big(|\pi|+1\big)\Delta\Big(M_0^{-1}\eta^{1+s},\pi^{\frac{k-1}{2},\frac{k-1}{2}}\Big)\Delta\Big(M_0^{-1}\nu^{1+s},\pi^{\frac{\ell-2}{2},\frac{\ell-2}{2}}\Big).$$

The identity, for odd k, follows immediately.

Suppose $f>0$. If μ is the restriction of M_0 to $o^\times$ then $\nu\eta\mu^{-1}=\chi_0$. According to (F), $R_{k,\ell}$ is equal to the product of

$$\big(1-|\pi|\big)|\pi|^{\frac{k+\ell+f}{2}}\frac{\Delta(\chi_0,\pi^{-f})}{\big|\Delta(\chi_0,\pi^{-f})\big|}\Delta(\chi_0,\pi^{-f})M(\Pi^{s(k+\ell)})=\chi_0(-1)|\pi|^{\frac{k+\ell}{2}+f}M(\Pi^{s(k+\ell)})$$

and

$$M(\Pi^{\ell-s\ell})\int_{\left\{\alpha\in O^\times\;\middle|\;\Pi^{f+k}\alpha+\Pi^{f+\ell}\in O^\times\right\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha.$$

On the other hand

$$\frac{\omega(\pi)}{|\pi|-1}T(M,\eta,k)T(M,\nu,\ell)-\frac{|\pi|}{|\pi|-1}\omega^2(\pi)T(M,\eta,k-1)T(M,\nu,\ell-1)$$

is equal to the product of $\bigl(1-|\pi|\bigr)^2|\pi|^{\frac{k+\ell}{2}+f}\chi_0(-1)M(\Pi^{\ell+1+s(k+1)})$ and

$$\frac{1}{|\pi|-1}\biggl\{\int_{\{\,\alpha\in O^\times\,|\,\Pi^{f+k}\alpha+\pi^{f+\ell}\in O\,\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha + \frac{|\pi|}{|\pi|-1}\int_{\{\,\alpha\in O^\times\,|\,\Pi^{f+k+1}\alpha+\pi^{f+\ell+1}\in O^\times\,\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha\biggr\}$$
$$-\frac{1}{|\pi|-1}\biggl\{\int_{\{\,\alpha\in O^\times\,|\,\Pi^{f+k-1}\alpha+\Pi^{f+\ell-1}\in O\,\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha + \frac{|\pi|}{|\pi|-1}\int_{\{\,\alpha\in O^\times\,|\,\Pi^{f+k}\alpha+\Pi^{f+\ell}\in O^\times\,\}} M_0^{-1}(\alpha)\eta(N\alpha)d\alpha\biggr\}.$$

Some simple rearrangements show that this is equal to

$$\frac{-1}{\bigl(|\pi|-1\bigr)^2}\int_{\{\,\alpha\in O^\times\,|\,\Pi^{f+k}\alpha+\Pi^{f+\ell}\in O^\times\,\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha + \frac{|\pi|}{\bigl(|\pi|-1\bigr)^2}\int_{\{\,\alpha\in O^\times\,|\,\Pi^{f+k+1}\alpha+\Pi^{f+\ell+1}\in O^\times\,\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha.$$

The identity follows immediately.

Finally we have to treat the case that $K=k\oplus k$. It is enough to verify that

$$\sum_{p=0}^{\infty}|\pi|^p\omega(\pi^p)\bigl\{L_{k+1-p,\ell+1-p}-\omega(\pi)L_{k-p,\ell-p}\bigr\} = \sum_{p=0}^{\infty}|\pi|^p\omega(\pi^p)\bigl\{R_{k+1-p,\ell+1-p}-\omega(\pi)R_{k-p,\ell-p}\bigr\}.$$

The left side is equal to

$$\frac{\omega(\pi)}{|\pi|-1}T(M,\eta,k)T(M,\nu,\ell).$$

For brevity set

$$\psi(n_1,n_2;m_1,m_2)=\int_{\{\,\alpha\in O^\times\,|\,\pi^{n_1,n_2}\alpha+\pi^{m_1,m_2}\in O^\times\,\}} M_0^{-1}(\alpha)\eta(N\alpha)\,d\alpha.$$

Apply formula (G) to see that $R_{k,\ell}$ is the product of $\bigl(1-|\pi|\bigr)^2|\pi|^{\frac{k+\ell}{2}}$ and

$$\sum_{n_1+n_2=k+\ell}M(\pi^{n_2,n_1})\sum_{m_1+m_2=0}\sum_{q\leqslant 0}\psi(n_1+m_1+q-\ell,n_2+m_2;m_1+q,m_2+\ell) + \frac{|\pi|}{|\pi|-1}\sum_{n_1+n_2=k+\ell}M(\pi^{n_2,n_1})\sum_{m_1+m_2=0}\psi(n_1+m_1+1-\ell,n_2+m_2;m_1+1,m_2+\ell).$$

Thus, $\sum_{p=0}^{\infty}|\pi|^p\omega(\pi^p)R_{k-p,\ell-p}$ is the product of $\bigl(1-|\pi|\bigr)^2|\pi|^{\frac{k+\ell}{2}}$ and

$$\sum_{n_1+n_2=k+\ell} M(\pi^{n_2,n_1}) \sum_{m_1+m_2=0} \sum_{p\leqslant 0}\sum_{q\leqslant 0}\psi(n_1+m_1+q-\ell, n_2+m_2+p; m_1+q, m_2+\ell+p)$$
$$+\frac{|\pi|}{|\pi|-1}\sum_{n_1+n_2=k+\ell}M(\pi^{n_2,n_1})$$
$$\cdot\sum_{m_1+m_2=0}\sum_{p\leqslant 0}\psi(n_1+m_1+1-\ell, n_2+m_2+p; m_1+1, m_2+\ell+p).$$

Now

$$\frac{\omega(\pi)}{|\pi|-1}T(M,\eta,k)T(M,\nu,\ell)$$

is equal to the product of $\bigl(|\pi|-1\bigr)^3\omega(\pi)|\pi|^{\frac{k+\ell}{2}}$ and

$$\sum_{(m_1+m_2=\ell)}\sum_{(n_1+n_2=k)} M(\pi^{n_2,n_1})M(\pi^{m_2,m_1})\Delta(M_0^{-1}\eta^{1+s},\pi^{n_1,n_2})\Delta(M_0^{-1}\nu^{1+s},\pi^{m_1,m_2}).$$

Replace n_1 by $n_1-m_2=n_1+m_1-\ell$, n_2 by $n_2-m_1=n_1+m_2-\ell$, and then m_2 by $m_2+\ell$ to obtain

$$\sum_{n_1+n_2=k+\ell}M(\pi^{n_2,n_1})\sum_{m_1+m_2=0}\Delta(M_0^{-1}\eta^{1+s},\pi^{n_1+m_1-\ell,n_2+m_2})\Delta(M_0^{-1}\nu^{1+s},\pi^{m_1,m_2+\ell}).$$

According to Lemma 1.4 this is the sum of

$$\sum_{(n_1+n_2=k+\ell)}\sum_{(m_1+m_2=0)}M(\pi^{n_2,n_1})\sum_{p\leqslant 0}\sum_{q\leqslant 0}\psi(n_1+m_1+q-\ell,n_2+m_2+p;m_1+q,m_2+\ell+p)$$

and

$$\frac{|\pi|}{|\pi|-1}\sum_{(n_1+n_2=k+\ell)}\sum_{(m_1+m_2=0)}M(\pi^{n_1,n_2})$$
$$\cdot\sum_{p\leqslant 0}\psi(n_1+m_1+1-\ell,n_2+m_2+p,m_1+1,m_2+\ell+p)$$

and

$$\frac{|\pi|}{|\pi|-1}\sum_{(n_1+n_2=k+\ell)}\sum_{(m_1+m_2=0)}M(\pi^{n_2,n_1})$$
$$\cdot\sum_{q\leqslant 0}\psi(n_1+m_1+q-\ell,n_2+m_2+1,m_1+q,m_2+\ell+1)$$

and

$$\left(\frac{|\pi|}{|\pi|-1}\right)^2\sum_{(n_1+n_2=k+\ell)}\sum_{(m_1+m_2=0)}M(\pi^{n_2,n_1})$$
$$\cdot\psi(n_1+m_1+1-\ell,n_2+m_2+1;m_1+1,m_2+\ell+1).$$

On the other hand

$$\sum_{p=0}^{\infty}|\pi|^p\omega(\pi^p)R_{k+1-p,\ell+1-p}-\omega(\pi)\sum_{p=0}^{\infty}|\pi|^p\omega(\pi^p)R_{k-p,\ell-p}$$

is equal to $\big(1-|\pi|\big)^2|\pi|^{\frac{k+\ell}{2}}\omega(\pi)$ times the sum of

$$\frac{|\pi|^2}{|\pi|-1}\sum_{n_1+n_2=k+\ell}M(\pi^{n_2,n_1})\sum_{m_1+m_2=0}\psi(n_1+m_1+1-\ell,n_2+m_2+1,m_1+1,m_2+\ell+1)$$

and

$$\frac{|\pi|^2}{|\pi|-1}\sum_{n_1+n_2=k+\ell}M(\pi^{n_2,n_1})\sum_{m_1+n_2=0}\sum_{p\leqslant 0}\psi(n_1+m_1+1-\ell,n_2+m_2+p;m_1+1,m_2+\ell+p)$$

and

$$|\pi|\sum_{n_1+n_2=k+\ell}M(\pi^{n_2,n_1})\sum_{m_1+m_2=0}\sum_{q\leqslant 0}\psi(n_2+m_1+q-\ell,n_2+m_2+1,m_1+q,m_2+\ell+1)$$

and

$$|\pi|\sum_{n_1+n_2=k+\ell}M(\pi^{n_2,n_1})$$
$$\cdot\sum_{m_1+m_2=0}\sum_{p\leqslant 0}\sum_{q\leqslant 0}\psi(n_1+m_1+q-\ell,n_2+m_2+p;m_1+q,m_2+\ell+p),$$

the contributions of the first infinite series, and

$$-\frac{|\pi|}{|\pi|-1}\sum_{n_1+n_2=k+\ell}M(\pi^{n_2,n_1})$$
$$\cdot\sum_{m_1+m_2=0}\sum_{p\leqslant 0}\psi(n_1+m_1+1-\ell,n_2+m_2+p;m_1+1,m_2+\ell+p)$$

and

$$-\sum_{n_1+n_2=k+\ell}M(\pi^{n_2,n_1})\sum_{m_1+m_2=0}\sum_{p\leqslant 0}\sum_{q\leqslant 0}\psi(n_1+m_1+q-\ell,n_2+m_2+p;m_1+q,m_2+\ell+p),$$

the contributions of the second. The identity can now be verified by inspection.

2. Representations of the general linear group in two variables over a non-archimedean field

This paragraph is, in its essentials, a recapitulation of work of Gelfand, Graev, and Kirillov. We adhere to the notation of the previous paragraph. Let $G_k=\mathrm{GL}(2,k)$ and let $G_O=\mathrm{GL}(2,O)$. A is the group of diagonal matrices and N is the group of matrices of the term $\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)$.

A representation σ of G_k on a vector space V, over $\mathbf{C}$, will be called quasi-simple if

(i) The stabilizer of every vector in V is an open subgroup of G_k.
(ii) If $\alpha\in k^\times$ then $\sigma\big(\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)\big)$ is a scalar multiple of the identity.

Lemma 2.1. *Suppose σ is a quasi-simple irreducible representation of G_k on the vector space V. V contains a non-zero vector invariant under N_k if and only if V is finite-dimensional.*

First of all suppose that V contains a non-zero vector v whose stabilizer contains N_k. Let $H = \{\, g \in G_k \mid \sigma(g)v = \lambda v \text{ with } \lambda \in \mathbf{C} \,\}$. Since V is spanned by the set $\{\, \sigma(g)v \mid g \in G_k \,\}$ it is sufficient to show that H is of finite index in G_k. Since H contains the diagonal matrices together with an open subgroup of G_k the image of H under the determinant function is of finite index in $k^\times$. Thus it is sufficient to show that $H_0 = \{\, g \in G_k \mid \sigma(g)v = v \,\}$ contains all matrices of determinant 1.

Let W be the space of column vectors of length 2 with entries from k. Let us show first that if $w \in W$ and $w \neq 0$ there is an h in H and an x in $k^\times$ such that

$$w = h\begin{pmatrix} x \\ 0 \end{pmatrix}.$$

If the second coordinate of w is zero this is clear. Since the stabilizer of v is open in G_k there is g in H_0 such that

$$g\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

with $\beta \neq 0$. Then

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha + \beta x \\ \beta \end{pmatrix}.$$

If the second coordinate of w is not 0 we can choose x so that w is a scalar multiple of the vector on the right.

In particular H_0 contains a matrix of the form $\left(\begin{smallmatrix} 0 & b \\ c & d \end{smallmatrix}\right)$. Since

$$\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & -d/c \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1/c \\ 1/b & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}\begin{pmatrix} x/b & 1/c \\ 1/b & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ cx/b & 1 \end{pmatrix},$$

H_0 contains all matrices of the form $\left(\begin{smallmatrix} 1 & 0 \\ y & 1 \end{smallmatrix}\right)$. Since

$$\begin{aligned}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} &= \begin{pmatrix} 1+xz & z \\ x & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ x/(1+xz) & 1 \end{pmatrix}\begin{pmatrix} 1+xz & 0 \\ 0 & 1/(1+xz) \end{pmatrix}\begin{pmatrix} 1 & z/(1+xz) \\ 0 & 1 \end{pmatrix},\end{aligned}$$

if $1 + xz \neq 0$, H_0 contains all diagonal matrices of determinant 1. Since

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \alpha x & \beta \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \alpha y \\ \alpha x & \beta + \alpha xy \end{pmatrix}$$

H_0 contains all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $a \neq 0$, which have determinant 1. Since

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & b \\ c & d \end{pmatrix} = \begin{pmatrix} cx & b + dx \\ c & d \end{pmatrix},$$

H_0 contains all matrices of determinant 1.

Conversely if V is finite-dimensional the kernel of σ is an open subgroup of G_k and there is an $\epsilon > 0$ such that $\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)$ belongs to this kernel if $|x| < \epsilon$. Since

$$\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{\alpha} & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & \alpha x\\ 0 & 1\end{pmatrix}$$

and for any x there is an α in $k^\times$ such that $|\alpha x| < \epsilon$, the kernel of σ contains N_k.

Corollary. *If σ is a finite-dimensional quasi-simple irreducible representation of G_k then σ is one-dimensional and there is a continuous homomorphism ρ of $k^\times$ into $\mathbf{C}^\times$ such that $\sigma(g) = \rho(\det g)$*

Since the kernel of σ contains N_k together with an open subgroup of G_k the above discussion shows that it contains every matrix of determinant 1. Also the inverse image of the group of non-zero matrices is of finite index in G_k. Thus if $g \in G_k$ there is a λ in $\mathbf{C}^\times$ and a positive integer n such that $\sigma(g)^n - \lambda = 0$. Thus $\sigma(g)$ is semi-simple. The corollary follows immediately.

Again we fix a character ξ_0 of the additive group of k such that the largest ideal on which ξ_0 is trivial is 0.

Lemma 2.2. *Suppose σ is an infinite-dimensional quasi-simple irreducible representation of G_k on V. Let W be the set of all vectors v in V such that for some ideal $\mathfrak{a}$ of k*

$$\int_{\mathfrak{a}} \overline{\xi_0(x)}\sigma\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) v\, dx = 0.$$

Then W is a subspace of V. Let $U = V/W$. If $v \in V$ let φ_v be the function $k^\times$ with values in U defined by

$$\varphi_v(\alpha) = \psi\left(\pi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) v\right)$$

where φ is the natural mapping from V to U. The map $v \to \varphi_v$ is an injection of V into the space of functions on $k^\times$ with values in U.

Since the stabilizer of v in N_k is an open subgroup of N_k the function $\sigma\big(\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\big)v$ takes only a finite number of values on $\mathfrak{a}$. Thus the integral involves no limiting processes and is well-defined. If $\mathfrak{a} \subset \mathfrak{b}$ then

$$\int_{\mathfrak{b}} \overline{\xi_0(x)}\sigma\left(\begin{pmatrix}1 & x\\ 0 & z\end{pmatrix}\right) v\, dx = \sum_y \overline{\xi_0(y)}\sigma\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)\left[\int_{\mathfrak{a}} \overline{\xi_0(x)}\sigma\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) v\, dx\right]$$

where the sum is taken over a system of representatives of $\mathfrak{b}/\mathfrak{a}$. It follows immediately that if the integral vanishes for a given ideal then it vanishes for all larger ideals. A simple argument now shows that W is a subspace of V.

If φ_v vanishes identically then for every α in $K^\times$ there is an ideal $\mathfrak{a}(\alpha)$ such that

$$\int_{\mathfrak{a}(\alpha)} \overline{\xi_0(\alpha x)}\sigma\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right) v\, dx = 0.$$

If $\beta \in o^\times$ and $\sigma\Big(\big(\begin{smallmatrix}\beta & 0\\ 0 & 1\end{smallmatrix}\big)\Big)v = v$ then

$$\int_{\mathfrak{a}} \overline{\xi_0(\alpha\beta x)}\sigma\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)v\,dx = \sigma\left(\begin{pmatrix}\beta^{-1} & 0\\ 0 & 1\end{pmatrix}\right)\int_{\mathfrak{a}} \overline{\xi_0(\alpha x)}\sigma\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)v\,dx.$$

Since the set of all β in $k^\times$ such that $\sigma\Big(\big(\begin{smallmatrix}\beta & 0\\ 0 & 1\end{smallmatrix}\big)\Big)v = v$ is an open subgroup of $k^\times$, there is for each integer n an ideal $\mathfrak{a}_n$ such that if $\mathfrak{a}_n \subseteq \mathfrak{a}$ and $\alpha \in o^\times$

$$\int_{\mathfrak{a}} \overline{\xi_0(\alpha\pi^n x)}\sigma\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)v\,dx = 0.$$

There certainly is an integer n_0 such that the function $\sigma\big(\big(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\big)\big)v$ is constant on cosets of $\mathfrak{p}^{-n_0}$. Let us show that if φ_v vanishes identically and this function is constant on left cosets of $\mathfrak{p}^{-n}$ then it is constant on left cosets of $\mathfrak{p}^{-n-1}$. This will show that $\sigma\big(\big(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\big)\big)v = v$ for all x. It will then follow from Lemma 2.1 that $v = 0$.

Take any ℓ such that $\ell \geqslant n+1$ and $\mathfrak{p}^{-\ell} \supseteq \mathfrak{a}_n$. If $x \in \mathfrak{p}^{-\ell}$ then

$$\sigma\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)v = |\pi|^\ell \sum_{m=n}^{\ell} \sum_{\alpha \in o^\times/o_{\ell-m}} \xi_0(\alpha\pi^m x)\int_{\mathfrak{p}^{-\ell}} \overline{\xi_0(\alpha\pi^n y)}\sigma\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)v\,dy.$$

By assumption the terms of this sum corresponding to $m = n$ are zero. Since $\xi_0(\alpha\pi^m x)$ is constant on left cosets of $\mathfrak{p}^{-n-1}$ if $m > n$ the assertion follows.

Lemma 2.3.

(*i*) *If* $w = \sigma\big(\big(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\big)\big)v$ *then* $\varphi_w(\beta) = \varphi_v(\beta\alpha)$

(*ii*) *If* $w = \sigma\big(\big(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\big)\big)v$ *then* $\varphi_w(\beta) = \xi_0(\beta\alpha)\varphi_v(\beta)$

(*iii*) *If* v *is in* V *there is an integer* k *and a non-negative integer* n *such that* $\varphi_v(\alpha) = 0$ *if* $|\alpha| > |\pi|^k$ *and* $\varphi_v(\beta\alpha) = \varphi_v(\alpha)$ *if* $\beta \in o_n^\times$.

The first assertion is a matter of definition. To prove the second we have to show that

$$\sigma\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)\sigma\left(\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)v - \xi_0(\alpha x)\sigma\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)v = z$$

is in W. Let $u = \sigma\big(\big(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\big)\big)v$. Then

$$\int_{\mathfrak{a}} \overline{\xi_0(y)}\sigma\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)z\,dy$$

is equal to

$$\int_{\mathfrak{a}} \overline{\xi_0(y)}\sigma\left(\begin{pmatrix}1 & y+\alpha x\\ 0 & 1\end{pmatrix}\right)u\,dy - \xi_0(\alpha x)\int_{\mathfrak{a}} \overline{\xi_0(y)}\sigma\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)u\,dy.$$

If $\alpha x \in \mathfrak{a}$ we can change variables in the first integral to see that it equals the second term. Finally it is clear that if $\sigma\big(\big(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\big)\big)v = v$ for $|x| \leqslant |\pi|^{-k}$ and $\sigma\Big(\big(\begin{smallmatrix}\beta & 0\\ 0 & 1\end{smallmatrix}\big)\Big)v = v$ for $\beta \in o_n^\times$ then $\varphi_v(\alpha) = 0$ if $|\alpha| > |\pi|^k$ and $\varphi_v(\beta\alpha) = \varphi_v(\alpha)$ if $\beta \in o_n^\times$.

Let ν be a character of $o^\times$ and let $V_\nu = \left\{ v \in V \;\middle|\; \sigma\big(\big(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\big)\big)v = \nu(\alpha)v \text{ for all } \alpha \in o^\times \right\}$. It is clear that V is the direct sum of the spaces V_ν. Let $\widehat{V}$ be the set of all v in V such that, for some $k \geqslant 0$, $\varphi_v(\alpha) = 0$ if $|\alpha| > |\pi|^{-k}$ or $|\alpha| < |\pi|^k$. Let $\widehat{V}_\nu = V_\nu \cap \widehat{V}$. It is also clear that $\widehat{V}$ is the direct sum of the spaces $\widehat{V}_\nu$. Finally let V^0 be the set of all v in V such that $\varphi_v(\alpha) \neq 0$ if $|\alpha| \neq 1$ and let $V_\nu^0 = V_\nu \cap V^0$. V^0 is the direct sum of the spaces V_ν^0.

Lemma 2.4.

