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General problem.

~~Class (Amalfi lecture)~~

$\mathcal{L}(s) = \sum_n \frac{a_n}{n^s}$; $a_1 = 1$, $a_n = O(n^\delta)$ for any $\delta > 0$.

"Euler product"

$\log \mathcal{L}(s) = \sum_n \frac{b_n}{n^s}$, $b_n = 0$ unless $n = p^\alpha$, $\alpha > 0$
$b_n = O(n^\theta)$ with $\theta < \frac{1}{2}$.

$(s-1)^m \mathcal{L}(s)$ integral function of finite order.

Functional equation

$$\Phi(s) = \varepsilon\, Q^{s} \prod_{j=1}^{n} \Gamma(\lambda_j s + \mu_j)\, \mathcal{L}(s)$$

constants
$|\varepsilon| = 1$, $Q > 0$; $\lambda_j > 0$, $\Re\mu_j \geq 0$.

$$\Phi(s) = \overline{\Phi(1-\bar{s})}$$; real for $s = \frac{1}{2}+it$; t real.

$\pi^{-\Lambda s}$ $\Lambda = \sum_1^n \lambda_j$

2Λ integer?

$\mathcal{L}_j(s)$ ... $j = 1 \ldots n$. same Γ factors.

Form

$$F(s) = \sum_{j=1}^{n} c_j\, \varepsilon_j\, Q_j^{s}\, \mathcal{L}_j(s)$$; $c_j \neq 0$; real.

$\prod_{j=1}^{n} \Gamma(\lambda_j s + \mu_j)\; F(s)$ real for $s = \frac{1}{2} + it$.

zeros of $F(s)$, apart from trivial zeros implied by Γ factors in functional equation, lie in same vertical strip $-A \leq \sigma \leq A$; number with imag. part in $(0, T)$

$$N(T, F) = \frac{\Lambda}{\pi} T(\log T + B) + O(\log T)$$

zeros with real part $\frac{1}{2}$; $N_0(T, F)$

Conjecture $N_0(T,F) \sim N(T,F)$. This can be proved if certain plausible conjectures are assumed.

What can we prove without any hypothesis.

Clear we can only expect to do something where we can prove some real results for the single $\mathcal{L}$'s. This has only been ~~possible~~ done for the case $\lambda = \frac{1}{2}$ and some cases with $\lambda = 1$.

I shall here look at the first case. The $\mathcal{L}$ - are then essentially Dirichlets L-functions $L(s, \chi)$.

Let χ be a primitive character mod q (including $q=1$ and $\chi(n)=1$ identically)

$$L(s,\chi) = \sum_n \frac{\chi(n)}{n^s}.$$

integral function for $q \neq 1$; $(s-1)L$ integral for $q=1$. write $a = \frac{1-\chi(-1)}{2}$

and $$\phi(s,\chi) = \varepsilon_\chi\, q^{\frac{s}{2}}\, \pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s+a}{2}\right) L(s,\chi)$$

then

$$\phi(s,\chi) = \overline{\phi(1-\bar{s},\chi)}$$

For simplicity consider case χ even (odd handled in quite similar way) - if we have n distinct characters χ_j and form

$$F(s) = \sum c_j\, \varepsilon_j\, q_j^{\frac{s}{2}}\, L(s,\chi_j)\ ;\quad c_j \text{ real} \neq 0$$

$$\left(\text{or } F(s) = \sum c_j\, \varepsilon_j \left(1 + q_j^{s-\frac{1}{2}}\right) L(s,\chi_j)\right)$$

$\underline{f(t)}$ = Then $\pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right) F(s)$ is real for $s = \frac{1}{2} + it$, t real

$$N(T,F) = \frac{T}{2\pi}(\log T + B) + O(\log T).$$

For single $L(s,\chi)$ proved that pos prop. of zeros have real part $\frac{1}{2}$, more precisely

$$N_0(T,L) > c\, T\log T \quad \text{for } T \geq Aq^2.$$

where c and A are absolute positive constants.

For general combination some results implied in literature. From Hardy-Littlewoods work follows $N_0(T,F) > cT$ for $T \geq T_0(F)$

Recently A. A Karatsuba has looked at combinations of the form:

$$F(s) = \varepsilon L(s,\chi) + \bar{\varepsilon}\, L(s,\bar{\chi})$$

where χ is a complex character and obtained the result (1994)

$$N_0(T,F) > T(\log T)^{\frac{1}{2}} e^{-c\sqrt{\log\log T}}; \quad T > T_0$$

with some constant c. for more general combination he has much weaker and more complicated result.

