# Topologies of the zero sets of random real projective hypersurfaces and monochromatic random waves

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Zurich, December 2017.

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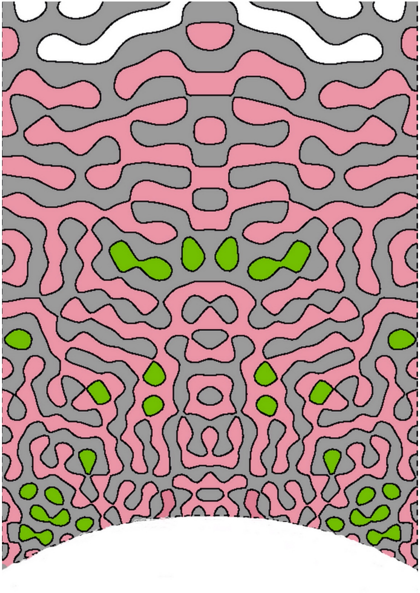
# Nodal portrait



# Setting

- **Monochromatic random waves** model the eigenfunctions of a quantization of a classically chaotic hamiltonian (M. Berry).
- **Random Fubini-Study ensembles** are a model for random real algebraic geometry.

Single variable:

$$f(x) = \sum_{j=0}^{t} a_j x^j \qquad a_j \in \mathbb{R}$$

$$Z(f) = \{x : \ f(x) = 0\}$$

Topology of $Z(f)$ is $|Z(f)|$.

$W_{1,t}$ = vector space of such polynomials $f$.

# What is random? (single variable)

We stick to centered Gaussian ensembles on a (finite) dimensional vector space $W$. This is **equivalent** to giving an inner product $\langle\ ,\ \rangle$ on $W$.

**'Naive' ensemble:**

$$\langle f, g\rangle = \sum_{j=0}^{t} a_j b_j \qquad \text{on } W_{1,t}.$$

- equivalent to choosing the $a_j's$ as i.i.d. standard Gaussians.
- **not natural** since it singles out $\pm 1$ as to where most the zeros locate themselves.

# What is random? (single variable)

**Real Fubini-Study ensemble:** $f(x,y) = \sum_{j=0}^{t} a_j x^j y^{t-j}$,

with

$$\langle f, g \rangle = \int_{\mathbb{R}^2} f(x) g(x) e^{-\frac{|x|^2}{2}} dx = * \int_{\mathbb{P}^1(\mathbb{R})} f(\theta) g(\theta) d\theta.$$

- In this ensemble $\{x^j y^{t-j} : \ j = 0, \ldots, t\}$ are not orthogonal, rather $\sin(\theta k)$ and $\cos(\theta k)$ are.

**Complex Fubini-Study ensemble on $W_{1,t}$:**

$$\langle f, g \rangle = \int_{\mathbb{P}^1(\mathbb{C})} \tilde{f}(z) \overline{\tilde{g}(z)} d\sigma(z).$$

- $\tilde{f}, \tilde{g}$ are complex extensions of $f, g$.
- In this ensemble $\{x^j y^{t-j} : \ j = 0, \ldots, t\}$ are orthogonal.

# Kac-Rice formulas (single variable)

Kac-Rice formulas give asymptotically the number of zeros of $f \in W_{1,t}$

- Naive ensemble: $\frac{2}{\pi}\log(t)$
- Real Fubini-Study : $t/\sqrt{3}$
- Complex Fubini-Study : $\sqrt{t}$
- Monochromatic (harmonic): $t$

$$\mathrm{Cov}_{f_t}(x,y) = \mathbb{E}(f_t(x), f_t(y)) =: K_t(x,y).$$

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} |\{x : |f(x)| < \varepsilon\}| = \sum_{a \in Z(f)} \frac{1}{|f'(a)|}.$$

$$\mathbb{E}(|Z(f)|) = \mathbb{E}\left( \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\{|f|<\varepsilon\}} |f'(y)| dy \right).$$

- This can be computed in terms of $K_t(x,y)$.
- Reduces problem to the asymptotics of $K_t(x,y)$ as $t \to \infty$.

# What is random? (several variables)

$W_{n,t}$: space of $f(x_0, x_1, \ldots, x_n)$ homogeneous of degree $t$.