(*i*) *For each* ν *the restriction of* ψ *to* V_ν^0 *defines an isomorphism of* V_ν^0 *and* U

(*ii*) $\widehat{V}_\nu$ *is the direct sum of the spaces* $\sigma\Big(\big(\begin{smallmatrix} \pi^k & 0 \\ 0 & 1 \end{smallmatrix}\big)\Big)V_\nu^0$, $k \in \mathbf{Z}$.

(*iii*) *If* v *is in* V_ν *there is a unique* v_k *in* V_ν^0 *such that if* $u = v - \sigma\Big(\big(\begin{smallmatrix} \pi^{-k} & 0 \\ 0 & 1 \end{smallmatrix}\big)\Big)v_k$ *then* $\varphi_u(\alpha) = 0$ *if* $|\alpha| = |\pi|^k$.

(*iv*) V *is spanned by* $\widehat{V}$ *and the vectors of the form*

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)v$$

with v *in* $\widehat{V}$.

We start with (iii) of which (ii) is an obvious consequence. The uniqueness of v_k is clear. If k is negative and $|k|$ is sufficiently large we can take $v_k = 0$. Thus the proof can proceed by induction on k. Set

$$w = v - \sum_{\ell<k} \sigma\left(\begin{pmatrix} \pi^{-\ell} & 0 \\ 0 & 1 \end{pmatrix}\right)v_\ell.$$

$\varphi_w(\alpha) = 0$ if $|\alpha| > |\pi^k|$ and $\varphi_w(\alpha) = \varphi_v(\alpha)$ if $|\alpha| \leqslant |\pi|^k$. Set

$$v_k = |\pi|^{-k-1}\sigma\left(\begin{pmatrix} \pi^k & 0 \\ 0 & 1 \end{pmatrix}\right)\int_{\mathfrak{p}^{-k-1}}\left\{w - \sigma\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)w\right\}dx.$$

Then,

$$\varphi_{v_k}(\pi^{-k}\alpha) = |\pi|^{-k-1}\int_{\mathfrak{p}^{-k-1}}\{1 - \xi_0(\alpha x)\}\,dx\,\varphi_w(\alpha).$$

The right side is zero if $|\alpha| \leqslant |\pi|^{k+1}$ or $|\alpha| > |\pi^k|$. It is $\varphi_w(\alpha)$ if $|\alpha| = |\pi^k|$. Part (iii) of the lemma follows.

It is clear that the restriction of ψ to V_ν^0 is an injection. It follows from (iii) that the restriction of ψ to V^0 is a surjection. Thus $U = \sum_\mu \psi(V_\mu^0)$. To prove part (i) it is sufficient to show that if $u \in U$ and $u = \psi(v)$ for a v in some V_μ^0 then there is a w in V_ν^0 such that $u = \psi(w)$. Given v set

$$z = \int_{o^\times}\overline{\nu}(\beta)\sigma\left(\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}\right)\sigma\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)v\,d\beta$$

where x is yet to be determined. Then z is in V_ν and since

$$\varphi_z(\alpha) = \varphi_v(\alpha)\int_{o^\times}\mu\overline{\nu}(\beta)\xi_0(\alpha\beta x)\,d\beta$$

it is in V_ν^0. In particular

$$\psi(z) = \varphi_z(1) = \left\{\int_{o^\times}\mu\overline{\nu}(\beta)\xi_0(\beta x)\,d\beta\right\}u.$$

Choose x so that this integral is not zero and set

$$w = \left\{ \int_{o^\times} \mu\overline{\nu}(\beta)\xi_0(\beta x)\, d\beta \right\}^{-1} z.$$

It follows from (i) that $\widehat{V} \neq \{0\}$. Choose w different from zero in $\widehat{V}$. Since σ is irreducible V is spanned by the vectors $\sigma(g)w$, $g \in G_k$. Either $g = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ or $g = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. In the first case $\sigma(g)w$ is in $\widehat{V}$. In the second case $\sigma(g)w$ is of the form $\sigma\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right)\sigma\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)u$ with u in $\widehat{V}$. It is easily seen that if v belongs to V and x belongs to k then $\sigma\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right)v - v$ belongs to $\widehat{V}$. The last assertion of the lemma follows.

If v is in V let $v = \sum_\nu v_\nu$ with v_ν in V_ν. Choose $v_{k,\nu}$ so that if $u = v_\nu - \sigma\left(\left(\begin{smallmatrix} \pi^{-k} & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)v_{k,\nu}$ then $\varphi_u(\alpha) = 0$ if $|\alpha| = |\pi^k|$. Set $u_{k,\nu} = \psi(v_{k,\nu})$ and write, purely formally,

$$v \sim \sum_\nu \sum_\ell u_{\ell,\nu} z^\ell.$$

Let $\sigma\left(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)\right) = \omega(\alpha)I$ for $\alpha \in k^\times$ and let ω_0 be the restriction of ω to $o^\times$. Let $\widetilde{v}(\alpha) = \omega_0(\alpha)\nu^{-1}(\alpha)$ if $\alpha \in o^\times$. If v is in V_ν^0, then $\sigma\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)v$ is in $V_{\widehat{\nu}}$. Let

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)v \sim \sum_k u_k z^k.$$

If $\psi(v) = u$ the map $u \to u_k$ is a linear transformation from U to U. Denote it by $T_{k,\nu}$. If v is in $\widehat{V}$ and

$$v \sim \sum_\nu \sum_\ell u_{\ell,\nu} z^\ell$$

then

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \sum_\nu \sum_\ell \left\{ \sum_{m-k=\ell} \omega(\pi^{-k}) T_{m,\widehat{\nu}} u_{k,\widetilde{\nu}} \right\} z^\ell.$$

It follows from the third part of Lemma 2.4 that if $v \in V$ there is a unique v_k in V^0 such that if $u = v - \sigma\left(\left(\begin{smallmatrix} \pi^{-k} & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)v_k$ then $\varphi_u(\alpha) = 0$ if $|\alpha| = |\pi|^k$. If $w = \sigma\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right)v$ then v_k is replaced by $w_k = \sigma\left(\left(\begin{smallmatrix} 1 & \pi^k x \\ 0 & 1 \end{smallmatrix}\right)\right)v_k$. If $v_k = \sum_\nu v_{k,\nu}$ with $v_{k,\nu}$ in V_ν^0 and $w_k = \sum_\nu w_{k,\nu}$ with $w_{k,\nu}$ in V_ν^0 then

$$w_{k,\mu} = \sum_\nu \int_{o^\times} \nu\overline{\mu}(\beta)\sigma\left(\begin{pmatrix} 1 & \beta\pi^k x \\ 0 & 1 \end{pmatrix}\right) v_{k,\nu}\, d\beta.$$

Consequently

$$\psi(w_{k,\mu}) = \sum_\nu \Delta(\nu\mu^{-1}, \pi^k x)\psi(v_{k,\nu}).$$

Thus if

$$v \sim \sum_\nu \sum_\ell u_{\ell,\nu} z^\ell,$$

$$w \sim \sum_\nu \sum_\ell \left\{ \sum_\mu \Delta(\mu\nu^{-1}, \pi^\ell x) u_{\ell,\mu} \right\}.$$

It is also easily seen that

$$\sigma\left(\begin{pmatrix} \pi^n & 0 \\ 0 & 1 \end{pmatrix}\right) v \sim \sum_\nu \sum_\ell u_{\ell+n,\nu} z^\ell.$$

The identity

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1/x & 0 \\ 0 & -x \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1/x \\ 0 & 1 \end{pmatrix}$$

for $x \neq 0$ is easily verified. If v is in $\widehat{V}$ and

$$v \sim \sum_\nu \sum_\ell u_{\ell,v} z^\ell$$

then

$$\pi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) v \sim \sum_\nu \sum_k \left\{ \sum_\ell \omega(\pi^{-\ell}) T_{k+\ell,\widetilde{\nu}} u_{\ell,\widetilde{\nu}} \right\} z^k,$$

$$\sigma\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) v \sim \sum_\mu \sum_k \left\{ \sum_\ell \sum_\nu \omega(\pi^{-\ell}) \Delta(\nu\mu^{-1}, \pi^k x) T_{k+\ell,\widetilde{\nu}} u_{\ell,\widetilde{\nu}} \right\} z^\ell.$$

If $\delta_\nu = 1$ if ν is trivial and $\delta_\nu = 0$ otherwise, then $\sigma\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right)\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right) v - \sigma\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right) v$ which belongs to $\widehat{V}$ corresponds to

$$\sum_\mu \sum_k \left\{ \sum_\ell \sum_\nu \left[\Delta(\nu\mu^{-1}, \pi^k x) - \delta_{\nu\mu^{-1}}\right] \omega(\pi^{-\ell}) T_{k+\ell,\widetilde{\nu}} u_{\ell,\widetilde{\nu}} \right\} z^\ell.$$

Finally $\sigma\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right)\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right) v$ corresponds to

$$\sum_\mu \sum_k \left\{ \sum_{\ell,m,\nu} \left[\Delta(\nu\widetilde{\mu}^{-1}, \pi^m x) - \delta_{\nu\widetilde{\mu}^{-1}}\right] \omega(\pi^{-\ell-m}) T_{k+m,\widetilde{\mu}} T_{m+\ell,\widetilde{\nu}} u_{\ell,\widetilde{\nu}} \right\} z^k + \sum_\mu \sum_k \omega(-1) u_{k,\mu} z^k.$$

On the other hand $\sigma\left(\left(\begin{smallmatrix} 1 & -1/x \\ 0 & 1 \end{smallmatrix}\right)\right) v$ corresponds to

$$\sum_\mu \sum_\ell \left\{ \sum_\nu \Delta(\nu\mu^{-1}, -\pi^\ell/x) u_{\ell,\nu} \right\} z^\ell$$

and $\sigma\left(\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}1&-1/x\\0&1\end{smallmatrix}\right)\right)v$ corresponds to

$$\sum_\mu\sum_k\left\{\sum_\ell\sum_k\omega(\pi^{-\ell})\Delta(\nu\widetilde{\mu}^{-1},-\pi^\ell/x)T_{k+\ell,\widetilde{\mu}}u_{\ell,\nu}\right\}z^k.$$

Letting $\sigma\left(\left(\begin{smallmatrix}1&-x\\0&1\end{smallmatrix}\right)\right)$ operate we obtain a vector corresponding to

$$\sum_\mu\sum_k\left\{\sum_{\ell,\nu,\eta}\omega(\pi^{-\ell})\Delta(\eta\mu^{-1},-\pi^kx)\Delta(\nu\widetilde{\eta}^{-1},-\pi^\ell/x)T_{k+\ell,\widetilde{\eta}}u_{\ell,\nu}\right\}z^k.$$

Finally if $\frac{1}{x}=\pi^r\beta$ with $\beta\in o^\times$ we apply $\sigma\left(\left(\begin{smallmatrix}-1/x&0\\0&-x\end{smallmatrix}\right)\right)$ to obtain

$$\sum_\mu\sum_k\left\{\sum_{\ell,\nu,\eta}\mu(\beta^2)\omega(-x)\omega(\pi^{-\ell})\Delta(\eta\mu^{-1},-\pi^{k+2r}x)\Delta(\nu\widetilde{\eta}^{-1},-\pi^\ell x)T_{k+2r+\ell,\widetilde{\eta}}u_{\ell,\nu}\right\}z^k.$$

Thus we obtain the identities

$$\begin{aligned}\sum_m\Big\{\Delta(\widetilde{\nu}\widetilde{\mu}^{-1},\pi^mx)\omega(\pi^{-\ell-m})T_{k+m,\widetilde{\mu}}T_{m+\ell,\nu}-\delta_{\widetilde{\nu}\widetilde{\mu}^{-1}}\omega(\pi^{-\ell-m})T_{k+m,\widetilde{\mu}}T_{m+\ell,\nu}\Big\}&\\ +\omega(-1)\delta_{\nu\mu^{-1}}\delta_{\ell,k}&\\ =\sum_\eta\mu(\beta^2)\omega(-x)\omega(\pi^{-\ell})\Delta(\eta\mu^{-1},-\pi^{k+2r}x)\Delta(\nu\widetilde{\eta}^{-1},-\pi^\ell/x)T_{k+2r+\ell,\widetilde{\eta}}.&\end{aligned}$$

For all we know at present both these sums are infinite. However all but a finite number of the operators on each side send a given vector in U to zero. Thus as an operational equation the identity has a sense.

We can rewrite the identities as

$$\begin{aligned}\sum_m\Big\{\Big[\Delta(\nu^{-1}\mu^{-1}\omega_0,\pi^m\beta^{-1})-\delta_{\nu\mu\omega_0^{-1}}\Big]\omega(\pi^{-\ell-m})T_{k+m,\mu}T_{m+\ell,\nu}\Big\}+\omega_0(-1)\delta_{\nu\mu\omega_0^{-1}}\Delta_{\ell,k}&\\ =\sum_\eta\widetilde{\mu}(\beta^2)\omega_0^{-1}(-\beta)\omega(\pi^{-\ell})\Delta(\mu\eta^{-1},-\pi^k\beta^{-1})\Delta(\nu\eta^{-1},-\pi^\ell\beta)T_{k+\ell,\eta}.&\end{aligned}$$

Recalling that $\Delta(\nu,\beta y)=\nu^{-1}(\beta)\Delta(\nu,y)$ we simplify the identities to

$$\begin{aligned}\sum_m\Big\{\Big[\Delta(\nu^{-1}\mu^{-1}\omega_0,\pi^m)-\delta_{\nu\mu\omega_0^{-1}}\Big]\omega(\pi^{-m})T_{k+m,\mu}T_{m+\ell,\nu}\Big\}+\omega_0(-1)\delta_{\nu\mu\omega_0^{-1}}\omega(\pi^\ell)&\\ =\mu\nu\omega_0^{-1}(-1)\sum_\eta\Delta(\mu\eta^{-1},\pi^k)\Delta(v\eta^{-1},\pi^\ell)T_{k+\ell,\eta}.&\end{aligned}$$

Making use of Lemma 1.1 we can simplify these identities further. If the order n of $\nu\mu\omega_0^{-1}$ is positive the identity becomes

$$\text{(A)}\quad \Delta(\nu^{-1}\mu^{-1}\omega_0, \pi^{-n})\omega(\pi^n)T_{k-n,\mu}T_{\ell-n,\nu} = \mu\nu\omega_0^{-1}(-1)\sum_\eta \Delta(\mu\eta^{-1}, \pi^k)\Delta(\nu\eta^{-1}, \pi^\ell)T_{k+\ell,\eta}.$$

If $\nu\mu\omega_0^{-1} = 1$ the identity becomes

$$\text{(B)}\quad \sum_{m=-\infty}^{-2} -\omega(\pi^{-n})T_{k+m,\mu}T_{m+\ell,\nu} + \frac{1}{|\pi|-1}\omega(\pi)T_{k-1,\mu}T_{\ell-1,\nu} + \omega_0(-1)\delta_{\ell,k}\omega(\pi^\ell) = \sum_\eta \Delta(\mu\eta^{-1}, \pi^k)\Delta(\nu\eta^{-1}, \pi^\ell)T_{k+\ell,\eta}.$$

Lemma 2.5.

(*i*) *For all* k, ℓ, μ, *and* ν, $T_{k,\mu}T_{\ell,\nu} = T_{\ell,\nu}T_{k,\mu}$.

(*ii*) *There is no non-trivial subspace of* u *left invariant by all the operators* $T_{k,\mu}$.

If $\nu\mu\omega_0^{-1}$ is not trivial the identity

$$T_{k,\mu}T_{\ell,\nu} = T_{\ell,\nu}T_{k,\mu}$$

follows immediately from (A). If $\nu\mu\omega_0^{-1}$ is trivial let u be in U. For a given k and ℓ and for $m \ll 0$ both $T_{k+m,\mu}u$ and $T_{\ell+m,\nu}u$ are zero. For such m

$$T_{k+m,\mu}T_{\ell+m,\nu}u = T_{\ell+m,\nu}T_{k+m,\mu}u.$$

Using the identity (B) and induction on m one shows readily that this relation is valid for all m.

Suppose that U' is a nontrivial subspace of U left invariant by all the operators $T_{k,\mu}$. Let V' be the set of all v in V such that $\varphi_v(\alpha) \in U'$ for all α. If $v \in V_\nu^0$ then $v \in V'$ if and only if $\psi(v) \in U'$. Thus V' is neither $\{0\}$ nor V and $V' \cap V_\nu^0 \neq \{0\}$. It is clear that V' and $V' \cap \widehat{V}$ are left fixed by the operators $\sigma\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right)$. Since V is irreducible it is spanned by $V' \cap \widehat{V}$ together with the set $\sigma\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right)\sigma\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)v$, $v \in V' \cap \widehat{V}$. Thus to obtain a contradiction we need only show that if v is in $V' \cap \widehat{V}$ then $\sigma\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)v$ is in V'. This is however an obvious consequence of the assumption.

It follows from this lemma that each $T_{k,\mu}$ is either zero or an invertible linear transformation. Thus for each μ there is an integer $k(\mu)$ such that $T_{k,\mu} = 0$ if $k < k(\mu)$. Moreover one of these operators can have a non-trivial eigenvector if and only if it is a scalar.

Now I would like to make some remarks which are not relevant to the main purpose of the letter. First of all let me observe that if k, ℓ, μ, ν are arbitrary there is a scalar a and scalars $a_{m,\rho}$ all but a finite number of which are zero such that

$$T_{k,\mu}T_{\ell,\nu} = a + \sum_\rho \sum_m a_{m,\rho}T_{m,\rho}.$$

If $\mu\nu\omega_0^{-1}$ is not trivial this follows immediately from identity (A). If $\mu\nu\omega_0^{-1}$ is trivial consider the set of integers p for which $T_{k+p,\mu}T_{\ell+p,\nu}$ is a linear combination of the identity and the operators $T_{m,\leqslant}$. If $p \ll 0$, $T_{k+p,\mu}T_{\ell+p,\nu} = 0$ and p belongs to this set. Using identity (B) and a simple induction argument one shows that the set contains all integers. It follows from this observation and the previous lemma that if $u \neq 0$ belongs to U then U is spanned by u and the set $\{T_{m,\rho}u\}$.

Choose a fixed ν and let the order n of μ be positive and so large that the orders of $\nu^{-1}\mu^{-1}\omega_0$ and $\nu\mu^{-1}$ are also n and $T_{r_0,\nu} \neq 0$ for some $r_0 \geqslant -n$. Take $\ell = r_0 + n \geqslant 0$ in identity (A) and cancel $T_{r_0,\nu}$ to obtain

$$\Delta(\nu^{-1}\mu^{-1}\omega_0, \pi^{-n})\omega(\pi^n)T_{k-n,\mu} = \mu\nu\omega_0^{-1}(-1)\Delta(\mu\nu^{-1}, \pi^k).$$

As a consequence for all but a finite number of characters of $o^\times$ the operator $T_{k,\mu}$ is a scalar for all k. If, for all ρ, $T_{m,\rho} = 0$ if $m \geqslant -1$ then there are only a finite number of operators in the set $\{T_{m,\rho}\}$ which are not scalars. Consequently U is finite-dimensional and each of the operators $T_{m,\rho}$ has a non-trivial eigenvector and is thus a scalar. It follows that U has dimension 1.

It is very unlikely that our assumptions (i) and (ii) together with irreducibility imply that U is one-dimensional. Consequently we make the further assumption which can certainly be useful in the case of interest to us at present.

(iii) No representation of G_o occurs more than a finite number of times in the restriction of σ to G_o.

If ρ is a representation of G_o let V_ρ be the set of all vectors in V which transform according to ρ. Any operator on V which commutes with all the operators $\sigma(g)$ must leave each of the finite-dimensional spaces V_ρ invariant. Thus it must have a non-trivial eigenvector and, because of the irreducibility, must be a scalar. It follows immediately from the first part of Lemma 2.4 that the map $v \to \varphi_v$ maps $\widehat{V}$ onto the set of all locally constant functions on $k^\times$ with values in U which vanish outside of some compact set. Suppose T is an operator on U which commutes with all the operators $T_{m,\rho}$. If φ is a function on $k^\times$ with values in U define $T\varphi$ by $(T\varphi)(\alpha) = T\big(\varphi(\alpha)\big)$. If $v \in \widehat{V}$ and

$$v \sim \sum_\mu \sum_k u_{k,\mu} z^k$$

then $T\varphi_v = \varphi_w$ where

$$w \sim \sum_\mu \sum_k T u_{k,\mu} z^k.$$

Then

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) v \sim \sum_\mu \sum_k \left\{ \sum_{m-\ell=k} \omega(\pi^{-\ell}) T_{m,\widetilde{\mu}} u_{\ell,\widetilde{\mu}} \right\} z^k,$$

$$\sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) w \sim \sum_\mu \sum_k T \left\{ \sum_{m-\ell=k} \omega(\pi^{-\ell}) T_{m,\widetilde{\mu}} u_{\ell,\widetilde{\mu}} \right\} z^k.$$

It follows immediately that T takes the image of V to itself. Thus it determines a linear transformation of V which is easily shown to commute with all the operators. As a consequence of assumption (iii) this linear transformation is a scalar. Thus T is a scalar. In particular all the operators $T_{m,\rho}$ are scalars and U is one-dimensional.

In the next two lemmas it is assumed that U is one-dimensional. Thus the operators $T_{m,\rho}$ are taken to be complex numbers.

Lemma 2.7. *Suppose that there is a character μ_1 of $o^\times$ and a $k_1 \geqslant -1$ such that $T_{k_1,\mu_1} \neq 0$. Let $K = k \oplus k$. There is a continuous homomorphism M of $K^\times$ into $\mathbf{C}^\times$ such that for all ℓ and ν*

$$T_{\ell,\nu} = T(M, \nu; \ell).$$

Let me observe immediately that it was shown in the previous paragraph that if the restriction of M to $k^\times$ is ω the identities (A) and (B) are satisfied if $T_{\ell,\nu}$ is replaced by $T(M, \nu; \ell)$. Set $\mu_2 = \mu_1^{-1}\omega_0$. It will perhaps require less mental effort if the cases $\mu_1 = \mu_2$ and $\mu_1 \neq \mu_2$ are treated separately.