I shall sketch a proof that for the general combination $F(s)$, we have:

(1) $N_0(T,F) > c(m)\, T\log T$ for $T > T_0(F)$

and

(2) if $\omega(t) \to \infty$ as $t \to \infty$, then $F(\frac{1}{2}+it)$ has a zero in the interval $\left(t, t+\frac{\omega(t)}{\log t}\right)$ for almost all t.

First we need to see how these results are proved for the single L-function.

$s = \frac{1}{2} + it; \ t > 0$, $\vartheta(t) = \arg \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2})$

$X(t, \chi) = \varepsilon_\chi q^{\frac{it}{2}} e^{i\vartheta(t)} L(s, \chi)$.

"approx. funct. equation"

$$X(t, \chi) = \varepsilon_\chi q^{\frac{it}{2}} e^{i\vartheta(t)} \sum_{n < \sqrt{\frac{tq}{2\pi}}} \chi(n) n^{-s} + (\qquad)$$

$$+ O\left(\left(\frac{q}{t}\right)^{\frac{1}{4}}\right).$$

write

$$\left(\zeta(s)\right)^{-\frac{1}{2}} = \sum_n \frac{\alpha_n}{n^s}; \ \alpha_1 = 1; \ \left(L(s, \chi)\right)^{-\frac{1}{2}} = \sum_n \chi(n) \frac{\alpha_n}{n^s}$$

and for $T \le t \le 2T$, $\xi = T^{\frac{1}{10}}$; $T > q^2$

write

$$\eta(s, \chi) = \sum_{n \le \xi} \chi(n) \frac{\alpha_n}{n^s} \left(1 - \frac{\log n}{\log \xi}\right)$$

will often write $\eta(t, \chi)$ for $\eta(\frac{1}{2} + it, \chi)$.

For $\frac{1}{\log T} \le H \le \frac{\log \log T}{\log T}$

consider three expressions.

$$I_\chi(t, H) = \int_t^{t+H} X(u, \chi) |\eta(u, \chi)|^2 \, du$$

$$M_\chi(t, H) = \int_t^{t+H} L(\tfrac{1}{2} + iu, \chi) \eta^2(u, \chi) \, du - H$$

$$J_\chi(t, H) = \int_t^{t+H} |X(u, \chi) \eta^2(u, \chi)| \, du$$

clear that if

$$J_\chi(t,H) > |I_\chi(t,H)|,$$

then $X(u,\chi)$ changes sign in $(t, t+H)$ and so has at least one zero there.

~~Also clearly~~

$$J_\chi(t,H) \geq H - |M_\chi(t,H)|,$$

~~so if~~

$$|M_\chi(t,H)| + |I_\chi(t,H)| < H$$

~~there is a zero in $(t, t+H)$.~~

~~Using "approx funct eqn"~~ for $L(s,\chi)$ or $X(t,\chi)$ we can show

$$(1) \quad \int_T^{2T} |I_\chi(t,H)|^2 \, dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right)$$

$$(2) \quad \int_T^{2T} |M_\chi(t,H)|^2 \, dt = O\left(T \frac{H^{\frac{3}{2}}}{\sqrt{\log T}}\right),$$

and as we shall use much later

$$(3) \quad \int_T^{2T} |X(t,\chi)\,\eta^2(t,\chi)|^2 \, dt = O(T).$$

We see that $|I_\chi(t,H)| \leq \frac{H}{3}$, $|M_\chi(t,H)| \leq \frac{H}{3}$ except in a subset of $(T, 2T)$ of measure $O\left(\frac{T}{\sqrt{H \log T}}\right)$

Thus for all t in $(T, 2T)$ except for a sub set of measure $O\left(\frac{T}{\sqrt{H \log T}}\right)$; $(t, t+H)$ contains a zero, choosing $H = \frac{\lambda}{\log T}$ with λ a large enough constant we get statement I and taking $\lambda = (\omega(T))^{\frac{1}{2}}$ you get statement II.

We shall now adapt this idea to the linear combination. We need for this some results on the value distribution of $\log|L(\frac{1}{2}+it, \chi)|$ or $\log|X(t,\chi)|$

For $T > 16q^2$, k a positive integer and $T^{\frac{1}{2k}} < x \leq T^{\frac{1}{k}}$, we can show

$$\int_T^{2T} \left| \log|X(t,\chi)| - R \sum_{p<x} \frac{\chi(p)}{p^{\frac{1}{2}+it}} \right|^{2k} dt$$

$$= O\left(Tk^{4k} e^{Ak}\right).$$

Constants implied by O are absolute.

