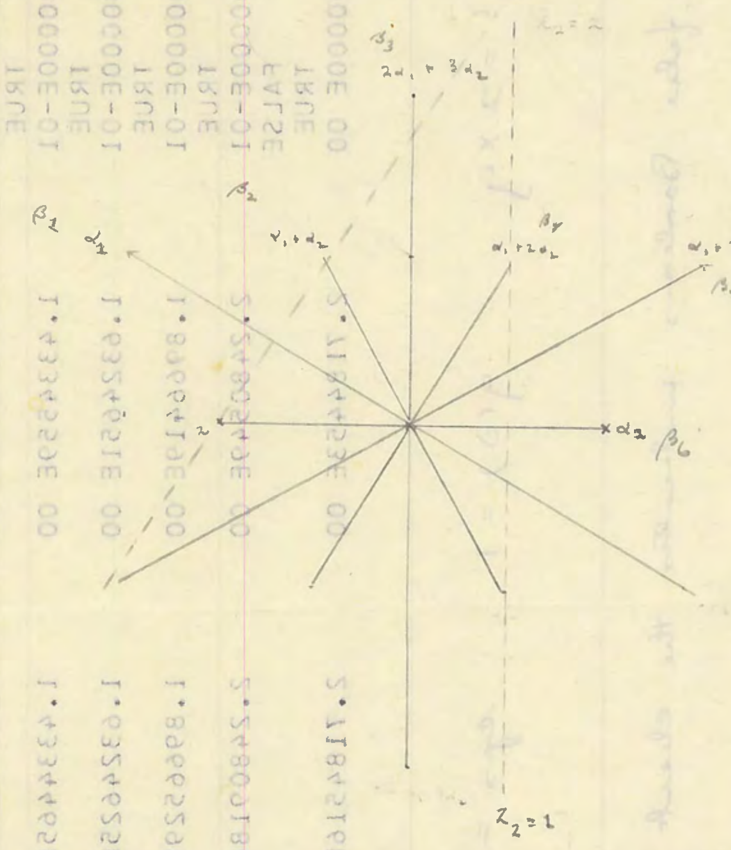
EXAMPLES IN THE THEORY OF EISENSTEIN SERIES FOR CHEVALLEY GROUPS.

1. The group G_2 over the field of rational numbers

The root diagram for G_2 is:



$\beta_1 = (-\frac{3}{2}, \frac{\sqrt{3}}{2})$

$\beta_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$

$\beta_3 = (0, \sqrt{3})$

$\beta_4 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

$\beta_5 = (\frac{3}{2}, \frac{\sqrt{3}}{2})$

$\beta_6 = (1, 0)$

As in the other notes on Eisenstein series the coordinates z_1, z_2 on the dual space to $\mathfrak{h}$ will be so chosen that $z_1(\lambda) = \lambda(H_{\alpha_1})$ and $z_2(\lambda) = \lambda(H_{\alpha_2})$. Since the determinant of the Cartan matrix is one the measure $|d\lambda|$ equals $dx_1\,dx_2$. Set,

$$\xi(t) = \pi^{-t/2}\,\Gamma(t/2)\,\zeta(t). \qquad a(t) = \frac{\xi(t)}{\xi(1+t)}$$

This notation is not the same as that used in the earlier notes. The notation used earlier was poorly chosen.

Of course in the case we are considering all ideals are just $\mathbb{Z}$ and the class number is one so

$$n(\sigma, \lambda) = \prod_{\substack{\alpha>0 \\ \sigma\alpha<0}} \frac{\xi(1-\lambda(H_\alpha))}{\xi(1+\lambda(H_\alpha))} = \prod_{\substack{\alpha>0 \\ \sigma\alpha<0}} \frac{\xi(\lambda(H_\alpha))}{\xi(1+\lambda(H_\alpha))}$$

Let us tabulate the elements of the Weyl group together with the positive roots they transform to negative roots. Let $\beta_1, \dots, \beta_6$ be an enumeration of the positive roots as in the diagram. Let 1 be the identity element of the group, let ρ_i be the reflection in the line $\langle\lambda, \beta_i\rangle = 0$ and let σ_θ be rotation through θ degrees in the positive direction. The possible values for θ are $\frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$

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(AVP)

σ	$\{\beta>0 \mid \sigma\beta<0\}$
1	
ρ_1	β_1
ρ_2	$\beta_1, \beta_2, \beta_3$
ρ_3	$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$
ρ_4	$\beta_2, \beta_3, \beta_4, \beta_5, \beta_6$
ρ_5	$\beta_4, \beta_5, \beta_6$
ρ_6	β_6
$\sigma_{\pi/3}$	β_1, β_2
$\sigma_{2\pi/3}$	$\beta_1, \beta_2, \beta_3, \beta_4$
σ_{π}	$\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$
$\sigma_{4\pi/3}$	$\beta_3, \beta_4, \beta_5, \beta_6$
$\sigma_{5\pi/3}$	β_5, β_6

~~As usual~~ the inner product of $\phi^1(\cdot)$ and $\psi^1(\cdot)$ is equal to

$$(\phi^1, \psi^1) = \sum_\sigma \frac{1}{(2\pi)^2} \int_{Re\lambda=\lambda_0} n(\sigma,\lambda)\, \Phi(\lambda)\, \overline{\Psi(-\sigma\bar\lambda)}\, dx_1 dx_2$$

If we replace this integral by an integral over the plane $Re\lambda = 0$ we have to add also the residues at the six complex lines $\lambda(H_{\beta_i}) = 1$, $1 \le i \le 6$. Thus (ϕ^1, ψ^1) is the sum of

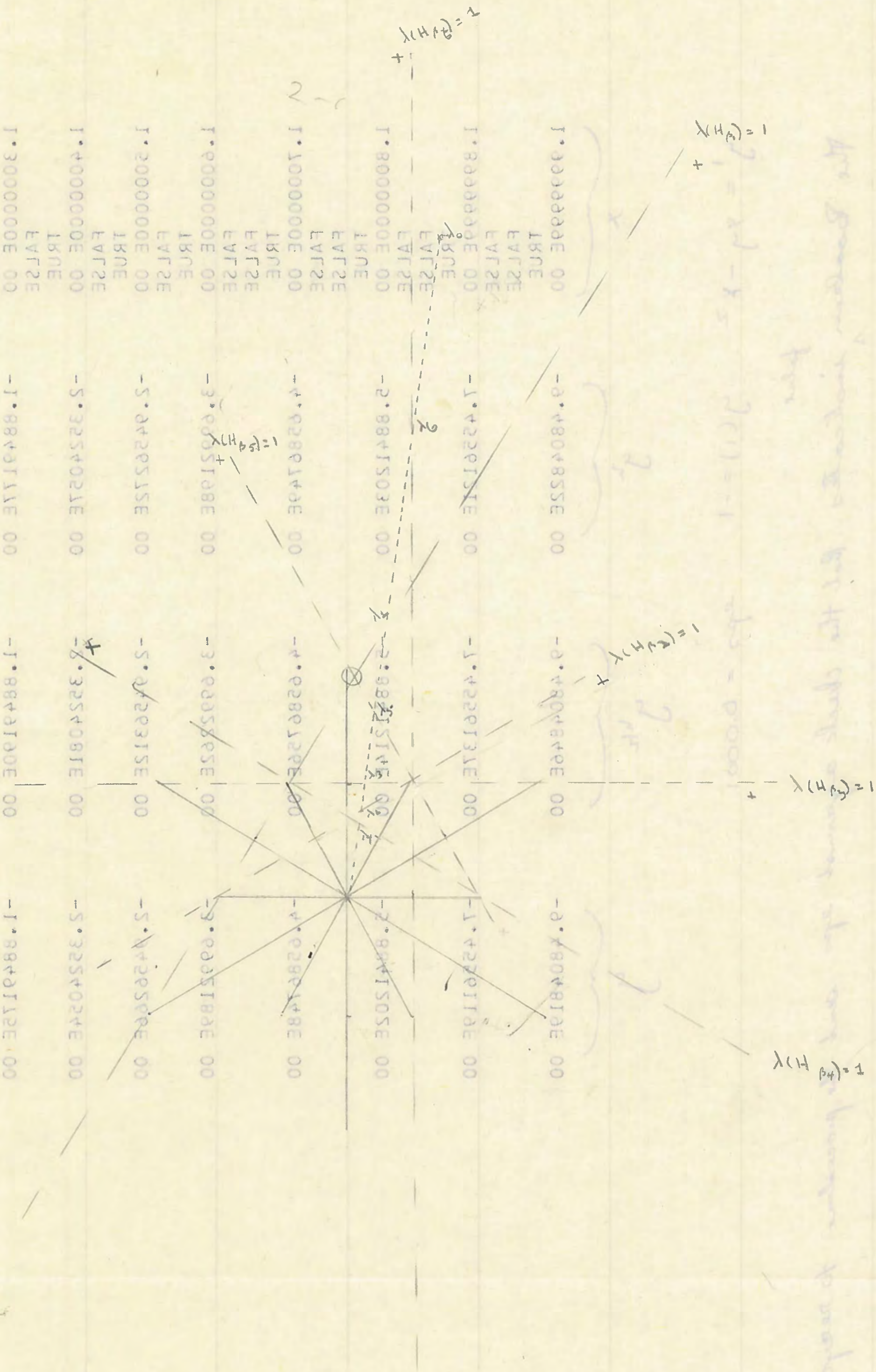
$$\frac{1}{(2\pi)^2} \int_{Re\lambda=0} n(\sigma,\lambda)\, \phi(\lambda)\, \overline{\psi(-\sigma\bar\lambda)}\, dx_1 dx_2.$$

and a remainder obtained by taking residues. Let us discuss this remainder more carefully. Denote the line $\lambda(H_{\beta_i})=1$ by $\underline{s}_i$. We have first to tabulate the sets $\Omega(s_i, s_j)$. The scheme of the table on p. 4 is this: In the first column of the square in the ith row and j^{th} column the elements of $\Omega(s_i, s_j)$ are listed. Opposite such an element $\bar\sigma$ are listed the elements of Ω whose restriction to $\underline{s}_i$ is σ and finally we give $n(\bar\sigma, \lambda)$. Of course $n(\bar\sigma,\lambda)$ depends on the coordinate and λ chosen on s_i. This will be:
hence on the basic one form

On s_1 the restriction of $\frac{3}{2} + z_2 = z$
" s_2 " " " $(\frac{3}{2} + \lambda(H_{\beta_2}) = z)$
" s_3 " " " $\frac{3}{2} - \lambda(H_{\beta_2}) = z$

On s_4 the restriction of $\frac{1}{2} - \lambda(H_{\beta_5}) = z$
" s_5 " " " $\frac{3}{2} - \lambda(H_{\beta_6}) = z$
" s_6 " " " $\frac{1}{2} + z_1 = z$

$\beta_4 (H_{\beta_4})$



In the following table use is made of the relations

(i) $H_{\beta_1} \simeq (-1, \frac{1}{\sqrt{3}}) \simeq H_{\alpha_1}$ $\qquad$ $H_{\beta_4} \simeq (1, \sqrt{3}) \simeq 3H_{\alpha_1} + 2H_{\alpha_2}$

$H_{\beta_2} \simeq (-1, \sqrt{3}) \simeq 3H_{\alpha_1} + H_{\alpha_2}$ $\qquad$ $H_{\beta_5} \simeq (1, \frac{1}{\sqrt{3}}) \simeq H_{\alpha_1} + H_{\alpha_2}$

$H_{\beta_3} \simeq (0, \frac{2}{\sqrt{3}}) \simeq 2H_{\alpha_1} + H_{\alpha_2}$ $\qquad$ $H_{\beta_6} \simeq (2, 0) = H_{\alpha_2}$