Suppose first that $\mu_1 = \mu_2$. In identity (A) take $\nu = \mu_1$, $\mu \neq \mu_1$, and take $\ell = k_1 + n \geqslant 0$ to obtain

$$\Delta(\mu_1\mu^{-1}, \pi^{-n})\omega(\pi^n)T_{k-n,\mu}T_{k_1,\mu_1} = \mu\mu_1^{-1}(-1)\Delta(\mu\mu_1^{-1}, \pi^k)T_{k+\ell,\mu_1}.$$

The right side is zero unless $k = -n$ but if $k = -n$ we can cancel T_{k_1,μ_1} from both sides to obtain

$$T_{-2n,\mu} = \frac{\mu\mu_1^{-1}(-1)}{\omega(\pi^n)}\ \frac{\Delta(\mu\mu_1^{-1}, \pi^{-n})}{\Delta(\mu_1\mu^{-1}, \pi^{-n})} = \big(1 - |\pi|\big)^2|\pi|^{-n}\omega(\pi^{-n})\big\{\Delta(\mu_1^{-1}\mu, \pi^{-n})\big\}^2.$$

Thus if ω_1 and ω_2 are two complex numbers such that $\omega_1\omega_2 = \omega(\pi)$ and M is defined by $M(\pi^p\alpha \oplus \pi^q\beta) = \omega_1^p\omega_2^q\mu_1(\alpha\beta)$ for $\alpha \in o^\times$, $\beta \in o^\times$ then $T_{k,\mu} = T(M, k, \mu)$ for $\mu \neq \mu_1$.

Take $\mu = \nu = \mu_1$ and $k = \ell$ in identity (B). If $k < -1$ the right side is a sum over those η such that the conductor of $\mu_1\eta^{-1}$ is $\mathfrak{p}^{-\ell}$. For such η

$$\Delta(\mu_1\eta^{-1}, \pi^k)\Delta(\mu_1\eta^{-1}, \pi^k)T_{2k,\eta}$$
$$= \omega(\pi^k)\mu\mu_1^{-1}(-1)\Delta(\eta\mu_1^{-1}, \pi^k)\Delta(\mu_1\eta^{-1}, \pi^k) = \frac{\omega(\pi^k)|\pi|^k}{\big(1 - |\pi|\big)^2}.$$

Since the number of such characters is $|\pi|^{-k}\big(1 - |\pi|\big)^2$ the right side of (B) is equal to $\omega(\pi^k)$. Since $\omega_0(-1) = \mu_1^2(-1) = 1$ we have, for $k < -1$,

$$\sum_{m=-\infty}^{-2} -\omega(\pi^{-m})T_{k+m,\mu_1}T_{k+m,\mu_1} + \frac{\omega(\pi)}{|\pi| - 1}T_{k-1,\mu_1}T_{k-1,\mu_1} = 0.$$

It follows by induction that $T_{m,\mu_1} = 0$ if $m < -2$.

Now take $\mu = \nu = \mu_1$, $\ell = -1$, and $k \geqslant 0$ in (B) to obtain

$$\frac{\omega(\pi)}{|\pi| - 1}T_{k-1,\mu_1}T_{-2,\mu_2} = \frac{|\pi|}{|\pi| - 1}T_{k-1,\mu_1}.$$

Since $T_{k-1,\mu_1} \neq 0$ for some $k \geqslant 0$ we conclude that $\omega(\pi)T_{-2,\mu_1} = |\pi|$.

Choose ω_1 and ω_2 to be the two solutions of the equation

$$\big(|\pi| - 1\big)\left\{\frac{x}{\omega(\pi)}|\pi|^{1/2} + \frac{|\pi|^{1/2}}{x}\right\} = T_{-1,\mu_1}.$$

It is easy to see that

$$\sum_{m=-\infty}^{\infty} T(M, \mu_1, m)x^m = \frac{\Big(1 - \omega_1^{-1}x^{-1}|\pi|^{1/2}\Big)\Big(1 - \omega_2^{-1}x^{-1}|\pi|^{1/2}\Big)}{\big(1 - \omega_1 x|\pi|^{1/2}\big)\big(1 - \omega_2 x|\pi|^{1/2}\big)}$$

if $|x| > 0$ and $|x|$ is sufficiently small. Thus $T_{m,\mu_1} = T(M, \mu_1, m)$ if $m < 0$. Taking $\mu = \nu = \mu_1$, $k = 0$, and $\ell \geqslant 0$ in (B) we obtain

$$\sum_{m=-\infty}^{-2} -\omega(\pi^{-m}) T_{m,\mu_1} T_{\ell+m,\mu_1} + \frac{\omega(\pi)}{|\pi|-1} T_{-1,\mu_1} T_{\ell-1,\mu_1} + \omega_0(-1)\delta_{\ell,k}\omega(\pi^\ell) = T_{\ell,\mu_1}.$$

Since the same formula is valid if T_{p,μ_1} is replaced by $T(M, \mu_1, p)$ we can show inductively that $T_{m,\mu_1} = T(M, \mu_1, m)$ for all m.

Now suppose $\mu_1 \neq \mu_2$. Let n be the order of $\mu_1\mu_2^{-1}$. Take $\mu = \nu = \mu_1$, $\ell = k_1 + 1$, and $k = -1$ in identity (A) to obtain

$$\Delta(\mu_2\mu_1^{-1}, \pi^{-n})\omega(\pi^n) T_{-1-n,\mu_1} T_{\ell-n,\mu_1} = \mu_1\mu_2^{-1}(-1)\frac{|\pi|}{|\pi|-1} T_{k_1,\mu_1}.$$

Thus $T_{-1-n,\mu_1} \neq 0$. Now take $\ell = k_1 + n$, $k < -1$ to obtain

$$\Delta(\mu_2\mu_1^{-1}, \pi^{-n})\omega(\pi^n) T_{k-n,\mu_1} T_{k,\mu_1} = 0.$$

Thus $T_{k-n,\mu_1} = 0$ if $k < -1$.

Now let us look at the identity (B) with $\mu = \mu_1$, $\nu = \mu_2$. If $k > -n$ and $\ell > -n$ the right side is zero because the order of either $\mu_1\eta^{-1}$ or $\mu_2\eta^{-1}$ is at least n. Thus in this case

$$\text{(C)} \qquad \sum_{m=-\infty}^{-2} -\omega(\pi^{-m}) T_{k+m,\mu_1} T_{\ell+m,\mu_2} + \frac{\omega(\pi)}{|\pi|-1} T_{k-1,\mu_1} T_{\ell-1,\mu_2} + \omega_0(-1)\delta_{\ell,k}\omega(\pi^\ell) = 0.$$

In particular take $\ell = n + 1$ to see that if $k > -n + 1$ and $T_{k-1,\mu_2} = 0$ so does T_{k-2,μ_2}.

If $k < -n$ and $\ell \neq k$ the right side of (B) is zero for $\mu = \mu_1$, $\nu = \mu_2$ because $k \leqslant -2$ and if the order of $\mu_1\eta^{-1}$ is $-k$ so is the order of $\mu_2\eta^{-1}$. Thus in this case

$$\sum_{m=-\infty}^{-2} -\omega(\pi^{-m}) T_{k+m,\mu_1} T_{\ell+m,\mu_2} + \frac{\omega(\pi)}{|\pi|-1} T_{k-1,\mu_1} T_{\ell-1,\mu_2} = 0.$$

The same result is valid if $\ell < -n$ and $k \neq \ell$. Take $k = -n$ to see that $T_{m,\mu_2} = 0$ if $m < -n - 1$.

Thus if $T_{m,\mu_2} = 0$ for all $m \geqslant -1$ then the only m for which $T_{m,\mu_2} \neq 0$ is $m = -n - 1$. Taking $\ell = -n - 1$ in (C) we would find that $T_{m,\mu_1} = 0$ for $m \geqslant -n$ which is contrary to assumption. At this point μ_1 and μ_2 play identical roles.

Taking $k = -n + 1$ in (C) we see that if $\ell \geqslant -n$

$$T_{\ell+1,\mu_2} T_{-n,\mu_1} = \big(|\pi| - 1\big)\omega(\pi) T_{\ell,\mu_2} T_{-1-n,\mu_1}.$$

Thus $T_{-n,\mu_1} \neq 0$ and $T_{\ell,\mu_2} \neq 0$ if $\ell \geqslant -n$. Set, if $\ell \geqslant -n$,

$$\omega_1|\pi|^{1/2} = T_{\frac{\ell+1,\mu_2}{T_{\ell,\mu_2}}} = \big(|\pi| - 1\big)\omega(\pi) T_{\frac{-1-n,\mu_1}{T_{-n,\mu_1}}}.$$

Similarly $T_{-n,\mu_2} \neq 0$ and $T_{\ell,\mu_1} \neq 0$ if $\ell \geqslant -n$. If $\ell \geqslant -n$, set

$$\omega_2|\pi|^{1/2} = T_{\frac{\ell+1,\mu_1}{T_{\ell,\mu_1}}} = \big(|\pi| - 1\big)\omega(\pi) T_{\frac{-1-n,\mu_2}{T_{-n,\mu_2}}}.$$

Now take $\nu = \mu = \mu_1$, $\ell \geqslant 0$, $k = -1$ in (A) to obtain

$$\Delta(\mu_1^{-1}\mu_2, \pi^{-n})\omega(\pi^n) T_{-1-n,\mu_1} T_{\ell-n,\mu_1} = \mu_1\mu_2^{-1}(-1)\frac{|\pi|}{|\pi|-1} T_{\ell-1,\mu_1}.$$

Thus

$$T_{-1-n,\mu_1} = \frac{|\pi|-1}{|\pi|^{\frac{n-1}{2}}}\frac{\omega_2^{n-1}}{\omega(\pi^n)}\Delta(\mu_1\mu_2^{-1},\pi^{-n}).$$

In the same way

$$T_{1-n,\mu_2} = \frac{|\pi|-1}{|\pi|^{\frac{n-1}{2}}}\frac{\omega_1^{n-1}}{\omega(\pi^n)}\Delta(\mu_2\mu_1^{-1},\pi^{-n}).$$

Thus if $\gamma = \frac{\omega_1\omega_2}{\omega(\pi)}$,

$$\begin{aligned}
T_{-1-n,\mu_1} &= \gamma^n\frac{\big(|\pi|-1\big)}{|\pi|^{\frac{n-1}{2}}}\frac{1}{\omega_2\omega_1^n}\Delta(\mu_2^{-1}\mu_1,\pi^{-n}),\\
T_{\ell,\mu_1} &= \gamma^{n-1}\big(1-|\pi|\big)^2|\pi|^{\frac{\ell}{2}}\omega_1^{\ell+n}\omega_1^{-n}\Delta(\mu_2^{-1}\mu_1,\pi^{-n}), \qquad \ell \geqslant -n,\\
T_{-1-n,\mu_2} &= \gamma^n\frac{\big(|\pi|-1\big)}{|\pi|^{\frac{n-1}{2}}}\frac{1}{\omega_2^n\omega_1}\Delta(\mu_1^{-1}\mu_2,\pi^{-n}),\\
T_{\ell,\mu_2} &= \gamma^{n-1}\big(1-|\pi|\big)^2|\pi|^{\frac{\ell}{2}}\omega_2^{-n}\omega_1^{\ell+n}\Delta(\mu_1^{-1}\mu_2,\pi^{-n}), \qquad \ell \geqslant -n.
\end{aligned}$$

If we take μ different from μ_1 and μ_2, $\nu = \mu_1$ and $\ell \geqslant 0$ in identity (A) we obtain

$$\Delta(\mu_2\mu_1^{-1},\pi^{-n_2})\omega(\pi^{n_2})T_{k-n_2,\mu}T_{\ell-n_2,\mu_1} = \mu\mu_2^{-1}(-1)\Delta(\mu\mu_1^{-1},\pi^k)T_{k+\ell,\mu_1}$$

if n_1 is the order of $\mu_1\mu^{-1}$ and n_2 is the order of $\mu_2\mu^{-1}$. Thus $T_{m,\mu} = 0$ if $m \neq -n_1-n_2$ but

$$T_{-n_1-n_2,\mu} = \big(1-|\pi|\big)^2|\pi|^{\frac{-n_1-n_2}{2}}\gamma^{n_2}\omega_2^{-n_1}\omega_1^{-n_2}\Delta(\mu_1^{-1}\mu,\pi^{-n_1})\Delta(\mu_2^{-1}\mu,\pi^{-n_2}).$$

If we can show that $\gamma = 1$ we will have proved that if $M(\pi^p\alpha \oplus \pi^q\beta) = \omega_1^p\omega_2^p\mu_1(\alpha)\mu_2(\beta)$ then $T(M,\mu,m) = T_{m,\mu}$ for all μ and all m.

Take $\mu = \mu_1$, $\nu = \mu_2$ and $k = \ell = -n$ in (B). If the order of both $\eta\mu_1^{-1}$ and $\eta\mu_2^{-1}$ is n, the value of the corresponding term on the right side is

$$\mu_1\mu_2(-1)\frac{|\pi|^n}{\big(1-|\pi|\big)^2}\ \frac{1}{\omega(\pi^n)}.$$

If $n > 1$ there are $\frac{1-|\pi|}{|\pi|^n}\big(1-2|\pi|\big)$ such characters η. The terms corresponding to the other characters are all zero so the right-hand side is $\frac{\mu_1\mu_2(-1)}{\omega(\pi^n)}\frac{1-2|\pi|}{1-|\pi|}$. If $n = 1$ there are $\frac{1-3|\pi|}{|\pi|}$ such characters. However the terms corresponding to $\eta = \mu_1$ and $\eta = \mu_2$ give a total contribution of

$$\frac{\mu_1\mu_2(-1)}{\omega(\pi)}\ \frac{|\pi|^2}{\big(1-|\pi|\big)^2} + \frac{\mu_1\mu_2(-1)}{\omega(\pi)}\frac{|\pi|^2}{\big(1-|\pi|\big)^2}.$$

Thus the right side is again $\frac{\mu_1\mu_2(-1)}{\omega(\pi^n)}\frac{1-2|\pi|}{1-|\pi|}$. The left side is

$$\frac{\gamma^{n-1}}{\omega(\pi^n)}\mu_1\mu_2(-1)\frac{|\pi|}{|\pi|-1} + \frac{\mu_1\mu_2(-1)}{\omega(\pi^n)}.$$

Consequently $\gamma^{n-1} = 1$. Now take $k = \ell = -n+1$ in (C) to obtain

$$\frac{-\mu_1\mu_2(-1)}{\omega(\pi^{n-1})}|\pi| + \frac{\mu_1\mu_2(-1)}{\omega(\pi^{n-1})}\gamma^{n-2}\big(|\pi|-1\big) + \frac{\mu_1\mu_2(-1)}{\omega(\pi^{n-1})} = 0.$$

Thus $\gamma^{n-2} = 1$ and $\gamma = 1$.

It will be convenient to record here the closed expressions for

$$\sum_{n=-\infty}^{\infty} T(M,\mu,n)z^n = T(M,\mu,z).$$

The series of course converges for $|z|>0$ and sufficiently small.

Lemma 2.8.

(i) Let $K = k\oplus k$ and let $M(\pi^p\alpha \oplus \pi^q\beta) = \omega_1^p\omega_2^p\mu_1(\alpha)\mu_2(\beta)$ if $\alpha\in o^\times$ and $\beta\in o^\times$.

(a) If $\mu_1=\mu_2$ then

$$T(M,\mu_1,z) = \frac{\left(1-\omega_1^{-1}z^{-1}|\pi|^{1/2}\right)}{\left(1-\omega_1 z|\pi|^{1/2}\right)}\ \frac{\left(1-\omega_2^{-1}z^{-1}|\pi|^{1/2}\right)}{\left(1-\omega_2 z|\pi|^{1/2}\right)}$$

and if $\mu\neq\mu_1$ and the order of $\mu^{-1}\mu_1$ is n

$$T(M,\mu,z) = \big(1-|\pi|\big)^2|\pi|^{-n}\omega_1^{-n}\omega_2^{-n}\Delta(\mu_1^{-1}\mu,\pi^{-n})\Delta(\mu_1^{-1},\mu,\pi^{-n})z^{-2n}.$$

(b) If $\mu_1\neq\mu_2$ then

$$T(M,\mu_1,z) = \big(1-|\pi|\big)|\pi|^{-n/2}\omega_1^{-n}\Delta(\mu_2^{-1}\mu_1,\pi^{-n})\frac{\left(1-\omega_2^{-1}z^{-1}|\pi|^{1/2}\right)}{\left(1-\omega_2 z|\pi|^{1/2}\right)}z^{-n};$$

$$T(M,\mu_2,z) = \big(1-|\pi|\big)|\pi|^{-n/2}\omega_2^{-n}\Delta(\mu_1^{-1}\mu_2,\pi^{-n})\frac{\left(1-\omega_1^{-1}z^{-1}|\pi|^{1/2}\right)}{\left(1-\omega_1 z|\pi|^{1/2}\right)}z^{-n}$$

if n is the order of $\mu_1^{-1}\mu_2$. If μ is different from μ_1 and μ_2 and the order of $\mu^{-1}\mu_1$ is n_1 and the order of $\mu^{-1}\mu_2$ is n_2 then

$$T(M,\mu,z) = \big(1-|\pi|\big)^2|\pi|^{\frac{-n_1-n_2}{2}}\omega_2^{-n_1}\omega_1^{-n_2}\Delta(\mu_1^{-1}\mu,\pi^{-n_1})\Delta(\mu_2^{-1}\mu,\pi^{-n_2})z^{-n_1-n_2}.$$

(ii) Let K be an unramified extension of k.

(a) Suppose there is a generalized character M_1 of $k^\times$ such that $M(\alpha)=M_1(\alpha^{1+s})$. Let $M_1(\pi^p\beta)=\omega_1^p\mu_1(\beta)$ for $\beta\in o^\times$. Then

$$T(M,\mu_1;z) = \frac{1-\omega_1^{-2}z^{-2}|\pi|}{1-\omega_1^2z^2|\pi|} = \frac{1-\omega_1^{-1}z^{-1}|\pi|^{1/2}}{1-\omega_1 z|\pi|^{1/2}}\ \frac{1+\omega_1^{-1}z^{-1}|\pi|^{1/2}}{1+\omega_1 z|\pi|^{1/2}}.$$

If $\mu\neq\mu_1$ and the order of $\mu\mu_1^{-1}$ is n so that the order of $\mu^{1+s}M_0^{-1}$ is also n then

$$\begin{aligned}T(M,\mu;z) &= \big(1-|\pi|^2\big)|\pi|^{-n}\omega_1^{-2n}\Delta(M_0^{-1}\mu^{1+s},\Pi^{-n})z^{-2n}\\ &= \big(1-|\pi|\big)^2|\pi|^{-n}\omega_1^{-n}(-\omega_1)^{-n}\big\{\Delta(\mu_1^{-1}\mu,\pi^{-n})\big\}^2z^{-2n}.\end{aligned}$$

(b) If there is no such character then for all μ

$$T(M,\mu,z) = \big(1-|\pi|^2\big)|\pi|^{-n}M(\Pi^{-n})\Delta(M_0^{-1}\mu^{1+s},\Pi^{-n})z^{-2n}$$

if n is the order of $M_0^{-1}\mu^{1+s}$.

(iii) Let K be a ramified extension of k.

(*a*) *Suppose there is a character* M_1 *of* $k^\times$ *such that* $M(\alpha) = M_1(\alpha^{1+s})$. *Let* $M_1(\pi^p\beta) = \omega_1^p\mu_1(\beta)$ *if* $\beta \in o^\times$. *Then*

$$T(M,\mu_1;z) = \big(1-|\pi|\big)|\pi|^{-f/2}\omega_1^{-f}\Delta(\chi_0,\pi^{-f})\frac{\Big(1-\omega_1^{-1}z^{-1}|\pi|^{1/2}\Big)}{\big(1-\omega_1 z|\pi|^{1/2}\big)}z^{-f}$$

$$T(M,\chi_0\mu_1,z) = \big(1-|\pi|\big)|\pi|^{-f/2}\omega_1^{-f}\Delta(\chi_0,\pi^{-f})\frac{\Big(1-\omega_1^{-1}z^{-1}|\pi|^{1/2}\Big)}{\big(1-\omega_1 z|\pi|^{1/2}\big)}z^{-f}$$

and if μ *is different from* μ_1 *and* $\chi_0\mu_1$ *then*

$$\begin{aligned} T(M,\mu,z) &= \big(1-|\pi|\big)^2\Delta(\chi_0,\pi^{-f})|\pi|^{-\frac{n-f}{2}}\omega_1^{-n-f}\Delta(M_0^{-1}\mu^{1+s},\Pi^{-n-f})z^{-n-f} \\ &= \big(1-|\pi|\big)^2|\pi|^{\frac{-n_1-n_2}{2}}\omega_1^{-n_1-n_2}\Delta(\mu_1^{-1}\mu,\pi^{-n_1})\Delta(\mu_2^{-1}\mu,\pi^{-n_2})z^{-n_1-n_2} \end{aligned}$$

if $\mu_2 = \chi_0\mu_1$, n *is the order of* $M_0^{-1}\mu^{1+s}$, n_1 *is the order of* $\mu_1^{-1}\mu$, *and* n_2 *is the order of* $\mu_2^{-1}\mu$.

(*b*) *If there is no such character* M_1 *then, for all* μ,

$$T(M,\mu,z) = \big(1-|\pi|\big)^2|\pi|^{\frac{-n-f}{2}}\Delta(\chi_0,\pi^{-f})M(\Pi^{-s(n+f)})\Delta(M_0^{-1}\mu^{1+s},\Pi^{-n-f})z^{-n-f}$$

if n *is the order of* $M_0^{-1}\mu^{1+s}$.

