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On the Noether-Lefschetz Theorem and Some Remarks on Codimension-two Cycles

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1. Introduction

The theorem referred to in the title of this paper is the

(*) **Theorem.** *If $d \geqq 4$, then a surface $S \subset \mathbb{P}^3$ of degree d having general moduli has Picard group $Pic(S) \simeq Z \cdot \mathcal{O}(1)$; i.e., every curve $C \subset S$ is a complete intersection.*

Here, "of general moduli" means that there is a countable union V of subvarieties of the space $\mathbb{P}^N$ of surfaces of degree d in $\mathbb{P}^3$, such that the statement $\operatorname{Pic}(S) = Z$ holds for $S \in \mathbb{P}^N - V$.

Noether, it would seem, stated this theorem but never completely proved it. Instead, he gave a plausibility argument, based on the following construction[1]: let $\mathcal{H}_{n,g}$ be the Hilbert scheme of curves of degree n and genus g in $\mathbb{P}^3$, $|\mathcal{O}_{\mathbb{P}^3}(d)| \simeq \mathbb{P}^N$ the space of surfaces of degree d in $\mathbb{P}^3$, and $\Sigma \subset \mathcal{H}_{n,g} \times |\mathcal{O}_{\mathbb{P}^3}(d)|$ the incidence correspondence

$$\Sigma_{n,g,d} = \{(C, S) : C \subset S\} .$$

Of course, all these schemes are projective. Moreover, the locus $\Sigma^C_{d,g} \subset \Sigma_{d,g}$ of pairs (C, S) such that C is a complete intersection with S is *open* in $\Sigma_{n,g,d}$ so that $\Sigma'_{n,g,d} = \Sigma_{n,g,d} - \Sigma^C_{n,g,d}$ is again projective, and, in particular, the projection map $\pi_2 : \Sigma'_{n,g,d} \to |\{\mathcal{O}_{\mathbb{P}^3}(d)\}|$ is proper. Thus we see first of all that the locus of surfaces $S \subset \mathbb{P}^3$ of degree d that contain curves other than complete intersections is the union $\bigcup_{n,g} \pi_2(\Sigma'_{n,g,d})$; in particular, it is a countable union of closed subvarieties of $\mathbb{P}^N$.

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1 We're interpreting rather broadly here

The question now is whether all the subvarieties $\pi_2(\Sigma'_{n,g,d}) \subset \mathbb{P}^N$ are proper, and here Noether uses a dimension count. If $d=1$, $g=0$, for instance, – that is, we are looking at surfaces containing lines – then since there is a 4-dimensional family of lines in $\mathbb{P}^3$, and it is $d+1$ conditions for a surface of degree d to contain a line ($\dim \mathscr{H}=4$, and the fiber dimension of π_1 is $N-d-1$, so $\dim \Sigma' = N-d+3$) it is clear that a general surface of degree $d \geqq 4$ contains no lines. In the case of plane conics, $\dim \mathscr{H}=8$ and the fiber dimension of π_1 is $N-2d-1$; with twisted cubics the numbers are 12 and $N-3d-1$. In every case that one checks (and Noether checked a lot!), the conclusion is the same: the general surface of degree $d \geqq 4$ contains no such curves.

Of course, there are a number of serious obstacles to making a proof along these lines: notably, we do not know $\dim \mathscr{H}_{d,g}$ in general; we do not know the general fiber dimension of π_1 (this is, in some cases, the subject of the maximal rank conjecture; cf. [6]); and additionally we do not know in what codimension the fiber dimension of π_1 may jump. Thus the theorem remained plausible but unproved until Lefschetz.

Lefschetz in [8] gave a proof of the theorem, along radically different lines and making essential use of Hodge theory. The second cohomology $H^2(S,\mathbb{Z})$ of a surface $S \subset \mathbb{P}^3$, Lefschetz argues, injects into $H^2(S,\mathbb{C})$, and the Picard group of S is just the intersection $H^{1,1}_{\mathbb{Z}}(S)$ of $H^2(S,\mathbb{Z})$ with the subspace $H^{1,1}(S) \subset H^2(S,\mathbb{C})$, which is a *proper* subspace of $H^2(S,\mathbb{Z})$ if $\deg S \geqq 4$. But the monodromy in the family of all surfaces of degree d acts irreducibly on the orthogonal complement of the class of $\mathcal{O}(1)$ in $H^2(S,\mathbb{Z})$; since the invariant sublattice $H^2(S,\mathbb{Z}) \cap H^{1,1}(S,\mathbb{C}) \cap c_1(\mathcal{O}(1))^{\perp}$ cannot be all of $c_1(\mathcal{O}(1))^{\perp}$, it must be zero in general. A somewhat sharper local version was recently given in [2] where it is shown that if $\gamma \in H^{1,1}_{\mathbb{Z}}(S)$ is any Hodge class other than a multiple of $c_1(\mathcal{O}(1))$, a general first-order deformation of S carries γ out of $H^{1,1}$.

What brought this whole matter to our attention was a related question having to do with curves on a threefold $X \subset \mathbb{P}^4$. Simply put, what we observed was this: applying Noether's set-up in this situation, one is led to question *whether a threefold $X \subset \mathbb{P}^4$ of degree $d \geqq 6$, having general moduli, contains any curves C other than complete intersections of X with a surface $S \subset \mathbb{P}^4$.*

If this strikes one as too extreme, one can give a succession of weaker statements, none of which is (to our knowledge) known to be true or false. Specifically, we ask:

If $X \subset \mathbb{P}^4$ is a general threefold of degree $d \geqq 6$, and $C \subset X$ any curve, is it necessarily true that

1) the degree of C is a multiple of d.

2) C is algebraically equivalent to a multiple mD of a plane section $D = X \cdot \mathbb{P}^2$ of X ($m \in \mathbb{Z}$),

3) C is rationally equivalent to mD,

or

4) C is a complete intersection of X with a surface $S \subset \mathbb{P}^4$.

These statements are in order of increasing strength; a fifth, which would be implied by 2), 3), or 4), is

5) $C - mD$ maps to zero in the intermediate Jacobian $J(X)$ ($m \in \mathbb{Q}$).

It should be noted here that we do not have an a priori notion of what the locus of threefolds X violating 1)–5) looks like, since a deformation of a proper intersection $C = X \cap S$ may no longer be proper. Thus, for example, it is possible that a general threefold X violates one of the statements 1)–5), but that some special X satisfy it.

In any event, in trying to establish any of the above it became clear that Noether's viewpoint, while suggestive, was not going to provide the basis for a proof. At the same time, it appeared that Hodge theory could not be applied directly, at least not in the manner of Lefschetz' proof: Hodge theory deals a priori with cohomology classes, and while the cohomology class of a curve on a surface in $\mathbb{P}^3$ determines whether or not it is a complete intersection, the cohomology class of a curve on a threefold X tells us very little: by the Lefschetz hyperplane theorem, all curves on X are homologous to a multiple of a plane section.

This left two possibilities. One was to ask whether one could give an algebraic proof of the Noether-Lefschetz theorem for surfaces, in the hope that such a proof might be applicable to higher codimension cycles as well. We were successful in the first part of this; it is the primary purpose of this paper to present an elementary algebraic proof of theorem (*).

(*Note:* Actually there is some "hidden Hodge theory" in the proof in that the essential step involves properties of the Jacobian of a general plane curve of degree $d-1$.)

As will be discussed following the proof, however, our lack of basic techniques for dealing with higher-codimensional cycles does not allow us to carry this argument over for threefolds.