- same definitions of the naive, real F-S, complex F-S, monochromatic.
- real F-S ($\alpha = 0$):

$$\langle f, g \rangle = \int_{P^n(\mathbb{R})} f(x)g(x)d\sigma(x).$$

- monochromatic random waves ($\alpha = 1$): same $\langle \ , \ \rangle$ but restricted to the subspace $H_{n,t}$ of $W_{n,t}$ consisting of harmonic polynomials.

Denote these two ensembles by $\mathcal{E}_{n,\alpha}$ with $\alpha = 0, 1$.

# Zero set: $Z(f) = \{x \in \mathbb{P}^n(\mathbb{R}) : \ f(x) = 0\}$

- For a random $f$, $Z(f)$ is smooth.
- Let $C(f)$ be the connected components of $Z(f)$. These are compact, $(n-1)$-dimensional manifolds.
- Let $\tilde{H}(n-1)$ be the countable collection of compact, $(n-1)$-dimensional manifolds **mod diffeos**.

$$Z(f) = \bigcup_{c \in C(f)} c, \qquad c \in \tilde{H}(n-1).$$

$$\mathbb{P}^n(\mathbb{R}) \backslash Z(f) = \bigcup_{\omega \in \Omega(f)} \omega.$$

the $\omega$'s are the nodal domains of $f$.

**What can we say** about the topologies of a random $f$ as $t \to \infty$?

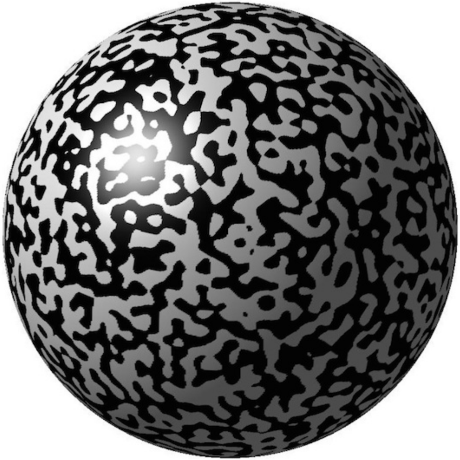
# Nesting of nodal domains

- Nesting tree $X(f)$ (Hilbert for ovals).
- The vertices of $X(f)$ are the nodal domains $\omega \in \Omega(f)$. Two vertices $\omega$ and $\omega'$ are joined if they have a common boundary $c \in \mathcal{C}(f)$ .
- $X(f)$ is a tree (Jordan-Brouwer).

$$|\Omega(f)| = |\mathcal{C}(f)| - 1.$$

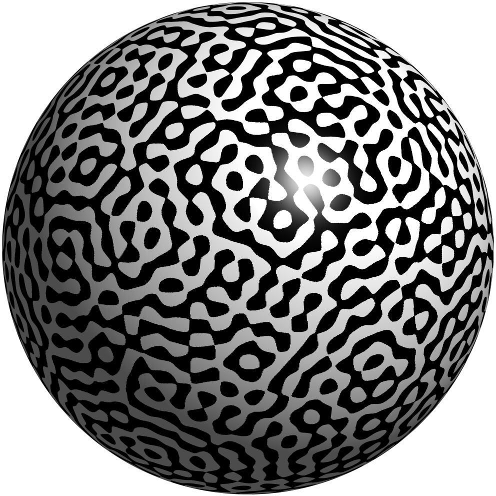
- $X(f)$ carries all the combinatorial information about the connectivities $m(\omega)$ for $\omega \in \Omega(f)$.

# Nodal portrait: Fubini-Study ensemble ($\alpha = 0$)



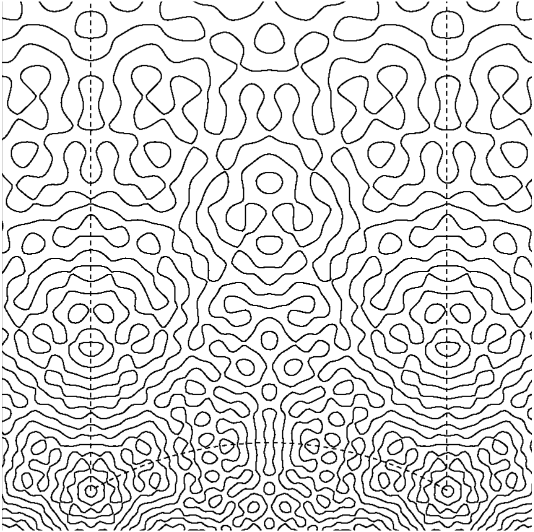
Sum of random spherical harmonics of degree $\leq 80$ (A. Barnett).