The formulas of this lemma follow from the definitions together with Lemmas 1.1 and 1.5. I would like to observe in cases (ii, a) and (iii, a) that if M' is the character of $(k\oplus k)^*$ defined by $M'(\alpha\oplus\beta) = M(\alpha)M_1(\beta)\chi(\beta)$, then, for all μ

$$T(M,\mu,z) = T(M',\mu,z).$$

It follows from Lemmas 2.7 and 2.8 that if the collection $\{T_{m,\mu}\}$ satisfies identities (A) and (B) the series

$$\sum_m T_{m,\mu}z^m$$

converges for $|z|>0$ and sufficiently small and its sum $T_\mu(z)$ is a rational function. If we return to the discussion of the representation σ we can choose some isomorphism of U with $\mathbf{C}$ and regard the functions φ_v as scalars. Let L' be the set of all locally constant complex-valued functions, i.e. invariant under some open subgroup, on $k^\times$. If ν is a character of $o^\times$ let L'_ν be the set of all functions φ in L' such that $\varphi(\beta\alpha) = \nu(\beta)\varphi(\alpha)$ if $\beta\in o^\times$. It is clear that L' is the direct sum of the spaces L'_ν. If $\varphi\in L'$ we write $\varphi = \sum_\nu \varphi_\nu$ with $\varphi_\nu\in L'_\nu$ and set $u_{k,\nu} = \varphi_\nu(\pi^k)$. Let L be the set of all functions φ in L' such that, for each ν, $u_{k,\nu}=0$ for $k\ll 0$ and

$$\varphi_\nu(z) = \sum_k u_{k,\nu}z^k$$

converges for $|z|>0$ and sufficiently small and represents a rational function. If $\widehat{H}$ is the set of all functions in L' with compact support then $\widehat{H}\subseteq L$. $\widehat{H}$ is clearly the image of $\widehat{V}$. By the way, it will not conflict with our previous notation if when $\varphi = \sum\varphi_\nu$ lies in L' and $u_{k,\nu} = \varphi_\nu(\pi^k)$ we set

$$\varphi \sim \sum_\nu\sum_k u_{k,\nu}z^k.$$

Now suppose $\{T_{m,\mu}\}$ is the collection corresponding to the representation σ. If $v \in \widehat{V}$ and

$$\varphi = \varphi_\nu \sum_\nu \sum_k u_{k,\nu} z^k$$

then

$$w = \sigma\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) v \sim \sum_\nu \sum_\ell \left\{ \sum_{m+k=\ell} \omega(\pi^k) T_{m,\widehat{\nu}} u_{-k,\widehat{\nu}} \right\} z^\ell.$$

Thus $\psi = \varphi_w$ is also in L and

$$\psi_\nu(z) = T_{\widehat{\nu}}(z) \varphi_{\widehat{\nu}}\big(\omega^{-1}(\pi) z^{-1}\big).$$

If $T_{m,\mu} = 0$ whenever $m \geqslant -1$ then $V = \widehat{V}$ so that

$$\varphi_\nu(z) = \omega_0(-1) T_{\widehat{\nu}}(z) \psi_{\widehat{\nu}}\big(\omega^{-1}(\pi) z^{-1}\big).$$

Thus, in this case,

$$\text{(D)} \qquad \omega_0(-1) T_\nu(z) T_{\widehat{\nu}}\big(\omega^{-1}(\pi) z^{-1}\big) = 1.$$

On the other hand if one notices that $MM^s = \omega^{1+s}$ so that $M^{-1}\omega^{1+s}\mu^{-1-s} = (M^{-1}\mu^{1+s})^{-s}$ one can verify by inspection that

$$\omega_0(-1) T(M, \nu, z) T\big(M, \widehat{\nu}, \omega^{-1}(\pi) z^{-1}\big) = 1.$$

Thus the identity (D) is valid whenever σ is an irreducible representation satisfying (i), (ii), and (iii).

Now let us suppose that ω is a continuous homomorphism of $k^\times$ into $\mathbf{C}^\times$ and that the family $\{T_{m,\mu}\}$ satisfies the relations (A), (B), and (D). If φ belongs to L' and $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ belongs to G_k let $\tau\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right)\varphi$ be the function whose value at α is $\omega(d)\xi_0\left(\frac{\alpha b}{d}\right)\varphi\left(\frac{\alpha a}{d}\right)$. τ is a representation of the group of upper triangular matrices in G_k on L'. $\widehat{H}$ is an invariant subspace of L' for τ. It is clear that the operators $\tau\left(\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)\right)$ leave L invariant. If $\varphi \in L$ then, for all $x \in k$, the function ψ defined by $\psi(\alpha) = \xi_0(\alpha x)\varphi(\alpha) - \varphi(\alpha)$ lies in $\widehat{H}$. Thus the operators $\tau\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right)$ leave any subspace of L containing $\widehat{H}$ invariant. Define $\tau\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)$ by the condition that if $\varphi \in L$ and $\psi = \tau\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)\varphi$ then

$$\psi_\nu(z) = T_{\widehat{\nu}}(z) \varphi_{\widehat{\nu}}\big(\omega^{-1}(\pi) z^{-1}\big).$$

It is easy to verify that

$$\tau\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) \tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) = \tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right) \tau\left(\begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}\right).$$

Thus the operators $\tau\left(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)\right)$ and $\tau\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)$ leave the space spanned by $\widehat{H}$ and the functions $\tau\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)\varphi$, $\varphi \in \widehat{H}$ invariant. Call this space H. Every matrix in G_k which is not supertriangular can be written in exactly one way as

$$g = \begin{pmatrix} \beta & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

Set $\tau(g) = \tau\left(\left(\begin{smallmatrix} \beta & 0 \\ 0 & \beta \end{smallmatrix}\right)\right)\tau\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right)\tau\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)\tau\left(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)\right)\tau\left(\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)\right)$. Thus $\tau(g)$ is defined for all g in G_k.

Let us verify that $\tau(g_1 g_2) = \tau(g_1)\tau(g_2)$. This is clear if $g_1 \sim g_2$ is a supertriangular matrix. Thus it is enough to verify this when

$$g_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

$$g_2 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The case $x + y = 0$ is taken care of by identity (D) so suppose $x + y = u \neq 0$. Then

$$g_1 g_2 = \begin{pmatrix} -\frac{1}{u} & 0 \\ 0 & -u \end{pmatrix}\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1/u \\ 0 & 1 \end{pmatrix},$$

$$\tau(g_1)\tau(g_2) = \tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\tau\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right),$$

$$\tau(g_1 g_2) = \tau\left(\begin{pmatrix} -\frac{1}{u} & 0 \\ 0 & -u \end{pmatrix}\right)\tau\left(\begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}\right)\tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\tau\left(\begin{pmatrix} 1 & -1/u \\ 0 & 1 \end{pmatrix}\right).$$

However if one examines the derivation of the identities (A) and (B) one sees that they are equivalent to the assertion that these two operators have the same effect on an element of $\widehat{H}$. To verify that the two operators are equal we need to show that if $\varphi \in \widehat{H}$ then

$$\tau(g_1 g_2)\tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi = \tau(g_1)\tau(g_2)\tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi.$$

The left side is equal to

$$\tau\left(g_1 g_2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi = \tau(g_1)\tau\left(g_2\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi = \tau(g_1)\tau(g_2)\tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi.$$

The representation τ on H certainly satisfies condition (ii). If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_0$ and $c \in \mathfrak{p}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{c}{a} & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d - \frac{bc}{a} \end{pmatrix}.$$

It is clear that for any φ in H the sets

$$\{\, g \mid \tau(g)\varphi = \varphi \,\},$$

$$\left\{ g \;\middle|\; \tau(g)\tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi = \tau\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi \right\}.$$

both contain an open subgroup of the group of upper triangular matrices. Thus the first set contains an open subgroup of the group of lower triangular matrices. It follows from the simple identity above that it contains an open subgroup of G_k.

To prove that the third condition is satisfied we need only show that if U is an open subgroup of the group of upper triangular matrices then the set X_0 of all φ in H such that U is contained in the isotropy group of both φ and $\tau\Big(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\Big)\varphi$ is finite-dimensional. If φ belongs to H then φ_ν has poles only at 0 and ∞. In general the poles of φ_ν at any point besides 0 and ∞ are of no higher order than those of $T_{\hat{\nu}}(z)$. It is clear that, if $\varphi \in X$, $\varphi_\nu = 0$ for all but a finite number of ν. Thus to prove the assertion all we need to do is obtain, for each ν, a bound on the order of the pole of φ_ν at 0 and ∞ which is valid for all φ in X. A glance at the form of the operator $\tau\big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\big)$ convinces one that there is a number N such that if U is in the isotropy group of φ then $\varphi(\alpha) = 0$ if $|\alpha| > |\pi|^N$. Thus the order of the pole of $\varphi_\nu(z)$ at 0 is at most $-N$. If φ is in X the order of the pole of

$$T_\nu(z)\varphi_\nu\big(\omega^{-1}(\pi)z^{-1}\big)$$

at 0 is also at most $-N$. The assertion follows.

Arguments similar to those used to prove Lemma 2.4 show that any invariant subspace of H different from $\{0\}$ contains a non-zero vector in $\widehat{H}$ and that $\widehat{H}$ is irreducible under the action of the upper triangular matrices. It follows immediately that τ is an irreducible representation of G_k on H.

Thus to completely classify all irreducible representations of G_k satisfying (i), (ii), and (iii) all we need to do is study the families $\{T_{m,\mu}\}$ of complex numbers which satisfy (A), (B), and (D) and have the property that, for all μ, $T_{m,\mu} = 0$ if $m \geqslant -1$. In this case, which is the case we shall discuss in the rest of this chapter, $H = \widehat{H}$.

Before going on let me observe that if ζ is another homomorphism of $k^\times$ into $\mathbf{C}^\times$ and ω is replaced by $\omega\zeta^2$ and $T_{m,\mu}$ is replaced by $\zeta(\pi^m)T_{m,\zeta_0^{-1}\mu}$ the relations (A), (B), and (D) continue to be satisfied. Thus, for our purposes, there is no harm in assuming that ω is a character.

Define an inner product on $\widehat{H}$ by

$$(\varphi,\psi) = \int_{k^\times} \varphi(\alpha)\psi(\alpha)\,d\alpha.$$

It is clear that, if g is an upper triangular matrix, $\big(\tau(g)\varphi,\tau(g)\psi\big) = (\varphi,\psi)$. It is also clear that if $\langle\varphi,\psi\rangle$ is another inner product with this property it is of the form

$$\langle\varphi,\psi\rangle = \sum_\nu a_\nu(\varphi_\nu,\psi_\nu).$$

Thus if T is the operator on $\widehat{H}$ defined by

$$T\Big(\sum\varphi_\nu\Big) = \sum a_\nu\varphi_\nu$$

$$\big(\tau(g)T\psi,\tau(g)\psi\big) = \big(T\tau(g)\varphi,\tau(g)\psi\big),$$

so that $\tau(g)T = T\tau(g)$ for all upper triangular matrices g. Thus each eigenspace of T is invariant under $\tau(g)$; so T is a scalar.

Let $\varphi_{\ell,\nu}$ be the function in $\widehat{H}_\nu = \widehat{H}\cap L'_\nu$ satisfying $\varphi_{\ell,\nu}(\pi^\ell) = 1$ and $\varphi_{\ell,\nu}(\alpha) = 0$ if $|\alpha| \neq |\pi|^\ell$. The collection $\{\varphi_{\ell,\nu}\}$ is an orthonormal basis of $\widehat{H}$. If φ is in $\widehat{H}$ and

$$\varphi \sim \sum_\nu\sum_\ell u_{\ell,\nu}z^\ell$$

then

$$\varphi = \sum_\nu \sum_\ell u_{\ell,\nu}\varphi_{\ell,\nu}.$$

If

$$g = \begin{pmatrix}\beta & 0\\ 0 & \beta\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\pi^n & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}$$

with $\alpha \in o^\times$ let us find the effect of $\tau(g)$ on φ. We iterate the effect of the various factors entering into the expression of $\tau(g)$ as a product.

$$\tau\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix}\right)\varphi = \sum_{\ell,\nu}\left\{\sum_\mu \Delta(\mu\nu^{-1}, \pi^\ell y)u_{\ell,\mu}\right\}\varphi_{\ell,\nu}.$$

Applying $\tau\big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)\big)$ to this one obtains

$$\sum_{\ell,\nu}\left\{\nu(\alpha)\sum_\mu \Delta(\mu\nu^{-1}, \pi^\ell y)u_{\ell,\mu}\right\}\varphi_{\ell,\nu}.$$

$\tau\Big(\left(\begin{smallmatrix}\pi^n & 0\\ 0 & 1\end{smallmatrix}\right)\Big)$ sends this to

$$\sum_{\ell,\nu}\left\{\nu(\alpha)\sum_\mu \Delta(\mu\nu^{-1}, \pi^{\ell+n} y)u_{\ell+n,\mu}\right\}\varphi_{\ell,\nu}.$$

Now apply $\tau\big(\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\big)$ to obtain

$$\sum_{\ell,\nu}\left\{\sum_{m-k=\ell} T_{m,\widehat{\nu}}\omega(\pi^{-k})\widehat{\nu}(\alpha)\sum_\mu \Delta(\mu\widehat{\nu}^{-1}, \pi^{k+n} y)u_{k+n,\mu}\right\}\varphi_{\ell,\nu}.$$

Finally $\tau\left(\left(\begin{smallmatrix}\beta & 0\\ 0 & \beta\end{smallmatrix}\right)\right)\tau\big(\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\big)$ transforms this to

$$\sum_{\ell,\nu}\left\{\omega(\beta)\sum_{m-k=\ell}\sum_{\rho,\mu} \widetilde{\rho}(\alpha)\omega(\pi^{-k})T_{m,\widetilde{\rho}}\Delta(\rho\nu^{-1}, \pi^\ell x)\Delta(\mu\widetilde{\rho}^{-1}, \pi^{k+n} y)u_{k+n,\mu}\right\}\varphi_{\ell,\nu}.$$

Thus if g has the above form the matrix element $\big(\tau(g)\varphi_{k,\mu}\varphi_{\ell,\nu}\big)$ is equal to

$$\omega(\beta)\omega(\pi^{n-k})\sum_\rho \widetilde{\rho}(\alpha)T_{k+\ell-n,\widetilde{\rho}}\Delta(\rho\nu^{-1}, \pi^\ell x)\Delta(\mu\widetilde{\rho}^{-1}, \pi^k y).$$

If $g = \Big(\begin{smallmatrix}\beta & 0\\ 0 & \beta\end{smallmatrix}\Big)\big(\begin{smallmatrix}\pi^n & 0\\ 0 & 1\end{smallmatrix}\big)\big(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\big)\big(\begin{smallmatrix}1 & y\\ 0 & 1\end{smallmatrix}\big)$ then $\big(\tau(g)\varphi_{k,\mu}, \varphi_{\ell,\nu}\big)$ is equal to 0 if $k \neq \ell + n$ but if $k = \ell + n$ it equals

$$\omega(\beta)\nu(\alpha)\Delta(\mu\nu^{-1}, \pi^k y).$$

A subset X of G_k will be called pseudo-compact if there is a compact subset Y of G_k such that $X \subseteq \bigcup_{\alpha\in k^\times}\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right)Y$.

Lemma 2.9. *If $T_{m,\mu} = 0$ for $m \geqslant 1$ the functions $\big(\tau(g)\varphi_{k,\mu}, \varphi_{\ell,\nu}\big)$ have their support in a pseudo-compact set.*

It is clear that the intersection of the support of $\bigl(\tau(g)\varphi_{k,\mu},\varphi_{\ell,\nu}\bigr)$ with the group of upper triangular matrices is a pseudo-compact set. If

$$g=\begin{pmatrix}\beta&0\\0&\beta\end{pmatrix}\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}\pi^n&0\\0&1\end{pmatrix}\begin{pmatrix}\alpha&0\\0&1\end{pmatrix}\begin{pmatrix}1&y\\0&1\end{pmatrix}$$

then

$$g=\begin{pmatrix}\beta&0\\0&\beta\end{pmatrix}\begin{pmatrix}-\pi^n x\alpha&1-\pi^n\alpha xy\\-\pi^n\alpha&-\pi^n\alpha y\end{pmatrix}.$$

Thus, if $N>0$ and n varies over $\{\, n \mid n \geqslant -N\,\}$ while x and y vary over

$$\Bigl\{\, z\in k \Bigm| |\pi|^{n/2}|z|<N \,\Bigr\}$$

and β varies over $k^\times$ the matrix g varies over a pseudo-compact set.

For a g of this form set

$$f_\rho(g)=\omega(\beta)\omega(\pi^{n-k})\widetilde{\rho}(\alpha)T_{k+\ell-n,\widetilde{\rho}}\Delta(\rho\nu^{-1},\pi^\ell x)\Delta(\mu\widetilde{\rho}^{-1},\pi^k y).$$

The support of f_ρ is certainly contained in a pseudo-compact set. As we saw some time ago, if the order of ρ is sufficiently large,

$$T_{k-m,\rho}=\omega(\pi^{-m})\ \frac{\Delta(\rho\nu_0^{-1},\pi^k)\Delta(\omega_0^{-1}\nu_0\rho,\pi^{-m})}{\bigl|\Delta(\omega_0^{-1}\nu_0\rho,\pi^{-m})\bigr|^2}$$

where ν_0 is a fixed character and m is the order of ρ. Thus, if the order of ρ is sufficiently large, $f_\rho(g)=0$ unless $n=k+\ell+2m$, $|\pi^{m+\ell}x|=1$, and $|\pi^{k+\ell}y|=1$. The lemma follows.

If ℓ and ν are fixed and $C_k=\Bigl\{\, \bigl(\left(\begin{smallmatrix}\alpha&0\\0&\alpha\end{smallmatrix}\right)\bigr) \Bigm| \alpha\in k^\times \,\Bigr\}$

$$\langle\varphi,\psi\rangle=\int_{G_k/C_k}\bigl(\tau(g)\varphi,\varphi_{\ell,\nu}\bigr)\Bigl(\overline{\tau(g)\psi,\varphi_{\ell,\nu}}\Bigr)\,dg$$

is a non-degenerate inner product on $\widehat{H}$. Clearly $\bigl\langle\tau(g)\varphi,\tau(g)\psi\bigr\rangle=\langle\varphi,\psi\rangle$ for all g in G_k and in particular for the upper triangular matrices. Thus there is a positive constant $C_{\ell,\nu}$ such that $\langle\varphi,\psi\rangle\equiv C_{\ell,\nu}(\varphi,\psi)$. Consequently the representation τ is unitary.

Lemma 2.10. *If the family* $\{T_{m,\mu}\}$ *of complex numbers satisfies the relations* (A), (B), *and* (D) *there is a two-dimensional semi-simple algebra K over k and a homomorphism M of $K^\times$ into $\mathbf{C}^\times$ such that*

$$T_{m,\mu}=T(M,\mu,m)$$

for all m and μ.

Because of Lemma 2.7 we need only prove this when the associated representation τ acts on $\widehat{H}$, is unitary and the matrix element $\bigl(\tau(g)\varphi_{k,\mu},\varphi_{\ell,\nu}\bigr)$ has compact support. To do this we need the Plancherel formula of Gelfand and Graev which will require a paragraph by itself. For now let us assume Lemma 2.10 and go on to its applications to the theory of automorphic forms.

3. The local functional equation for non-archimedean fields

For the sake of brevity we shall call an irreducible representation σ of G_k which satisfies (i), (ii), and (iii) of the previous chapter a simple representation.

If η is a continuous homomorphism of A_k, the group of diagonal matrices in G_k, into $\mathbf{C}^\times$, let $L(\eta)$ be the space of all locally constant functions on G_k satisfying $\varphi(ag) \equiv \eta(a)\varphi(g)$ for all a in A. Since $L(\eta)$ is invariant under right translations we obtain a representation $g \to \rho(g)$ of G_k on $L(\eta)$.

Lemma 3.1. *No infinite-dimensional simple representation of G_k is contained more than once in the restriction of ρ to $L(\eta)$.*

We may take the simple representation to be the representation τ on H considered in the previous paragraph. Suppose V is a subspace of $L(\eta)$ and T is an isomorphism of H with V such that

$$T\big(\tau(g)\varphi\big) = \rho(g)T\varphi$$

for all φ. Set $\lambda(\varphi) = T\varphi(1)$. Then $T\varphi(g) = \big(\rho(g)T\varphi\big)(1) = \lambda\big(\tau(g)\varphi\big)$. Thus T is completely determined by λ. If $a \in A_k$ then

$$\lambda\big(\tau(a)\varphi\big) = \eta(a)\lambda(\varphi).$$

Let us verify that up to a scalar factor there is at most one linear function on H with this property. Let $\eta\Big(\big(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\big)\Big) = \eta_1(a)\eta_2(b)$ and, assumed, let $\tau\big(\big(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\big)\big) = \omega(\alpha)I$. There is no such function unless $\eta_1\eta_2 = \omega$. If $\varphi \in H_\nu = H \cap L'_\nu$ and $\alpha \in o^\times$ then $\eta_1(\alpha)\lambda(\varphi) = \nu(\alpha)\lambda(\varphi)$. Thus λ vanishes on H_ν unless $\nu = \nu_0$, the restriction of η_1 to $o^\times$. If $\varphi \in H_{\nu_0}$ and $\psi = \tau\Big(\big(\begin{smallmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\big)\Big)\varphi - \eta_1(\pi)^{-1}\varphi$ or, what is the same, if

$$\psi(z) = \big(z - \eta_1^{-1}(\pi)\big)\varphi(z) \tag{A}$$

then $\lambda(\psi) = 0$.

If $H_{\nu_0} = \widehat{H}_{\nu_0}$ then $\big\{\, \psi(z) \;\big|\; \psi \in H_{\nu_0} \,\big\}$ consists of all rational functions with poles nowhere but at 0 and ∞. Then $\psi(z)$ can be put in the above form if and only if $\eta_1^{-1}(\pi)$ is a zero of $\psi(z)$. The assertion follows in this case. If $H_{\nu_0} \neq \widehat{H}_{\nu_0}$ either

$$T_{\widetilde{\nu}_0}(z) = cz^{-k}\frac{(z-\gamma_1)}{(z-\delta_1)}\frac{(z-\gamma_2)}{(z-\delta_2)}$$

or

$$T_{\widetilde{\nu}_0}(z) = cz^{-k}\frac{z-\gamma_1}{z-\delta_1}.$$

Here c is a complex constant, k is an integer, and γ_1, γ_2, δ_1, δ_2 are complex constants. In the first case we may suppose that $\gamma_i \neq \delta_j$ for $i, j = 1$ or 2 and in the second case we may suppose that $\gamma_1 \neq \delta_1$. In the first case $\big\{\, \psi(z) \;\big|\; \psi \in H_{\nu_0} \,\big\}$ consists of all rational functions with poles of arbitrary order at 0 and ∞, poles of order at most 1 at δ_1 and δ_2 and no other poles. In the second case it consists of all rational functions with poles of arbitrary order at 0 and ∞, a pole of order at most 1 at δ_1, and no other poles. In any case $\psi(z)$ is of the form (A) if and only if the order of its pole at $\eta_1^{-1}(\pi)$ is 1 less than the maximum allowable. This completes the proof of the lemma.