	1	2	3	4	5	6
1	ρ^+, ρ_1: $\frac{1}{\xi(2)}$ ρ^-, $\sigma_\mp$: $\frac{1}{\xi(2)} a(z-\tfrac32)a(z-\tfrac12)a(z+\tfrac12)a(z+\tfrac32)a(2z)$ $= \frac{1}{\xi(2)} \frac{\xi(z-3/2)\,\xi(2z)}{\xi(z+5/2)\,\xi(2z+1)}$		σ^+, ρ_2: $\frac{1}{\xi(2)} a(z+1/2)\,a(z+3/2)$ $= \frac{1}{\xi(2)} \frac{\xi(z+1/2)}{\xi(z+5/2)}$ σ^-, $\sigma_{2\pi/3}$: $\frac{1}{\xi(2)} a(z+1/2)\,a(z+\tfrac32)\,a(2z)$ $= \frac{1}{\xi(2)} \frac{\xi(z+1/2)}{\xi(z+5/2)} \frac{\xi(2z)}{\xi(2z+1)}$		ν^+, ρ_3: $\frac{1}{\xi(2)} a(z-\tfrac12)a(z+\tfrac12)a(z+\tfrac32)a(2z)$ $= \frac{1}{\xi(2)} \frac{\xi(z-1/2)\,\xi(2z)}{\xi(z+5/2)\,\xi(2z+1)}$ ν^-, $\sigma_{\pi/3}$: $\frac{1}{\xi(2)} \frac{\xi(z+3/2)}{\xi(z+\frac52)}$	p. 293, 294 (AVP) 225, 226
2		ρ_2: $\frac{1}{\xi(2)} a(\tfrac12-z)\,a(\tfrac12+z)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12-z)\,\xi(\frac12+z)}{\xi(\frac32-z)\,\xi(3/2+z)}$ $\sigma_\mp$: $\frac{1}{\xi(2)} a(\tfrac12-z)a(\tfrac12+z)a(3z+1/2)a(2z)a(3z-\tfrac12)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12-z)\,\xi(\frac12+z)}{\xi(\frac32-z)\,\xi(\frac32+z)} \frac{\xi(3z-1/2)}{\xi(3z+\frac32)} \frac{\xi(2z)}{\xi(2z+1)}$		ρ_2: $\frac{1}{\xi(2)} a(\tfrac12-z)a(\tfrac12+z)a(3z+\tfrac12)a(2z)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12-z)}{\xi(\frac32-z)} \frac{\xi(\frac12+z)}{\xi(\frac32+z)} \frac{\xi(2z)}{\xi(2z+1)} \frac{\xi(3z+1/2)}{\xi(3z+3/2)}$ $\sigma_{2\pi/3}$: $\frac{1}{\xi(2)} a(\tfrac12-z)a(\tfrac12+z)a(3z+1/2)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12-z)}{\xi(\frac32-z)} \frac{\xi(\frac12+z)}{\xi(\frac32+z)} \frac{\xi(3z+1/2)}{\xi(3z+3/2)}$		ρ_4: $\frac{1}{\xi(2)} a(\tfrac12+z)a(3z+1/2)a(2z)a(3z-1/2)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12+z)}{\xi(\frac32+z)} \frac{\xi(2z)}{\xi(2z+1)} \frac{\xi(3z-1/2)}{\xi(3z+\frac32)}$ $\sigma_{\pi/3}$: $\frac{1}{\xi(2)} a(\tfrac12-z)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12-z)}{\xi(\frac32-z)}$
3	ρ_2: $\frac{1}{\xi(2)} a(\tfrac12-z)\,a(\tfrac32-z)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12-z)}{\xi(5/2-z)}$ $\sigma_{4\pi/3}$: $\frac{1}{\xi(2)} a(z+\tfrac12)a(z+3/2)a(2z)$ $= \frac{1}{\xi(2)} \frac{\xi(z+1/2)}{\xi(z+5/2)} \frac{\xi(2z)}{\xi(2z+1)}$		ρ_3: $\frac{1}{\xi(2)} a(\tfrac12-z)a(\tfrac32-z)a(z+1/2)a(z+\tfrac32)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12-z)}{\xi(5/2-z)} \frac{\xi(\frac12+z)}{\xi(5/2+z)}$ $\sigma_\mp$: $\frac{1}{\xi(2)} a(\tfrac12-z)a(\tfrac32-z)a(z+1/2)a(z+\tfrac32)a(2z)$ $\frac{1}{\xi(2)} \frac{\xi(\frac12-z)\,\xi(\frac12+z)\,\xi(2z)}{\xi(5/2-z)\,\xi(5/2+z)\,\xi(1+2z)}$		ρ_4: $\frac{1}{\xi(2)} a(\tfrac32-z)a(z+1/2)a(z+3/2)a(2z)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac32-z)\,\xi(z+1/2)\,\xi(2z)}{\xi(\frac52-z)\,\xi(z+\frac52)\,\xi(2z+1)}$ $\sigma_{2\pi/3}$: $\frac{1}{\xi(2)} a(\tfrac12-z)a(\tfrac32-z)a(z+3/2)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12-z)\,\xi(3/2+z)}{\xi(5/2-z)\,\xi(5/2+z)}$	
4		ρ_3: $\frac{1}{\xi(2)} a(2z)a(3z+\tfrac12)a(z+1/2)a(\tfrac12-z)$ $= \frac{1}{\xi(2)} \frac{\xi(2z)}{\xi(2z+1)} \frac{\xi(3z+1/2)}{\xi(3z+3/2)} \frac{\xi(\frac12+z)\,\xi(\frac12-z)}{\xi(\frac32+z)\,\xi(\frac32-z)}$ $\sigma_{4\pi/3}$: $\frac{1}{\xi(2)} a(z+1/2)a(\tfrac12-z)a(\tfrac12-3z)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12+z)\,\xi(\frac12-z)\,\xi(\frac12-3z)}{\xi(3/2+z)\,\xi(\frac32-z)\,\xi(\frac32-3z)}$		ρ_4: $\frac{1}{\xi(2)} a(3z+1/2)a(z+1/2)a(\tfrac12-z)a(\tfrac12-3z)$ $= \frac{1}{\xi(2)} \frac{\xi(3z+1/2)}{\xi(3z+3/2)} \frac{\xi(z+1/2)}{\xi(z+3/2)} \frac{\xi(\frac12-z)}{\xi(\frac32-z)} \frac{\xi(\frac12-3z)}{\xi(\frac32-3z)}$ $\sigma_\mp$: $\frac{1}{\xi(2)} a(2z)a(3z+1/2)a(\tfrac12-3z)a(\tfrac12+z)a(\tfrac12-z)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12+z)\,\xi(\frac12-z)\,\xi(2z)\,\xi(\frac12+3z)\,\xi(\frac12-3z)}{\xi(\frac32+z)\,\xi(\frac32-z)\,\xi(1+2z)\,\xi(\frac32+3z)\,\xi(\frac32-3z)}$		ρ_5: $\frac{1}{\xi(2)} a(\tfrac12-z)\,a(\tfrac12-3z)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12-z)}{\xi(\frac32-z)} \frac{\xi(\frac12-3z)}{\xi(\frac32-3z)}$ $\sigma_{2\pi/3}$: $\frac{1}{\xi(2)} a(2z)\,a(3z+1/2)\,a(z+1/2)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac12+z)}{\xi(\frac32+z)} \frac{\xi(2z)}{\xi(2z+1)} \frac{\xi(\frac12+3z)}{\xi(\frac32+3z)}$
5	ρ_3: $\frac{1}{\xi(2)} a(z-\tfrac12)a(2z)a(\tfrac12+z)a(\tfrac32+z)$ $= \frac{1}{\xi(2)} \frac{\xi(z-1/2)\,\xi(2z)}{\xi(z+5/2)\,\xi(2z+1)}$ $\sigma_{5\pi/3}$: $\frac{1}{\xi(2)} a(3/2-z)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac32-z)}{\xi(\frac52-z)}$		ρ_4: $\frac{1}{\xi(2)} a(2z)a(\tfrac12+z)a(\tfrac32+z)a(\tfrac32-z)$ $= \frac{1}{\xi(2)} \frac{\xi(3/2-z)\,\xi(\frac12+z)\,\xi(2z)}{\xi(\frac52-z)\,\xi(\frac52+z)\,\xi(2z+1)}$ $\sigma_{4\pi/3}$: $\frac{1}{\xi(2)} a(z+1/2)a(z+3/2)a(3/2-z)$ $= \frac{1}{\xi(2)} \frac{\xi(z+1/2)}{\xi(z+5/2)} \frac{\xi(3/2-z)}{\xi(5/2-z)}$		ρ_5: $\frac{1}{\xi(2)} a(z+\tfrac32)\,a(z-3/2)$ $= \frac{1}{\xi(2)} \frac{\xi(z+3/2)\,\xi(3/2-z)}{\xi(z+5/2)\,\xi(\frac52-z)}$ $\sigma_\mp$: $\frac{1}{\xi(2)} a(\tfrac32-z)a(z-\tfrac12)a(z+\tfrac12)a(z+\tfrac32)a(2z)$ $= \frac{1}{\xi(2)} \frac{\xi(\frac32-z)}{\xi(\frac52-z)} \frac{\xi(z-1/2)}{\xi(z+5/2)} \frac{\xi(2z)}{\xi(2z+1)}$	
6		ρ^+, ρ_4: $\frac{1}{\xi(2)} a(3z-1/2)a(2z)a(3z+1/2)a(z+1/2)$ $= \frac{1}{\xi(2)} \frac{\xi(z+1/2)\,\xi(2z)\,\xi(3z-1/2)}{\xi(z+3/2)\,\xi(2z+1)\,\xi(3z+3/2)}$ ρ^-, $\sigma_{5\pi/3}$: $\frac{1}{\xi(2)} a(z+1/2)$ $= \frac{1}{\xi(2)} \frac{\xi(z+1/2)}{\xi(z+3/2)}$		σ^+, ρ_5: $\frac{1}{\xi(2)} a(3z+1/2)\,a(z+1/2)$ $= \frac{1}{\xi(2)} \frac{\xi(3z+1/2)}{\xi(3z+3/2)} \frac{\xi(z+1/2)}{\xi(z+3/2)}$ σ^-, $\sigma_{\pi/3}$: $\frac{1}{\xi(2)} a(2z)\,a(3z+1/2)\,a(z+1/2)$ $= \frac{1}{\xi(2)} \frac{\xi(z+1/2)}{\xi(z+3/2)} \frac{\xi(2z)}{\xi(2z+1)} \frac{\xi(3z+1/2)}{\xi(3z+3/2)}$		ν^+, ρ_6: $\frac{1}{\xi(2)}$ ν^-, $\sigma_\mp$: $\frac{1}{\xi(2)} a(z-1/2)a(3z-1/2)a(2z)a(3z+1/2)a\ldots$ $= \frac{1}{\xi(2)} \frac{\xi(z-1/2)}{\xi(z+3/2)} \frac{\xi(2z)}{\xi(1+2z)} \frac{\xi(3z-1/2)}{\xi(3z+3/2)}$

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(ii) $H_{\beta_1} = H_{\beta_1}$ $\qquad$ $H_{\beta_4} = 2H_{\beta_2} - 3H_{\beta_1}$

$H_{\beta_2} = H_{\beta_2}$ $\qquad$ $H_{\beta_5} = H_{\beta_2} - 2H_{\beta_1}$

$H_{\beta_3} = H_{\beta_2} - H_{\beta_1}$ $\qquad$ $H_{\beta_6} = H_{\beta_2} - 3H_{\beta_1}$

(iii) $H''_{\beta_1} = H_{\beta_2} - H_{\beta_3}$ $\qquad$ $H_{\beta_4} = -H_{\beta_2} + 3H_{\beta_3}$

$H_{\beta_2} = H_{\beta_2}$ $\qquad$ $H_{\beta_5} = -H_{\beta_2} + 2H_{\beta_3}$

$H_{\beta_3} = H_{\beta_3}$ $\qquad$ $H_{\beta_6} = -2H_{\beta_2} + 3H_{\beta_3}$

(iv) $H_{\beta_1} = H_{\beta_4} - 2H_{\beta_5}$ $\qquad$ $H_{\beta_4} = H_{\beta_4}$

$H_{\beta_2} = -3H_{\beta_5} + 2H_{\beta_4}$ $\qquad$ $H_{\beta_5} = H_{\beta_5}$

$H_{\beta_3} = H_{\beta_4} - H_{\beta_5}$ $\qquad$ $H_{\beta_6} = 3H_{\beta_5} - H_{\beta_4}$

(v) $H_{\beta_1} = H_{\beta_5} - H_{\beta_6}$ $\qquad$ $H_{\beta_4} = 3H_{\beta_5} - H_{\beta_6}$

$H_{\beta_2} = 3H_{\beta_5} - 2H_{\beta_6}$ $\qquad$ $H_{\beta_5} = H_{\beta_5}$

$H''_{\beta_3} = 2H_{\beta_5} - H_{\beta_6}$ $\qquad$ $H_{\beta_6} = H_{\beta_6}$

(vi)

The collection $\{\underline{s}_1, \underline{s}_2, \underline{s}_3, \underline{s}_4, \underline{s}_5, \underline{s}_6\}$ breaks up into two equivalence classes $\{\underline{s}_1, \underline{s}_3, \underline{s}_5\}$ and $\{\underline{s}_2, \underline{s}_4, \underline{s}_6\}$. The principal element in the first is $\underline{s}_1$ and in the second $\underline{s}_6$. The corresponding matrices $M(H)$, as defined on p. 7.29 of Eisenstein Series, are written as $M(z)$ where z is the coordinate on $\underline{s}_1$ or $\underline{s}_6$. The entries of these matrices are given in the tables below. The entry in the box in the row labeled σ_+ and the column labeled ρ_- is $M(\sigma_+ \rho_+ \rho_-^{-1}, \rho_- \rho_+ H)$ if $H = H(z)$.