# Nodal portrait: Random spherical harmonic ($\alpha = 1$)

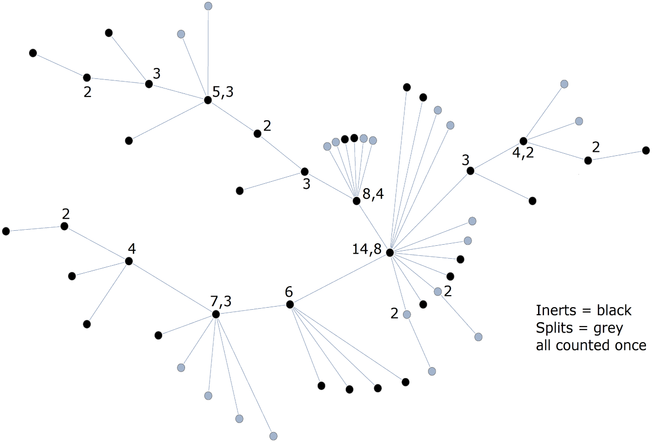


random spherical harmonic of degree $= 80$. (A. Barnett)

# Zero set



# Nesting tree



# Local and global quantities

For a Gaussian ensemble the Kac-Rice formula allows for the explicit computation of the expected values of **local** quantities.

- $|Z(f)|$ the induced $(n-1)$ dimensional volume of $Z(f)$.
- The Euler number $\chi(Z(f))$.
- The number of critical points of $f$.

The question of **global** topology of $Z(f)$ is much more difficult.

Nazarov and Sodin [NS] have introduced some powerful “soft" techniques to study the problem of the number of connected components of $Z(f)$ for random $f$.

Their methods show that most of the components $c \in C(f)$ are small occuring at a scale of $1/t$ and thus semi-localising this count.

# Nazarov-Sodin

**Theorem (Nazarov-Sodin 2013,2016)**

*There are positive constants $\beta_{n,\alpha}$ such that*

$$|C(f)| \sim \beta_{n,\alpha} t^n \qquad \text{as } t \to \infty$$

*for the random $f$ in $\mathcal{E}_{n,\alpha}(t)$, for $\alpha = 0, 1$.*

- Their 'soft' proof offers no effective lower bounds for these N-S constants $\beta_{n,\alpha}$.
- Their barrier method (2008) can be made effective but the resulting bounds are extremely small.
  - $\beta_{2,0} \geq 10^{-320}$ Nastasescu,
  - $\beta_{2,0} \geq 10^{-70}$ deCourcy-Ireland,
  - $\beta_{n,0} \geq e^{-e^{257 n^3/2}}$ Gayet-Welschinger
- For a random $f$ the set $Z(f)$ has many components and we can ask about their topologies.

# Topologies and Nestings

For $f \in \mathcal{E}_{n,\alpha}(t)$ set

$$(A) \quad \mu_{\mathcal{C}(f)} := \frac{1}{|\mathcal{C}(f)|} \sum_{c \in \mathcal{C}(f)} \delta_{t(c)}$$

where $t(c)$ is the topological type of $c$ in $\tilde{H}(n-1)$ and $\delta_{t(c)}$ is the point mass at $t(c)$.

$\mu_{\mathcal{C}(f)}$ is a probability measure on $\tilde{H}(n-1)$.

$$(B) \quad \mu_{\mathcal{X}(f)} := \frac{1}{|\mathcal{C}(f)|} \sum_{c \in \mathcal{C}(f)} \delta_{e(c)}$$

where $e(c)$ is the smallest of the two rooted trees that one gets from $X(f)$ after removing the edge $c \in \mathcal{C}(f)$.

$\mu_{\mathcal{X}(f)}$ is a probability measure on $\mathcal{T}$ (the space of finite rooted trees).

# Topologies and Nestings: main result

**Theorem**[Wigman-S 2015, Canzani-S 2017]

**(i)** There are **probability** measures $\mu_{C,n,\alpha}$ on $\tilde{H}(n-1)$ and $\mu_{X,n,\alpha}$ on $\mathcal{T}$ such that for random $f \in \mathcal{E}_{n,\alpha}(t)$

$$\mu_{C(f)} \to \mu_{C,n,\alpha}, \qquad \mu_{X(f)} \to \mu_{X,n,\alpha}$$

as $t \to \infty$, and the convergence is tight.