If $\xi(x)$ is a non-trivial character of k let $L(\xi)$ be the set of all locally constant functions on G_k satisfying $\varphi\big(\big(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\big)g\big) \equiv \xi(x)\varphi(g)$ for all $x \in k$. Let $\rho(\xi)$ be the restriction of the right regular representation to $L(\xi)$.

Lemma 3.2. *Every infinite-dimensional simple representation of G_k occurs exactly once in $\rho(\xi)$.*

Choose γ in $k^\times$ so that $\xi(x) = \xi_0(\gamma x)$. Let the simple representation τ act on H, as before. Suppose there is a homomorphism T of H into $L(\xi)$ such that $T\big(\tau(g)\varphi\big) = \rho(g)(T\varphi)$. Set $\lambda(\varphi) = T\varphi(1)$. Then $\lambda\Big(\tau\big(\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\big)\varphi\Big) = \xi_0(\gamma x)\lambda(\varphi)$. Since $T\varphi(g) = \lambda\big(\tau(g)\varphi\big)$, T is determined by λ. Conversely if λ is such a linear function and $T\varphi$ is defined by $T\varphi(g) = \lambda\big(\tau(g)\varphi\big)$ then T commutes with the action of G_k.

Such a linear function must annihilate all functions in H of the form

$$\psi(\alpha) = \big\{\xi_0(\gamma x) - \xi_0(\alpha x)\big\}\varphi(\alpha)$$

with φ in H. Since any function in H which vanishes at γ is a linear combination of such functions the assertion follows.

Suppose τ is a simple representation of G_k. Let K be a two-dimensional semi-simple algebra over k and let M be a homomorphism of $K^\times$ into $\mathbf{C}^\times$. Suppose τ is associated to the family $\big\{T(M,\mu,n)\big\}$. Let the restriction of M to $k^\times$ be $\chi\omega$. Suppose ζ is a continuous homomorphism of A_k into $\mathbf{C}^\times$ such that $\zeta\big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right)\big)\omega(\alpha) \equiv 1$. Let $\zeta\Big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)\Big) = \zeta_1(\alpha)\zeta_2(\beta)$. Let ζ_0 be the restriction of ζ_1 to $o^\times$ and let $\zeta_1(\alpha\pi^n) = \zeta_0(\alpha)|\pi|^s$ for $\alpha \in o^\times$. ζ is uniquely determined by ζ_0 and s and we shall occasionally write $\zeta = \zeta(s,\zeta_0)$. Let $L(\xi,\tau)$ be the unique subspace of $L(\xi)$ which transforms according to the representation τ.

If η is any continuous homomorphism of A_k into $\mathbf{C}^\times$ let $\widetilde{\eta}$ be the homomorphism defined by $\widetilde{\eta}\Big(\left(\begin{smallmatrix}a & 0\\ 0 & b\end{smallmatrix}\right)\Big) = \eta\Big(\left(\begin{smallmatrix}b & 0\\ 0 & a\end{smallmatrix}\right)\Big)$.

Lemma 3.3. *If τ is given there is a number N such that if φ belongs to $L(\xi,\tau)$ and $\zeta = \zeta(s,\zeta_0)$ the integral*

$$\Phi(g,\zeta,\varphi) = \int_{k^\times} \varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right)\zeta\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\right) d\alpha$$

is defined for $\mathrm{Re}(s) > N$.

(*i*) *Suppose $K = k \oplus k$ and $M(\pi^p\alpha \oplus \pi^q\beta) = \omega_1^p\omega_2^q\mu_1(\alpha\beta)$ if α, $\beta \in o^\times$. Suppose also that neither $\frac{\omega_1}{\omega_2}$ nor $\frac{\omega_2}{\omega_1}$ is equal to $|\pi|$. If $\mu_1 = \zeta_0^{-1}$ set*

$$\Phi'(g,\zeta,\varphi) = \Big(1 - \omega_1|\pi|^{s+1/2}\Big)\Big(1 - \omega_2|\pi|^{s+1/2}\Big)\Phi(g,\zeta,\varphi).$$

Then, for each g, $\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover if $\mathfrak{p}^{-d}$ is the largest ideal on which ξ is trivial

$$\zeta_1(\pi^d)\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g,\eta,\varphi\right) = \widetilde{\zeta}_1(\pi^d)\Phi'(g,\widetilde{\zeta},\varphi).$$

If $\mu_1 \neq \zeta_0^{-1}$ set $\Phi'(g,\zeta,\varphi) = \Phi(g,\zeta,\varphi)$. Then, for each g, $\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^{-s}$ and $|\pi|^s$ and, for a suitable choice of g and φ it is a constant. Moreover, if

$\xi(x) = \xi_0(\gamma x),$

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, \zeta, \varphi\right) = \bigl(1 - |\pi|\bigr)^2 |\pi|^{-n-2ns} \omega_1^{-n} \omega_2^{-n} \widetilde{\zeta}_1(\gamma) \bigl\{\Delta(\mu_1\zeta_0, \pi^{-n})\bigr\}^2 \Phi'(g, \widetilde{\zeta}, \varphi)$$

if n is the order of $\mu_1\zeta_0$.

(*ii*) *Suppose $K = k \oplus k$ and $M(\pi^p\alpha \oplus \pi^q\beta) = \omega_1^p\omega_2^q\mu_1(\alpha\beta)$ if α, $\beta \in o^\times$. Suppose also that $\frac{\omega_1}{\omega_2} = |\pi|$. If $\mu_1 = \zeta_0^{-1}$ set*

$$\Phi'(g, \zeta, \varphi) = \Bigl(1 - \omega_1|\pi|^{s+1/2}\Bigr)\Phi(g, \zeta, \varphi).$$

Then, for each g, $\Phi'(g, \zeta, \varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\pi^d)\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, \zeta, \varphi\right) = -\frac{|\pi|^{-\frac{1}{2}-s}}{\omega_2}\widetilde{\zeta}_1(\pi^d)\Phi'(g, \widetilde{\zeta}, \varphi).$$

If $\mu_1 \neq \zeta_0^{-1}$ set $\Phi'(g, \zeta, \varphi) = \Phi(g, \zeta, \varphi)$. Then, for each g, $\Phi'(g, \zeta, \varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} g, \zeta, \varphi\right) = \bigl(1 - |\pi|\bigr)^2 |\pi|^{-n-2ns} \omega_1^{-n} \omega_2^{-n} \widetilde{\zeta}_1(\gamma) \bigl\{\Delta(\mu_1\zeta_0, \pi^{-n})\bigr\}^2 \Phi'(g, \widetilde{\zeta}, \varphi).$$

(*iii*) *Suppose $K = k \oplus k$ and $M(\pi^p\alpha \oplus \pi^q\beta) = \omega_1^p\omega_2^q\mu_1(\alpha\beta)$ if α, $\beta \in o^\times$. Suppose also that $\frac{\omega_2}{\omega_1} = |\pi|$. If $\mu_1 = \zeta_0^{-1}$ set*

$$\Phi'(g, \zeta, \varphi) = \Bigl(1 - \omega_2|\pi|^{s+1/2}\Bigr)\Phi(g, \zeta, \varphi).$$

Then, for each g, $\Phi'(g, \zeta, \varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\pi^d)\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, \zeta, \varphi\right) = -\frac{|\pi|^{-1/2-s}}{\omega_1}\widetilde{\zeta}_1(\pi^d)\Phi'(g, \widetilde{\zeta}, \varphi).$$

If $\mu_1 \neq \eta_0^{-1}$ set $\Phi'(g, \zeta, \varphi) = \Phi(g, \zeta, \varphi)$. Then, for each g, $\Phi'(g, \zeta, \varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, \zeta, \varphi\right) = \bigl(1 - |\pi|\bigr)^2 |\pi|^{-n-2ns} \omega_1^{-n} \omega_2^{-n} \widetilde{\zeta}_1(\gamma) \bigl\{\Delta(\mu_1\zeta_0, \pi^{-n})\bigr\}^2 \Phi'(g, \widetilde{\zeta}, \rho).$$

(*iv*) *Suppose $K = k \oplus k$ and $M(\pi^p\alpha \oplus \pi^q\beta) = \omega_1^p\omega_2^q\mu_1(\alpha)\mu_2(\beta)$ if α, $\beta \in o^\times$ where $\mu_1 \neq \mu_1$. If $\mu_1 = \zeta_0^{-1}$ set*

$$\Phi'(g, \zeta, \varphi) = \Bigl(1 - \omega_1|\pi|^{s+1/2}\Bigr)\Phi(g, \zeta, \varphi).$$

Then, for each g, $\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g,\zeta,\varphi\right) = \big(1-|\pi|\big)|\pi|^{-\frac{n}{2}-ns}\omega_2^{-n}\Delta(\zeta_0\mu_2,\pi^{-n})\widetilde{\zeta}_1(\gamma)\Phi'(g,\widetilde{\zeta},\varphi)$$

if n is the order of $\mu_1^{-1}\mu_2$. If $\mu_2=\zeta_0^{-1}$ set

$$\Phi'(g,\zeta,\varphi) = \Big(1-\omega_2|\pi|^{s+1/2}\Big)\Phi(g,\zeta,\varphi).$$

Then, for each g, $\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g,\zeta,\varphi\right) = \big(1-|\pi|\big)|\pi|^{-\frac{n}{2}-ns}\widetilde{\zeta}_1(\gamma)\omega_1^{-n}\Delta(\zeta_0\mu_1,\pi^{-n})\Phi'(g,\widetilde{\zeta},\varphi).$$

If ζ_0^{-1} is different from both μ_1 and μ_2 set $\Phi'(g,\zeta,\varphi)=\Phi(g,\zeta,\varphi)$. Then, for each g, $\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g,\zeta,\varphi\right)$$

is equal to

$$\big(1-|\pi|\big)^2|\pi|^{(-n_1-n_2)\left(\frac{1}{2}+s\right)}\widetilde{\zeta}_1(\gamma)\omega_1^{-n_2}\omega_2^{-n_1}\Delta(\mu_2\zeta_0,\pi^{-n_1})\Delta(\mu_1\zeta_0,\pi^{-n_2})\Phi'(g,\widetilde{\zeta},\varphi)$$

if n_1 is the order of $\mu_2\zeta_0$ and n_2 is the order of $\mu_1\zeta_0$.

(*v*) *Suppose K is an unramified extension of k and there is a homomorphism M_1 of $k^\times$ into $\mathbf{C}^\times$ such that $M(\alpha)=M_1(N\alpha)$. Let $M_1(\pi^p\beta)=\omega_1^p\mu_1(\beta)$ for p in $o^\times$. If $\mu_1=\zeta_0^{-1}$ set*

$$\Phi'(g,\zeta,\varphi) = \big(1-\omega_1^2|\pi|^{2s+1}\big)\Phi(g,\zeta,\varphi).$$

Then, for each g, $\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\pi^d)\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g,\zeta,\varphi\right) = \widetilde{\zeta}_1(\pi^d)\Phi'(g,\widetilde{\zeta},\varphi).$$

If $\mu_1\neq\zeta_0^{-1}$ set $\Phi'(g,\zeta,\varphi)=\Phi(g,\zeta,\varphi)$. Then, for each g, $\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g,\zeta,\varphi\right) = \widetilde{\zeta}_1(\gamma)\big(1-|\pi|^2\big)|\pi|^{-n-2ns}\omega_1^{-2n}\Delta\big((\mu_1\zeta_0)^{1_s},\Pi^{-n}\big)\Phi'(g,\widetilde{\zeta},\varphi)$$

if n is the order of $\mu_1\zeta_0$.

(*vi*) *Suppose K is an unramified extension of k and there is no homomorphism M_1 of $k^\times$ into $\mathbf{C}^\times$ such that $M(\alpha)\equiv M_1(N\alpha)$. Set $\Phi'(g,\zeta,\varphi)=\Phi(g,\zeta,\varphi)$. Then, for all g,*

$\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g,\zeta,\varphi\right)$$
$$= \widetilde{\zeta}_1(\gamma)\big(1-|\pi|^2\big)|\pi|^{-n-2ns}M(\Pi^{-n})\Delta\big(M_0^{-1}(\omega_0\zeta_0)^{1+s},\Pi^{-n}\big)\Phi'(g,\widetilde{\zeta},\varphi).$$

(*vii*) *Suppose K is a ramified extension of k and there is a generalized character M_1 of $k^\times$ such that $M(\alpha)\equiv M_1(N\alpha)$. Let $M_1(\pi^p\beta)=\omega_1^p\mu_1(\beta)$ if $\beta\in o^\times$. If $\mu_1=\zeta_0^{-1}$ set*

$$\Phi'(g,\zeta,\varphi)=\Big(1-\omega_1|\pi|^{s+1/2}\Big)\Phi(g,\zeta,\varphi).$$

Then, for each g, $\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g,\zeta,\varphi\right)=\big(1-|\pi|\big)|\pi|^{-\frac{f}{2}-fs}\omega_1^{-f}\widetilde{\zeta}_1(\gamma)\Delta(\chi_0,\pi^{-f})\Phi'(g,\widetilde{\zeta},\varphi).$$

If $\mu_1\chi_0=\zeta_0^{-1}$ set

$$\Phi'(g,\zeta,\varphi)=\Big(1-\omega_1|\pi|^{s+1/2}\Big)\Phi(g,\zeta,\varphi).$$

Then, for for each g, $\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g,\zeta,\varphi\right)=\big(1-|\pi|\big)|\pi|^{-\frac{f}{2}-fs}\omega_1^{-f}\widetilde{\zeta}_1(\gamma)\Delta(\chi_0,\pi^{-f})\Phi'(g,\widetilde{\zeta},\varphi).$$

If ζ_0^{-1} is equal to neither μ_1 nor $\mu_1\chi_0$ set $\Phi'(g,\zeta,\varphi)=\Phi(g,\zeta,\varphi)$. Then, for each g, $\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$ and for a suitable choice of g and φ it is a constant. Moreover,

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g,\zeta,\varphi\right)$$

is equal to

$$\big(1-|\pi|\big)^2|\pi|^{-\frac{n-f}{2}-(n+f)s}\omega_1^{-n-f}\widetilde{\zeta}_2(\gamma)\Delta\Big(M_0^{-1}(\omega_0\zeta_0)^{1+s},\Pi^{-n-f}\Big)\Delta(\chi_0,\pi^{-f})\Phi'(g,\widetilde{\zeta},\varphi).$$

(*viii*) *Suppose K is a ramified extension of k and there is no homomorphism M_1 of $k^\times$ into $\mathbf{C}^\times$ such that $M(\alpha)\equiv M_1(N\alpha)$. Set $\Phi'(g,\zeta,\varphi)=\Phi(g,\zeta,\varphi)$. Then, for each g, $\Phi'(g,\zeta,\varphi)$ is a polynomial in $|\pi|^s$ and $|\pi|^{-s}$. Moreover,*

$$\zeta_1(\gamma)\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g,\zeta,\varphi\right)$$

is equal to

$$\big(1-|\pi|\big)^2|\pi|^{-\frac{n-f}{2}-(n-f)s}\widetilde{\zeta}_1(\gamma)M(\Pi^{-s(n+f)})$$
$$\cdot\,\Delta(\chi_0,\pi^{-f})\Delta\Big(M_0^{-1}(\omega_0\zeta_0)^{1+s},\Pi^{-n-f}\Big)\Phi'(g,\zeta,\varphi).$$

Of course, $\widetilde{\zeta}\Big(\big(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\big)\Big) = \widetilde{\zeta}_1(a)\widetilde{\zeta}_2(b)$. Thus $\widetilde{\zeta}_1 = \zeta_2$. Since $\zeta_1\zeta_2 = \omega^{-1}$, $\widetilde{\zeta}_1 = \omega^{-1}\zeta_1^{-1}$. In particular, $\widetilde{\zeta}_0 = \omega_0^{-1}\zeta_0^{-1}$ so that $\widetilde{\zeta}_0 = \zeta_0$ if $\zeta_0^{-2} = \omega_0$. If $\xi(x) = \xi_0(\gamma x)$ then the map $\varphi \to \psi$ with $\psi(g) = \varphi\Big(\big(\begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix}\big)g\Big)$ is an isomorphism of $L(\xi_0, \tau)$ with $L(\xi, \tau)$.

$$\int_{k^\times} \psi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)\zeta\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) d\alpha = \zeta_1^{-1}(\gamma)\int_{k^\times} \varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)\zeta\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) d\alpha.$$

This, together with the previous observation that $\widetilde{\zeta}_0 = \zeta_0$ if $\zeta_0^{-1} = \omega_0$, makes it clear that it is enough to prove the lemma for $\xi = \xi_0$.

Since $L(\xi_0, \tau)$ is invariant under right translations it is enough to prove the assertions of the lemma for $g = 1$. The map $\psi \to T\psi$ where

$$T\psi(g) = \big(\tau(g)\psi\big)(1)$$

is an isomorphism of H and $L(\xi_0, \tau)$. If $\varphi = T\psi$ then

$$\int_{k^\times} \varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right)\zeta\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}\right) d\alpha = \int_{k^\times} \psi(\alpha)\zeta_1(\alpha)\, d\alpha.$$

Since $H \subseteq L$ the integral on the right converges if $\mathrm{Re}(s)$ is sufficiently large and

$$\Phi(1, \zeta, \varphi) = \psi_{\zeta_0^{-1}}\big(|\pi|^s\big).$$

The proof of Lemma 3.1, together with Lemma 2.8, shows that there are at most two points, which are independent of ν and ψ, besides 0 and ∞ where $\psi_\nu(z)$ can have a pole. This shows that for $\mathrm{Re}(s)$ sufficiently large the integral on the right converges for all ψ. Let $\psi' = \tau\Big(\big(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\big)\Big)\psi$. Then

$$\begin{aligned} \Phi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, \zeta, \varphi\right) &= \psi'_{\zeta_0^{-1}}\big(|\pi|^s\big) \\ &= T_{\omega_0\zeta_0}\big(|\pi|^s\big)\psi_{\omega_0\zeta_0}\big(\omega^{-1}(\pi)|\pi|^{-s}\big) \end{aligned}$$

and

$$\Phi(1, \widetilde{\zeta}, \varphi) = \psi_{\omega_0\zeta_0}\big(\omega^{-1}(\pi)|\pi|^{-s}\big).$$

The lemma follows from these two relations, the formulae of Lemma 2.8, and the observations about $\{\, \psi(z) \mid \psi \in H_\nu \,\}$ made while proving Lemma 3.1. It is a matter of inspection which must be left to the reader.

Lemma 3.4. *There is a vector in H whose isotropy group contains G_o only if ω_0 is the trivial character. If ω_0 is trivial the only cases of the previous lemma for which H contains such a vector are (i) and (v). In cases (i) and (v) H contains such a vector if and only if μ_1 is trivial.*

It is clear that such a function (or vector) can exist only if ω_0 is trivial and that if ν_0 is the trivial character of $o^\times$, it must lie in $H\nu_0$. Suppose there is a function φ in $H\nu_0$ invariant under G_o. Then $\varphi(z)$ has no pole at zero and

$$\varphi(z) = T\nu_0(z)\varphi\big(\omega^{-1}(\pi)z^{-1}\big).$$

In all cases, $T_{\nu_0}(Z)$ has a pole of order at least two at 0. Thus $\varphi\big(\omega^{-1}(\pi)z^{-1}\big)$ has a zero of order at least two at 0 and $\varphi(z)$ has a zero of order at least two at ∞. Consequently it has at least two poles in the finite plane. The discussion during the proof of Lemma 3.1 shows that this is possible only in the cases mentioned. Besides these two poles there can be no others. Thus the only zeros are at infinity and $\varphi(z)$ is a constant multiple of

$$\frac{1}{\left(1-\omega_1 z|\pi|^{1/2}\right)}\ \frac{1}{\left(1-\omega_2 z|\pi|^{1/2}\right)}$$

in the first case and of

$$\frac{1}{1-\omega_1^2 z^2|\pi|}$$

in the fifth.

Conversely if ω_0 is trivial, φ lies in $H\nu_0$ and $\varphi(z)$ has this form, the isotropy group of φ contains $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and the upper triangular matrices in G_o. However G_o is generated by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ ad the upper triangular matrices in it.

Lemma 3.5. *No one-dimensional simple representation of G_k is continual in $\rho(\xi)$.*

According to the corollary to Lemma 2.1 any function on G_k which transformed according to a one-dimensional simple representation of G_k would be invariant on the right, and therefore on the left, under the group of matrices in G_k of determinant 1. In particular it would satisfy $\varphi\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)g\right) \equiv \varphi(g)$ for all x in k. Such a function could not possibly lie in $L(\xi)$.

Let L_0 be the space of all functions on $N_k\backslash G_k$ which are G_o finite on the right.

Lemma 3.6.

(*i*) *Let $K = k \oplus k$, let $M(\alpha_1 \oplus \alpha_2) = \chi_1(\alpha_1)\chi_2(\alpha_2)$ be a continuous homomorphism of $K^\times$ into $\mathbf{C}^\times$, and let τ be the representation associated with the family $\{T(M,\mu,m)\}$.*

(*a*) *Suppose $\chi_1\chi_2^{-1}$ is not one of the characters $\alpha \to 1$, $\alpha \to |\alpha|$, $\alpha \to |\alpha|^{-1}$. Then there are two subspaces H_1 and H_2 of L_0 which transform according to the representation τ and have the property that*

$$\varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} g\right) \equiv \left|\frac{\alpha}{\beta}\right|^{1/2} \chi_1(\alpha)\chi_2(\beta)\varphi(g)$$

if $\varphi \in H_1$ and

$$\varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} g\right) \equiv \left|\frac{\alpha}{\beta}\right|^{1/2} \chi_1(\beta)\chi_2(\alpha)\varphi(g)$$

if $\varphi \in H_2$. Moreover, any subspace H of L_0 which transforms according to τ is contained in $H_1 + H_2$.