A second possibility in trying to answer the questions above was to try to apply Hodge theory, indirectly: For example, one could realize a general threefold $X \subset \mathbb{P}^4$ as a general member of a pencil of hyperplane sections of a general fourfold Y in $\mathbb{P}^5$ and then hope to apply the theory of normal functions to this fourfold – in this way the study of curves on X has implications on the Hodge structure of $H^4(\tilde{Y}, \mathbb{C})$ for suitable branched coverings $\tilde{Y} \to Y$. Using this idea we are able to arrive at a conjecture [cf. (3.4) below] and support it with a couple of plausibility arguments.

Via the mechanism of normal functions, there is a close connection between curves on a general threefold in $\mathbb{P}^4$ and surfaces lying on a fourfold $Y \subset \mathbb{P}^5$. In the last section we complement our previous paper [5] by establishing a stable variational form of the Hodge conjecture for such Y's – here "stable" has the somewhat perverse interpretation that the degree of Y should be large relative to the degree of the (effective) algebraic 2-cycle we are trying to construct.

2. Algebraic Proof of the Noether-Lefschetz Theorem

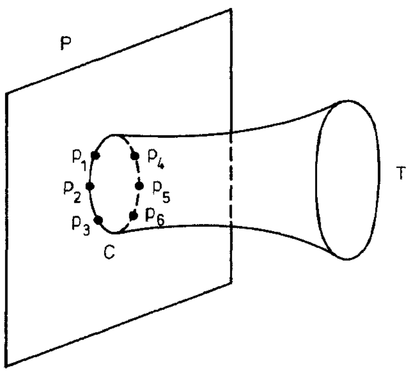
a) Construction of a Family of Surfaces

The line of argument will be this: first, we will construct a particular family $\pi: X \to \Delta$ of surfaces parametrized by a disc Δ with parameter t, whose fibers $X_t = \pi^{-1}(t)$ for $t \neq 0$ will all be smooth surfaces of degree d in $\mathbb{P}^3$, and whose fiber X_0 over $t = 0$ will be a reducible surface in $\mathbb{P}^3$. After modifying the family somewhat to obtain a smooth family $\tilde{X} \to \Delta$, we will then investigate i) the group of line bundles/divisor classes on the central fiber $\tilde{X}_0$ of $\tilde{X}$; and ii) the behaviour of a family W_t of curves on X_t as $t \to 0$. Given our knowledge of $\mathrm{Pic}(X_0)$, we will conclude that the W_t are all complete intersections. Finally, we have to consider the case of a "multi-valued" family of curves on X_t; i.e., investigate the effect of a base change on the family $\tilde{X} \to \Delta$.

To construct our family, let T be an arbitrary smooth surface of degree $d-1$ in $\mathbb{P}^3$, given by the homogeneous equation $F(X)=0$. Let $P \subset \mathbb{P}^3$ then be a plane, chosen generically with respect to T, with equation $L(X)=0$. Then choose a surface $U \subset \mathbb{P}^3$ of degree d, generically with respect to T and P; let $G(X)=0$ be the equation of U. Our initial family of surfaces is then the pencil containing U and $T+P$; that is, the threefold $X \subset \mathbb{P}^3 \times \Delta$ given by the equation

$$L(X)\,F(X) - t \cdot G(X) = 0 .$$

For notation, let $\pi: X \to \Delta$ be the projection; let $C \cong T \cap P$ be the double curve of the fiber $X_0 = T \cup P$ (P having been chosen generically with respect to T, they will meet transversely), and let $p_1, \ldots, p_{d(d-1)} \in \mathbb{P}^3$ be the points of intersection of the curve $T \cap P$ with the surface U [we will also, by abuse of notation, denote by p_i the point $(p_i, 0) \in C \subset X_0 \subset X \subset \mathbb{P}^3 \times \Delta$]. The picture of X_0 is:

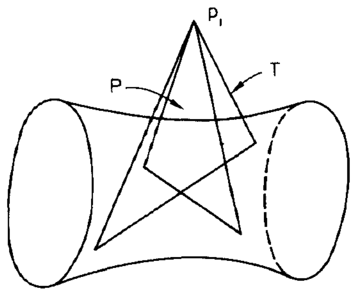


There is one essential modification of the family $X \xrightarrow{\pi} \Delta$ we must make in order to use it for our purposes. As indicated above, we want to calculate $\mathrm{Pic}(X_0)$, and use this to say something about line bundles on the general fiber X_t. The problem is that X is not smooth, so that the limit on X_0 of a family of line bundles L_t on X_t for $t \neq 0$ need not be a line bundle. Specifically, X will be singular at the points p_i; since U was chosen generically with respect to P and T, X will have ordinary double points at the p_i. Thus X will look near p_i like a cone over a quadratic surface; in local coordinates x, y, z on $\mathbb{P}^5$, X will be given by

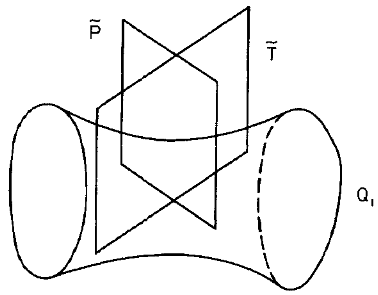
$$xy - tz = 0$$

with $(x)=P$, $(y)=T$, and $z=(U)$ in $\mathbb{P}^3$.

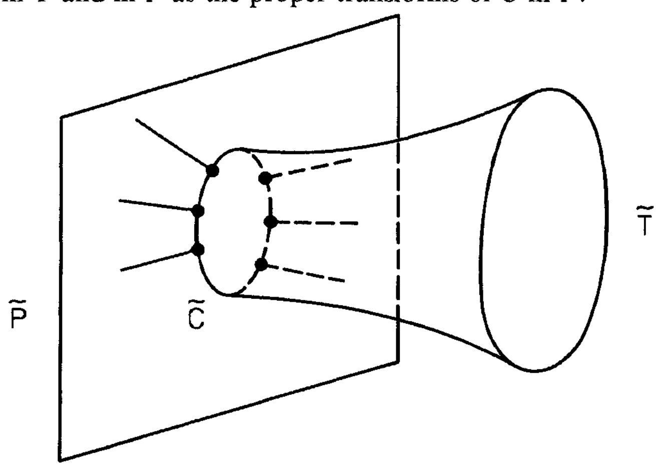
The picture is



To resolve the singularity of X at p_i, we first blow it up. This now introduces a surface Q_i in the fiber over $t=0$, isomorphic to a quadric surface; at the same time, T and P are each blown up at p_i, their exceptional divisors meeting Q_i in lines of opposite rulings of Q_i. The picture of the fiber over $t=0$ is



Now, as is well known, the quadric Q_i arising from the blow-up of an ordinary double point of a 3-fold can be blown down along either ruling without making the threefold singular. We do this in our present circumstance, *blowing down each Q_i along the ruling containing its intersection with the proper transform of T.* The resulting (smooth) threefold we will call $\tilde{X}$, and the map to Δ we will call $\tilde{\pi}$. The central fiber $\tilde{X}_0$ of $\tilde{\pi}$ now consists of two components, one isomorphic to T (which we will also call $\tilde{T}$) and one isomorphic to the plane P blown up at each of the $d(d-l)$ points p_i (which we will call $\tilde{P}$) meeting in a curve $\tilde{C}$ isomorphic to C, sitting in $\tilde{T}$ as C in T and in $\tilde{P}$ as the proper transforms of C in P:



b) The Proof

Having constructed $\tilde{X}$, we may now proceed with our argument. Suppose that, for general t, the surface $X_t \subset \mathbb{P}^3$ contains a curve that is not a complete intersection. Of course, it does not follow that we can find such a curve $W_t \subset X_t$ rationally defined over the base Δ – we may have to make a base change – but for clarity *we will assume first that we can*, dealing with the general case later. We thus have a line bundle L on $\tilde{X}^* = \tilde{X} - \tilde{X}_0$ which restricts to $\mathcal{O}_{X_t}(W_t)$ on each fiber X_t, and since $\tilde{X}$ is smooth, L extends to a line bundle on all of $\tilde{X}$. We want to study the restriction of L to $\tilde{X}_0$; the key calculation is the

Lemma. $\mathrm{Pic}(\tilde{X}_0) = \mathbb{Z} \oplus \mathbb{Z}$.