**(ii)** $\quad \mathrm{supp}(\mu_{C,n,\alpha}) = H(n-1) \quad$ and $\quad \mathrm{supp}(\mu_{X,n,\alpha}) = \mathcal{T}$.

**Obs.** $H(n-1)$ is the subset of diffeomorphism types in $\tilde{H}(n-1)$ that can be embedded into $\mathbb{R}^n$.

**Obs.** These give universal laws for the distributions of the topologies of the components of random real hypersurfaces ($\alpha = 0$) and monochromatic waves ($\alpha = 1$), as well as for nesting ends.

# Betti numbers and connectivities

The theorem implies universal laws for the distribution of the Betti numbers of the components as well as for the connectivities of the domains.

For $f \in \mathcal{E}_{n,\alpha}(t)$ set

$$(A) \quad \nu_{Betti(f)} := \frac{1}{|\mathcal{C}(f)|} \sum_{c \in \mathcal{C}(f)} \delta_{B(c)}$$

where $B(c) = (b_1(c), \ldots, b_{n-2}(c))$ is the collection of Betti numbers.

$$(B) \quad \nu_{con(f)} := \frac{1}{|\Omega(f)|} \sum_{\omega \in \Omega(f)} \delta_{m(\omega)}$$

where $m(\omega)$ is the number of boundary components of $\omega$.
The universal limits are

$$\nu_{Betti,n,\alpha} \text{ on } (\mathbb{Z}_{\geq 0})^{n-2}, \qquad \nu_{con,n,\alpha} \text{ on } \mathbb{N}.$$

# Remarks

- The existence of the universal measures follows the 'soft' methods of N-S. However, the tightness of the convergence (with the consequence that all universal measures are probability measures) and the determination of their supports (especially when $\alpha = 1$) is a challenge.

- Gayet and Welschinger (2013) used the barrier method, in the context of the Kostlan distribution and its generalizations, to show that every topological type $c \in H(n-1)$ occurs with positive probability.

- Lelario-Lunderberg (2013) used the barrier method to give lower bounds for the number of connected components for random Fubini-Study ($\alpha = 0$).

# How do the universal measures look like?

Barnett/Jin (2013, 2017) carried out Monte-Carlo simulations $n = 2, 3$.

- When $n = 2$ we have $H(1)$ is a point.
- The connectivity measures $\nu_{con(f)}$ satisfy

$$\mathbb{E}\left(\nu_{con(f)}\right) = \sum_{m=1}^{\infty} m \cdot \nu_{con(f)}(m) = \sum_{\omega \in \Omega(f)} \frac{m(\omega)}{|\Omega(f)|} = 2 + o(1).$$

| m | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\nu_{con,2,0}$ | 0.973 | 0.027 | 0.009 | 0.003 | 0.002 | 0.002 | 0.001 | 0.001 |

| m | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\nu_{con,2,1}$ | 0.906 | 0.055 | 0.010 | 0.006 | 0.003 | 0.002 | 0.001 | 0.0009 |

# Observations

- It appears that
$$\mathbb{E}(\nu_{con,\alpha,2}) < 2$$
corresponding to the persistence of many domains of large connectivity.

- The N-S constants $\beta_{2,\alpha}$ are of order $10^{-2}$ and for $\alpha = 2$ the random plane curve is 4% Harnack (that is, it has 4% of the maximum number of ovals that such a curve can have). M. Natasescu(2012).

- When $n = 3$ we have $H(2)$ is the set of compact orientable surfaces; determined by their genus $g \in \mathbb{Z}_{\geq 0}$. So $\mu_{\mathcal{C},3,\alpha}$ is a probability measure on $Z_{\geq 0}$.