From this we can prove that

$$\frac{\log|X(t,\chi)|}{\sqrt{\pi \log\log t}}$$

has a normal gaussian distribution, more precisely, let $\chi_{a,b}$ denote the characteristic function of the interval a, b, then

$$\int_T^{2T} \chi_{a,b}\left(\frac{\log|X(t,\chi)|}{\sqrt{\pi \log\log T}}\right) dt = T \int_a^b e^{-\pi u^2} du + O\left(T \frac{(\log\log\log T)^2}{\sqrt{\log\log T}}\right)$$

(rather less would also do.)

Also for two distinct characters χ and χ' similar results hold for

~~difference~~

$$\log|X(t,\chi)| - \log|X(t,\chi')|$$

only here we must divide by $\sqrt{2\pi \log\log t}$ to get the normal gaussian distribution.

From this we see for instance that the set of t ~~where~~ in $(T, 2T)$ where $\frac{1}{2} > \delta > 0$

$$\left|\log|X(t,\chi)| - \log|X(t,\chi')|\right| \le (\log\log T)^{\delta}$$

has measure

$$O\left(T\,(\log\log T)^{-\frac{1}{2}+\delta}\right)$$

Thus most of time ~~some~~ $X(t,\chi_j)$ dominates all the others substantially.

This dominance is somewhat persistant over intervals somewhat long compared to $\frac{1}{\log T}$.

Define

$$\Delta_H(t,\chi) = \frac{1}{H}\int_t^{t+H} \log|X(u,\chi)|\,du,$$

For $0 \le h \le H$, we can show for any positive integer k that

$$\int_T^{2T} \left(\Delta_H(t,\chi) - \log|X(t+h,\chi)|\right)^{2k} dt$$

$$= O\left(T e^{Ak}\left(k^k \log^k(H\log T) + k^{4k}\right)\right)$$

Taking k large enough we see that

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We now have everything needed.

For $x_1 \dots x_n$ we now

$S_{j,k}$ as the subset of $(T, 2T)$

where
$$|\Delta_H(t, x_j) - \Delta_H(t, x_k)| \le (\log\log T)^{\delta}$$

this has ~~meas.~~ $m(S_{j,k}) = O(T(\log\log T)^{-\frac{1}{2}+\delta})$

If we exclude all of these subsets the rest of $(T, 2T)$ consists of n sets S_j such that in S_j and $k \ne j$
$$\Delta_H(t, x_j) > \Delta_H(t, x_k) + (\log\log T)^{\delta}$$

If we also from each S_j exclude all t for which $w(t, x_k) > \frac{H}{(\log\log T)^N}$ for any k

we get that $(T, 2T)$ except for a subset of measure $O(T(\log\log T)^{-\frac{1}{2}+\delta})$ is divided into n subsets S_j^* such that $\sum m(S_j^*) = T - O(T(\log\log T)^{-\frac{1}{2}+\delta})$

for each t in S_j^* we have
$$\log|X(t+h, x_j)| > \log|X(t+h, x_k)| + (\log\log T)^{\delta} - 2(\log\log T)^{\frac{\delta}{2}}$$
$$> \tfrac{1}{2}(\log\log T)^{\delta}$$

except for a set of h in $(0, H)$ of measure at most $O\left(\frac{H}{(\log\log T)^N}\right)$.

Now look at

$$I^*_{\chi_j}(t,H),\ M^*_{\chi_j}(t,H),\ J^*_{\chi_j}(t,H)$$

which are for t in S_j^* the integrals $I_{\chi_j}(t,H)$ etc. with the bad subset of h removed.

We have

$$\int_T^{2T} |X(t,\chi)\,\eta^2(t,\chi)|^2\,dt = O(T)$$

from which we see that in $(T, 2T)$

$$\int_t^{t+H} |X(u,\chi_j)\,\eta^2(u,\chi_j)|^2\,du = O\left(H(\log\log T)^{[illegible]}\right)$$

except for a subset of t of measure $O\left(\frac{T}{\log\log T}\right)$; we exclude also these t from S_j^* and see then that in the remaining part ~~of~~ S_j^{**} the I, M, J differ from the I^*, M^*, J^* at most by

$$O\left(\sqrt{\frac{H}{(\log\log T)^{N+1}}}\cdot\sqrt{H\log\log T}\right) = O\left(\frac{H}{(\log\log T)^{\frac{N}{2}}}\right)\ f(t)\ldots$$

and we see we get a ~~zero~~ sign change of $e\,f(\tfrac{1}{2}+it)$ in $(t, t+H)$ for t in S_j^{**} and $J^*_{\chi_j}(t,H) > |I^*_{\chi_j}(t,H)|$

which is equivalent to

$$H > |I_{\chi_j}(t,H)| + |M_{\chi_j}(t,H)| + O\left(\frac{H}{(\log\log T)^N}\right)$$

For T large enough this holds outside a measure of $O\left(\frac{T}{\sqrt{H\log T}}\right)$ in $(T,2T)$ and so in most of S_j^{**} if $H = \frac{\lambda n^2}{\log T}$ with λ a large enough constant.