$\{\underline{s}_1, \underline{s}_3, \underline{s}_5\}$

	ρ_+	ρ_-	σ_+	σ_-	z_+	z_-
ρ_+	$\frac{1}{\xi(2)}$	$\frac{1}{\xi(2)}\frac{\xi(-z-\frac{3}{2})\xi(-2z)}{\xi(-z+\frac{5}{2})\xi(-2z+1)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)}{\xi(\frac{5}{2}-z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(-2z)}{\xi(\frac{5}{2}-z)\xi(1-2z)}$	$\frac{1}{\xi(2)}\frac{\xi(-z-1/2)\xi(-2z)}{\xi(-z+\frac{5}{2})\xi(-2z+1)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{3}{2}-z)}{\xi(\frac{5}{2}-z)}$
ρ_-	$\frac{1}{\xi(2)}\frac{\xi(z-\frac{3}{2})\xi(2z)}{\xi(z+\frac{5}{2})\xi(2z+1)}$	$\frac{1}{\xi(2)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}+z)}{\xi(z+\frac{5}{2})}\frac{\xi(2z)}{\xi(2z+1)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}+z)}{\xi(\frac{5}{2}+z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{3}{2}+z)}{\xi(\frac{5}{2}+z)}$	$\frac{1}{\xi(2)}\frac{\xi(z-\frac{1}{2})\xi(2z)}{\xi(z+\frac{5}{2})\xi(2z+1)}$
σ_+	$\frac{1}{\xi(2)}\frac{\xi(z+1/2)}{\xi(z+5/2)}$	$\frac{1}{\xi(2)}\frac{\xi(-z+1/2)\xi(-2z)}{\xi(-z+5/2)\xi(-2z+1)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(\frac{1}{2}+z)}{\xi(\frac{5}{2}-z)\xi(\frac{5}{2}+z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}+z)\xi(\frac{1}{2}-z)\xi(-2z)}{\xi(\frac{5}{2}+z)\xi(\frac{5}{2}-z)\xi(1-2z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{3}{2}+z)\xi(\frac{1}{2}+z)\xi(-2z)}{\xi(\frac{5}{2}+z)\xi(\frac{5}{2}-z)\xi(-2z+1)}$	$\frac{1}{\xi(2)}\frac{\xi(z+\frac{1}{2})\xi(\frac{3}{2}+z)}{\xi(z+\frac{5}{2})\xi(\frac{5}{2}-z)}$
σ_-	$\frac{1}{\xi(2)}\frac{\xi(z+1/2)\xi(2z)}{\xi(z+\frac{5}{2})\xi(2z+1)}$	$\frac{1}{\xi(2)}\frac{\xi(-z+1/2)}{\xi(-z+\frac{5}{2})}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(\frac{1}{2}+z)\xi(2z)}{\xi(\frac{5}{2}-z)\xi(\frac{5}{2}+z)\xi(2z+1)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}+z)\xi(\frac{1}{2}-z)}{\xi(\frac{5}{2}+z)\xi(\frac{5}{2}-z)}$	$\frac{1}{\xi(2)}\frac{\xi(-z+1/2)\xi(\frac{3}{2}+z)}{\xi(-z+\frac{5}{2})\xi(\frac{5}{2}+z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{3}{2}-z)\xi(\frac{1}{2}+z)\xi(2z)}{\xi(\frac{5}{2}-z)\xi(\frac{5}{2}+z)\xi(2z+1)}$
z_+	$\frac{1}{\xi(2)}\frac{\xi(z-1/2)\xi(2z)}{\xi(z+\frac{5}{2})\xi(2z+1)}$	$\frac{1}{\xi(2)}\frac{\xi(-z+3/2)}{\xi(-z+\frac{5}{2})}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{3}{2}-z)\xi(\frac{1}{2}+z)\xi(2z)}{\xi(\frac{5}{2}-z)\xi(\frac{5}{2}+z)\xi(1+2z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}+z)\xi(\frac{3}{2}-z)}{\xi(\frac{5}{2}+z)\xi(\frac{5}{2}-z)}$	$\frac{1}{\xi(2)}\frac{\xi(z+\frac{3}{2})\xi(\frac{3}{2}+z)}{\xi(z+\frac{5}{2})\xi(\frac{5}{2}+z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{3}{2}-z)\xi(z+1/2)\xi(2z)}{\xi(\frac{5}{2}-z)\xi(z+\frac{5}{2})\xi(2z+1)}$
z_-	$\frac{1}{\xi(2)}\frac{\xi(z+3/2)}{\xi(z+5/2)}$	$\frac{1}{\xi(2)}\frac{\xi(-z-1/2)\xi(-2z)}{\xi(-z+\frac{5}{2})\xi(-2z+1)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(\frac{3}{2}+z)}{\xi(\frac{5}{2}-z)\xi(\frac{5}{2}+z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{3}{2}+z)\xi(\frac{1}{2}-z)\xi(-2z)}{\xi(\frac{5}{2}+z)\xi(\frac{5}{2}-z)\xi(1-2z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{3}{2}+z)\xi(-z+1/2)\xi(-2z)}{\xi(\frac{5}{2}+z)\xi(-z+\frac{5}{2})\xi(-2z+1)}$	$\frac{1}{\xi(2)}\frac{\xi(z+\frac{3}{2})\xi(\frac{3}{2}-z)}{\xi(\frac{5}{2}-z)\xi(\frac{5}{2}+z)}$

$\{\underline{s}_2, \underline{s}_4, \underline{s}_6\}$

	ρ_+	ρ_-	σ_+	σ_-	z_+	z_-
ρ_+		$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(\frac{1}{2}+z)\xi(3z-1/2)\xi(2z)}{\xi(\frac{3}{2}-z)\xi(\frac{3}{2}+z)\xi(3z+\frac{3}{2})\xi(2z+1)}$	$\frac{1}{\xi(2)}\frac{\xi(2z)\xi(\frac{1}{2}+3z)\xi(\frac{1}{2}+z)\xi(\frac{1}{2}-z)}{\xi(1+2z)\xi(\frac{3}{2}+3z)\xi(\frac{3}{2}+z)\xi(\frac{3}{2}-z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(\frac{1}{2}+z)\xi(\frac{1}{2}+3z)}{\xi(\frac{3}{2}-z)\xi(\frac{3}{2}+z)\xi(\frac{3}{2}+3z)}$	$\frac{1}{\xi(2)}\frac{\xi(z+1/2)\xi(2z)\xi(3z-1/2)}{\xi(z+3/2)\xi(2z+1)\xi(3z+3/2)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)}{\xi(3/2-z)}$
ρ_-		$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(\frac{1}{2}+z)}{\xi(\frac{3}{2}-z)\xi(\frac{3}{2}+z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}+z)\xi(\frac{1}{2}-z)\xi(\frac{1}{2}-3z)}{\xi(\frac{3}{2}+z)\xi(\frac{3}{2}-z)\xi(\frac{3}{2}-3z)}$	$\frac{1}{\xi(2)}\frac{\xi(-2z)\xi(\frac{1}{2}-3z)\xi(\frac{1}{2}-z)\xi(\frac{1}{2}+z)}{\xi(-2z+1)\xi(\frac{3}{2}-3z)\xi(\frac{3}{2}-z)\xi(\frac{3}{2}+z)}$	$\frac{1}{\xi(2)}\frac{\xi(1/2+z)}{\xi(3/2+z)}$	$\frac{1}{\xi(2)}\frac{\xi(1/2+z)\xi(-2z)\xi(-3z-1/2)}{\xi(3/2-z)\xi(-2z+1)\xi(-3z+3/2)}$
σ_+		$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(\frac{1}{2}+z)\xi(\frac{1}{2}+3z)}{\xi(\frac{3}{2}-z)\xi(\frac{3}{2}+z)\xi(3z+\frac{3}{2})}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}+3z)\xi(\frac{1}{2}+z)\xi(\frac{1}{2}-z)\xi(\frac{1}{2}-3z)}{\xi(\frac{3}{2}+3z)\xi(\frac{3}{2}+z)\xi(\frac{3}{2}-z)\xi(\frac{3}{2}-3z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(\frac{1}{2}+z)\xi(-2z)\xi(\frac{1}{2}-3z)\xi(\frac{1}{2}+3z)}{\xi(\frac{3}{2}-z)\xi(\frac{3}{2}+z)\xi(1-2z)\xi(\frac{3}{2}+3z)\xi(\frac{3}{2}-3z)}$	$\frac{1}{\xi(2)}\frac{\xi(3z+1/2)\xi(z+1/2)}{\xi(3z+3/2)\xi(z+3/2)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(-2z)\xi(\frac{1}{2}-3z)}{\xi(3/2-z)\xi(1-2z)\xi(3/2-3z)}$
σ_-		$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(\frac{1}{2}+z)\xi(2z)\xi(3z+1/2)}{\xi(\frac{3}{2}-z)\xi(\frac{3}{2}+z)\xi(1+2z)\xi(3z+\frac{3}{2})}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}+z)\xi(\frac{1}{2}-z)\xi(2z)\xi(\frac{1}{2}+3z)\xi(\frac{1}{2}-3z)}{\xi(\frac{3}{2}+z)\xi(\frac{3}{2}-z)\xi(1+2z)\xi(\frac{3}{2}+3z)\xi(\frac{3}{2}-3z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-3z)\xi(\frac{1}{2}-z)\xi(\frac{1}{2}+z)\xi(\frac{1}{2}+3z)}{\xi(\frac{3}{2}-3z)\xi(\frac{3}{2}-z)\xi(\frac{3}{2}+z)\xi(\frac{3}{2}+3z)}$	$\frac{1}{\xi(2)}\frac{\xi(z+1/2)\xi(2z)\xi(3z+1/2)}{\xi(z+3/2)\xi(2z+1)\xi(3z+3/2)}$	$\frac{1}{\xi(2)}\frac{\xi(-3z+1/2)\xi(-z+1/2)}{\xi(-3z+3/2)\xi(-z+3/2)}$
z_+		$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)}{\xi(\frac{3}{2}-z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(\frac{1}{2}+3z)}{\xi(\frac{3}{2}-z)\xi(\frac{3}{2}+3z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}-z)\xi(-2z)\xi(\frac{1}{2}-3z)}{\xi(3/2-z)\xi(1-2z)\xi(\frac{3}{2}-3z)}$	$\frac{1}{\xi(2)}$	$\frac{1}{\xi(2)}\frac{\xi(-z-1/2)\xi(-2z)\xi(-3z-1/2)}{\xi(-z+\frac{3}{2})\xi(-2z+1)\xi(-3z+3/2)}$
z_-		$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}+z)\xi(2z)\xi(3z-1/2)}{\xi(\frac{3}{2}+z)\xi(1+2z)\xi(3z+3/2)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}+z)\xi(2z)\xi(\frac{1}{2}+3z)}{\xi(\frac{3}{2}+z)\xi(1+2z)\xi(\frac{3}{2}+3z)}$	$\frac{1}{\xi(2)}\frac{\xi(\frac{1}{2}+z)\xi(\frac{1}{2}+3z)}{\xi(\frac{3}{2}+z)\xi(\frac{3}{2}+3z)}$	$\frac{1}{\xi(2)}\frac{\xi(z-1/2)\xi(2z)\xi(3z-1/2)}{\xi(z+3/2)\xi(1+2z)\xi(3z+3/2)}$	$\frac{1}{\xi(2)}$