Proof. Since the components $\tilde{T}$ and $\tilde{P}$ of $\tilde{X}_0$ intersect transversely along $\tilde{C}$, a line bundle on $\tilde{X}_0$ consists of a line bundle on both $\tilde{T}$ and $\tilde{P}$, whose restrictions to $\tilde{C}$ are isomorphic. We accordingly look at the restriction maps

$$r_1 : \mathrm{Pic}(\tilde{T}) \to \mathrm{Pic}(\tilde{C})$$

and

$$r_2 : \mathrm{Pic}(\tilde{P}) \to \mathrm{Pic}(\tilde{C}).$$

We make the following observations:

i) $\mathrm{Pic}(\tilde{T})$ [and hence also its image in $\mathrm{Pic}(\tilde{C})$] is finitely generated.

ii) r_1 *is injective*, since the curve $\tilde{C}$ is a generically chosen hyperplane section of the non-ruled surface $\tilde{T}$.

iii) $\mathrm{Pic}(\tilde{P})$ is freely generated by the line bundles $\mathcal{O}(1)$ [by which we mean the pullback of $\mathcal{O}(1)$ from the plane $P \subset \mathbb{P}^3$] and the line bundles $\mathcal{O}(E_i)$ associated to the exceptional divisors of the blow up map $\tilde{P} \to P$. Of course, $r_2(\mathcal{O}(1)) = \mathcal{O}_C(1)$, and $r_2(\mathcal{O}(E_i)) = \mathcal{O}_C(p_i)$.

iv) Our final assertion is twofold: that $\ker r_2$ consists solely of the line bundle $\mathcal{O}_P(-d)(-E_1 - \ldots - E_{d(d-1)})$ and its powers, and that the image of r_2 intersects the image of r_1 solely in the line bundles $\mathcal{O}_C(n)$. Since $\mathrm{Pic}(\tilde{X}_0) = \mathrm{Pic}(\tilde{P}) \times_{\mathrm{Pic}(C)} \mathrm{Pic}(\tilde{T})$, the Lemma will follow from assertion ii) and this final assertion. The assertion itself follows immediately from assertions i) and iii), the construction of $\tilde{X}_0$, and the

Sublemma. *Let $C \subset \mathbb{P}^2$ be any plane curve of degree $m \geqq 3$, $\Sigma \subset \mathrm{Pic}(C)$ any countably generated subgroup containing $\mathcal{O}(1)$, and $D \subset \mathbb{P}^2$ a general curve of degree n meeting C in points $p_1, \ldots, p_{mn}$. Then no nontrivial linear combination of the points p_i lies in the subgroup Σ of* $\mathrm{Pic}(C)$, *except $\mathcal{O}(p_1 + \ldots + p_{mn}) = \mathcal{O}(n)$.*

Proof. This is a straightforward application of the monodromy principle (the "uniform position lemma") of [7]; in the present circumstances this states that as D varies, the monodromy/Galois group $\mathcal{M}$ acts on the points p_i as the full symmetric group. The point is, if for general D we have a relation

$$a_1 p_1 + \ldots + a_{mn} p_{mn} \equiv \eta$$

for some $\eta \in \Sigma$, then the same relation must hold when the p_i are permuted by the monodromy group $\mathcal{M}$. Since $\mathcal{M}$ contains all simple transpositions, we conclude that

$$(a_j - a_i)(p_j - p_i) = 0$$

in $\mathrm{Pic}^0(C)$; and since a general point of C does not differ from any other point of C by a torsion class in $\mathrm{Pic}(C)$, we conclude that $a_j = a_i$; i.e., $a_1 = \ldots = a_{mn} = a$ and $\eta = \mathcal{O}(na)$. Q.E.D. for sublemma + lemma

Observe that the proof of the lemma allows us to identify a pair of generators for $\mathrm{Pic}(\tilde{X}_0)$: one, which we may call $\mathcal{O}(1)$, is just the line bundle $\mathcal{O}(1)$ on both components $\tilde{P}$ and $\tilde{T}$; the other, which we call M, arising from the kernel of r_2, is trivial on $\tilde{T}$ and isomorphic to $\mathcal{O}(d)(-E_1 - \ldots - E_{d(d-1)}) \simeq \mathcal{O}_{\tilde{P}}(1) \otimes \mathcal{O}_{\tilde{P}}(\tilde{C})$ on $\tilde{P}$. Note that M is represented by an effective divisor, namely the proper transform $\tilde{Y}_0$ in $\tilde{P}$ of the intersection Y_0 of P with U in $\mathbb{P}^3$. Finally, we point out that the bundle N, defined to be the restriction

$$N = \mathcal{O}_{\tilde{X}}(\tilde{P}) \otimes \mathcal{O}_{\tilde{X}_0}$$

to $\tilde{X}_0$ of the bundle $\mathcal{O}_{\tilde{X}}(\tilde{P})$, is isomorphic to $\mathcal{O}(\tilde{C})$ on $\tilde{T}$ and to $\mathcal{O}(-\tilde{C})$ on $\tilde{P}$, so that

$$N = \mathcal{O}_{\tilde{X}_0}(1) \otimes M^{-1}.$$

Thus, we can interpret the Lemma as saying that, modulo the ambiguity introduced by the reducibility of $\tilde{X}_0 = \tilde{P} \cup \tilde{T}$, every line bundle on $\tilde{X}_0$ is $\mathcal{O}(n)$ for some n. Nor is this ambiguity just a technical matter: limits Y_0 on X_0 of complete intersection curves Y_t on X_t need not be complete intersections on X_0. For example, if $Y_t = P \cap X_t$ for $t \neq 0$, the "limiting position" of the (constant) curve Y_t is the curve $Y_0 = U \cap P$ above, which is the divisor of a section of M. To put it another way, the equation $L(X)$ defines a section of the bundle $\mathcal{O}_{\tilde{X}}(1)$, whose divisor Y' contains the component $\tilde{P}$ of $\tilde{X}_0$. Removing the component $\tilde{P}$ from Y', we get a divisor $Y = Y' - P$ whose associated line bundle

$$\mathcal{O}_{\tilde{X}}(Y) = \mathcal{O}_{\tilde{X}}(1) \otimes \mathcal{O}_{\tilde{X}}(-P)$$

restricts to $\mathcal{O}_{\tilde{X}_0}(1) \otimes N^{-1} = M$ on $\tilde{X}_0$. Similarly, the equation $F(X)$ gives a section of $\mathcal{O}_{\tilde{X}}(d-1)$ whose divisor Z' contains $\tilde{T}$; letting $Z = Z' - \tilde{T}$, we get a divisor on $\tilde{X}$ meeting X_t in the curve $X_t \cap T$ for $t \neq 0$ and meeting $\tilde{X}_0$ in a divisor with associated line bundle $\mathcal{O}_{\tilde{X}}(d-1)(-\tilde{T}) \otimes \mathcal{O}_{\tilde{X}_0}(d-1) \otimes N$.

To conclude our argument, recall that we have a family of curves $W_t \subset X_t$ for $t \neq 0$, corresponding to a line bundle L on $\tilde{X} - \tilde{X}_0 = X^*$, and a section $\sigma \in \Gamma(X^*, L)$. We can extend L to a line bundle on all of $\tilde{X}$, and (possibly after multiplying σ by a power of t) extend σ to a holomorphic section of L over $\tilde{X}$. The divisor W of σ may then contain a component – say $\tilde{P}$ – of $\tilde{X}_0$; if so, we can replace L by $L \otimes \mathcal{O}_{\tilde{X}}(-m\tilde{P})$ for some m and thereby insure that $\sigma | \tilde{X}_0$ vanishes only on a curve $\tilde{W}_0 \subset \tilde{X}_0$. The image W_0 of $\tilde{W}_0$ in $\mathbb{P}^3$ will be, of course, the limiting position of the curves W_t.