(*b*) *Suppose $\chi_1 = \chi_2$. Then there are two subspaces H_1 and H_2 of L_0 which transform according to the representation τ and an isomorphism T of H_2 into H_1 which commutes with the action of G_k and is such that*

$$\varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} g\right) \equiv \left|\frac{\alpha}{\beta}\right|^{1/2} \chi_1(\alpha)\chi_2(\beta)\varphi(g)$$

if $\varphi \in H_1$ and

$$\varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} g\right) \equiv \left|\frac{\alpha}{\beta}\right|^{1/2} \chi_1(\alpha)\chi_2(\beta)\left\{\varphi(g) + T\varphi(g)\log\left|\frac{\alpha}{\beta}\right|\right\}$$

if $\varphi \in H_2$. Moreover, any subspace of L_0 which transforms according to τ is contained in $H_1 + H_2$.

(*c*) *Suppose $\chi_1\chi_2^{-1}(\alpha) \equiv |\alpha|$. Then there is a subspace H_1 of L_0 which transforms according to the representation τ and has the property that*

$$\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} g\right) \equiv \left|\frac{\alpha}{\beta}\right|^{1/2} \chi_1(\alpha)\chi_2(\beta)\varphi(g)$$

if $\varphi \in H$. Moreover H_1 is the only subspace of L_0 which transforms according to τ.

(*ii*) *Let K be a separable extension of k and let M be a continuous homomorphism of $K^\times$ into $\mathbf{C}^\times$. Let τ be the representation associated to the family $\{T(M, m, \mu)\}$. If there is no continuous homomorphism M_1 of $k^\times$ into $\mathbf{C}^\times$ such that $M(\alpha) \equiv M_1(N\alpha)$ then there is no subspace of L_0 which transforms according to τ.*

As in the proofs of Lemmas 3.1 and 3.2, there is a one:one correspondence between G-invariant homomorphisms T of H, the space on which τ acts as in paragraph 2, into L_0 and linear functions λ on H satisfying $\lambda\Big(\tau\big(\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\big)\varphi\Big) = \lambda(\varphi)$ for all φ in H and all $x \in k$. Given such a linear function $(T\varphi)(g) = \lambda\big(\tau(g)\varphi\big)$. A linear function λ is of the required type if and only if it annihilates all functions of the form

$$\psi(\alpha) = \big(\xi(\alpha x) - 1\big)\varphi(\alpha) \qquad\qquad \varphi \in H,\ x \in k.$$

The space spanned by such functions is just $\widehat{H}$. Now $\widehat{H}_\nu = H_\nu$ for all but one or two characters ν. Moreover if $H_\nu \neq \widehat{H}_\nu$ then $\lambda(\varphi_\nu)$, $\varphi_\nu \in H_\nu$ can depend only on the coefficients of the principal parts of $\varphi_\nu(z)$ at the finite poles different from 0.

Part (ii) of the lemma follows immediately. For part (i) let $\chi_i(\pi^p a) = \omega_i^p \mu_i(\alpha)$ if $\alpha \in o^\times$. If we are in case (i, a) set

$$\lambda_1(\varphi) = \lambda_1\Big(\sum \varphi_\nu\Big) = \operatorname{Re}(s)\varphi_{\mu_1}(z)\Big|_{\frac{1}{\omega_1|\pi|^{1/2}}},$$

$$\lambda_2(\varphi) = \lambda_2\Big(\sum \varphi_\nu\Big) = \operatorname{Re}(s)\varphi_{\mu_2}(z)\Big|_{\omega_2|\pi|^{1/2}}.$$

Then λ is a linear combination of λ_1 and λ_2. If we are in case (i, b) let

$$\frac{a_1}{\Big(z - \frac{1}{\omega_2|\pi|^{1/2}}\Big)^2} + \frac{a_2}{\Big(z - \frac{1}{\omega_1|\pi|^{1/2}}\Big)}$$

be the principal part of $\varphi_{\mu_1}(z)$ at $\frac{1}{\omega_1|\pi|^{1/2}}$ and set $\lambda_1(\varphi) = \lambda_1\big(\sum \varphi_\nu\big) = a_1$ and $\lambda_2(\varphi) = \lambda_2\big(\sum \varphi_\nu\big) = a_2$. Then λ is a linear combination of λ_1 and λ_2. If we are in case (i, c) let

$$\lambda_1(\varphi) = \lambda_1\Big(\sum \varphi_\nu\Big) = \operatorname{Re}(s)\varphi_{\mu_1}(z)\Big|_{\frac{1}{\omega_1|\pi|^{1/2}}}.$$

In all cases H_i is the image of H under the map T_i associated to λ_i. In case (i, b) take $T_0 = -\frac{1}{\log|\pi|}T_1T_2^{-1}$. The other assertions of the lemma follow from the form of the mapping associated to a given linear function, the fact that $\tau\big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right)\big)\varphi = \chi_1(\alpha)\chi_2(\alpha)\varphi$, and the fact that if $\psi = \tau\Big(\left(\begin{smallmatrix}\pi^p\alpha & 0\\ 0 & 1\end{smallmatrix}\right)\Big)\varphi$ with $\alpha \in o^\times$ then $\psi_\nu(z) = \nu(\alpha)z^{-p}\varphi_\nu(z)$.

4. The local functional equations reconsidered

In mathematics also "our beginnings never know our ends." In order to give the main theorem a more striking form than was previously possible I want to reformulate the local functional equations. First of all let me recall the functional equations of the Hecke L-series.

Suppose K is a local field. We shall associate to each generalized character χ of $K^\times$ a function $\xi(s,\chi)$ of the complex variable s. We shall introduce a local factor $\epsilon(s,\chi)$. $\epsilon(s,\chi)$ will depend upon the choice of a character ξ of K. (Notice that the symbol ξ, like the symbol s, is used to denote two different objects.)

If K is a global field, χ a generalized character of $K^\times\backslash I$, and ξ a character of $K\backslash \mathbf{A}$ let $\chi_\mathfrak{p}$ and $\xi_\mathfrak{p}$ be the restrictions of χ and ξ to $K_\mathfrak{p}^\times$ and $K_\mathfrak{p}$, respectively. Define $\xi(s,\chi_\mathfrak{p})$ and $\epsilon(s,\chi_\mathfrak{p})$ to be the local factors corresponding to $\xi_\mathfrak{p}$. The (modified) zeta function associated to χ will be

$$\prod_\mathfrak{p} \xi(s,\chi_\mathfrak{p}) = \xi(s,\chi).$$

It will satisfy the functional equation

$$\xi(s,\chi) = \epsilon(s,\chi)\xi(1-s,\chi^{-1}),$$
$$\epsilon(s,\chi) = \prod_\mathfrak{p} \epsilon(s,\chi_\mathfrak{p}),$$

both products are taken over all primes, both finite and infinite.

Let us describe the functions $\xi(s,\chi)$ and $\epsilon(s,\chi)$ for local fields.

(i) $K=\mathbf{R}$. Let $\chi(\alpha) = (\operatorname{sgn}\alpha)^M|\alpha|^r$, with $m=0$ or 1, and let $\xi(x) = e^{2\pi i u x}$. Then

$$\xi(s,\chi) = \pi^{-\frac{1}{2}(s+r+m)}\Gamma\left(\frac{s+r+m}{2}\right),$$
$$\epsilon(s,\chi) = \frac{(i\operatorname{sgn} n)^m}{|u|^{1/2-s-r}}.$$

(ii) $K=\mathbf{C}$. Let $|\alpha|$ be the square of the ordinary absolute value. Let $\chi(\alpha) = |\alpha|^r\left(\frac{\alpha^m\bar\alpha^n}{|\alpha|^{\frac{m+n}{2}}}\right)$ with $mn=0$ and $m+n\geqslant 0$. Let $\xi(z) = e^{4\pi i \operatorname{Re} wz}$. Then

$$\xi(s,\chi) = 2(2\pi)^{-\left(s+r+\frac{m+n}{2}\right)}\Gamma\left(s+r+\frac{m+n}{2}\right)$$
$$\epsilon(s,\chi) = i^{m+n}\chi(w)|w|^{s-\frac{1}{2}}.$$

(iii) K is non-archimedean. Let $\mathfrak{P}^{-d}$ be the largest ideal on which ξ is trivial. If Π is a generator of $\mathfrak{P}$ and the conductor of χ is 0

$$\xi(s,\chi) = \frac{1}{1-\chi(\Pi)|\Pi|^s}$$
$$\epsilon(s,\chi) = \chi(\Pi^d)|\Pi^d|^{s-\frac{1}{2}}.$$

If the conductor of χ is $\mathfrak{P}^n$ with $n>0$

$$\xi(s,\chi) = 1,$$
$$\epsilon(s,\chi) = \chi(\Pi^{d+n})|\Pi^d|^{s-\frac{1}{2}}\frac{1-|\Pi|}{|\Pi|^{n/2}}\int_{O^\times}\xi\left(\frac{\alpha}{\Pi^{d+n}}\right)\chi^{-1}(\alpha)\,d\alpha.$$

Before restating the local functional equations let me introduce some conventions. Let k be a local field. Let us introduce some language which, though rather bizarre, will be useful. If $k = \mathbf{R}$ a simple representation of G_k is an irreducible quasi-simple representation of $\{\sigma, \mathfrak{A}\}$ (the notation is that of paragraph 2 of my letter to Weil). If $k = \mathbf{C}$ a simple representation of G_k is an irreducible quasi-simple representation of $\mathfrak{A}$ (the notation is that of paragraph 4 of my letter). If k is non-archimedean the simple representations of G_k have been introduced in the previous paragraph. If τ is an infinite-dimensional simple representation of G_k and ξ is a character of k the space $L(\xi, \tau)$ has been defined.

If χ is a homomorphism of $k^\times$ into $\mathbf{C}^\times$ and s a complex number and φ belongs to $L(\xi, \tau)$ set

$$\Phi(g, s, \varphi, \chi) = \int_{k^\times} \varphi\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(\alpha)|\alpha|^s \, d\alpha.$$

The integral converges for $\mathrm{Re}(s)$ sufficiently large. We shall introduce factors[1] $\xi(s, \tau, \chi)$ and $\epsilon(s, \tau, \chi)$ and set

$$\Phi'(g, s, \varphi, \chi) = \frac{\Phi(g, s, \varphi, \chi)}{\xi(s, \tau, \chi)}.$$

Then the local functional equation will be

$$\Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g, -s, \varphi, (\eta\chi)^{-1}\right) = \epsilon(s, \tau, \chi)\Phi'(g, s, \varphi, \chi)$$

if[2] $\tau\big(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)\big) \equiv \eta(\alpha)I$. I shall write down the factors $\xi(s, \tau, \chi)$ and $\epsilon(s, \tau, \chi)$ but I will leave to the reader the task of verifying that the local functional equation takes the above form. He will probably require paper and pencil. Since the analytical properties of the functions $\Phi'(g, s, \tau, \chi)$ follow immediately from previous results I shall not formulate them explicitly either.

(i) $k = \mathbf{R}$

(a) Let M be a continuous homomorphism of $\mathbf{R}^\times \times \mathbf{R}^\times$ into $\mathbf{C}^\times$. Let

$$M\big((t_1, t_2)\big) = |t_1|^{s_1}|t_2|^{s_2}\left(\frac{t_1}{|t_1|}\right)^{m_1}\left(\frac{t_2}{|t_2|}\right)^{m_2}$$

with m_1 and m_2 equal to 0 or 1. Suppose $(s_1 - s_2) - (m_1 - m_2)$ is not an odd integer. Set $\chi_1(t) = M\big((t, 1)\big)$, $\chi_2(t) = M\big((1, t)\big)$. Let $\tau = \tau_M$ be the simple representation π_M introduced in paragraph 2 of my letter to Weil. Set

$$\xi(s, \tau, \chi) = \xi\left(\frac{1}{2} + s, \chi_1\chi\right)\xi\left(\frac{1}{2}, s, \chi_2\chi\right),$$

$$\epsilon(s, \tau, \chi) = \epsilon\left(\frac{1}{2} + s, \chi_1\chi\right)\epsilon\left(\frac{1}{2} + s, \chi_2\chi\right).$$

(Notice when verifying this that there is an error in part (i) on page 3.34 of the letter to Weil.[3] The second factor in the denominator on the right should be $\Gamma\big(z + |m_2 - \ell| + \frac{1}{2} - \frac{s}{2}\big)$.)

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[1]They, too, will depend on the choice of a character of k.

[2]I leave it to the reader to give a meaning to $\tau\big(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)\big)$ in the case of the real or complex field.

[3]in Lemma 3.6 (1998)

(b) Let M be a continuous homomorphism of $\mathbf{C}^\times$ into $\mathbf{C}^\times$. Suppose

$$M(\alpha) = (N\alpha)^r \frac{\alpha^m \overline{\alpha}^{-n}}{|\alpha|^{\frac{m+n}{2}}}$$

with $mn = 0$, $m + n \geqslant 0$. Let ω be the homomorphism

$$(t_1, t_2) \to |t_1 t_2|^r \left|\frac{t_1}{t_2}\right|^{\frac{m+n}{2}} \operatorname{sgn} t_1$$

of $\mathbf{R}^\times \times \mathbf{R}^\times$ into $\mathbf{C}^\times$ and let $\tau = \tau_M$ be the unique infinite-dimensional irreducible representation deducible from π_ω. If ξ is a character of $\mathbf{R}$ then $\xi'(z) = \xi(z + \overline{z})$ is a character of $\mathbf{C}$. If χ is a homomorphism of $\mathbf{R}^\times$ into $\mathbf{C}^\times$ then $\chi'(\alpha) = \chi(N\alpha) = \chi(\alpha\overline{\alpha})$ is a homomorphism of $\mathbf{C}$ into $\mathbf{C}^\times$. Set

$$\xi(s, \tau, \chi) = \xi\left(s + \frac{1}{2}, M\chi'\right),$$

$$\epsilon(s, \tau, \chi) = (i \operatorname{sgn} u)\epsilon\left(s + \frac{1}{2}, M\chi'\right).$$

Of course the expressions on the left are for the character ξ and those on the right are for the character ξ'.

(c) Suppose M is a continuous homomorphism of $\mathbf{R}^\times \times \mathbf{R}^\times$ into $\mathbf{C}^\times$ of the form $(t_1, t_2) \to |t_1 t_2|^r \operatorname{sgn} t_1$ or $(t_1, t_2) \to |t_1 t_2|^r \operatorname{sgn} t_2$. In the first case let $\chi_1(t) = |t|^r \operatorname{sgn} t$, $\chi_2(t) = |t|^r$; in the second case set $\chi_1(t) = |t|^r$, $\chi_2(t) = |t|^r \operatorname{sgn} t$. The representation π_M introduced in paragraph 2 of my letter to Weil is irreducible. Let $\tau = \tau_M$ be π_M. Set

$$\xi(s, \tau, \chi) = \xi\left(\frac{1}{2} + s, \chi_1\chi\right)\xi\left(\frac{1}{2} + s, \chi_2\chi\right),$$

$$\epsilon(s, \tau, \chi) = \epsilon\left(\frac{1}{2} + s, \chi_1\chi\right)\epsilon\left(\frac{1}{2} + s, \chi_2\chi\right).$$

(ii) $k = \mathbf{C}$. Let M be continuous homomorphism of $\mathbf{C}^\times \times \mathbf{C}^\times$ into $\mathbf{C}^\times$. Let $M\big((t_1, t_2)\big) = |t_1|^{s_1}|t_2|^{s_2}\left(\frac{t_1}{|t_1|^{1/2}}\right)^{m_1}\left(\frac{t_2}{|t_2|^{1/2}}\right)^{m_2}$ and suppose that neither $\frac{s_1 - s_2}{2} - \left(1 + \frac{|m_1 - m_2|}{2}\right)$ nor $\frac{s_2 - s_1}{2} - \left(1 + \frac{|m_1 - m_2|}{2}\right)$ is a non-negative integer. The representation π_M introduced in paragraph 4 of my letter to Weil is irreducible. Let $\tau = \tau_M$ be π_M. Set

$$\xi(s, \tau, \chi) = \xi\left(s + \frac{1}{2}, \chi_1\chi\right)\xi\left(s + \frac{1}{2}, \chi_2\chi\right),$$

$$\epsilon(s, \tau\chi) = \epsilon\left(s + \frac{1}{2}, \chi_1, \chi\right)\epsilon\left(s + \frac{1}{2}, \chi_2\chi\right),$$

if $\chi_1(t) = M\big((t, 1)\big)$ and $\chi_2(t) = M\big((1, t)\big)$.

(iii) k is a non-archimedean field.

(a) Let M be a continuous homomorphism of $k^\times \times k^\times$ into $\mathbf{C}^\times$. Let $M\big((\alpha, \beta)\big) = \chi_1(\alpha)\chi_2(\beta)$. Suppose that neither $\chi_1\chi_2^{-1}$ nor $\chi_1\chi_1^{-1}$ is the character $\alpha \to |\alpha|$. Let

$\tau = \tau_M$ be the simple representation associated to the family $\{T(M, \mu, m)\}$. Set

$$\xi(s, \tau, \chi) = \xi\left(s + \frac{1}{2}, \chi_1\chi\right)\xi\left(s_{\frac{1}{2}}, \chi_2\chi\right),$$
$$\epsilon(s, \tau, \chi) = \epsilon\left(s + \frac{1}{2}, \chi_1\chi\right)\epsilon\left(s + \frac{1}{2}, \chi_2\chi\right).$$

(b) Suppose K is an unramified extension of k and M is a continuous homomorphism of $K^\times$ into $\mathbf{C}^\times$. Let $\tau = \tau_M$ be the representation associated to the family $\{T(M, \mu, m)\}$. If ξ is a character of k then $\xi'(x) = \xi(Sx)$ is a character of K. If χ is a continuous homomorphism of $k^\times$ into $\mathbf{C}^\times$ let χ' be the homomorphism $\alpha \to \chi(N\alpha)$ of $K^\times$ into $\mathbf{C}^\times$. Set[4]

$$\xi(s, \tau, \chi) = \xi\left(s + \frac{1}{2}, M\chi'\right)$$
$$\epsilon(s, \tau, \chi) = \rho(K/k)\epsilon\left(s + \frac{1}{2}, M\chi'\right).$$

The factors on the left are taken with respect to ξ and those on the right with respect to ξ'.

(c) Suppose K is a ramified extension of k and M is a continuous homomorphism of $K^\times$ into $\mathbf{C}^\times$. Let $\tau = \tau_M$ be the representation associated to the family $\{T(M, \mu, m)\}$. If ξ is a character of k then $\xi'(x) = \xi(Sx)$ is a character of K. If χ is a continuous homomorphism of $k^\times$ into $\mathbf{C}^\times$ let χ' be the homomorphism $\alpha \to \chi(N\alpha)$ of $K^\times$ into $\mathbf{C}^\times$. Set

$$\xi(s, \tau, \chi) = \xi\left(s + \frac{1}{2}, M\chi'\right),$$
$$\epsilon(s, \tau, \chi) = \rho(K/k)\epsilon\left(s + \frac{1}{2}, M\chi'\right),$$
$$\rho(K/k) = \big(1 - |\pi|\big)|\pi|^{-f/2}\chi_0(\pi^{f+d}) \int_{O^\times} \xi\left(\frac{\alpha}{\pi^{f+d}}\right)\chi_0^{-1}(\alpha)\, d\alpha,$$

if $\mathfrak{p}^{-d}$ is the largest ideal on which ξ is trivial. Notice that this expression is independent of the choice of π but not of ξ. χ_0 is of course the unique non-trivial character of $k^\times/NK^\times$.

(d) Suppose $M\big((t_1, t_2)\big) = \chi_1(t_1)\chi_2(t_2)$ is a continuous homomorphism of $k^\times \times k^\times$ into $\mathbf{C}^\times$ and suppose $\chi_1\chi_2^{-1}(\alpha) \equiv |\alpha|$. Let τ be the representation associated to the family $\{T(M, \mu, m)\}$,

$$\xi(s, \tau, \chi) = \xi\left(s + \frac{1}{2}, \chi_1\chi\right),$$
$$\epsilon(s, \tau, \chi) = -\chi\chi_1(\pi^{2d+1})|\pi|^{(2d+1)(s-1/2)},$$

[4]If $\mathfrak{p}^{-d}$ is the largest ideal of k on which ξ is trivial and if χ_0 is the unique non-trivial character of $k^\times/NK^\times$ then $\rho(K/k) = \chi_0(\pi^d)$. It is independent of the choice of π.

if the conductor of $\chi\chi_1$ is o and

$$\epsilon(s,\tau,\chi)=\bigl(1-|\pi|\bigr)^2|\pi|^{-d-2n}\chi\chi_1(\pi^{2d+2n})|\pi|^{(2d+2n)s}\left\{\int_{o^\times}\xi\left(\frac{\alpha}{\pi^{d+n}}\right)\chi^{-1}\chi_1^{-1}(\alpha)\,d\alpha\right\}^2$$

if the order of $\chi\chi_1$ is n.

(e) Suppose $M\bigl((t_1,t_2)\bigr)=\chi_1(t_1)\chi_2(t_2)$ is a continuous homomorphism of $k^\times\times k^\times$ into $\mathbf{C}^\times$ and suppose $\chi_1^{-1}\chi_2(\alpha)\equiv|\alpha|$. Let τ be the representation associated to the family $\bigl\{T(M,\mu,m)\bigr\}$. Set

$$\xi(s,\tau,\chi)=\xi\left(s+\frac{1}{2},\chi_2\chi\right),$$
$$\epsilon(s,\tau,\chi)=-\chi\chi_2(\pi^{2d+1})|\pi|^{(2d+1)(s-1/2)},$$

if the conductor of $\chi\chi_1$ is o and

$$\epsilon(s,\tau,\chi)=\bigl(1-|\pi|\bigr)^2|\pi|^{-d-2n}\chi\chi_2(\pi^{2d+2n})|\pi|^{(2d+2n)s}\left\{\int_{o^\times}\xi\left(\frac{\alpha}{\pi^{d+n}}\right)\chi^{-1}\chi_2^{-1}(\alpha)\,d\alpha\right\}^2$$

if the order of $\chi\chi_2$ is n.