The remainder mentioned on p 2 is, of course,

$$\Sigma_{i=1}^{6}\left\{\Sigma_{j=1}^{6}\ \Sigma_{\sigma\in\Omega(\mathfrak{s}_i,\mathfrak{s}_j)}\ \frac{1}{2\pi i}\int_{\mathrm{Re}\,\lambda=\lambda_i} n(\sigma,\lambda)\,\Phi(\lambda)\,\overline{\Psi(-\sigma\bar\lambda)}\,dz\right.$$

dz is of course the differential of the coordinate z described on p. 2. Following the general procedure of the theory of Eisenstein series we replace the integral over the line $\mathrm{Re}\,\lambda=\lambda_i$ by an integral over the line $\mathrm{Re}\,\lambda=\lambda_i'$ if λ_i' is the point on $\mathfrak{s}_i$ whose coordinate is zero. Since none of the functions $n(\sigma,\lambda)$ have a singularity on the line $\mathrm{Re}\,\lambda=\lambda_i'$ in $\mathfrak{s}_i$ we can treat each of the integrals

$$\int_{\mathrm{Re}\,\lambda=\lambda_i} n(\sigma,\lambda)\,\Phi(\lambda)\,\overline{\Psi(-\sigma\bar\lambda)}\,dz$$

separately. We now compute the remainder in each case.

(i) Take $i=1$. At the point λ_1, $-\frac{3}{2}<z<\frac{5}{2}$.

(a) $$R(\rho_+)=\int_{\mathrm{Re}\,\lambda=\lambda_i} n(\rho_+,\lambda)\,\Phi(\lambda)\,\overline{\Psi(-\rho_+\bar\lambda)}\,dz-\int_{\mathrm{Re}\,\lambda=\lambda_i'} n(\rho_+,\lambda)\,\Phi(\lambda)\,\overline{\Psi(-\rho_+\bar\lambda)}\,dz$$

$$=0$$

(b)

$$R(\rho_-)=-\frac{\xi(3)}{\xi(2)\,\xi^2(4)}\,\Phi(\beta_3)\overline{\Psi(\beta_3)}+\frac{1}{2\,\xi(2)\,\xi(3)}\,\Phi(\beta_2)\overline{\Psi(\beta_2)}$$

(c)

$$R(\sigma_+)=\frac{1}{\xi(2)\,\xi(3)}\,\Phi(\beta_2)\overline{\Psi(\beta_2)}$$

(d) Suppose that, in the neighbourhood of $z=1$

$$\xi(z)=\frac{1}{z-1}+a+b(z-1)+O((z-1)^2)$$

then

$$R(\sigma_-)=\frac{1}{\xi(2)}\mathrm{Res}\left[\left(\frac{1}{z-1/2}+a\right)\left(\frac{1}{2}\frac{1}{(z-1/2)}+a\right)\left(\frac{1}{\xi(3)}-\frac{(z-\frac{1}{2})\,\xi'(3)}{\xi^2(3)}\right)\left(\frac{1}{\xi(2)}-2(z-1/2)\frac{\xi'(2)}{\xi^2(2)}\right)\left(\Phi(\beta_2)+(z-1/2)\,D_{\beta_5}\Phi(\beta_2)\right)\right.$$

4?

$$\left.\left(\bar\Psi(\beta_4)-(z-1/2)\,D_{\beta_1}\bar\Psi(\beta_4)\right)\right]$$

6?

If μ is any linear function then $D_\mu\Phi(\lambda)=\frac{d}{dt}\Phi(\lambda+t\mu)\Big|_{t=0}$. Since $\left(\frac{1}{z-1/2}+a\right)\left(\frac{1}{2(z-1/2)}+a\right)=$ $\frac{1}{2(z-1/2)^2}+\frac{3a}{2(z-1/2)}+O(1)$ and $\left(\frac{1}{\xi(3)}-\frac{(z-1/2)\,\xi'(3)}{\xi^2(3)}\right)\left(\frac{1}{\xi(2)}-2(z-1/2)\frac{\xi'(2)}{\xi^2(2)}\right)=\frac{1}{\xi(2)\,\xi(3)}-(z-1/2)\left(\frac{2\xi'(2)}{\xi(3)\,\xi^2(2)}+\frac{\xi'(3)}{\xi(2)\,\xi^2(3)}\right)+O((z-1/2)^2)$

the residue equals

$$\frac{1}{2\,\xi^2(2)\,\xi(3)}\left[+\Phi(\beta_2)\,D_{\beta_2}\overline{\Phi}(\beta_4)+D_{\beta_4}\Phi(\beta_2)\overline{\Phi}(\beta_4)\right]+\left[\frac{3a}{2\xi^2(2)\xi(3)}-\frac{\xi'(2)}{\xi(3)\xi^3(2)}-\frac{\xi'(3)}{2\xi^2(2)\xi^2(3)}\right]\Phi(\beta_2)\overline{\Phi}(\beta_4)$$

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(e) $R(z_+) = \frac{\xi(3)}{\xi(2)\,\xi^2(4)}\,\Phi(\beta_3)\overline{\Phi}(\beta_3) +$

$$\frac{1}{\xi(2)}\operatorname{Res}\left[\left(\frac{-1}{z-1/2}+a\right)\left(\frac{1}{2(z-1/2)}+a\right)\left(\frac{1}{\xi(3)}-(z-1/2)\frac{\xi'(3)}{\xi^2(3)}\right)\left(\frac{1}{\xi(2)}-2(z-1/2)\frac{\xi'(2)}{\xi^2(2)}\right)\left(\Phi(\beta_2)+(z-1/2)D_{\beta_2}\Phi(\beta_2)\right)\right.$$

$$\left.\cdot\left(\overline{\Phi}(\beta_4)+(z-1/2)D_{\beta_4}\overline{\Phi}(\beta_4)\right)\right]$$

Since $\left(-\frac{1}{z-1/2}+a\right)\left(\frac{1}{2(z-1/2)}+a\right)=\frac{-1}{2(z-1/2)^2}-\frac{a}{2(z-1/2)}+O(1)$ and $\left(\frac{1}{\xi(3)}-(z-1/2)\frac{\xi'(3)}{\xi^2(3)}\right)\left(\frac{1}{\xi(2)}-2(z-1/2)\frac{\xi'(2)}{\xi^2(2)}\right)$

$=\frac{1}{\xi(2)\,\xi(3)}-(z-1/2)\left[\frac{\xi'(3)}{\xi^2(3)\,\xi(2)}+\frac{2\,\xi'(2)}{\xi^2(2)\,\xi(3)}\right]$ we have

$$R(z_+)=\frac{\xi(3)}{\xi(2)\,\xi^2(4)}\,\Phi(\beta_3)\overline{\Phi}(\beta_3)-\frac{1}{2\xi^2(2)\,\xi(3)}\left[\Phi(\beta_2)\,D_{\beta_2}\overline{\Phi}(\beta_4)+D_{\beta_4}\Phi(\beta_2)\overline{\Phi}(\beta_4)\right]$$

$$+\left[\left(-\frac{a}{2\xi^2(2)\,\xi(3)}+\frac{1}{2}\frac{\xi'(3)}{\xi^2(2)\,\xi^2(3)}+\frac{\xi'(2)}{\xi^3(2)\,\xi(3)}\right)\Phi(\beta_2)\overline{\Phi}(\beta_4)\right]$$

(f)

$R(z_-)=0$

Adding we see that the total residue from the integral on S_1 is

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(i) $\;+\frac{3}{2\,\xi(2)\,\xi(3)}\Phi(\beta_2)\overline{\Phi}(\beta_2)+\frac{a}{\xi^2(2)\,\xi(3)}\Phi(\beta_2)\overline{\Phi}(\beta_4)+\frac{1}{2\xi^2(2)\,\xi(3)}\Phi(\beta_2)\,D_{\beta_2}\overline{\Phi}(\beta_4)$

(ii) The coordinate of the point λ_2 satisfies the inequality, $\frac{1}{6}<z<1/2$

p 300 (AVP)

(a)

$R(\rho_+ z_+\rho_+^{-1})=0$

(b)

$$R(\rho_- z_+ \rho_+^{-1}) = \frac{-1}{3\xi(2)} \frac{\xi(\frac{1}{3})\,\xi(\frac{2}{3})}{\xi(\frac{4}{3})\,\xi(\frac{5}{3})} \frac{\xi(\frac{1}{3})}{\xi(\frac{4}{3})} \frac{1}{\xi(2)} \bar{\Phi}\left(\frac{\beta_3}{3}\right) \overline{\bar{\Phi}\left(\frac{\beta_3}{3}\right)}$$

(c)

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$$R(\sigma_+ z_+ \rho_+^{-1}) = \frac{1}{3\xi(2)} \frac{\xi(\frac{1}{3})}{\xi(\frac{4}{3})} \frac{\xi(\frac{2}{3})}{\xi(\frac{5}{3})} \frac{\xi(\frac{1}{3})}{\xi(\frac{4}{3})} \frac{1}{\xi(2)} \bar{\Phi}\left(\frac{\beta_3}{3}\right) \overline{\bar{\Phi}\left(\frac{\beta_3}{3}\right)}$$

(d)

$$R(\sigma_- z_+ \rho_+^{-1}) = \frac{1}{3\xi(2)} \frac{\xi(\frac{1}{3})}{\xi(\frac{4}{3})} \frac{\xi(\frac{2}{3})}{\xi(\frac{5}{3})} \frac{1}{\xi(2)} \bar{\Phi}\left(\frac{\beta_3}{3}\right) \overline{\bar{\Phi}\left(\frac{\beta_5}{3}\right)}$$

(e)

$$R(z_+ z_+ \rho_+^{-1}) = \frac{-1}{3\xi(2)} \frac{\xi(\frac{2}{3})}{\xi(\frac{5}{3})} \frac{\xi(\frac{1}{3})}{\xi(\frac{4}{3})} \frac{1}{\xi(2)} \bar{\Phi}\left(\frac{\beta_3}{3}\right) \overline{\bar{\Phi}\left(\frac{\beta_5}{3}\right)}$$

(f)

$$R(z_- z_+ \rho_+^{-1}) = 0$$

The total residue from the integral in s_2 is consequently zero.

(iii) It is clear from the diagram on p. 3 that the total residue from the integral in s_3 is zero.

(iv) It is also clear that the total residue from the integral in s_4 is also zero.