The point is that we can, by the lemma above, identify the line bundle $L_0 = L \otimes \mathcal{O}_{\tilde{X}_0}$ on $\tilde{X}_0$ associated to $\tilde{C}_0$. A priori we have

$$L_0 \cong \mathcal{O}_{\tilde{X}_0}(a) \otimes N^b$$

for some a and b. Suppose $b \geqq 0$. Then consider the divisor $V = W + bY$ on $\tilde{X}$ where Y is as introduced above. We have

$$\mathcal{O}_{\tilde{X}}(W + bY) \otimes \mathcal{O}_{\tilde{X}_0} \cong \mathcal{O}_{\tilde{X}_0}(a+b)$$

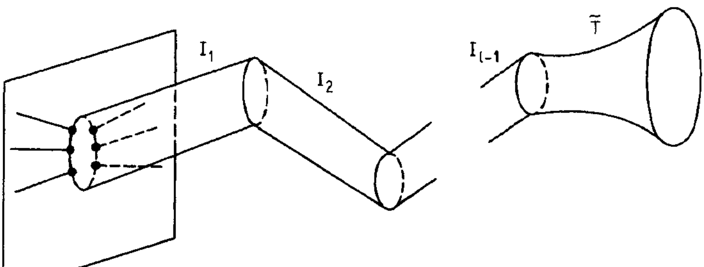
so that the curve V_0 is a complete intersection of X_0 with a surface in $\mathbb{P}^3$; and hence so is $V_t = W_t + b \cdot Y_t$ for $t \neq 0$. But Y_t is already the complete intersection of X_t with a plane; and by Noether's $AF+BG$ theorem it follows that W_t must likewise be a complete intersection with X_t.

Similarly, if $b \leqq 0$, let $V = W - bZ$ on $\tilde{X}$. Here,

$$\mathcal{O}_X(V) \otimes \mathcal{O}_{\tilde{X}_0} \cong \mathcal{O}_{\tilde{X}_0}(a - b(d-1))$$

so again V_0, and hence V_t for general t, is a complete intersection. Finally, since $Z_t = X_t \cap T$ is a complete intersection with X_t, it follows that W_t is too.

We have seen that if we have a family of curves $W_t \subset X_t$, the W_t must all be complete intersections. To complete the proof of the theorem, we have to deal with the possibility that each X_t may have divisor classes other than $\mathcal{O}(n)$, but none rationally defined over Δ. In this case we have to make a base change, pulling back the family $\tilde{X} \to \Delta$ via the map $\Delta \to \Delta$ given by $t \to t^l$, before we an assume that we have a family of non-complete intersection curves $W_t \subset X_t$. This base change introduces singularities in $\tilde{X}$ exactly along the double locus $\tilde{C}$ of the central fiber $\tilde{X}_0$; after resolving these singularities the central fiber of our new family looks like



$l-1$ intermediate components I_α

The proof of this assertion will be deferred to the following section.

The rest of the argument is exactly analogous to the above. To begin with, since modulo $\mathcal{O}_{I_\alpha}(C_\alpha)$, every divisor class on the ruled surface I_α has the same restriction to $C_{\alpha-1}$ and C_α, we see as before that $\mathrm{Pic}(\tilde{X}_0) = Z^{l+1}$, the generators being $\mathcal{O}(1)$, $N_0 = \mathcal{O}_{\tilde{X}}(\tilde{P}) \otimes \mathcal{O}_{X_0}$, and $N_\alpha = \mathcal{O}_{\tilde{X}}(I_\alpha) \otimes \mathcal{O}_{\tilde{X}_0}$ for $\alpha = 1, \ldots, l-1$.

Again, these "extra" divisor classes I_α do arise naturally: for example, if $M(X)$ is a general linear functional, the section $\sigma_\alpha = L(X) - t^\alpha M(X)$ of $\mathcal{O}_{\tilde{X}}(1)$ has divisor

$$(\sigma_\alpha) = Y_\alpha + \alpha\tilde{P} + \alpha I_1 + \ldots + \alpha I_{l-\alpha} + (\alpha - 1) I_{l-\alpha+1} + \ldots + I_{l-1},$$

where Y_α meets X_0 in a curve; thus Y_α represents a family of divisors $(Y_\alpha)_t \in |\mathcal{O}_{X_t}(1)|$ tending to a divisor $(Y_\alpha)_0 \in |\mathcal{O}_{X_0}(1) \otimes N_0^{-\alpha} \otimes \ldots \otimes N_{l-1}^{-1}|$. Similarly, if $H(X)$ is a general polynomial of degree $d-1$, the section $\tau_\alpha = F(X) - t^\alpha H(X)$ of $\mathcal{O}_{\tilde{X}}(1)$ has divisor

$$(\tau_\alpha) = Z_\alpha + I_1 + 2I_2 + \ldots + \alpha I_\alpha + \alpha I_{\alpha+1} + \ldots + \alpha I_{l-1}$$

where Z_α meets X_0 properly. Again, the proof is deferred to Sect. 2c.

The result, in any event, is the same: if $\{W_t\}$ is any family of curves on X_t, then by adding a suitable combination of the divisors Y_α and Z_α to W we arrive at a divisor V on $\tilde{X}$ meeting $\tilde{X}_0$ properly and with $\mathcal{O}_{\tilde{X}}(V) \otimes \mathcal{O}_{\tilde{X}_0} \cong \mathcal{O}_{\tilde{X}_0}(n)$ for some $n \ldots$, i.e., with $V_0 = V \cap X_0$ a complete intersection. It follows then that the nearby $V_t \subset X_t$ must likewise be complete intersections; and since V_t differs from W_t only by the addition of complete intersections with X_t, it follows by Noether's $AF+BG$ theorem that W_t must likewise be a complete intersection.

c) "Appendix": Applying Base Change to $\tilde{X}$

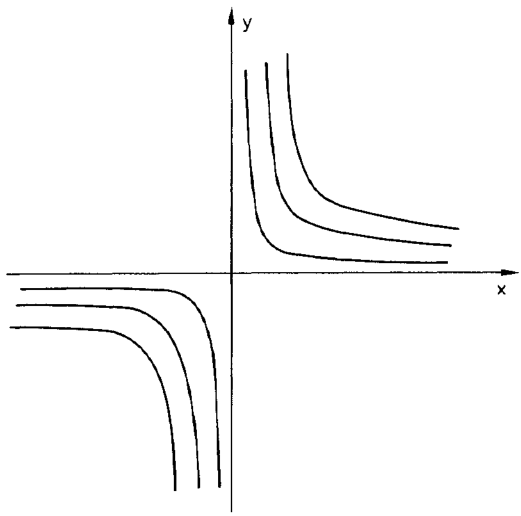
The purpose of this appendix is to verify a couple of the statements made in the course of the argument above, namely that when we apply a base change $t \mapsto t^l$ to the 3-fold $\tilde{X} \to \Delta$ constructed above (that is, take the fibre product $\tilde{X} x_\omega \Delta$ where $\omega: \Delta \to \Delta$ sends t to t^l and then minimally resolve the resulting singularities),

i) the resulting family over Δ has central fiber as pictured above; and

ii) the divisors of the pullbacks of the functions $L(X) - t^\alpha M(X)$ and $F(X) - t^\alpha H(X)$ are as stated above.

