Dear Millson,

Now I brought me a copy of your paper with Goldman on deformations of flat bundles. It seems to me that it contains a number of inaccuracies, but also that, when corrected, it gives the stronger results you ask for in [sic] 10 & 11. I begin with the latter.

The philosophy, which I had not realized before reading your paper, seems the following: in characteristic 0, a deformation problem is controlled by a differential graded Lie algebra $d\mathfrak{g}_0$, with quasi-isomorphic $d\mathfrak{g}$ Lie algebras giving the same deformation theory. If the $d\mathfrak{g}$ Lie algebra controlling a problem is "formal," i.e., quasi-isomorphic to $\oplus H^i$, then the usual (formal) deformation space is that of $\oplus H^i$, i.e., to the completion at 0 of the subscheme of $H^2$ defined by the equations $[\nu_\mu] = 0$.

If one believes in this, what you do becomes transparent:

- **Deformation problem**: $G$ is a given reductive group, and one wants to deform a given $G$-torsor $P$. Heart: $G(C)$ torsor, or, equivalently, $G(\mathbb{R})$-torsor, with integrable connection $\nabla$.
- **Controlling $d\mathfrak{g}$ algebra**: (one is over $C$):
  
  $d\mathfrak{g}^\# (\text{Lie } G^p)$

If $P$ can be reduced to a maximal compact $K \subset G$ (i.e., to a compact form of $G$), Hodge theory shows that the controlling
algebra is "formal": one is quasi-isomorphisms of DG Lie algebras

\( \Omega^* \text{Lie}(E) \leftarrow K_{d'}(d') \rightarrow K_{d''}(d'') \)

and \( K_{d''}(d'') \) has \( d = 0 \). To check the above, it is easier to use the
principle of the 2-types in the form that the double complex \( \Omega^* \text{Lie}(E) \)
io (forgetting \( E \)) is a

1. a double complex with \( d' = d'' = 0 \) ("harmonic part")
2. squares of isomorphisms:

\[
\begin{array}{c c c}
\ast & \rightarrow & \ast \\
\uparrow & & \downarrow \\
\downarrow & & \downarrow \\
\ast & \rightarrow & \ast
\end{array}
\]

(0 elsewhere)

As far as formal deformations are concerned, this concludes the
story. It is not clear to me how to use Antoni in the analytic case
[see, for example] to get the same for analytic valued. The algebraic
case is OK, but Antoni is more powerful there.

Here is how to give a meaning to the philosophy (and check it)

Construction: How to attach to a DG Lie algebra \( L^* \) (over a

certain field \( k \)) a fibered \( L^* \)-Lie algebra over the category of

\( \text{Sp}(A) \), a \( k \)-\( k \)-algebra of finite dimension with constant field \( k \).

Object over \( \text{Sp}(A) \): \( \lbrack k = A/m \rbrack \)

\( \omega \in L^0 \otimes m, \) such that \( d\omega + \frac{1}{2} [\omega, \omega] = 0 \)

Map: The Lie algebra \( L^0 \) give rise to a formal group \( \hat{L}^0 \); the

\( A \)-point of this formal group are the \( L^0 \otimes m \), viewed as a nilpotent

Lie algebra, and equipped with Campbell-Hausdorff group law.

Notation: \( \hat{L}^0(A) \).
The formal group $\widehat{\mathbb{G}}$ acts on $L^1$ (thanks to $\otimes: L^1 \otimes L^1 \to L^1$), hence $\widehat{\mathbb{G}}(A)$ on $L^1 \otimes A$. One has also a map

$$g^* dq^* : \widehat{\mathbb{G}} \to L^1 : \exp(\lambda) \mapsto 1 - \exp(-\alpha A) \left[ C \right]$$

which $g \in \widehat{\mathbb{G}}(A)$ and that

$$w' = g dq^{-1} + q(w)$$

One has to check that $d w' + \frac{1}{2} [w, w] = g(d w + \frac{1}{2} [w, w])$ when $w$ and $w'$ are related by (1).

To be checked: for a square zero ideal $I$ in $A$, and an object $w$ on $A/I$, there is an outer obstruction in $H^2(A/I)$ to extend $w$ to $A$. If the obstruction vanish, extensions form a principal homogeneous space under $H^*(A/I)$. Automorphisms of any extension (trivial on $A/I$) are $H^0(A/I \otimes I)$. This is functorial in $L^1$, hence the fact that quasi-automorphisms induce equivalence of fibered categories.

Application: let $X$ be an analytic variety and $(V, \nabla)$ be a holomorphic vector bundle with connection. The deformation problem is $A \mapsto$ category of vector bundles, with connection in the $X$-direction, on $X \otimes \mathfrak{sp}(A)$, deforming $(V, \nabla)$.

It is given by the previous construction applied to $\mathcal{S}^\infty(\text{End} V)$. 
Ring: In your application, because you have the explicit quasi-isomorphism (1), you have an explicit equivalence of fibred categories between

* deformations of $(V, V)$ as above
* category with objects over $Sp(A)$ the maps

$$Sp(A) \longrightarrow \text{scheme of } H^1 \text{ with } [u, w] = 0$$

and arrows: \( g \in (H^1)^\vee(A) \) from \( u \) to \( g(u) \)

The functor \( \mathfrak{F} \) is such: a cohomology class \( u \in H^1 \circ \mathfrak{m} \), \( \mathfrak{m} \in \mathfrak{M} \), can be uniquely written as \( u \in H^1 \circ \mathfrak{m} \)

with \( \partial'w = d'w = 0 \). This \( w \) can be connected by a \( \alpha \) to have zero curvature, if \([u, u] = 0\). The connected \( \tilde{w} \) is unique mod.

\( \exp(H^1 \circ \mathfrak{m}) \).

Ring: (1) is a filtered quasi-isomorphism, for the Hodge filtration. This should translate into saying that the push of spaces

$$(\text{deformation of } (V, V)) \circ (\text{deformation of } V \text{ on a fixed } V)$$

is isomorphic to the intersection of the push of the pair $$(\text{deformation of } H^1 \circ F^0 H^2) \circ \mathfrak{m}$$

and, \([u, u] = 0\).
II. Comments on your paper:

p. 118 after co-B: no [take $C = U_d$]

p. 118 holomorphic? I guess you want to start with $\gamma$ such that $d'^{-1} \gamma = \delta^{a} \gamma = 0$

p? Why closed $\Rightarrow (\theta, 0)$ - compact closed?

III. Generalization

Hodge theory is available for variation of Hodge structure (complex variations on compact Kähler manifolds). The useful bijection of $\Omega^{\infty}(E_d V)$ is obtained by mixing that of $\Omega$ and that of $E_d V$. It is useful now that (i) is a filtered quasi-isomorphism. One get

1) The same result as before on deformation of a realization of $\pi$ given by a complex variation.

2) Looking at the deformation theory controlled by $F^0(\cdots)$: same result for the usual moduli of a filtered bundle with connection coming from a complex variation. One insist that the deformed connection must continue to satisfy "transversality": $\nabla F^0 \subset \pi' \otimes F'^{-1}$.

Yours sincerely,

P. Deligne

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