# $\mu_{\mathcal{C}(f)}$

A Kac-Rice computation (Podkoytov 2001) gives

$$\mathbb{E}\left(|\chi(Z(f)|\right) \sim \begin{cases} \frac{t^3}{3^{3/2}}, & \alpha = 0 \\ \frac{t^3}{5^{3/2}}, & \alpha = 1. \end{cases}$$

Thus,

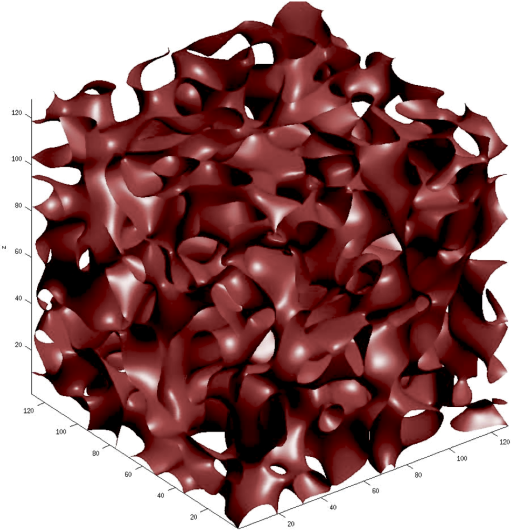
$$\mathbb{E}\left(\mu_{\mathcal{C}(f)}\right) = \sum_{g=0}^{\infty} g \cdot \mu_{C(f)}(g) \sim \begin{cases} 2 + \frac{1}{3^{3/2}\beta_{3,0}} = A_0, & \alpha = 0 \\ 2 + \frac{1}{5^{3/2}\beta_{3,1}} = A_1, & \alpha = 1. \end{cases}$$

In particular,

$$\mathbb{E}\left(\mu_{C,3,\alpha}\right) \leq A_\alpha.$$

What Barnett-Jin find for $\mu_{\mathcal{C}(f)}$ is dramatic.

# Zero set



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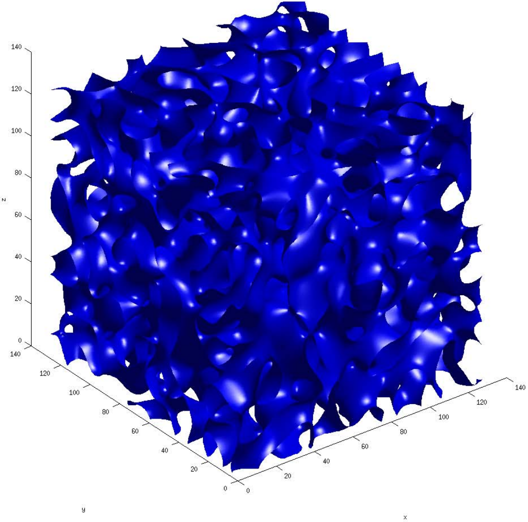

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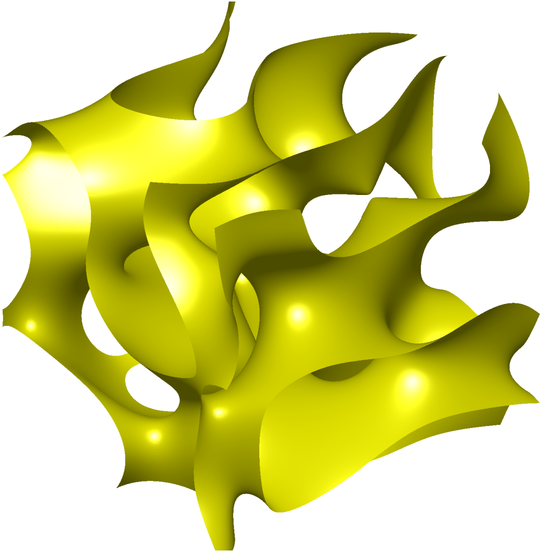

# Observations

- Apparently we are in a super critical regime with a unique giant percolating component $\pi(f) \in \mathcal{C}(f)$.

- The N-S constants $\beta_{3,0}, \beta_{3,1}$ are very small ($\approx 10^{-7}$) and the feasibility of observing $\mu_{\mathcal{C},3,\alpha}, \mu_{\mathcal{X},3,\alpha}$ is problematic.

- $A_0, A_1$ are very large so there is a dramatic loss of mean in going from the finite measures to their limits.

- In the main equidistribution theorems each topological component is counted with equal weight. So there is no contradiction as $\pi(f)$ is treated as equal to others.

- Clearly, to complete the basic understanding of $Z(f)$, the topology of $\pi(f)$ needs to be examined.

# Speculations/Questions

- As an element of the discrete $H(n-1)$, $\pi(f) \to \infty$ as $t \to \infty$ for random $f$.