To do both, we observe that it is sufficient to look at a neighborhood of a general point p of the double curve $\tilde{C}$ of $\tilde{X}_0 \subset \tilde{X}$, in a normal slice of $\tilde{C}$ in $\tilde{X}$. Thus we will look simply at a surface, given by $xy - t$, and apply a base change (we may assume it has even order $2l$) to arrive at a surface S with equation $xy - t^{2l}$, and its minimal resolution $\tilde{S}$.

To see accurately the picture of the resolution of the surface $xy = t^{2l}$,



it is helpful to make a change of variables

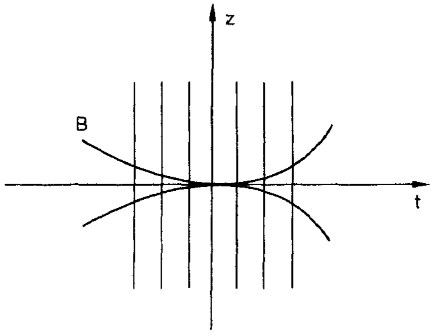
$$x = z + w$$

$$y = z - w$$

so that the equation of the surface becomes

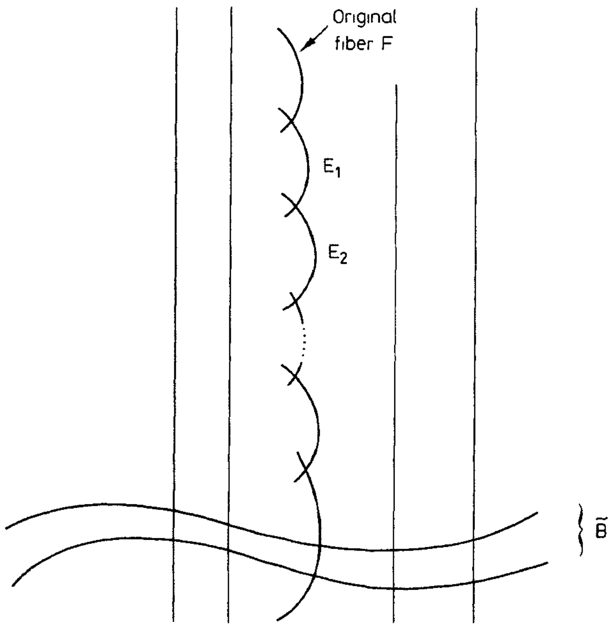
$$w^2 = -t^{2l} + z^2 .$$

This we can think of as the double cover $S \to \mathbb{A}^2$ of the (z, t)-plane, branched along the curve B given by $-t^{2l} + z^2 = 0$:



Of course, when we take the double cover of a smooth surface, singularities appear exactly over the singularities of the branch curve; so what we have to do here is to resolve B in the (z, t)-plane, and *then* take the double cover.

We resolve B by blowing up. Each time we do, the exponent of t' in the new coordinate system $t' = t$, $z' = z/t$ is reduced by 2, so that after l blow-ups we have the picture

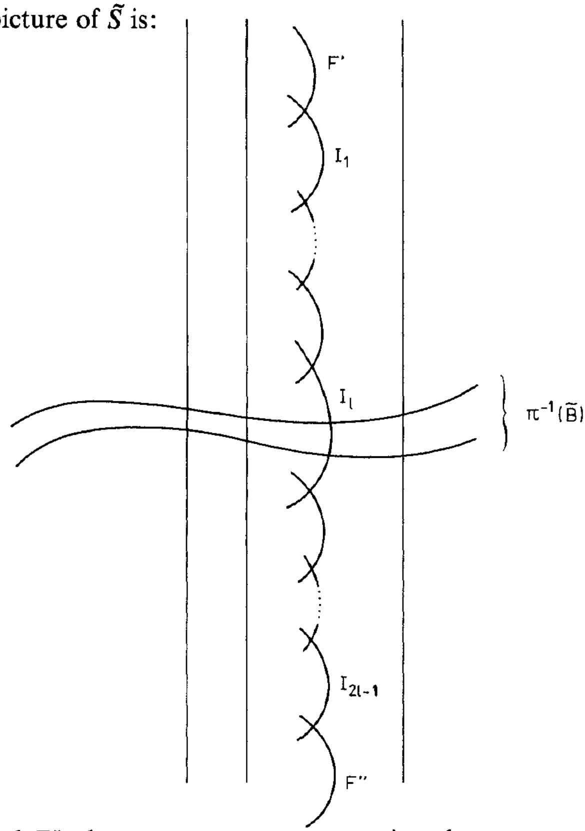


Note that since all the exceptional divisors E_i have *even* multiplicity in the total transform of B, they do not appear in the branch locus of the double cover $\tilde{S} = S \times_{\mathbb{A}^2} \tilde{\mathbb{A}}^2$ after we normalize. Thus the branch curve $\tilde{B}$ of our new cover $\tilde{S} \to \tilde{\mathbb{A}}^2$ is smooth, and so $\tilde{S}$ is. Moreover, we can describe exactly what happens in the central fiber $t = 0$:

i) E_l is doubly covered by a single, irreducible $\mathbb{P}^1$ (call it I_l), branched over the two points of $E_l \cap \tilde{B}$;

ii) all the other exceptional divisors E_i and the original fiber F, since they do not meet the branch locus, are each covered by 2 disjoint copies of themselves.

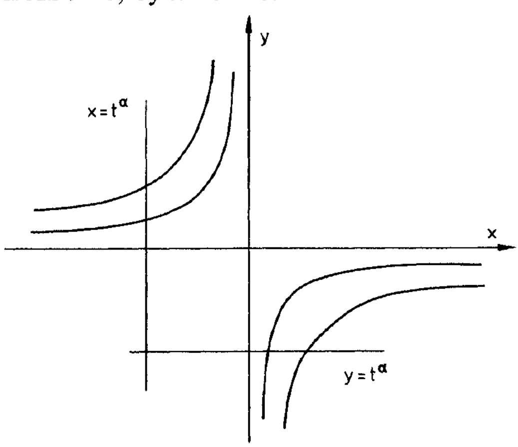
Thus the picture of $\tilde{S}$ is:



Here F' and F'', the two components covering the proper transform of the original fiber $t=0$ in $\mathbb{A}^2$, correspond to the two original curves $y=0$, $x=0$ in X, and the curves I_α, $I_{2l-\alpha}$ covering E_α form a simple chain of length $2l-1$ connecting them.

This justifies the picture of the resolution of our threefold $\tilde{X}$ after the base change.

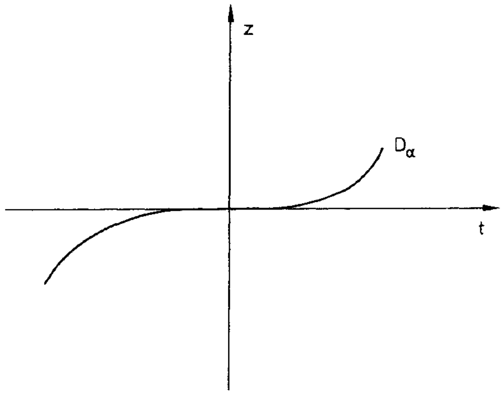
We now want to consider various curves in $\tilde{S}$, and their divisor classes; specifically, we want to look at the curves C_α, defined to be the closure in $\tilde{S}$ of the curve given, away from $t=0$, by $x-t^\alpha=0$.