(v). The coordinate of λ_5 satisfies the inequality, $\frac{1}{2} < z < \frac{3}{2}$

(a)

$$R(\rho_+ \rho_+ z_+^{-1}) = \frac{1}{\xi(2)} \operatorname{Res}\Big[\left(\frac{-1}{z-\frac{1}{2}} + a\right)\left(\frac{1}{2(z-\frac{1}{2})} + a\right)\left(\frac{1}{\xi(3)} - (z-\tfrac{1}{2})\frac{\xi'(3)}{\xi^2(3)}\right)\left(\frac{1}{\xi(2)} - 2(z-\tfrac{1}{2})\frac{\xi'(2)}{\xi^2(2)}\right)\left(\bar{\Phi}(\beta_4) + (z-\tfrac{1}{2})D_{\beta_2}\bar{\Phi}(\beta_4)\right)$$

$$\left(\overline{\bar{\Phi}}(\beta_2) + (z-\tfrac{1}{2})D_{\beta_4}\overline{\bar{\Phi}(\beta_2)}\right)\Big]$$

Since $\left(\frac{-1}{z-1/2}+a\right)\left(\frac{1}{2(z-1/2)}+a\right) = \frac{-1}{2(z-1/2)^2} - \frac{a}{2(z-1/2)} + O(1)$ and the product of the second two terms is

$\frac{1}{\xi(2)\xi(3)} - (z-1/2)\left[\frac{\xi'(3)}{\xi(2)\xi^2(3)} + 2\frac{\xi'(2)}{\xi^2(2)\xi(3)}\right]$ the residue equals

3 4 5 5 1

$$\frac{-1}{2\xi^2(2)\xi(3)}\left\{\Phi(\beta_4)D_{\beta_4}\bar\Phi(\beta_2) + D_{\beta_2}\Phi(\beta_4)\bar\Phi(\beta_2)\right\} + \left\{\frac{-a}{2\xi^2(2)\xi(3)} + \frac{\xi'(3)}{2\xi^2(2)\xi^2(3)} + \frac{\xi'(2)}{\xi^3(2)\xi(3)}\right\}\Phi(\beta_4)\bar\Phi(\beta_2)$$

(b)

1

$$R(\rho_-\rho_+z_+^{-1}) = \frac{-1}{\xi^2(2)}\Phi(\beta_4)\bar\Phi(\beta_6)$$ p. 301 (AVP)

(c)

$$R(\sigma_+\rho_+z_+^{-1}) = \frac{1}{\xi(2)}\mathrm{Res}\Big[\Big(\frac{-1}{z-1/2}+a-b(z-1/2)\Big)\Big(\frac{1}{z-1/2}+a+b(z-1/2)\Big)\Big(\frac{1}{2(z-1/2)}+a+2b(z-1/2)\Big)\Big(\frac{1}{\xi(2)}+(z-1/2)\frac{\xi'(2)}{\xi^2(2)} - \frac{(z-1/2)^2}{2}\Big(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\Big)\Big)$$

$$\times\Big(\frac{1}{\xi(3)} - (z-1/2)\frac{\xi'(3)}{\xi^2(3)} - \frac{(z-1/2)^2}{2}\Big(\frac{\xi''(3)}{\xi^2(3)} - \frac{2(\xi'(3))^2}{\xi^3(3)}\Big)\Big)\Big(\frac{1}{\xi(2)} - 2(z-1/2)\frac{\xi'(2)}{\xi^2(2)} - \frac{4(z-1/2)^2}{2}\Big(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\Big)\Big)$$

$$\times\Big(\Phi(\beta_4)+(z-1/2)D_{\beta_2}\Phi(\beta_4)+\frac{(z-1/2)^2}{2}D^2_{\beta_2}\Phi(\beta_4)\Big)\Big(\bar\Phi(\beta_4)+(z-1/2)D_{\beta_6}\bar\Phi(\beta_4)+\frac{(z-1/2)^2}{2}D^2_{\beta_6}\bar\Phi(\beta_4)\Big)$$

$$= \frac{1}{\xi(2)}\mathrm{Res}\Big[\Big\{\frac{-1}{2(z-1/2)^3} + \frac{1}{(z-1/2)^2}\Big(\frac a2 - \frac a2 - a\Big) + \frac{1}{(z-1/2)}\Big(-\frac b2 - \frac b2 - 2b\Big) + \frac{1}{z-1/2}\Big(-a^2+a^2+\frac{a^2}{2}\Big)\Big\}$$

$$\Big\{\frac{1}{\xi^2(2)\xi(3)} + (z-1/2)\Big(\frac{-2\xi'(2)}{\xi^3(2)\xi(3)} - \frac{\xi'(3)}{\xi^2(2)\xi^2(3)} + \frac{\xi'(2)}{\xi^3(2)\xi(3)}\Big) + (z-1/2)^2\Big(2\frac{\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)} + 2\frac{(\xi'(2))^2}{\xi^4(2)\xi(3)} - \frac{\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)}\Big)$$

$$+(z-1/2)^2\Big(\frac{-1}{2\xi(2)\xi(3)}\Big(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\Big) - \frac{1}{2\xi^2(2)}\Big(\frac{\xi''(3)}{\xi^2(3)} - \frac{2(\xi'(3))^2}{\xi^3(3)}\Big) - \frac{2}{\xi(2)\xi(3)}\Big(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\Big)\Big)\Big\}$$

$$\Big\{\Phi(\beta_4)\bar\Phi(\beta_4) + (z-1/2)\big(D_{\beta_2}\Phi(\beta_4)\bar\Phi(\beta_4) + \Phi(\beta_4)D_{\beta_6}\bar\Phi(\beta_4)\big) + (z-1/2)^2 D_{\beta_2}\Phi(\beta_4)D_{\beta_6}\bar\Phi(\beta_4)$$

$$+\frac{(z-1/2)^2}{2}\big(\Phi(\beta_4)D^2_{\beta_6}\bar\Phi(\beta_4) + D^2_{\beta_2}\Phi(\beta_4)\bar\Phi(\beta_4)\big)\Big\}$$

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$$= \frac{1}{\xi(2)}\Big[\frac{-1}{2\xi^2(2)\xi(3)}\{D_{\beta_2}\Phi(\beta_4)D_{\beta_6}\bar\Psi(\beta_4) + \tfrac{1}{2}\Phi(\beta_4)D^2_{\beta_6}\bar\Psi(\beta_4) + \tfrac{1}{2}D^2_{\beta_2}\Phi(\beta_4)\bar\Psi(\beta_4)\}$$

$$+\frac{1}{2}\left(\frac{\xi'(2)}{\xi^3(2)\xi(3)} + \frac{\xi'(3)}{\xi^2(2)\xi^2(3)}\right)\{D_{\beta_2}\Phi(\beta_4)\bar\Psi(\beta_4) + \Phi(\beta_4)D_{\beta_6}\bar\Psi(\beta_4)\}$$

$$-\frac{a}{\xi^2(2)\xi(3)}\{D_{\beta_2}\Phi(\beta_4)\bar\Psi(\beta_4) + \Phi(\beta_4)D_{\beta_6}\bar\Psi(\beta_4)\} - \frac{1}{2}\left(\frac{\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)} - \frac{2(\xi'(2))^2}{\xi^4(2)\xi(3)}\right)\Phi(\beta_4)\bar\Psi(\beta_4)$$

$$+a\left(\frac{\xi'(2)}{\xi^3(2)\xi(3)} + \frac{\xi'(3)}{\xi^2(2)\xi^2(3)}\right)\Phi(\beta_4)\bar\Psi(\beta_4) + \frac{1}{\xi^2(2)\xi(3)}\left(\tfrac{a}{2} - 3b\right)\Phi(\beta_4)\bar\Psi(\beta_4)$$

omit

$$+\left\{\frac{15}{4\xi(2)\xi(3)}\left(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\right) + \frac{1}{4\xi^2(2)}\left(\frac{\xi''(3)}{\xi^2(3)} - \frac{2(\xi'(3))^2}{\xi^3(3)}\right)\right\}\Phi(\beta_4)\bar\Psi(\beta_4)\Big]$$

(d)

$$R(\sigma_-\rho_+z_+^{-2}) = \frac{1}{\xi(2)}\mathrm{Res}\Big[\left(\frac{1}{z-\frac12}+a\right)\left(\frac{-1}{z-\frac12}+a\right)\left(\frac{1}{\xi(3)} - (z-\tfrac12)\frac{\xi'(3)}{\xi^2(3)}\right)\left(\frac{1}{\xi(2)} + (z-\tfrac12)\frac{\xi'(2)}{\xi^2(2)}\right)\left(\Phi(\beta_4) + (z-\tfrac12)D_{\beta_2}\Phi(\beta_4)\right)$$

$$\cdot\left(\bar\Psi(\beta_2) - (z-\tfrac12)D_{\beta_6}\bar\Psi(\beta_2)\right)\Big]$$

$$= \frac{1}{\xi(2)}\mathrm{Res}\Big[\frac{-1}{(z-\frac12)^2}\left(\frac{1}{\xi(2)\xi(3)} + (z-\tfrac12)\left(\frac{\xi'(2)}{\xi^2(2)\xi(3)} - \frac{\xi'(3)}{\xi^2(3)\xi(2)}\right)\right)\left(\Phi(\beta_4)\bar\Psi(\beta_2) + (z-\tfrac12)\left(D_{\beta_2}\Phi(\beta_4)\bar\Psi(\beta_2) - \Phi(\beta_4)D_{\beta_6}\bar\Psi(\beta_2)\right)\right)\Big]$$

$$= \frac{1}{\xi^2(2)\xi(3)}\left(\Phi(\beta_4)D_{\beta_6}\bar\Psi(\beta_2) - D_{\beta_2}\Phi(\beta_4)\bar\Psi(\beta_2)\right) + \left(\frac{\xi'(3)}{\xi(2)\xi^2(3)} - \frac{\xi'(2)}{\xi^3(2)\xi(3)}\right)\Phi(\beta_4)\bar\Psi(\beta_2).$$

(e)

$$R(z_+\rho_+z_+^{-1}) = \frac{-1}{\xi(2)\xi(3)}\Phi(\beta_4)\bar\Psi(\beta_6)$$

(f)

$$R(z_-\rho_+z_+^{-1}) = \frac{1}{\xi(2)}\mathrm{Res}\Big[\left(\frac{-1}{z-\frac12}+a-b(z-\tfrac12)\right)\left(\frac{-1}{z-\frac12}+a-b(z-\tfrac12)\right)\left(\frac{1}{2(z-\frac12)}+a+2b(z-\tfrac12)\right)\left(\frac{1}{\xi(2)} + (z-\tfrac12)\frac{\xi'(2)}{\xi^2(2)} - \frac{(z-\frac12)^2}{2}\left(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\right)\right)$$

$$\times\left(\frac{1}{\xi(3)} - (z-\tfrac12)\frac{\xi'(3)}{\xi^2(3)} - \frac{(z-\frac12)^2}{2}\left(\frac{\xi''(3)}{\xi^2(3)} - \frac{2(\xi'(3))^2}{\xi^3(3)}\right)\right)\left(\frac{1}{\xi(2)} - 2(z-\tfrac12)\frac{\xi'(2)}{\xi^2(2)} - \frac{4(z-\frac12)^2}{2}\left(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\right)\right)$$

$$\times\left(\Phi(\beta_4) + (z-\tfrac12)D_{\beta_2}\Phi(\beta_4) + \frac{(z-\frac12)^2}{2}D^2_{\beta_2}\Phi(\beta_4)\right)\left(\bar\Psi(\beta_4) + (z-\tfrac12)D_{\beta_2}\bar\Psi(\beta_4) + \frac{(z-\frac12)^2}{2}D^2_{\beta_2}\bar\Psi(\beta_4)\right)\Big]$$

$$= \frac{1}{\xi(2)} \operatorname{Re}\Bigg[\Bigg\{ \frac{1}{2(z-1/2)^3} + \cancel{\left(a - \frac{a}{2} - \frac{a}{2}\right)} \frac{1}{(z-1/2)^2} + \frac{1}{(z-1/2)}\left(-a^2 - a^2 + \frac{a^2}{2}\right) + \frac{1}{z-1/2}\left(2b + \frac{b}{2} + \frac{b}{2}\right) \Bigg\}$$

$$\times \Bigg\{ \frac{1}{\xi^2(2)\xi(3)} + (z-1/2)\left(\frac{-2\xi'(2)}{\xi^3(2)\xi(3)} - \frac{\xi'(3)}{\xi^2(2)\xi^2(3)} + \frac{\xi'(2)}{\xi^3(2)\xi(3)} \right)$$

$$+ (z-1/2)^2 \left(\frac{2\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)} - \frac{2(\xi'(2))^2}{\xi^4(2)\xi(3)} - \frac{\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)} \right)$$

$$+ (z-1/2)^2 \left(-\frac{1}{2\xi(2)\xi(3)}\left(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)} \right) - \frac{1}{2\xi^2(2)}\left(\frac{\xi''(3)}{\xi^2(3)} - \frac{2(\xi'(3))^2}{\xi^3(3)} \right) - \frac{2}{\xi(2)\xi(3)}\left(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)} \right)\right)\Bigg\}$$

$$\times \Big\{ \Phi(\beta_4)\bar\Psi(\beta_4) + (z-1/2)\left(D_{\beta_2}\Phi(\beta_4)\bar\Psi(\beta_4) + \Phi(\beta_4) D_{\beta_2}\bar\Psi(\beta_4) \right) + (z-1/2)^2 \left(D_{\beta_2}\Phi(\beta_4) D_{\beta_2}\bar\Psi(\beta_4) + \tfrac{1}{2} D^2_{\beta_2}\Phi(\beta_4)\bar\Psi(\beta_4) + \tfrac{1}{2}\Phi(\beta_4) D^2_{\beta_2}\bar\Psi(\beta_4) \right)\Big\}\Bigg]$$

$$= \frac{1}{\xi(2)}\Bigg[\frac{1}{2\xi^2(2)\xi(3)}\left(D_{\beta_2}\cancel{\Phi}(\beta_4) D_{\beta_2}\bar\Psi(\beta_4) + \tfrac{1}{2} D^2_{\beta_2}\cancel{\Phi}(\beta_4)\bar\Psi(\beta_4) + \tfrac{1}{2}\Phi(\beta_4) \cancel{D}^2_{\beta_2}\bar\Psi(\beta_4) \right) +$$