The representations of (d) and (e) are anomalous. I do not know if they have any role to play in the theory of automorphic forms. Before coming to the main theorem there is an observation we should make. Suppose k is a local field, K a two-dimensional semi-simple algebra over k, and ξ a character of k. If k is non-archimedean and K is a field we have introduced the symbol $\rho(K/k)=\rho(K/k,\xi)$. If $k=\mathbf{R}$ and $K=\mathbf{C}$ and $\xi(x)=e^{2\pi iux}$ set $\rho(K/k,\xi)=i\operatorname{sgn}u$. If K is not a field set $\rho(K/k,\xi)=1$. Now let k be a global field, K a two-dimensional semi-simple algebra over k, and ξ a character of $\mathbf{A}/k$. If $\mathfrak{p}$ is a prime of k let $K_\mathfrak{p}=K\otimes_k k_\mathfrak{p}$ and let $\xi_\mathfrak{p}$ be the restriction of ξ to $k_\mathfrak{p}$. I claim that

$$\prod_\mathfrak{p}\rho(K_\mathfrak{p}/k_\mathfrak{p},\xi_\mathfrak{p})=1.$$

This is clear if K is not a field. If K is a field the (modified) zeta function of K is

$$\prod_\mathfrak{P}\xi(s,1_\mathfrak{P})=\xi_K(s,1).$$

On the other hand if χ is the unique non-trivial character of $I_k/k^\times NI_K$ it is

$$\prod_\mathfrak{p}\xi(s,1_\mathfrak{p})\xi(s,\chi_\mathfrak{p}).$$

Taking as our character of $\mathbf{A}_K/K$ the character $x\to\xi(Sx)$ we find that

$$\frac{\xi_K(s,1)}{\xi_K(1-s,1)}=\prod_\mathfrak{P}\epsilon(s,1_\mathfrak{P})=\prod_\mathfrak{p}\epsilon(s,1_\mathfrak{p})\epsilon(s,\chi_\mathfrak{p}).$$

Checking things case by case we find that, for all $\mathfrak{p}$,

$$\left\{\prod_{\mathfrak{P}|\mathfrak{p}}\epsilon(s,1_\mathfrak{P})\right\}\rho(K_\mathfrak{p}/k_\mathfrak{p},\xi_\mathfrak{p})=\epsilon(s,1_\mathfrak{p})\epsilon(s,\chi_\mathfrak{p}).$$

The result follows. It is of course well known. I remark it because it shows immediately that the main theorem is applicable to the Hecke L-series over a quadratic extension of the ground field.

Appendix

There are a few facts which it will be useful to have at our disposal when proving the main theorem. For lack of a better place I record them here. Suppose $\tau\big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right)\big) = \eta(\alpha)I$. Let $\zeta = \zeta(\chi, s)$ be defined by

$$\zeta\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right) = \eta(\beta)\chi(\beta\alpha^{-1})|\beta\alpha^{-1}|^s.$$

Then the map

$$\varphi \to \Phi'(\cdot, s, \varphi, \chi)$$

is a homomorphism of $L(\xi, \tau)$ into the unique subspace of $L(\zeta)$ transforming according to the representation τ (cf. Lemma 3.1 and Lemmas 3.1 and 5.1 of the previous letter). Since we know that, for a suitable choice of g and φ, $\Phi'(g, s, \varphi, \chi)$ is a non-zero exponential in s, this homomorphism can never be zero.

On the other hand we know (cf. Lemma 3.5 and the appendix to paragraph 7 of the previous letter) that for some τ and some continuous homomorphisms ω of

$$A_k = \left\{ \begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix} \;\middle|\; \alpha \in k^\times, \beta \in k^\times \right\}$$

into $\mathbf{C}^\times$ there is a "G_k-invariant" map of $L(\xi, \tau)$ into the space of function on G_k satisfying $\varphi\Big(\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)g\Big) = \omega\Big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)\Big)\varphi(g)$. The image of $L(\xi, \tau)$ will, in particular lie in $L(\omega')$ if $\omega'\Big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)\Big) = \left|\frac{\alpha}{\beta}\right|^{1/2}\omega\Big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)\Big)$. Thus if $\omega' = \zeta(s, \chi)$ it must be a constant multiple of the map $\varphi \to \Phi'(\cdot, s, \varphi, \chi)$.

Suppose $L(\zeta, \tau)$ is an invariant subspace of $L(\omega')$ which transforms according to the representation τ. Suppose N_1 and N_2 are two spaces of functions on G_k invariant under the right regular representation (of $\{\sigma, \mathfrak{A}\}$, $\mathfrak{A}$, or G_k according as k is real, complex, or non-archimedean). Suppose N_1 and N_2 are irreducible and transform according to τ. Suppose also that there are isomorphisms T_1 and T_2 of N_1 and N_2, respectively, with $L(\zeta, \tau)$ such that if $\varphi \in N_i$

$$\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}g\right) = \zeta\left(\begin{pmatrix}\alpha & 0\\ 0 & \beta\end{pmatrix}\right)\left\{\varphi(g) + c_1 \log\left|\frac{\alpha}{\beta}\right| T_1\varphi(g)\right\}$$

where c_i, $i = 1$, 2 is a non-zero constant. Set $T = T_2^{-1}T_1$. Then, if $\varphi \in H_1$, $c_2\varphi - c_1T\varphi \in L(\zeta, \tau)$. Thus $N_1 + L(\zeta, \tau) = N_2 + L(\zeta, \tau)$. If $\zeta\Big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)\Big) = \eta(\beta)\chi(\beta\alpha^{-1})|\beta\alpha^{-1}|^2$ then the set of functions

$$\frac{d}{ds}\Phi'(\cdot, s, \varphi, \chi), \qquad \varphi \in L(\zeta, \tau),$$

would be a possible choice for N_1. On the other hand if $\tau = \tau_M$ where M is a homomorphism of $k^\times \times k^\times$ into $\mathbf{C}^\times$ of the form $M\big((\alpha, \beta)\big) = \chi(\alpha\beta)$ and $\omega'\Big(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right)\Big) = \left|\frac{\alpha}{\beta}\right|^{1/2}\chi(\alpha\beta)$ then both

$L(\omega', \tau)$ and N_2 can be taken[5] to lie in the space of functions on G_k satisfying $\varphi\big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) g\big) \equiv \varphi(g)$.

5. The Main Theorem

Now let k be a global field and let $\mathbf{A}$ be the adèle ring of k. The corrected form of Lemma 7.1 of the previous letter is

Lemma 5.1. *There is a constant c_0 such that if g belongs to $G_{\mathbf{A}}$ there is a γ in G_k such that $\prod_{\mathfrak{p}} \max\big(|c|_{\mathfrak{p}}, |d_{\mathfrak{p}}|\big) \leqslant c_2 |\det g|^{1/2}$ if $\gamma_g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.*

There seems little point in including a proof of this.

Let us take the space $\mathcal{L}$ as in the previous letter except for making the modification in condition (iii) required by the change in Lemma 7.1.

Suppose that V is a complex vector space and for each real prime $\mathfrak{p}$ we have a representation of $\{\sigma_{\mathfrak{p}}, \mathfrak{A}_{\mathfrak{p}}\}$ in V, for each complex prime a representation of $\mathfrak{A}_{\mathfrak{p}}$ on V. If any two operators associated to distinct primes commute we shall, for the purposes of this paragraph, say that we have a "representation" of $G_{\mathbf{A}}$ on V.

Suppose in particular that for each prime $\mathfrak{p}$ we are given a simple representation $\tau_{\mathfrak{p}}$ of $G_{k_{\mathfrak{p}}}$ (in the sense of the previous paragraph) on a vector space $V_{\mathfrak{p}}$. Suppose moreover that for almost all non-archimedean primes $V_{\mathfrak{p}}$ contains a non-zero vector invariant under $G_{o_{\mathfrak{p}}}$. Since this vector is determined up to a scalar factor we have in all but finitely many of the $V_{\mathfrak{p}}$ a distinguished one-dimensional subspace and we can form the tensor product $\bigotimes_{\mathfrak{p}} V_{\mathfrak{p}}$. The natural "representation" of $G_{\mathbf{A}}$ on V will be denoted $\bigotimes_{\mathfrak{p}} \tau_{\mathfrak{p}}$. A "representation" of $G_{\mathbf{A}}$ equivalent to such a representation will be called a simple representation of $G_{\mathbf{A}}$.

Certainly we have a "representation" of $G_{\mathbf{A}}$ on $\mathcal{L}$. An invariant subspace of $\mathcal{L}$ which transforms according to a simple representation of $G_{\mathbf{A}}$ will be called a characteristic space of automorphic forms. Suppose L is a characteristic space of automorphic forms and let ξ be a character of $k\backslash\mathbf{A}$. If $\varphi \in L$ set

$$\varphi_0(g) = \frac{1}{\text{measure}(k\backslash\mathbf{A})} \int_{k\backslash\mathbf{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx,$$

$$\varphi_1(g) = \frac{1}{\text{measure}(k\backslash\mathbf{A})} \int_{k\backslash\mathbf{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\xi(x)}\, dx.$$

As before

$$\varphi(g) = \varphi_0(g) + \sum_{\alpha \in k^\times} \varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right).$$

Suppose the "representation" of $G_{\mathbf{A}}$ on L is equivalent to $\bigotimes_{\mathfrak{p}} \tau_{\mathfrak{p}}$. If one of the $\tau_{\mathfrak{p}}$ is finite-dimensional it follows rather easily from Lemma 3.5 of this letter and the corollaries to Lemma 3.2 and 5.4 of the previous letter that, for all φ in L, $\varphi_1(g) \equiv 0$. Then $\varphi(hg) \equiv \varphi(g)$ if $h \in G_k$ or $h = \big(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\big)$ with $x \in \mathbf{A}$. The argument used in the proof of Lemma 2.1 shows rather easily that, if G^0 is the group of matrices of determinant 1 in G, φ is a function on $G_{\mathbf{A}} \backslash G^0_{\mathbf{A}}$. Consequently L is one-dimensional. We exclude this case from the following discussion.

[5]Notice that in part (ii) of Lemma A in the appendix to paragraph 7 of the previous letter one should have $s = 0$ and $m = 0$.

With this case excluded the function φ_1 can never vanish identically. For a suitable choice of φ it is of the form

$$\varphi_1(g) = \varphi_1\left(\prod_{\mathfrak{p}} g_{\mathfrak{p}}\right) = \prod_{\mathfrak{p}} \varphi_{\mathfrak{p}}(g_{\mathfrak{p}})$$

with $\varphi_{\mathfrak{p}}$ in $L(\xi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$. Moreover we can suppose that for almost all non-archimedean primes $\varphi_{\mathfrak{p}}(1) = 1$.

Lemma 5.2. *Suppose φ_0 is different from zero for some φ in L. Then there is a continuous homomorphism M of $k^\times\backslash I \times k^\times\backslash I$ such that $\tau_{\mathfrak{p}} = \tau_{M_{\mathfrak{p}}}$ for any prime for which $\tau_{M_{\mathfrak{p}}}$ is defined. If $\tau_{M_{\mathfrak{p}}}$ is not defined and $\mathfrak{p}$ is archimedean then $\tau_{\mathfrak{p}}$ is the unique infinite-dimensional simple representation deducible from $\pi_{M_{\mathfrak{p}}}$. If $\tau_{M_{\mathfrak{p}}}$ is not defined and $\mathfrak{p}$ is non-archimedean $\tau_{\mathfrak{p}}$ is the simple representation associated to the family $\{T(M_{\mathfrak{p}}, \mu, m)\}$. Let $M\big((\alpha, \alpha)\big) = \eta(\alpha)$.*

(*i*) *Suppose $M\big((\alpha, \beta)\big) = \eta(\beta)\chi(\beta\alpha^{-1})|\beta\alpha^{-1}|^{s_0}$. If $M\big((\alpha, 1)\big) \not\equiv M\big((1, \alpha)\big)$ there are constants c_1 and c_2 such that when φ_1 is of the above form*

$$\varphi_0(g) = c_1 \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\right) + c_2 \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, -\frac{1}{2} - s_0, \varphi_{\mathfrak{p}}, (\eta^{-1}\chi^{-1})_{\mathfrak{p}}\right).$$

If $M\big((\alpha, 1)\big) \equiv M\big((1, \alpha)\big)$ there are constants c_1 and c_2 such that when φ_1 is of the above form

$$\varphi_0(g) = c_1 \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, s_0 + \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\right) + c_2 \frac{d}{ds} \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\right).$$

(*ii*) *Suppose $M\big((\beta, \alpha)\big) = \eta(\beta)\chi(\beta\alpha^{-1})|\beta\alpha^{-1}|^{s_0+1/2}$. If $M\big((\alpha, 1)\big) \not\equiv M\big((1, \alpha)\big)$ there are constants*[6] *c_1 and c_2 such that when φ_1 is of the above form*

$$\varphi_0(g) = c_2 \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\right) + c_1 \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, -\frac{1}{2} - s_0, \varphi_{\mathfrak{p}}, (\eta^{-1}\chi^{-1})_{\mathfrak{p}}\right).$$

If $M\big((\alpha, 1)\big) \equiv M\big((1, \alpha)\big)$ there are constants c_1 and c_2 such that when φ_1 is of the above form

$$\varphi_0(g) = c_2 \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\right) + c_1 \frac{d}{ds} \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\right).$$

The proof of this lemma will be based on the appendix to paragraph 4 and Lemma E of the appendix to paragraph 7 of the previous letter. However the proof of that lemma was written up rather hastily so I do not have complete confidence in it. I will examine it more carefully later. If it turns out to be unsatisfactory I shall let you know. In order to get on to the main point I will take Lemma 5.2 for granted.

In proving the main theorem I shall not enter into questions of convergence. Anything which is not discussed in the previous letter is taken care of by Lemma 5.2 Thus if χ is a continuous homomorphism of $k^\times\backslash I$ into $\mathbf{C}^\times$ and φ_1 is of the above form

$$\int_I \varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(\alpha)|\alpha|^s \, d\alpha$$

[6]The constants of parts (i) and (ii) are the same.

converges absolutely for $\mathrm{Re}(s)$ sufficiently large. It is equal to

$$\left\{\prod_{\mathfrak{p}} \xi(s, \tau_{\mathfrak{p}}, \chi_{\mathfrak{p}})\right\}\left\{\prod_{\mathfrak{p}} \Phi'(g_{\mathfrak{p}}, s, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}})\right\}.$$

On the other hand it is equal to

$$\int_{k^\times\backslash I}\left\{\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right) - \varphi_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right)\right\}\chi(\alpha)|\alpha|^s\,d\alpha.$$

This is equal to the sum of[7]

$$\int_{\{\,\alpha\,\mid\,|\alpha|\geqslant 1\,\}}\left\{\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right) - \varphi_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right)\right\}\chi(\alpha)|\alpha|^s\,d\alpha$$

and

$$\int_{\{\,\alpha\,\mid\,|\alpha|\leqslant 1\,\}}\left\{\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right) - \varphi_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right)\right\}\chi(\alpha)|\alpha|^s\,d\alpha.$$

The first of these integrals is an entire function of s.

On the other hand if $\varphi\left(\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha\end{smallmatrix}\right)g\right) \equiv \eta(\alpha)\varphi(g)$ for $\alpha \in I$

$$\int_{\{\,\alpha\,\mid\,|\alpha|\geqslant 1\,\}}\left\{\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g\right) - \varphi_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g\right)\right\}(\eta\chi)^{-1}(\alpha)|\alpha|^s\,d\alpha$$

$$= \int_{\{\,\alpha\,\mid\,|\alpha|\leqslant 1\,\}}\Bigg\{\varphi\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g\right)$$

$$- \varphi_0\left(\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g\right)\Bigg\}\eta\chi(\alpha)|\alpha|^s\,d\alpha$$

$$= \int_{\{\,\alpha\,\mid\,|\alpha|\leqslant 1\,\}}\left\{\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right) - \varphi_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right)\right\}\chi(\alpha)|\alpha|^s\,d\alpha$$

$$+ \int_{\{\alpha|\alpha|\leqslant 1\}}\left\{\varphi_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right) - \eta(\alpha)\varphi_0\left(\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}g\right)\right\}\chi(\alpha)|\alpha|^s\,d\alpha.$$

Let us suppose that φ_0 is not zero for all φ and consider the last integral. Let M be the homomorphism of Lemma 5.2 and let $M\big((\alpha,\beta)\big) = \chi_1(\alpha)\chi_2(\beta)$. If neither $\chi_1\chi$ nor $\chi_2\chi$ is trivial on the idèles of norm one this integral is zero. Suppose that $\chi_1\chi$ is trivial on the idèles

[7] At first we shall discuss the case of a number field. Afterwards the necessary modifications for a function field will be indicated. The argument of the previous letter was not correct for a function field.

of norm one but $\chi_2\chi$ is not. Let $\chi_1\chi(\alpha) = |\alpha|^{-s_0}$. Then the integral is equal to

$$c_1 \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\Big) \int_0^1 t^{s-s_0+\frac{1}{2}} \frac{dt}{t}$$

$$- c_2 \prod_{\mathfrak{p}} \Phi'\Big(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, (\eta^{-1}\chi^{-1})_{\mathfrak{p}}\Big) \int_0^1 t^{s-s_0-\frac{1}{2}} \frac{dt}{t}$$

$$= \frac{c_1}{\frac{1}{2} + s - s_0} \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\Big)$$

$$+ \frac{c_2}{\frac{1}{2} - s + s_0} \prod_{\mathfrak{p}} \Phi'\Big(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s_0 - \frac{1}{2}\varphi_{\mathfrak{p}}, (\eta^{-1}\chi^{-1})_{\mathfrak{p}}\Big).$$

On the other hand if $\chi_2\chi$ is trivial on the idèles of norm one and $\chi_1\chi$ is not, let $\chi_2\chi(\alpha) = |\alpha|^{-s_0}$. Then the integral is equal to

$$\frac{c_2}{\frac{1}{2} + s - s_0} \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\Big)$$

$$+ \frac{c_1}{\frac{1}{2} - s + s_0} \prod_{\mathfrak{p}} \Phi'\Big(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, (\eta^{-1}\chi^{-1})_{\mathfrak{p}}\Big).$$

Since

$$\prod_{\mathfrak{p}} \Phi'\Big(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\Big) = \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\Big)$$

it is clear, in this case at least, that these expressions do not change if g is replaced by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) g$, χ by $\eta^{-1}\chi^{-1}$ and s by $-s$.

Now suppose that $\chi_1 \neq \chi_2$ but both $\chi_1\chi$ and $\chi_2\chi$ are trivial on the idèles of norm 1. Let $\chi_1\chi(\alpha) = |\alpha|^{-s_1}$, $\chi_2\chi(\alpha) = |\alpha|^{-s_2}$. Then the integral is equal to

$$\frac{c_1}{\frac{1}{2} + s - s_1} \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, s_1 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\Big) + \frac{c_2}{\frac{1}{2} + s - s_2} \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, s_2 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\Big)$$

$$+ \frac{c_1}{\frac{1}{2} - s + s_2} \prod_{\mathfrak{p}} \Phi'\Big(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s_2 - \frac{1}{2}, \varphi_{\mathfrak{p}}, (\eta^{-1}\chi^{-1})_{\mathfrak{p}}\Big)$$

$$+ \frac{c_2}{\frac{1}{2} - s + s_1} \prod_{\mathfrak{p}} \Phi'\Big(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s_1 - \frac{1}{2}, \varphi_{\mathfrak{p}}, (\eta^{-1}\chi^{-1})_{\mathfrak{p}}\Big).$$

When χ is replaced by $\eta^{-1}\chi^{-1}$, s_1 is replaced by $-s_2$ and s_2 is replaced by $-s_1$. Thus this expression is not changed if s is replaced by $-s$, χ by $\eta^{-1}\chi^{-1}$, and g by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) g$.

Finally suppose that $\chi_1 = \chi_2$ and $\chi_1\chi(\alpha) = |\alpha|^{-s_0}$. Then the integral is equal to

$$c_2 \frac{d}{ds} \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\Big) \int_0^1 t^{\frac{1}{2}+s-s_0} \frac{dt}{t}$$
$$+ \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\Big) \int_0^1 (c_1 - c_2 \log t) t^{\frac{1}{2}+s-s_0} \frac{dt}{t}$$
$$- c_2 \frac{d}{ds} \prod_{\mathfrak{p}} \Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, (\eta\chi)_{\mathfrak{p}}^{-1}\right) \int_0^1 t^{s-s_0-\frac{1}{2}} \frac{dt}{t}$$
$$- \prod_{\mathfrak{p}} \Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s_0 - \frac{1}{2} \varphi_{\mathfrak{p}}, (\eta\chi)_{\mathfrak{p}}^{-1}\right) \int_0^1 (c_1 + c_2 \log t) t^{s-s_0-\frac{1}{2}} \frac{dt}{t}.$$

This is of course equal to

$$\frac{c_2}{\frac{1}{2}+s-s_0} \frac{d}{ds} \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\Big)$$
$$+ \frac{c_2}{\frac{1}{2}-s+s_0} \frac{d}{ds} \prod_{\mathfrak{p}} \Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, (\eta\chi)_{\mathfrak{p}}^{-1}\right)$$
$$+ \left\{ \frac{c_1}{\frac{1}{2}+s-s_0} + \frac{c_2}{\left(\frac{1}{2}+s-s_0\right)^2} \right\} \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\Big)$$
$$+ \left\{ \frac{c_1}{\frac{1}{2}-s+s_0} + \frac{c_2}{\left(\frac{1}{2}-s+s_0\right)^2} \right\} \prod_{\mathfrak{p}} \Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, (\eta\chi)_{\mathfrak{p}}^{-1}\right).$$

It is clear that this does not change if s is replaced by $-s$, χ by $\eta^{-1}\chi^{-1}$, and g by $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) g$.