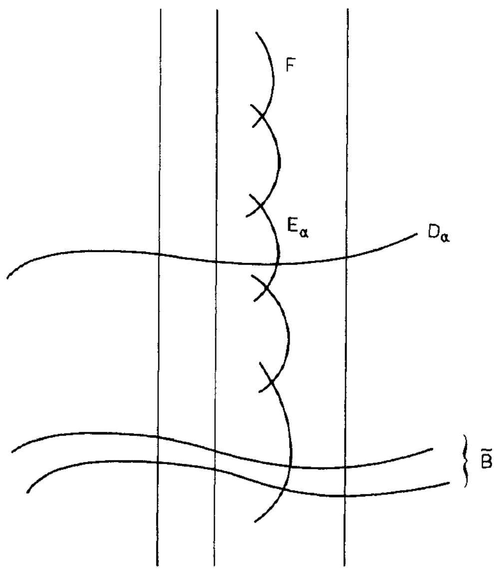
To see what these look like, we pass as before to the picture of $\tilde{S}$ as a double cover of the blown-up (z, t)-plane. In the (z, t)-plane, the image D_α of the curve C_α is given by

$$2z = t^\alpha + t^{2l-\alpha}$$

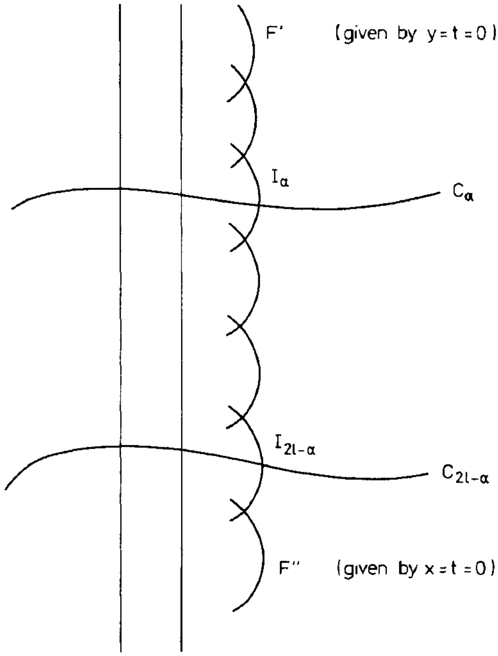
(note that the other component of the inverse image of this curve is just $C_{2l-\alpha}$)



which is separated from B after α blow-ups. Thus in $\tilde{\mathbb{A}}^2$ we have the picture



and in the double cover, the inverse image of D_α decomposes into C_α and $C_{2l-\alpha}$, which meet the components I_α and $I_{2l-\alpha}$ respectively:



Now, we can use this picture to determine the divisors on $\tilde{S}$ of the pullbacks of the functions $x-t^\alpha$ and $y-t^\alpha$ on S. Bearing in mind that the pullback to $\tilde{S}$ of any line bundle on S must be trivial on I_α – in other words, the pullback to $\tilde{S}$ of a Cartier divisor on $\tilde{S}$ will have intersection number 0 with I_α – we see that when we write

$$(x-t^\alpha)=C_\alpha+a_0F'+a_1I_1+\ldots+a_{2l-1}I_{2l-1}+a_{2l}F''$$

the coefficients a_i must satisfy

$$a_{i-1}+a_{i+1}+(C_\alpha\cdot I_i)=2a_i\,.$$

Using this, the fact that $(C_\alpha\cdot I_i)=\delta_{\alpha,i}$, and the observed fact that

$$\mathrm{mult}_{F'}(x-t^\alpha)=0$$

$$\mathrm{mult}_{F''}(x-t^\alpha)=\alpha$$

[near F', x and t are local coordinates with $F'=(t)$; near F'', y and t are local coordinates with $F'=(t)$ and $(x)=2l\cdot F'$], we can solve to find that

$$(x-t^\alpha)=C_\alpha+I_1+2I_2+3I_3+\ldots+\alpha I_\alpha+\alpha I_{\alpha+1}+\ldots+\alpha I_{2l-1}+\alpha F''\,.$$

and similarly

$$(y-t^\alpha)=C_{2l-\alpha}+\alpha F'+\alpha I_1+\ldots+\alpha I_{2l-\alpha}+(\alpha-1)I_{2l-a+1}+\ldots+I_{2l-1}\,.$$

Observe, as a check, that the total transform of the divisor $D_\alpha=(2z-t^\alpha-t^{2l-\alpha})\subset\mathbb{A}^2$ in $\tilde{\mathbb{A}}^2$ is linearly equivalent to

$$\tilde{D}_\alpha+E_1+2E2+3E_3+\ldots+\alpha E_\alpha+\alpha E_{\alpha+1}+\ldots+\alpha E_l$$

so that in the double cover $\tilde{X}$ we have

$$\begin{aligned}(2z-t^{\alpha}-t^{2l-\alpha})=(C_{\alpha}+C_{2l-\alpha}+I_1+2I_2+\ldots\\ \ldots+\alpha I_{\alpha}+\alpha I_{\alpha+1}+\ldots+\alpha I_{2l-\alpha}+(\alpha-1)I_{2l-\alpha+1}+\ldots\\ \ldots+I_{2l-1};\end{aligned}$$

on the other hand, we have

$$\begin{aligned}(x-t^{\alpha})+(y-t^{\alpha})&=(xy-xt^{\alpha}-yt^{\alpha}+t^{2\alpha})\\ &=(t^{2l}-xt^{\alpha}-yt^{\alpha}+t^{2\alpha})\\ &=(t^{\alpha})+(x+y-t^{\alpha}-t^{2l-\alpha})\\ &=\alpha\cdot(t)+(2z-t^{\alpha}-t^{2l-\alpha})\end{aligned}$$

which agrees with our computation of $(x-t^{\alpha})$.

Returning to our threefold $\tilde{X}$, we recall that at a general point of the double curve $\tilde{C}$ of $\tilde{X}_0$, the map $\tilde{\pi}:\tilde{X}\to\Delta$ has local equation $xy=t$, where x is a local equation for $\tilde{P}$ and y a local equation for $\tilde{T}$. Thus, if we make a base change of order $2l$ and resolve the resulting singularities as indicated, the pullback of the function

$$L(X)-t^{\alpha}M(X)$$

[where $M(X)$ is a general linear function] by the above computation will have divisor

$$\begin{aligned}(L-t^{\alpha}M)=Y_{\alpha}+\alpha\tilde{P}+\alpha I_1+\ldots+\alpha I_{2l-\alpha}\\ +(\alpha-1)I_{2l-\alpha+1}+\ldots+I_{2l-1}\end{aligned}$$

where the divisor Y_{α} meets X_0 in a curve $(Y_{\alpha})_0$; and similarly for $F-t^{\alpha}H$.

d) Applying this Argument to Threefolds

It would seem natural, in considering the questions raised in the first section about curves on a general threefold, to try and mimic the proof of the Noether-Lefschetz theorem just given, using the second Chow cohomology group A^2 instead of Pic or A^1. The problem here is that many of the basic properties of line bundles/divisor classes – the ones that make it so convenient to deal with the Picard group – are not known to hold for higher Chow cohomology groups. We mention here five of these properties.

1) At the outset of our argument, we used the fact that if we have a family $X\xrightarrow{\pi}\Delta$ of varieties with X smooth, and a line bundle L_t on X_t for $t\neq 0$ varying holomorphically – that is, a class α in $A^1(X-X_0)$ – then L_t would have as a limit an honest line bundle L_0 on X_0 – that is, α could be extended to a class in $A^1(X)$, and then of course restricted to give a class $\alpha_0\in A^1(X_0)$. Is this true for the higher Chow cohomology groups, in particular A^2?

2) In computing $\mathrm{Pic}(X_0)$ in our argument, we started with the observation that if a variety X is the union of two irreducible components Y and Z, then to give a line bundle on X one just had to give a line bundle on each of Y and Z, together with an isomorphism of their restriction to the (scheme-theoretic) intersection

$Y \cap Z$. Thus, if $Y \cap Z$ is connected and reduced, we have a fiber square

$$\begin{tikzcd}
 & A^1 Y \arrow[dl] & \\
A^1(Y \cap Z) & & A^1(Y \cap Z) \arrow[ul] \arrow[dl] \\
 & A^1(z) \arrow[ul] &
\end{tikzcd}$$

Is any such "Mayer-Vietoris" statement true for A^k in general?