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$$+ \frac{1}{2}\left(\frac{-\xi'(2)}{\xi^3(2)\xi(3)} - \frac{\xi'(3)}{\xi^2(2)\xi(3)} \right)\left(D_{\beta_2}\cancel{\Phi}(\beta_4)\bar\Psi(\beta_4) + \Phi(\beta_4)\cancel{D}_{\beta_2}\bar\Psi(\beta_4) \right) + \frac{1}{2}\left(\cancel{\frac{\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)} - \frac{2(\xi'(2))^2}{\xi^4(2)\xi(3)}} \right)\Phi(\beta_4)\bar\Psi(\beta_4)$$

$$+ \frac{1}{2}\left(\frac{-5}{2\xi(2)\xi(3)}\left(\cancel{\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}} \right) - \frac{1}{2\xi^2(2)}\left(\cancel{\frac{\xi''(3)}{\xi^2(3)} - \frac{2(\xi'(3))^2}{\xi^3(3)}} \right)\right)\Phi(\beta_4)\bar\Psi(\beta_4) + \left(3\cancel{b} - \frac{3\cancel{b}}{2}\right)\frac{1}{\xi^2(2)\xi(3)}\Phi(\beta_4)\bar\Psi(\beta_4)$$

Adding these up we see that the total contribution from the integral in 35 is

$$-\frac{1}{\xi^2(2)}\Phi(\beta_4)\cancel{\bar\Phi}(-\beta_6) - \frac{1}{\xi(2)\xi(3)}\Phi(\cancel{\beta_4})\bar\Psi(\beta_6) - \frac{1}{2\xi^2(2)\xi(3)}\Phi(\beta_4)\cancel{D}_{\beta_2}\bar\Psi(\beta_2) - \frac{3}{2\xi^2(2)\xi(3)} D_{\beta_2}\cancel{\Phi}(\beta_4)\bar\Psi(\beta_2) + \frac{1}{\xi^2(2)\xi(3)}\left(\frac{-a}{2} + \frac{3\xi'(3)}{2\cancel{\xi}(3)} \right)\Phi(\beta_4)\bar\Phi(\beta_2)$$

$$+ \frac{1}{4\xi^2(2)\xi(3)}\Phi(\beta_4)\left(\cancel{D^2_{\beta_2} - D^2_{\beta_6}} \right)\bar\Psi(\beta_4) + \cancel{\frac{1}{2\xi^2(2)\xi(3)}} D_{\beta_2}\Phi(\beta_4) D_{\beta_2}\bar\Psi(\beta_4) - \frac{1}{2\xi(2)}\left(\cancel{\frac{\xi'(2)}{\xi^3(2)\xi(3)} + \frac{\xi'(3)}{\xi^2(2)\xi^2(3)}} \right)\Phi(\beta_4) D_{\beta_2}\bar\Psi(\beta_4) - \frac{a}{\xi^3(2)\xi(3)} D_{\beta_2}\cancel{\Phi}(\beta_4)\bar\Psi(\beta_4)$$

$$- \frac{a}{\xi^3(2)\xi(3)}\Phi(\beta_4)\cancel{D}_{\beta_6}\bar\Psi(\beta_4) - \frac{a^2}{\xi^3(2)\xi^2(3)}\Phi(\beta_4)\bar\Psi(\beta_4) + \frac{a}{\xi(2)}\left(\cancel{\frac{\xi'(2)}{\xi^3(2)\xi(3)} + \frac{\xi'(3)}{\xi^2(2)\xi^2(3)}} \right)\Phi(\beta_4)\bar\Psi(\beta_4)$$

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remove

(vi) The coordinate of λ_6 is greater than 3/2.

(a)

$$R(\rho_+) = \frac{1}{\xi(2)} \operatorname{Res}\Big[\Big(\frac{1}{z-1/2} + a + b(z-1/2)\Big)\Big(\frac{1}{2(z-1/2)} + a + 2b(z-1/2)\Big)\Big(\frac{1}{3(z-1/2)} + a + 3b(z-1/2)\Big)\Big(\frac{1}{\xi(2)} - (z-1/2)\frac{\xi'(2)}{\xi^2(2)} - \frac{(z-1/2)^2}{2}\Big(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\Big)\Big)$$

$$\Big(\frac{1}{\xi(2)} - 2(z-1/2)\frac{\xi'(2)}{\xi^2(2)} - \frac{4(z-1/2)^2}{2}\Big(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\Big)\Big)\Big(\frac{1}{\xi(3)} - 3(z-1/2)\frac{\xi'(3)}{\xi^2(3)} - \frac{9(z-1/2)^2}{2}\Big(\frac{\xi''(3)}{\xi^2(3)} - \frac{2(\xi'(3))^2}{\xi^3(3)}\Big)\Big)$$

$$\Big(\bar\Phi(\beta_4) + (z-1/2)D_{\beta_3}\Phi(\beta_4) + \frac{(z-1/2)^2}{2}D^2_{\beta_3}\Phi(\beta_4)\Big)\Big(\bar\Psi(\beta_4) + (z-1/2)D_{\beta_5}\bar\Psi(\beta_4) + \frac{(z-1/2)^2}{2}D^2_{\beta_5}\bar\Psi(\beta_4)\Big)\Big]$$

$$- \frac{1}{3\xi(2)}\frac{\xi(\frac{2}{3})}{\xi(5/3)}\frac{\xi(\frac{1}{3})}{\xi(\frac{4}{3})}\frac{1}{\xi(2)}\Phi\Big(\frac{\beta_5}{3}\Big)\bar\Psi\Big(\frac{\beta_3}{3}\Big)$$

$$= \frac{1}{\xi(2)}\operatorname{Res}\Big[\Big\{\frac{1}{6(z-1/2)^3} + \frac{1}{(z-1/2)^2}\Big(\frac{a}{2} + \frac{a}{3} + \frac{a}{6}\Big) + \frac{1}{z-1/2}\Big(a^2 + \frac{a^2}{2} + \frac{a^2}{3}\Big) + \frac{1}{z-1/2}\Big(\frac{3b}{2} + \frac{2b}{3} + \frac{b}{6}\Big)\Big\}$$

$$\Big\{\frac{1}{\xi^2(2)\xi(3)} + (z-1/2)\Big(-\frac{3\xi'(3)}{\xi^2(2)\xi^2(3)} - \frac{2\xi'(2)}{\xi^3(2)\xi(3)} - \frac{\xi'(2)}{\xi^3(2)\xi(3)}\Big) + (z-1/2)^2\Big(\frac{+6\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)} + \frac{3\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)} + \frac{2(\xi'(2))^2}{\xi^4(2)\xi(3)}\Big)$$

$$+ (z-1/2)^2\Big(-\frac{9}{2\xi^2(2)}\Big(\frac{\xi''(3)}{\xi^2(3)} - \frac{2(\xi'(3))^2}{\xi^3(3)}\Big) - \frac{2}{\xi(2)\xi(3)}\Big(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\Big) - \frac{1}{2\xi(2)\xi(3)}\Big(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\Big)\Big)\Big\}$$

$$\Big\{\bar\Phi(\beta_4)\bar\Psi(\beta_4) + (z-1/2)\big(D_{\beta_3}\Phi(\beta_4)\bar\Psi(\beta_4) + \bar\Phi(\beta_4)D_{\beta_5}\bar\Psi(\beta_4)\big) + (z-1/2)^2\Big(D_{\beta_3}\Phi(\beta_4)D_{\beta_5}\bar\Psi(\beta_4) + \frac{1}{2}D^2_{\beta_3}\Phi(\beta_4)\bar\Psi(\beta_4) + \frac{1}{2}\Phi(\beta_4)D^2_{\beta_5}\bar\Psi(\beta_4)\Big)\Big\}$$

$$- \frac{\xi(2/3)\xi(1/3)}{3\xi^2(2)\xi(5/3)\xi(4/3)}\Phi\Big(\frac{\beta_5}{3}\Big)\bar\Psi(\beta_3/3)$$

8 4 7

$$= \frac{1}{\xi(2)}\Big[\frac{1}{6\xi^2(2)\xi(3)}\Big\{D_{\beta_3}\Phi(\beta_4)D_{\beta_5}\bar\Psi(\beta_4) + \frac{1}{2}D^2_{\beta_3}\Phi(\beta_4)\bar\Psi(\beta_4) + \frac{1}{2}\Phi(\beta_4)D^2_{\beta_5}\bar\Psi(\beta_4)\Big\}$$

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5 9 10 11

$$+ \frac{1}{2}\Big(-\frac{\xi'(3)}{\xi^2(2)\xi^2(3)} - \frac{\xi'(2)}{\xi^3(2)\xi(3)}\Big)\big(D_{\beta_3}\Phi(\beta_4)\bar\Psi(\beta_4) + \Phi(\beta_4)D_{\beta_5}\bar\Psi(\beta_4)\big) + \frac{a}{\xi^2(2)\xi(3)}\big(D_{\beta_3}\Phi(\beta_4)\bar\Psi(\beta_4) + \Phi(\beta_4)D_{\beta_5}\bar\Psi(\beta_4)\big)$$

6 7 13

$$+ \Big\{\frac{1}{6}\Big(\frac{+9\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)} + \frac{2(\xi'(2))^2}{\xi^4(2)\xi(3)}\Big) + \frac{1}{6}\Big(\frac{-9}{2\xi^2(2)}\Big(\frac{\xi''(3)}{\xi^2(3)} - \frac{2(\xi'(3))^2}{\xi^3(3)}\Big) - \frac{5}{2\xi(2)\xi(3)}\Big(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)}\Big)\Big) + a\Big(\frac{-3\xi'(3)}{\xi^2(2)\xi^2(3)} - \frac{3\xi'(2)}{\xi^3(2)\xi(3)}\Big)\Big\}\Phi(\beta_4)\bar\Psi(\beta_4)$$

12 1 3

$$+ \frac{(11/6\, a^2 + 7/3\, b)}{\xi^2(2)\xi(3)}\Phi(\beta_4)\bar\Psi(\beta_4)\Big] - \frac{\xi(2/3)\xi(1/3)}{3\xi^2(2)\xi(5/3)\xi(4/3)}\Phi\Big(\frac{\beta_5}{3}\Big)\bar\Psi(\beta_3/3)$$

(b)

1

$$R(\rho_-) = \frac{1}{\xi^2(2)}\bar\Phi(\beta_4)\bar\Psi(-\beta_6)$$

(c)