This produces more than $\frac{c}{n^3} T\log T$ sign changes in S_j^{**} adding up for all j we get in all

$$> \frac{c}{n^2} T\log T \text{ sign changes.}$$

proves statement (i), second follows taking $\lambda = (\omega(T))^{\frac{1}{2}}$.

Dependence on n of $c(n)$ can be improved by sharpening estimation

$$T\frac{H^2}{\sqrt{H\log T}} \quad \text{to} \quad T\frac{H^2}{(H\log T)^\alpha}$$

for any $\alpha < 1$; $> c_\alpha n^{-\frac{1}{\alpha}} T\log T$; $\alpha = 1$?

Case $\lambda = 1$; missing part

$$\int_T^{2T} |M_\chi(t+\lambda)|^2\, dt$$

$A'' = 1.076834832\cdots$

~~1.2920~~

~~1.9999~~

Riemann Siegel formula.

$s = \frac{1}{2} + it \quad ; \; t > 0$

$\theta(t) = \arg \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$

$$e^{i\theta(t)} \xi(s) = 2 \sum_{n < \sqrt{\frac{t}{2\pi}}} \frac{\cos(\theta(t) - t \log n)}{\sqrt{n}} + R.$$

Hardy-Littlewood ca 1920 $\quad R(t) = O(t^{-\frac{1}{4}})$

Riemann had asymptotic expansion for R with error $O(t^{-N})$ for any N.

$$\cos^2 \frac{t_0 \log p}{2} < \frac{1}{\log^N p}$$

$$e^{\pi \frac{2n+1}{t_0} - \frac{\alpha}{(2n+1)^{\frac{N}{2}}}} < p < e^{\pi \frac{2n+1}{t_0} + \frac{\alpha}{(2n+1)^{\frac{N}{2}}}}$$

$$\sum \frac{1}{p} \leq \frac{3\alpha}{(2n+1)^{\frac{N}{2}}} \quad ; \; N > 2$$

convergence.

contradiction.

$\frac{2}{\sqrt{\xi}} \frac{|a_n|}{n^{2\sigma}}$ $\quad \frac{1}{\sqrt{\xi}} \quad T \frac{|a_n|^2}{n^{4\sigma}} \quad T\, \xi^{-\frac{3}{2}}$

$T\, \xi^{-(\sigma-\frac{1}{2})} \quad T\, \xi^{-\frac{1}{2}(\sigma-\frac{1}{2})}$

We write

$$M(t,H) = \int_t^{t+H} \left(\alpha\left(\tfrac{1}{2}+iu\right)\eta^2\left(\tfrac{1}{2}+iu\right)-1\right)du$$

$$\leq \left|\int_{\frac{1}{2}+it}^{2+it} \left(\alpha(s)\,\eta^2(s)-1\right)ds\right| + \left|\int_{\frac{1}{2}+i(t+H)}^{2+i(t+H)} \left(\alpha(s)\,\eta^2(s)-1\right)ds\right|$$

$$+\left|\int_{2+it}^{2+i(t+H)} \left(\alpha^2(s)\eta^2(s)-1\right)ds\right|$$

$$O\left(\xi^{-\frac{1}{3}}\right)$$

$$\left|\int_{\frac{1}{2}+it}^{2+it}\right|^2 \leq \int_{\frac{1}{2}}^{2} \xi^{\frac{1}{4}(\frac{1}{2}-\sigma)}d\sigma \int_{\frac{1}{2}}^{2} \xi^{\frac{1}{4}(\sigma-\frac{1}{2})} \left|\alpha(\sigma+it)\eta^2(\sigma+it)-1\right|^2 d\sigma$$

$$O\left(\frac{1}{\log \xi}\right) \quad O\left(T\, \xi^{\frac{1}{4}(\sigma-\frac{1}{2})}\, \xi^{-\frac{1}{2}(\sigma-\frac{1}{2})} d\sigma\right.$$

$$O\left(\frac{T}{\log^2 \xi}\right.$$

ξ

$\xi^{\frac{1}{2}-} \cdot \sum \frac{1}{n^4}$

$\xi^{-\frac{3}{2}} \quad T.$

$\xi^{-1} \quad \xi \quad \xi^{-\frac{3}{2}}$