3) Given this fact about Pic, our computation of $\mathrm{Pic}(X_0)$ then rested on two facts relating the Picard group $A^1(X)$ of a variety to a general hyperplane sections $Y \subset X$: we had

i) the restriction map $A^1(X) \to A^1(Y)$ is injective; and

ii) the push-forward map $A^0(Y) \to A^1(X)$ (defined if X is smooth) is also injective (the sublemma above).

Are the analogues of these statements true, specifically for the map $A^2(X) \to A^2(Y)$ if X is a general threefold in $\mathbb{P}^4$, and the map $A^1(Y) \to A^2(X)$ if X is a general surface in $\mathbb{P}^3$?

4) Finally, to conclude the Noether-Lefschetz theorem from the computation of $\mathrm{Pic}(X_0)$, we use a variant of the upper-semi-continuity of the Picard number – that is, in a simple case, the statement that if $\{X_t\}$ is a family of smooth surfaces and $A^1(X_0) = \mathbb{Z}$, then $A^1(X_t) = \mathbb{Z}$ for general small t. Is such a statement true, for example, for $A^2(X_t)$ in a family $\{X_t\}$ of smooth threefolds?

3. Hodge-Theoretic Considerations for Codimension-two Algebraic Cycles on Hypersurfaces in $\mathbb{P}^n$ for $n = 4, 5$

a) Remarks on Normal Functions Depending Algebraically on $X \subset \mathbb{P}^4$

For a smooth threefold X we denote by

$$J(X) = F^2 H^3(X, \mathbb{C})^* / H_3(X, \mathbb{Z})$$

the middle intermediate Jacobian, by $z_h^2(X)$ the algebraic 1-cycles on X that are homologous to zero, and by

$$u : z_h^2(X) \to J(X)$$

the Abel-Jacobi mapping.

Suppose now that $X \subset \mathbb{P}^4$ is a smooth hypersurface of degree d. For any algebraic curve $C \subset X$ we shall define

$$u(C) \in J(X)/(\text{subgroup of } d\text{-torsion points}).$$

Let $\Gamma = X \cap \mathbb{P}^3 \cap \mathbb{P}'^3$ be a general complete intersection and recall that

$$H_2(X, \mathbb{Z}) \cong H_2(\mathbb{P}^4, \mathbb{Z}) \cong \mathbb{Z}.$$

If $\deg C = md$ for some integer m, then

$$C - m\Gamma \in z_h^2(X) \tag{3.1}$$

and we set

$$u(C) = u(C - m\Gamma).$$

In general we will have (3.1) for some $m = p/d \in \mathbb{Q}$ and we set

$$u(C) = \frac{1}{d} u(qC - p\Gamma). \tag{3.2}$$

Since as a group

$$J(X) \cong \mathbb{R}^k/\mathbb{Z}^k,$$

the right hand side of (3.2) is a well-defined element of $\mathbb{R}^k \Big/ \left(\frac{1}{d}\mathbb{Z}\right)^k$ $\cong J(X)/(d\text{-torsion})$.

It is general yoga that:

If $X \subset \mathbb{P}^4$ is a hypersurface with general moduli of degree $d \geqq 3$ and $C \subset X$ is an "interesting" curve (e.g., C is not a complete intersection $S \cap X$ of X with any surface $S \subset \mathbb{P}^4$), then

$$u(C) \neq 0 \tag{3.3}$$

Examples of this yoga abound – cf. [3].

Motivated by the questions 2)–4) of the introduction, we shall consider the following variant of 5):

(3.4) *For an $X \subset \mathbb{P}^4$ of general moduli and of degree $d \geqq 6$, does there exist a non-torsion point*

$$u(X) \in J(X)$$

depending algebraically on X?

More precisely, by a *normal function depending algebraically on X* we shall mean that we are given a variety S together with a dominant equidimensional mapping

$$S \to \mathbb{P}H^0(\mathbb{P}^4, \mathcal{O}(d)),$$

denoted by

$$t \to X_t,$$

and a holomorphic cross-section

$$u(t) \in J(X_t)$$

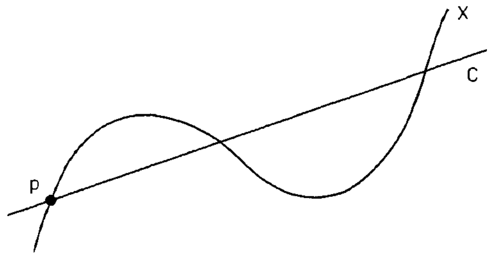
of the family of the intermediate Jacobians; we assume that $u(t)$ satisfies the two additional technical conditions (quasi-horizontality and moderate growth at infinity) required in the definition of a normal function [4].

It is our feeling that the answer to (3.4) is *no*. To motivate this feeling we shall give a few remarks.

i) The corresponding question for a general smooth curve $X \subset \mathbb{P}^2$ has an affirmative answer. For example, let $C \subset \mathbb{P}^2$ be any curve; write

$$C \cdot X = p + D$$

where $p \in X$ depends algebraically on X



and set

$$u(X) = u(dp - H)$$

where $H \in \mathrm{Div}^d(X)$ is a hyperplane section.

ii) This construction fails for $X \subset \mathbb{P}^4$ with an irreducible surface $S \subset \mathbb{P}^4$ replacing $C \subset \mathbb{P}^2$, since a general hypersurface section $X \cdot S$ will be an irreducible curve.

iii) A special case of (3.4) is when $u(X) \in J(X)$ depends *rationally* on X; i.e., when $S \subset \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^4}(d))$ is a Zariski open set. We shall show that:

There is no non-zero normal function depending rationally on X if $\deg X \geqq 3$.

Proof. Let $|X_t|$ be a general pencil with base locus $B = X_0 \cdot X_\infty$ a smooth surface of degree d^2 in $\mathbb{P}^4$. By the Noether theorem, the Picard number $\varrho(B) = 1$ if $d \geqq 3$ and X_0, X_∞ are chosen generically. Let

$$\tilde{P} = \text{blow up of } \mathbb{P}^4 \text{ along } B.$$

Then we have a diagram

$$\begin{array}{ccc} \tilde{P} & \longrightarrow & \mathbb{P}^4, \\ {\scriptstyle\pi}\downarrow & & \\ \mathbb{P}^1 & & , \quad \pi^{-1}(t) = X_t \end{array}$$

and by the above remark

$$\begin{aligned} H^{2,2}_{\mathbb{Z}}(\tilde{P}) &\cong H^{2,2}_{\mathbb{Z}}(P) \oplus H^{1,1}_{\mathbb{Z}}(B), \qquad P = \mathbb{P}^4, \\ &\cong \mathbb{Z} \oplus \mathbb{Z}. \end{aligned}$$

On the other hand, if we have a normal function

$$u(t) \in J(X_t),$$

then its fundamental class is a primitive Hodge class

$$\lambda \in \ker\{H^{2,2}_{\mathbb{Z}}(\tilde{P}) \to H^{2,2}(X)\}/\mathrm{image}\{H^{1,1}_{\mathbb{Z}}(X) \xrightarrow{\gamma} H^{2,2}_{\mathbb{Z}}(\tilde{P})\}$$

where X is a general X_t and γ is the Gysin mapping (cf. [9]). Moreover, $\lambda \neq 0$ if $u \neq 0$. Since, by construction

$$H_{\mathbb{Z}}^{1,1}(X) \xrightarrow{\sim} H_{\mathbb{Z}}^{1,1}(B)$$

is an isomorphism, it follows that $\lambda = 0$, and consequently also $u = 0$. Q.E.D.

b) A Complement to [5]

We consider the situation

$$C_0 \subset S_0 \subset \mathbb{P}^3$$

where C_0 is a smooth curve of genus g and degree d, S_0 is a smooth surface of degree $m \geqq m_0(g, d)$, and we set

$$\begin{cases} \gamma = \text{fundamental class of } C_0 \\ \omega = c_1(\mathcal{O}_{S_0}(1)). \end{cases}$$

Then

$$\begin{cases} \gamma \in H_{\mathbb{Z}}^{1,1}(S_0) \text{ is a Hodge class} \\ \gamma \cdot \omega = d \\ \gamma^2 = 2g - 2 - (m-4)d. \end{cases} \tag{3.5}$$

Now suppose that $S \subset \mathbb{P}^3$ is any smooth surface of degree m and $\gamma \in H^2(S, \mathbb{Z})$ is any smooth surface of degree m and $\gamma \in H^2(S, \mathbb{Z})$ satisfies (3.5). Then in [5] it was shown that γ is the fundamental class of a curve C as above. Thus, not only is γ an algebraic cycle [which we know anyway by the Lefschetz (1, 1) theorem], it is actually effective. This is in contrast to the usual way of making a class effective, which is to fix m and replace γ by $\gamma + n\omega$ for large n.)