2 2

$$R(\sigma_+) = \frac{1}{\xi(2)}\frac{\xi(2)}{\xi(3)}\frac{1}{\xi(2)}\Phi(\beta_4)\bar\Psi(\beta_6) + \frac{1}{3\xi(2)}\frac{\xi(4/3)}{\xi(2)\xi(5/3)}\Phi\Big(\frac{\beta_5}{3}\Big)\bar\Psi\Big(\frac{\beta_5}{3}\Big)$$

(d) $$R(\sigma_-) = \frac{1}{\xi(2)} \operatorname{Res}\Big[\Big(\frac{1}{z-1/2}+a\Big)\Big(\frac{1}{2(z-1/2)}+a\Big)\big(\xi(2)+3(z-1/2)\xi'(2)\big)\Big(\frac{1}{\xi(2)}-(z-1/2)\frac{\xi'(2)}{\xi^2(2)}\Big)\Big(\frac{1}{\xi(2)}-2(z-1/2)\frac{\xi'(2)}{\xi^2(2)}\Big)\Big(\frac{1}{\xi(3)}-3(z-1/2)\frac{\xi'(3)}{\xi^2(3)}\Big)$$

$$\cdot\big(\bar\Phi(\beta_4)+(z-1/2)D_{\beta_3}\bar\Phi(\beta_4)\big)\big(\bar\Psi(\beta_2)+(z-1/2)D_{\beta_2}\bar\Psi(\beta_2)\big)\Big] + \frac{1}{3\xi(2)}\frac{\xi(\frac{2}{3})\,\xi(\frac{1}{3})}{\xi(\frac{5}{3})\,\xi(\frac{4}{3})}\frac{1}{\xi(2)}\bar\Phi\Big(\frac{\beta_5}{3}\Big)\bar\Psi\Big(\frac{\beta_3}{3}\Big)$$

$$= \frac{1}{\xi(2)}\operatorname{Res}\Big[\Big(\frac{1}{2(z-1/2)^2}+\frac{3a}{2(z-1/2)}\Big)\Big(\frac{1}{\xi(2)\xi(3)}+(z-1/2)\Big(\frac{3\xi'(2)}{\xi^2(2)\xi(3)}-\frac{\xi'(2)}{\xi(3)\xi^2(2)}-\frac{2\xi'(2)}{\xi(3)\xi^2(2)}-\frac{3\xi'(3)}{\xi(2)\xi^2(3)}\Big)\Big)$$

$$\big(\bar\Phi(\beta_4)\bar\Psi(\beta_2)+(z-1/2)\big(D_{\beta_3}\bar\Phi(\beta_4)\bar\Psi(\beta_2)+\bar\Phi(\beta_4)D_{\beta_2}\bar\Psi(\beta_2)\big)\Big] + \frac{\xi(1/3)\xi(2/3)}{2\,\xi(4/3)\xi(5/3)\,\xi^2(2)}\bar\Phi\Big(\frac{\beta_5}{3}\Big)\bar\Psi\Big(\frac{\beta_3}{3}\Big)$$

$$= \frac{1}{2\xi^2(2)\xi(3)}\big(D_{\beta_3}\overset{5}{\cancel{\bar\Phi(\beta_4)}}\bar\Psi(\beta_2)+\bar\Phi(\beta_4)\overset{4}{\cancel{D_{\beta_2}}}\bar\Psi(\beta_2)\big) + \frac{1}{2\xi(2)}\overset{6}{\cancel{\Big(\frac{-3\xi'(3)}{\xi(2)\xi^2(3)}\Big)}}\bar\Phi(\beta_4)\bar\Psi(\beta_2)$$

$$+ \frac{3a}{2\xi^2(2)\overset{6}{\cancel{\xi(3)}}}\bar\Phi(\beta_4)\bar\Psi(\beta_2) + \frac{\xi(1/3)\xi(2/3)}{3\,\xi(4/3)\xi(5/3)\overset{3}{\cancel{\xi^2(2)}}}\bar\Phi\Big(\frac{\beta_5}{3}\Big)\bar\Psi\Big(\frac{\beta_3}{3}\Big)$$

(e)

$R(z_+) = 0$

(f) $R(z_-)$ is the sum of two terms, one coming from two simple poles of $n(z_-,\lambda)$ the other coming from a pole of $n(z,\lambda)$ of order three. The first term is, if ρ is one-half the sum of the positive roots,

$$-\frac{1}{3\xi(2)}\frac{\xi(\frac{4}{3})}{\xi(\frac{5}{3})}\overset{2}{\cancel{\frac{\xi(\frac{1}{3})}{\xi(\frac{4}{3})}}}\frac{1}{\xi(2)}\bar\Phi\Big(\frac{\beta_5}{3}\Big)\bar\Psi\Big(\frac{\beta_5}{3}\Big)^{\times} + \frac{\xi(3)\,\xi(4)}{\xi(2)\,\xi(3)\,\xi(4)\overset{3}{\cancel{\xi(1)}}}\bar\Phi(\rho)\bar\Psi(\rho) \qquad \text{N. B. } \xi(\tfrac{1}{3}) = \xi(\tfrac{2}{3})$$

The second term is

$$\frac{1}{\xi(2)}\operatorname{Res}\Big[\Big(\frac{-1}{z-1/2}+a-b(z-1/2)\Big)\Big(\frac{1}{2(z-1/2)}+a+2b(z-1/2)\Big)\Big(\frac{1}{3(z-1/2)}+a+3b(z-1/2)\Big)\Big(\frac{1}{\xi(2)}-(z-1/2)\frac{\xi'(2)}{\xi^2(2)}-\frac{(z-1/2)^2}{2}\Big(\frac{\xi''(2)}{\xi^2(2)}-\frac{2(\xi'(2))^2}{\xi^3(2)}\Big)\Big)$$

$$\Big(\frac{1}{\xi(2)}-2(z-1/2)\frac{\xi'(2)}{\xi^2(2)}-\frac{4(z-1/2)^2}{2}\Big(\frac{\xi''(2)}{\xi^2(2)}-\frac{2(\xi'(2))^2}{\xi^3(2)}\Big)\Big)\Big(\frac{1}{\xi(3)}-3(z-1/2)\frac{\xi'(3)}{\xi^2(3)}-\frac{9(z-1/2)^2}{2}\Big(\frac{\xi''(3)}{\xi^2(3)}-\frac{2(\xi'(3))^2}{\xi^3(3)}\Big)\Big)$$

$$\Big(\bar\Phi(\beta_4)+(z-1/2)D_{\beta_3}\bar\Phi(\beta_4)+\frac{(z-1/2)^2}{2}D^2_{\beta_3}\bar\Phi(\beta_4)\Big)\Big(\bar\Psi(\beta_4)+(z-1/2)D_{\beta_3}\bar\Psi(\beta_4)+\frac{(z-1/2)^2}{2}D^2_{\beta_3}\bar\Psi(\beta_4)\Big)$$

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$$= \frac{1}{\xi(2)} \operatorname{Res} \Big[\Big\{ \frac{-1}{6(z-1/2)^3} + \frac{1}{(z-1/2)^2}\left(\frac{a}{2} - \frac{a}{3} + \frac{a}{6}\right) + \frac{1}{z-1/2}\left(-a^2 + \frac{a^2}{2} + \frac{a^2}{3}\right) + \frac{1}{z-1/2}\left(\frac{3b}{2} - \frac{2b}{3} - \frac{b}{6}\right)\Big\}$$

$$\Big\{ \frac{1}{\xi^2(2)\xi(3)} + (z-1/2)\left(-\frac{3\xi'(3)}{\xi^2(2)\xi^2(3)} - \frac{2\xi'(2)}{\xi^3(2)\xi(3)} - \frac{\xi'(2)}{\xi^3(2)\xi(3)} \right) + (z-1/2)^2 \left(\frac{6\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)} + \frac{3\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)} + \frac{2(\xi'(2))^2}{\xi^4(2)\xi(3)} \right)$$

$$+ (z-1/2)^2 \left(-\frac{1}{2\xi(2)\xi(3)}\left(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)} \right) - \frac{2}{\xi(2)\xi(3)}\left(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)} \right) - \frac{9}{2\xi^2(2)}\left(\frac{\xi''(3)}{\xi^2(3)} - \frac{2(\xi'(3))^2}{\xi^3(3)} \right) \right) \Big\}$$

$$\Big\{ \Phi(\beta_4)\overline{\Phi}(\beta_4) + (z-1/2)\left(\Phi(\beta_4) D_{\beta_3}\overline{\Phi}(\beta_4) + D_{\beta_3}\Phi(\beta_4)\overline{\Phi}(\beta_4) \right) + (z-1/2)^2 \left(D_{\beta_3}\Phi(\beta_4) D_{\beta_3}\overline{\Phi}(\beta_4) \right)$$

$$+ (z-1/2)^2 \left(-\frac{1}{2} D^2_{\beta_3}\Phi(\beta_4)\overline{\Phi}(\beta_4) + \frac{1}{2}\Phi(\beta_4) D^2_{\beta_3}\overline{\Phi}(\beta_4) \right) \Big\}$$

$$= \frac{1}{\xi(2)} \Big[-\frac{1}{6\xi^2(2)\xi(3)} \left(\cancel{D_{\beta_3}\Phi(\beta_4) D_{\beta_3}\overline{\Phi}(\beta_4)} + \cancel{\tfrac{1}{2} D^2_{\beta_3}\Phi(\beta_4)\overline{\Phi}(\beta_4)} + \cancel{\tfrac{1}{2}\Phi(\beta_4) D^2_{\beta_3}\overline{\Phi}(\beta_4)} \right)$$

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$$+ \frac{1}{2}\left(\frac{\xi'(3)}{\xi^2(2)\xi^2(3)} + \frac{\xi'(2)}{\xi^3(2)\xi(3)} \right) \left(\cancel{\Phi(\beta_4) D_{\beta_3}\overline{\Phi}(\beta_4)} + \cancel{D_{\beta_3}\Phi(\beta_4)\overline{\Phi}(\beta_4)} \right)$$

$$- \frac{2a}{3\xi^2(2)\xi(3)} \left(\cancel{\Phi(\beta_4) D_{\beta_3}\overline{\Phi}(\beta_4)} + \cancel{D_{\beta_3}\Phi(\beta_4)\overline{\Phi}(\beta_4)} \right) - \frac{1}{6}\left(\cancel{\frac{9\xi'(2)\xi'(3)}{\xi^3(2)\xi^2(3)} + \frac{2(\xi'(2))^2}{\xi^4(2)\xi(3)}} \right) \Phi(\beta_4)\overline{\Phi}(\beta_4)$$

$$+ 2a \left(\cancel{\frac{\xi'(3)}{\xi^2(2)\xi^2(3)} + \frac{\xi'(2)}{\xi^3(2)\xi(3)}} \right) \Phi(\beta_4)\overline{\Phi}(\beta_4) + \frac{\left(-\frac{a^2}{6} - \frac{7}{3}b\right)}{\xi^2(2)\xi(3)} \Phi(\beta_4)\overline{\Phi}(\beta_4) - \frac{1}{6\xi(2)}\left\{ \cancel{\frac{-5}{2\xi(2)\xi(3)}\left(\frac{\xi''(2)}{\xi^2(2)} - \frac{2(\xi'(2))^2}{\xi^3(2)} \right)} - \cancel{\frac{9}{2\xi^2(2)}\left(\frac{\xi''(3)}{\xi^2(3)} - \frac{2(\xi'(3))^2}{\xi^3(3)} \right)} \right\} \Phi(\beta_4)\overline{\Phi}(\beta_4) \Big]$$

remove

Adding these up we see that the total contribution from the integral in $\mathcal{S}_6$ is