- Betti$(\pi(f))$:

$$\lim_{t\to\infty} \frac{B(\pi(f))}{t^n} = \begin{cases} 0 \in (\mathbb{Z}_{\geq 0})^{n-2} & n-1 \text{ odd} \\ (0,\ldots,0,\delta_{\frac{n-1}{2}},0,\ldots,0) & n-1 \text{ even} \end{cases}$$

with $\delta_{\frac{n-1}{2},\alpha} > 0$.

That is, for $n-1$ even the homology of the percolating component is $\delta\%$ of the homology of that of a complex hypersurface $f = 0$.

To explain the source of the super critical percolation we need to go into some of the analysis.

# Brief comments about proofs

Covariance:

$$K_{n,\alpha}(t;x,y) = \mathbb{E}_{f\in\mathcal{E}_{n,\alpha}(t)}(f(x)f(y)).$$

As $t \to \infty$ one shows using well known asymptotics of special functions and micro-local analysis in the more general setting of 'band limited functions' on a manifold, Canzani-Hanin (2015)

$$\frac{K_{n,\alpha}(t;x,y)}{\dim\mathcal{E}_{n,\alpha}(t)} = \begin{cases} B_{n,\alpha}(t\,d(x,y)) + O(1/t), & td(x,y) \leq 1, \\ O(1/t), & td(x,y) \geq 1, \end{cases}$$

where

$$B_{n,\alpha}(\omega) = B_{n,\alpha}(|\omega|) = \frac{1}{|\Omega_\alpha|}\int_{\Omega_\alpha} e^{i\langle\omega,\xi\rangle}d\xi$$

with $\Omega_\alpha = \{\omega :\ \alpha \leq |\omega| \leq 1\}$.

# Brief comments about proofs

- Following N-S we show that our quantities can be studied semi locally, i.e. in neighborhoods of size $1/t$.

- After scaling one arrives at a Gaussian translation invariant isotropic field on $\mathbb{R}^n$ (with slow decay of spatial correlations).

- The existence of the limiting measures, as well as the convergence in measure, follows from soft ergodic theory of the action of $\mathbb{R}^n$.

- The properties of the universal $\mu$'s, that of being probability measures (i.e. no escape of topology for them) and that they charge every admissible atom positively, is much harder earned.

# Brief comments about proofs

- To control the escape of topology, that is the tightness of the convergence, we show that **most** components of the scaled Gaussian are geometrically controlled (specifically their curvatures) and eventually apply a form of Cheeger finiteness.

- To show that the support is full in the case $\alpha = 1$ requires one to prescribe topological configurations locally for "1-harmonic'" entire functions
$$\Delta\psi + \psi = 0 \qquad \text{on } \mathbb{R}^n.$$
For this we prove versions of Runge type approximation/interpolation theorems for such $\psi$'s.

- The nesting prescription is the most challenging and is achieved in $n = 3$ by deformation
$$f = f_0 + \varepsilon f_1$$
$f_0 = \sin(x)\sin(y)\sin(z)$ and $f_1$ a suitable 1-harmonic function.

# Percolating component

To end we explain the source of the dominant percolating $\pi(f)$. For $\alpha = 1$ and $n = 3$ the scaling limit mean zero Gaussian field on $\mathbb{R}^3$ has

$$\mathrm{Cov}(x, y) = K(x, y) = *\frac{\sin(|x - y|)}{|x - y|} \qquad x, y \in \mathbb{R}^3$$

for this field or any similar Gaussian field define the critical level $h_*$ by:

- For $h > h_*$ the set $\{x : f(x) \geq h\}$ has no infinite component with probability 1.
- For $h < h_*$ the set $\{x : f(x) \geq h\}$ has an infinite component with probability 1.

$h_*$ is a function of the field.

# Conjecture

**Conjecture:** If $n \geq 3$, then $h_* > 0$.

- In particular, the zero levels $h = 0$ are supercritical. Note that for $n = 2$ it is known that $h_* = 0$ (Alexander '96).

- Evidence towards this conjecture is provided by the recent proof (Rodriguez, Drewitz, Prevost) of the 1987 conjecture of Brimont-Lebowitz-Maes, that for the discrete analogue on $\mathbb{Z}^3$ of the Gaussian free field ($K(x,y) = \frac{1}{|x-y|}$) one has $h_* > 0$.

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