An obvious question is whether the analogous result remains true for the situation

$$S^2 \subset X^4 \subset \mathbb{P}^5$$

(the superscripts denote dimensions) where $\deg X = m \geqq m_0$ (numerical invariants of S)? Of course, we don't know the answer; however, we can prove the corresponding variational result.

Theorem. *Let $\sigma \in H^4(X, \mathbb{Z})$ be the fundamental class of S, and suppose that $\{\dot{X}_\varepsilon\}$, $\varepsilon^2 = 0$, is an infinitesimal variation of X under which σ remains of type* (2, 2). *Then there exists a variation $\{S_\varepsilon \subset X_\varepsilon\}$ of $S \subset X$.*

Remark. Although we have not checked the details, it seems likely that the analogous result is true for any situation

$$Z^n \subset X^{2n} \subset Y^{2n+1}$$

where X is sufficiently ample relative to Z and Y.

Proof. From the normal bundle sequences

$$0 \to N_{X/P} \otimes \mathcal{I}_S \to N_{X/P} \to N_{X/P} \otimes \mathcal{O}_S \to 0$$

$$0 \to N_{S/X} \to N_{S/P} \to N_{X/P} \otimes \mathcal{O}_S \to 0,$$

where $P=\mathbb{P}^5$ and $\mathcal{I}_S$ is the ideal sheaf of S, we obtain a commutative cohomology diagram

(3.6)
$$\begin{tikzcd}
H^3(\Omega_X^1) & H^1(N_{S/P}) \\
H^1(\Omega_X^3\otimes\mathcal{O}_S)^* \arrow[u, "\alpha"] & H^1(N_{S/X}) \arrow[l, "\beta"'] \arrow[u] \\
H^0(N_{X/P}) \arrow[r, "\gamma"] & H^0(N_{X/P}\otimes\mathcal{O}_S) \arrow[u, "\delta"] \\
& H^0(N_{S/P}). \arrow[u, "\lambda"]
\end{tikzcd}$$

Here, α is the dual of the restriction mapping

$$H^1(\Omega_X^3)\xrightarrow{\alpha^*}H^1(\Omega_X^3\otimes\mathcal{O}_S)$$

and β is the dual of the mapping

$$H^1(\Omega_X^3\otimes\mathcal{O}_S)\xrightarrow{\beta^*}H^1(N^*_{(S/X)}\otimes\Omega_S^2)$$

induced from the cohomology sequence of

(3.7) $$0\to\Lambda^2N^*_{S/X}\otimes\Omega_S^1\to\Omega_X^3\otimes\mathcal{O}_S\to N^*_{S/X}\otimes\Omega_S^2\to 0.$$

We note the interpretations (cf. Bloch [1])

$$\gamma^{-1}(\ker\delta)=\begin{cases}\text{infinitesimal deformations of}\\ X\subset\mathbb{P}^5\quad\text{under which}\\ S\text{ moves}\end{cases}$$

$$\ker(\alpha\circ\beta\circ\delta\circ\gamma)=\begin{cases}\text{infinitesimal deformations of}\\ X\subset\mathbb{P}^5\text{ such that }\sigma\text{ remains}\\ \text{of Hodge type }(2,2).\end{cases}$$

To prove our result we must show that these are the same subspaces of $H^0(N_{X/P})$, and this follows from the two assertions:

(3.8) *β is an isomorphism*

(3.9) *α is injective.*

Proof of (3.8). From the second normal bundle sequence and

$$N_{X/P}\otimes\mathcal{O}_S=\mathcal{O}_S(m)$$
$$\det N_{S/P}=\mathcal{O}_S(6)\otimes K_S$$

we infer that

$$\Lambda^2N^*_{S/X}=K_S^{-1}\otimes\mathcal{O}_S(m-6).$$

For $m \gg 0$ we then have

$$h^i(\Lambda^2 N^*_{S/X} \otimes \Omega^1_S) = 0 \qquad i = 1, 2,$$

and (3.8) follows from the exact cohomology sequence of (3.7).

Proof of (3.9). We will show that

$$H^1(\Omega^3_X) \xrightarrow{\alpha^*} H^1(\Omega^3_X \otimes \mathcal{O}_S)$$

is surjective. The dual of the normal bundle sequence of X in $\mathbb{P}^5$ plus the dual of the Euler sequence give a commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_X(-m) \arrow[r] & \Omega^1_P \otimes \mathcal{O}_X \arrow[r] \arrow[d] & \Omega^1_X \arrow[r] & 0 \\
 & & \overset{6}{\oplus}\, \mathcal{O}_X(-1) \arrow[d] & & \\
 & & \mathcal{O}_X \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

A piece of the cohomology diagram is [using $h^3(\mathcal{O}_X(-1)) = h^2(\mathcal{O}_X) = 0$]

$$0 \to H^3(\Omega^1_X) \to H^4(\mathcal{O}_X(-m)),$$

whose dual then gives [using $K_X = \mathcal{O}(m-6)$]

(3.10) $$H^0(\mathcal{O}_X(2m-6)) \xrightarrow{\varrho} H^1(\Omega^3_X) \to 0.$$

This mapping ϱ is the well-known representation of the cohomology of X by residues – cf. [2]. Next we have

$$H^1(\Omega^3_X \otimes \mathcal{O}_S)^* = H^1(\Theta_X \otimes K_X \otimes \mathcal{O}_S)^*$$

(3.11) $$= H^1(\Omega^1_X \otimes K_S(6-m)).$$

From

$$0 \to K_S(6-2m) \to \Omega^1_P(6-m) \otimes K_S \to \Omega^1_X(6-m) \otimes K_S \to 0$$

and $h^1(\Omega^1_P(6-m) \otimes K_S) = 0$ for $m \gg 0$, we obtain

$$0 \to H^1(\Omega^1_X(6-m) \otimes K_S) \to H^2(K_S(6-2m))$$

whose dual is, using (3.11),

(3.12) $$H^0(\mathcal{O}_S(2m-6)) \xrightarrow{\varrho'} H^1(\Omega^3_X \otimes \mathcal{O}_S) \to 0.$$

Clearly (3.10) and (3.12) fit into a commutative diagram, where r is the obvious restriction,

$$\begin{tikzcd} H^0(\mathcal{O}_X(2m-6)) \arrow[r, "\varrho"] \arrow[d, "r"] & H^1(\Omega_X^3) \arrow[r] \arrow[d, "\alpha^*"] & 0 \\ H^0(\mathcal{O}_S(2m-6)) \arrow[r, "\varrho"] & H^1(\Omega_X^3 \otimes \mathcal{O}_S) \arrow[r] & 0 \end{tikzcd},$$

and since r is surjective for $m \gg 0$ we obtain (3.9). Q.E.D.

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