Putting everything together we see that

$$\left\{ \prod_{\mathfrak{p}} \xi(s, \tau_{\mathfrak{p}}, \chi_{\mathfrak{p}}) \right\} \left\{ \prod_{\mathfrak{p}} \Phi'(g_{\mathfrak{p}}, -s, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}) \right\}$$

is meromorphic in the whole complex plane and equals

$$\left\{ \prod_{\mathfrak{p}} \xi\Big(-s, \tau_{\mathfrak{p}}, (\eta\chi)_{\mathfrak{p}}^{-1}\Big) \right\} \left\{ \prod_{\mathfrak{p}} \Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s, \varphi_{\mathfrak{p}}, (\eta\chi)_{\mathfrak{p}}^{-1}\right) \right\}.$$

The second factor is equal to

$$\left\{ \prod_{\mathfrak{p}} \epsilon(s, \tau_{\mathfrak{p}}, \chi_{\mathfrak{p}}) \right\} \left\{ \prod_{\mathfrak{p}} \Phi'(g_{\mathfrak{p}}, s, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}) \right\}.$$

Thus if

$$\xi(s, L, \chi) = \prod_{\mathfrak{p}} \xi(s, \tau_{\mathfrak{p}}, \chi_{\mathfrak{p}}),$$

$$\epsilon(s, L, \chi) = \prod_{\mathfrak{p}} \epsilon(s, \tau_{\mathfrak{p}}, \chi_{\mathfrak{p}}),$$

$\xi(s, L, \chi)$ is meromorphic in the entire complex plane and satisfies the functional equation

$$\xi\big(-s, L, (\eta\chi)^{-1}\big)\epsilon(s, L, \chi) = \xi(s, L, \chi).$$

To investigate its poles we use the fact that for a suitable choice of φ and g

$$\prod_{\mathfrak{p}} \Phi'(g_{\mathfrak{p}}, s, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}})$$

is an exponential in s. Thus if neither $\chi_1\chi$ nor $\chi_2\chi$ is trivial on the idèles of norm 1 it has no poles. If $\varphi_0 = 0$ for all φ in L then it has no poles for any choice of χ. To find the principal parts at the poles in the other cases we observe that

$$\frac{1}{\frac{1}{2} + s - s_0}\left\{\prod_{\mathfrak{p}} \Phi'(g_{\mathfrak{p}}, s, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}) - \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\right)\right\}$$

and

$$\frac{1}{\left(\frac{1}{2} + s - s_0\right)^2}\left\{\prod_{\mathfrak{p}} \Phi'(g_{\mathfrak{p}}, s, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}) - \prod_{\mathfrak{p}}\left(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\right) - \left(\frac{1}{2} + s - s_0\right)\frac{d}{ds}\prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\right)\right\}$$

are entire functions of s.

Thus if $\chi_1\chi$ is trivial on the idèles of norm 1 there are simple poles at $s_0 - \frac{1}{2}$ and $s_0 + \frac{1}{2}$ with residues $-c_1$ and $c_2\epsilon\big(s_0 + \frac{1}{2}, L, \chi\big)$ respectively. If $\chi_2\chi$ is trivial on the idèles of norm 1 but $\chi_1\chi$ is not there are simple poles at $s_0 - \frac{1}{2}$ and $s_0 + \frac{1}{2}$ with residues $-c_2$ and $c_1\epsilon\big(s_0 + \frac{1}{2}, L, \chi\big)$ respectively. If $\chi_1 \neq \chi_2$ but both $\chi_1\chi$ and $\chi_2\chi$ are trivial on the idèles of norm 1 there are simple poles at $s_1 - \frac{1}{2}$, $s_2 - \frac{1}{2}$, $s_1 + \frac{1}{2}$, and $s_2 + \frac{1}{2}$ with residues $-c_1$, $-c_2$, $c_2\epsilon\big(s_1 + \frac{1}{2}, L, \chi\big)$, $c_1\epsilon\big(s_2 + \frac{1}{2}, L, \chi\big)$ respectively. If $\chi_1 = \chi_2$ there are poles of order two at $s_0 - \frac{1}{2}$ and $s_0 + \frac{1}{2}$. The principal part at $s_0 - \frac{1}{2}$ is

$$-\frac{c_2}{(s - s_0 + 1/2)^2} - \frac{c_1}{s - s_0 + 1/2}.$$

The principal part at $s_0 + \frac{1}{2}$ is determined by the functional equation.

For a function field we write our integral as the sum of

$$\int_{\{\alpha\,|\,|\alpha|>1\}}\left\{\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right)-\varphi_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right)\right\}\chi(\alpha)|\alpha|^s\,d\alpha$$

$$+\int_{\{\alpha\,|\,|\alpha|>1\}}\left\{\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g\right)-\varphi_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g\right)\right\}(\eta\chi)^{-1}(\alpha)|\alpha|^{-s}\,d\alpha$$

and

$$\int_{|\alpha|\leqslant 1}\varphi\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right)\chi(\alpha)|\alpha|^s\,d\alpha$$

and

$$-\int_{\{\alpha\,|\,|\alpha|=1\}}\varphi_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right)\chi(\alpha)|\alpha|^s d\alpha+\int_{\{\alpha\,|\,|\alpha|<1\}}\varphi_0\left(\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}g\right)\eta(\alpha)\chi(\alpha)|\alpha|^s d\alpha.$$

The first two of these expressions are clearly entire functions of s which do not change when g is replaced by $\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)g$, s by $-s$, and χ by $\eta^{-1}\chi^{-1}$.

Again let us consider the last expression when φ_0 is not zero for all φ and at least one of $\chi_1\chi$ or $\chi_2\chi$ is trivial on the idèles of norm 1. If $\chi_1\chi$ is but $\chi_2\chi$ is not, let $\chi_1\chi(\alpha)=|\alpha|^{-s_0}$. The expression equals

$$-\frac{c_1}{1-q^{-\frac{1}{2}-s+s_0}}\prod_{\mathfrak{p}}\Phi'\left(g_{\mathfrak{p}},s_0-\frac{1}{2},\varphi_{\mathfrak{p}},\chi_{\mathfrak{p}}\right)$$
$$-\frac{c_2}{1-q^{-\frac{1}{2}+s-s_0}}\prod_{\mathfrak{p}}\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g_{\mathfrak{p}},-s_0-\frac{1}{2},\varphi_{\mathfrak{p}},(\eta^{-1}\chi^{-1})_{\mathfrak{p}}\right).$$

If $\chi_2\chi$ is trivial but $\chi_1\chi$ is not, and $\chi_2\chi(\alpha)=|\alpha|^{-s_0}$ it equals

$$\frac{c_2}{1-q^{-\frac{1}{2}-s+s_0}}\prod_{\mathfrak{p}}\Phi'\left(g_{\mathfrak{p}},s_0-\frac{1}{2},\varphi_{\mathfrak{p}},\chi_{\mathfrak{p}}\right)$$
$$-\frac{c_1}{1-q^{-\frac{1}{2}+s-s_0}}\prod_{\mathfrak{p}}\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g_{\mathfrak{p}},-s_0-\frac{1}{2},\varphi_{\mathfrak{p}},(\eta^{-1}\chi^{-1})_{\mathfrak{p}}\right).$$

If $\chi_1\neq\chi_2$ but both $\chi_1\chi$ and $\chi_2\chi$ are trivial on the idèles of norm 1 let $\chi_1\chi(\alpha)=|\alpha|^{-s_1}$ and $\chi_2\chi(\alpha)=|\alpha|^{-s_2}$. The expression equals

$$-\frac{c_1}{1-q^{-\frac{1}{2}-s+s_1}}\prod_{\mathfrak{p}}\Phi'\left(g_{\mathfrak{p}},s_1-\frac{1}{2},\varphi_{\mathfrak{p}},\chi_{\mathfrak{p}}\right)-\frac{c_2}{1-q^{-\frac{1}{2}-s+s_2}}\prod_{\mathfrak{p}}\Phi'\left(g_{\mathfrak{p}},s_2-\frac{1}{2},\varphi_{\mathfrak{p}},\chi_{\mathfrak{p}}\right)$$
$$-\frac{c_1}{1-q^{-\frac{1}{2}+s-s_2}}\prod_{\mathfrak{p}}\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g_{\mathfrak{p}},-s_2-\frac{1}{2},\varphi_{\mathfrak{p}}(\eta^{-1}\chi^{-1})_{\mathfrak{p}}\right)$$
$$-\frac{c_2}{1-q^{-\frac{1}{2}+s-s_1}}\prod_{\mathfrak{p}}\Phi'\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g_{\mathfrak{p}},-s_1-\frac{1}{2},\varphi_{\mathfrak{p}},(\eta^{-1}\chi^{-1})_{\mathfrak{p}}\right).$$

Finally suppose that $\chi_1 = \chi_2$ and $\chi_1\chi(\alpha) = |\alpha|^{-s_0}$. The expression yields

$$- \frac{c_2}{1 - q^{-\frac{1}{2}-s+s_0}} \frac{d}{ds} \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\right)$$

$$- \frac{c_2}{1 - q^{-\frac{1}{2}+s-s_0}} \frac{d}{ds} \prod_{\mathfrak{p}} \Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s_0 \frac{1}{2}, \varphi_{\mathfrak{p}}, (\eta\chi)_{\mathfrak{p}}^{-1}\right)$$

$$- \left\{ \frac{c_1}{1 - q^{-\frac{1}{2}-s+s_0}} + c_2 \frac{\frac{d}{ds}\left(1 - q^{-\frac{1}{2}-s+s_0}\right)}{\left(1 - q^{-\frac{1}{2}-s+s_0}\right)^2} \right\} \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, s_0 - \frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}}\right)$$

$$- \left\{ \frac{c_1}{1 - q^{-\frac{1}{2}+s-s_0}} - c_2 \frac{\frac{d}{ds}\left(1 - q^{-\frac{1}{2}+s-s_0}\right)}{(1 - q^{-1/2+s-s_0})^2} \right\}.$$

The functional equation follows as before. The principal parts at the poles can also be determined. Since I am principally interested in the case of a number field I shall not bother to discuss them explicitly. Moreover for the converse theorem I shall limit myself to the case of a number field. The statement and the proof for a function field will differ only in minor points.

For the converse theorem we suppose that, for each prime $\mathfrak{p}$, we are given an infinite-dimensional simple representation $\tau_{\mathfrak{p}}$ of $G_{k_{\mathfrak{p}}}$ on $V_{\mathfrak{p}}$. We suppose that for almost all non-archimedean primes there is a non-zero vector in $V_{\mathfrak{p}}$ whose isotropy group contains $G_{o_{\mathfrak{p}}}$. For such a prime there will be a continuous homomorphism $M_{\mathfrak{p}}\big((\alpha, \beta)\big) = \chi'_{\mathfrak{p}}(\alpha)\chi'_{\mathfrak{p}}(\beta)$ of $k_{\mathfrak{p}}^\times \times k_{\mathfrak{p}}^\times$ into $\mathbf{C}^\times$ such that $\tau_{\mathfrak{p}} = \tau_{M_{\mathfrak{p}}}$. We suppose that there is a constant $N > 0$ such that for all such $\mathfrak{p}$

$$\left|\chi'_{\mathfrak{p}}(\pi)\right| \leqslant |\pi|^{-N} \qquad \left|\chi''_{\mathfrak{p}}(\pi)\right| = |\pi|^{-N}$$

if π is a generator of the maximal ideal of $o_{\mathfrak{p}}$. Let $\tau_{\mathfrak{p}}\big(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha \end{smallmatrix}\right)\big) = \eta_{\mathfrak{p}}(\alpha)I$ if $\alpha \in k_{\mathfrak{p}}^\times$. We suppose that

$$\eta(\alpha) = \eta\left(\prod_{\mathfrak{p}} \alpha_{\mathfrak{p}}\right) = \prod_{\mathfrak{p}} \eta_{\mathfrak{p}}(\alpha_{\mathfrak{p}})$$

which is a continuous homomorphism of I into $\mathbf{C}^\times$ is trivial on $k^\times$.

If χ is a continuous homomorphism of $k^\times \backslash I$ into $\mathbf{C}^\times$ the product

$$\prod_{\mathfrak{p}} \xi(s, \tau_{\mathfrak{p}}, \chi_{\mathfrak{p}}) = \xi_0(s, \chi)$$

converges for $\mathrm{Re}(s)$ sufficiently large. We suppose that for each χ it is meromorphic in the whole plane, that it has only a finite number of poles, that it is bounded in the regions obtained by removing circles about its poles from any vertical strip of finite width, and that the functional equations

$$\xi_0\big(-s, (\eta\chi)^{-1}\big)\xi_0(s, \chi) = \xi_0(s, \chi),$$

with

$$\epsilon_0(s, \chi) = \prod_{\mathfrak{p}} \epsilon(s, \tau_{\mathfrak{p}}, \chi_{\mathfrak{p}}),$$

are satisfied.

We suppose that there are two continuous homomorphisms χ_1 and χ_2 of $k^\times\backslash I$ into $\mathbf{C}^\times$ with $\chi_1\chi_2 = \eta$ and two complex numbers c_1 and c_2 such that $\xi_0(s,\chi)$ has no poles unless either $\chi_1\chi$ or $\chi_2\chi$ is trivial on the idèles of norm 1.

(i) If $\chi_1\chi$ is trivial on the idèles of norm 1 but $\chi_2\chi$ is not and if $\chi_1\chi(\alpha) = |\alpha|^{-s_0}$ there are simple poles at $s_0 - \frac{1}{2}$ and $s_0 + \frac{1}{2}$ with residues $-c_1$ and $c_2\epsilon_0\bigl(s_0 + \frac{1}{2}, \chi\bigr)$ respectively.

(ii) If $\chi_2\chi$ is trivial on the idèles of norm 1 but $\chi_1\chi$ is not and $\chi_2\chi(\alpha) = |\alpha|^{-s_0}$ there are simple poles at $s_0 - \frac{1}{2}$ and $s_0 + \frac{1}{2}$ with residues $-c_2$ and $c_1\epsilon_0\bigl(s_0 + \frac{1}{2}, \chi\bigr)$ respectively.

(iii) If $\chi_1\chi(\alpha) = |\alpha|^{-s_1}$ and $\chi_2\chi(\alpha) = |\alpha|^{-s_2}$ with $s_1 \neq s_2$ there are simple poles at $s_1 - \frac{1}{2}$, $s_2 - \frac{1}{2}$, $s_1 + \frac{1}{2}$, $s_2 + \frac{1}{2}$ with residues $-c_1$, $-c_2$, $c_2\epsilon_0\bigl(s_1 + \frac{1}{2}, \chi\bigr)$, $c_1\epsilon_0\bigl(s_2 + \frac{1}{2}, \chi\bigr)$ respectively.

(iv) If $\chi_1\chi(\alpha) = \chi_2\chi(\alpha) = |\alpha|^{-s_0}$ there are poles of order two at $s_0 - \frac{1}{2}$ and $s_0 + \frac{1}{2}$. The principal part at $s_0 - \frac{1}{2}$ is

$$-\frac{c_2}{\bigl(s - s_0 + \frac{1}{2}\bigr)^2} - \frac{c_1}{s - s_0 + 1/2}.$$

The principal part at $s_0 + \frac{1}{2}$ is determined by the functional equation.

We allow the possibility that c_1 or c_2 or both are zero. In particular if

$$\psi_1(g) = \psi_1\left(\prod_{\mathfrak{p}} g_{\mathfrak{p}}\right) = \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, -\frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}}^{-1}\right)$$

is not, for any choice of the collection $\{\varphi_{\mathfrak{p}}\}$ with $\varphi_{\mathfrak{p}}$ in $L(\xi_{\mathfrak{p}}, \tau_{\mathfrak{p}})$ such that $G_{o_{\mathfrak{p}}}$ lies in the isotropy group of $\varphi_{\mathfrak{p}}$ for almost all non-archimedean primes and $\varphi_{\mathfrak{p}}(1) = 1$ for almost all non-archimedean primes, a function satisfying $\psi_1\bigl(\bigl(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\bigr)\bigr) = \psi_1(g)$ for all x in $\mathbf{A}$ we demand that $c_1 = 0$. Also if $\chi_1 \neq \chi_2$ we demand that $c_2 = 0$ if for the same choices of the collection $\{\varphi_{\mathfrak{p}}\}$ the functions

$$\psi_2(g) = \prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, -\frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{2,\mathfrak{p}}^{-1}\right)$$

do not all satisfy $\psi_2\bigl(\bigl(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\bigr)g\bigr) \equiv \psi_2(x)$ for all $x \in \mathbf{A}$. If $\chi_1 = \chi_2$ we demand that $c_2 = 0$ if

$$\psi_2(g) = \frac{d}{ds}\prod_{\mathfrak{p}} \Phi'\left(g_{\mathfrak{p}}, -\frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}}^{-1}\right)$$

does not satisfy this condition. Notice that given χ_1 and χ_2 and the collection $\{\tau_{\mathfrak{p}}\}$ we can, according to the appendix to the previous paragraph, decide whether or not ψ_1 and ψ_2 satisfy these conditions. Notice also that our theorem will be most interesting when both c_1 and c_2 are zero.

In any case the converse theorem states that when all these conditions are satisfied there is a characteristic space of automorphic forms which transforms according to the "representation" $\bigotimes_{\mathfrak{p}} \tau_{\mathfrak{p}}$. To prove it we show that if the collection $\{\psi_{\mathfrak{p}}\}$ is chosen as above and

$$\varphi_1(g) = \prod_{\mathfrak{p}} \varphi_{\mathfrak{p}}(g_{\mathfrak{p}})$$

while

$$\varphi_0(g) = c_1 \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, -\frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}}^{-1}\Big) + c_1 \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, -\frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{2,\mathfrak{p}}^{-1}\Big)$$

if $\chi_1 \neq \chi_2$ and

$$\varphi_0(g) = c_1 \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, -\frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}}^{-1}\Big) + c_2 \frac{d}{ds} \prod_{\mathfrak{p}} \Phi'\Big(g_{\mathfrak{p}}, -\frac{1}{2}, \varphi_{\mathfrak{p}}, \chi_{1,\mathfrak{p}}^{-1}\Big)$$

if $\chi_1 = \chi_2$ then

$$\varphi(g) = \varphi_0(g) + \sum_{\alpha \in k^\times} \varphi_1\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right)$$

is a function on $G_k \backslash G_{\mathbf{A}}$.

By its very construction it is invariant under left translations by upper triangular matrices in G_k so the only problem is to show that $\varphi\Big(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) g\Big) \equiv \varphi(g)$. Let us show that for each g the functions $\varphi\Big(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) g\Big)$ and $\varphi\big(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) g\big)$ on I are equal. Let $\varphi_1(\alpha)$ be the function obtained from the second of these functions by subtracting $\varphi_0\big(\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) g\big)$ if $|\alpha| \geqslant 1$ and $\varphi_0\Big(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right) g\Big)$ if $|\alpha| \leqslant 1$. Let $\psi_2(\alpha)$ be the function obtained from the other function by the same process. It is enough to show that $\psi_1(\alpha) \equiv \psi_2(\alpha)$. Now if χ is any character of $k^\times \backslash I$

$$\int_{k^\times \backslash I} \psi_1(\alpha)\chi(\alpha)|\alpha|^s \, d\alpha = \mu_1(s, \chi)$$

is defined for $\mathrm{Re}(s)$ sufficiently large and, as we shall see,

$$\int_{k^\times \backslash I} \psi_2(\alpha)\chi(\alpha)|\alpha|^s \, d\alpha = \mu_2(s, \chi)$$

is defined for $\mathrm{Re}(s)$ sufficiently small. It is enough to show that, for each χ, $\mu_1(s, \chi)$ and $\mu_2(s, \chi)$ are entire functions of s which equal each other. We must also show that they are bounded in vertical strips.

The first integral is equal to the sum of

$$\xi_0(s, \chi) \prod_{\mathfrak{p}} \Phi'(g_{\mathfrak{p}}, s, \varphi_{\mathfrak{p}}, \chi_{\mathfrak{p}})$$

and

$$\int_{|\alpha| \leqslant 1} \varphi_0\left(\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} g\right) \chi(\alpha)|\alpha|^s d\alpha - \int_{|\alpha| \leqslant 1} \varphi_0\left(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g\right) \eta(\alpha)\chi(\alpha)|\alpha|^s \, d\alpha.$$

The second integral is the sum of

$$\int_{k^\times \backslash I} \varphi_1\left(\begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g\right) \eta\chi(\alpha)|\alpha|^s \, d\alpha,$$

which equals

$$\xi_0(-s, \eta^{-1}\chi^{-1}) \prod_{\mathfrak{p}} \Phi'\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_{\mathfrak{p}}, -s, \varphi_{\mathfrak{p}}, (\eta\chi)_{\mathfrak{p}}^{-1}\right),$$

and of

$$\int_{|\alpha|\geqslant 1} \varphi_0\left(\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g\right)\eta\chi(\alpha)|\alpha|^s - \varphi_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g\right)\chi(\alpha)|\alpha|^s\,d\alpha,$$

which equals the sum of

$$-\int_{|\alpha|\leqslant 1} \varphi_0\left(\begin{pmatrix}\alpha^{-1} & 0\\ 0 & 1\end{pmatrix}g\right)\chi^{-1}(\alpha)|\alpha|^{-s}\,d\alpha$$

and

$$\int_{|\alpha|\leqslant 1} \varphi_0\left(\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}g\right)\eta^{-1}\chi^{-1}(\alpha)|\alpha|^{-s}\,d\alpha.$$

The functional equation assumed for $\xi_0(s,\chi)$ together with the local functional equations show that the first term in the expression for $\mu_2(s,\chi)$ is the same as the first term in the expression for $\mu_2(s,\chi)$. The second term in the expression for $\mu_1(s,\chi)$ is an integral we have already investigated. We know that its poles cancel the assumed poles of the first term and that it is given by an analytical expression which does not change when g is replaced by $\left(\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\right)g$, s is replaced by $-s$, and χ is replaced by $\eta^{-1}\chi^{-1}$. But the second term in the expression for $\mu_2(s,\chi)$ is given by the same analytical expression except that s is replaced by $-s$, g by $\left(\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\right)g$, and χ by $\eta^{-1}\chi^{-1}$. One shows as in the previous letter that $\mu_1(s,\chi)$ and $\mu_2(s,\chi)$ are bounded in vertical strips. The converse theorem is thus proved.

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