$$\frac{1}{\xi^2(2)}\Phi(\beta_4)\overline{\Phi}(-\beta_6) + \frac{1}{\xi(2)\xi(3)}\Phi(\beta_4)\overline{\Phi}(\beta_6) + \frac{1}{\xi(2)\xi(6)}\Phi(\beta)\overline{\Phi}(\beta) + \frac{1}{2\xi^2(2)\xi(3)}\Phi(\beta_4) D_{\beta_3}\overline{\Phi}(\beta_2) + \frac{1}{2\xi^2(2)\xi(3)} D_{\beta_3}\Phi(\beta_4)\overline{\Phi}(\beta_2)$$

$$+ \frac{3}{2\xi^2(2)\xi(3)}\left(a - \frac{\xi'(3)}{\xi(3)} \right)\Phi(\beta_4)\overline{\Phi}(\beta_2) + \frac{1}{12\xi^3(2)\xi(3)}\Phi(\beta_4)\left(D^2_{\beta_5} - D^2_{\beta_3} \right)\overline{\Phi}(\beta_4) - \frac{1}{6\xi^3(2)\xi(3)} D_{\beta_3}\Phi(\beta_4) D_{\beta_2}\overline{\Phi}(\beta_4) + \frac{1}{2\xi(2)}\left(\frac{\xi'(3)}{\xi^2(2)\xi^2(3)} + \frac{\xi'(2)}{\xi^3(2)\xi(3)} \right)\Phi(\beta_4) D_{\beta_2}\overline{\Phi}(\beta_4)$$

$$+ \frac{a}{3\xi^3(2)\xi(3)} D_{\beta_3}\Phi(\beta_4)\overline{\Phi}(\beta_4) + \frac{a}{\xi^3(2)\xi(3)}\Phi(\beta_4)\left(D_{\beta_6} - \tfrac{1}{3} D_{\beta_2} \right)\overline{\Phi}(\beta_4) + \frac{5a^2}{3\xi^3(2)\xi(3)}\Phi(\beta_4)\overline{\Phi}(\beta_4) - \frac{a}{\xi(2)}\left(\frac{\xi'(2)}{\xi^3(2)\xi(3)} + \frac{\xi'(3)}{\xi^2(2)\xi^2(3)} \right)\Phi(\beta_4)\overline{\Phi}(\beta_4)$$

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$D_5 - \frac{2}{3} D_3$

Now we have add together the residues from each of the six integrals. In matrix notation, the result is

$$\begin{bmatrix} \Psi(\rho) \\ \Psi(\beta_2) \\ \Psi(\beta_4) \\ D_{\alpha_1}\Psi(\beta_4) \end{bmatrix}^{*} \begin{bmatrix} \frac{1}{\xi(2)\xi(6)} & 0 & 0 & 0 \\ 0 & \frac{3}{2\,\xi(2)\xi(3)} & \frac{a}{\xi^2(2)\xi(3)} & \frac{-1}{2\,\xi^2(2)\xi(3)} \\ 0 & \frac{a}{\xi^2(2)\xi(3)} & \frac{2a^2}{3\,\xi^3(2)\xi(3)} & \frac{-a}{3\,\xi^3(2)\xi(3)} \\ 0 & \frac{-1}{2\,\xi^2(2)\xi(3)} & \frac{-a}{3\,\xi^3(2)\xi(3)} & \frac{4}{6\,\xi^3(2)\xi(3)} \end{bmatrix} \begin{bmatrix} \Phi(\rho) \\ \Phi(\beta_2) \\ \Phi(\beta_4) \\ D_{\alpha_1}\Phi(\beta_4) \end{bmatrix}$$

Notice that the 3×3 block in the lower right hand corner is of rank one.

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2 The group A_2 over the field of rational numbers.

The root diagram is

$\beta_1 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$

$\beta_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$

$\beta_3 = (1, 0)$

$H_{\beta_1} = (-1, \sqrt{3})$

$H_{\beta_2} = (1, \sqrt{3})$

$H_{\beta_3} = (2, 0)$

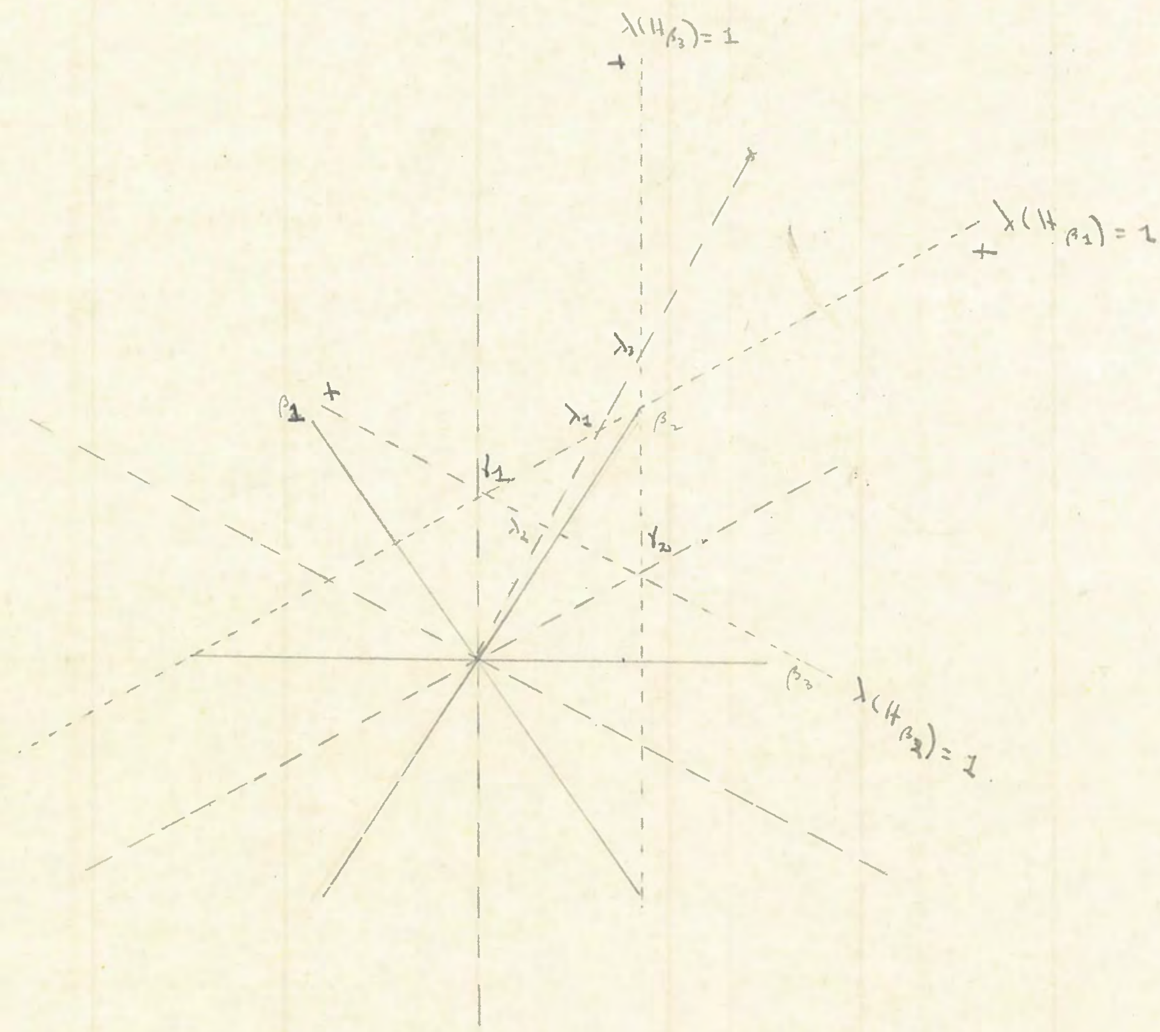
If the Lie algebra of A_2 is taken to be the Lie algebra of all 3×3 matrices of trace zero then a Cartan subalgebra is the set of all diagonal matrices $D(x) = D(x_1, x_2, x_3)$ with diagonal entries x_1, x_2, x_3 and $x_1 + x_2 + x_3 = 0$. We identify $D(x)$ with the triple (x_1, x_2, x_3). Then we may suppose $\beta_1(x) = x_1 - x_2$, $\beta_2(x) = x_1 - x_3$, $\beta_3(x) = x_2 - x_3$. The elements of the Weyl group act by permuting x_1, x_2, x_3. We tabulate the elements of the Weyl group together with the positive roots they transform to negative roots. The formula on p. 1 allows us to read off the functions $n(\sigma, \lambda)$ from this table. ρ_i is a reflection in the line $\beta_i = 0$ and σ_θ is a rotation through the angle θ.

σ	$\sigma(x_1, x_2, x_3)$	$\{\beta \mid \sigma\beta < 0\}$
ρ_1	(x_2, x_1, x_3)	β_1
ρ_2	(x_3, x_2, x_1)	$\beta_1, \beta_2, \beta_3$
ρ_3	(x_1, x_3, x_2)	β_3
$1 = \sigma_0$	(x_1, x_2, x_3)	
$\sigma_{2\pi/3}$	(x_2, x_3, x_1)	β_1, β_2
$\sigma_{4\pi/3}$	(x_3, x_1, x_2)	β_2, β_3

The inner product $(\phi^\wedge(\cdot), \psi^\wedge(\cdot))$ is as usual equal to

$$\sum_\sigma \frac{1}{(2\pi)^2} \int_{Re\,\lambda=0} n(\sigma,\lambda)\,\Phi(\lambda)\,\overline{\Psi}(-\bar\sigma\lambda)\,d\lambda$$

plus a remainder. If we work with the simply connected form of A_2 and use the measure introduced in the notes on the volume of fundamental domain for Chevalley groups then $d\lambda = dx_1dx_2$ if z_1 and z_2 are the coordinates defined by Z. Let s_i be the complex line $\lambda(H_{\alpha_i}) = 1$. The remainder is a sum of integrals over vertical lines in $\underline{s}_1$, $\underline{s}_2$, and $\underline{s}_3$. If $\lambda_1, \lambda_2, \lambda_3$ are as in the diagram below these are the lines $Re\,\lambda = \lambda_i$.



Before describing the remainder we have to tabulate the sets $\Omega(\underline{s}_i, \underline{s}_j)$ and find the functions $n(\sigma,\lambda)$ for σ in $\Omega(s_i, s_j)$. The coordinates we shall use are:

On s_1 $\quad z = \lambda(H_{\beta_3}) + \frac{1}{2}$

On s_2 $\quad z = -\lambda(H_{\beta_3}) + \frac{1}{2}$

On s_3 $\quad z = \lambda(H_{\beta_1}) + \frac{1}{2}$

The following table is to be read in the same way as that on page 4.

	1	2	3
1	$\rho\rho\rho^{-1} = \rho_2$: $\frac{1}{\xi(2)}$	$\sigma\rho\rho^{-1} = \sigma_{2\pi/3}$: $\frac{1}{\xi(2)}\, a(z+1/2) = \frac{1}{\xi(2)}\, \frac{\xi(z+1/2)}{\xi(z+3/2)}$	$z\rho\rho^{-1} = \rho_2$: $\frac{1}{\xi(2)}\, a(z+1/2)\, a(z-1/2) = \frac{1}{\xi(2)}\, \frac{\xi(z-1/2)}{\xi(z+3/2)}$
2	$\rho\rho\sigma^{-1} = \sigma_{4\pi/3}$: $\frac{1}{\xi(2)}\, a(\frac{1}{2}-z) = \frac{1}{\xi(2)}\, \frac{\xi(1/2-z)}{\xi(3/2-z)}$	$\sigma\rho\sigma^{-1} = \rho_2$: $\frac{1}{\xi(2)}\, a(\frac{1}{2}-z)\, a(\frac{1}{2}+z) = \frac{1}{\xi(2)}\, \frac{\xi(\frac{1}{2}-z)}{\xi(\frac{3}{2}-z)}\, \frac{\xi(\frac{1}{2}+z)}{\xi(3/2+z)}$	$z\rho\sigma^{-1} = \sigma_{2\pi/3}$: $\frac{1}{\xi(2)}\, a(\frac{1}{2}+z) = \frac{1}{\xi(2)}\, \frac{\xi(1/2+z)}{\xi(3/2+z)}$
3	$\rho\rho z^{-1} = \rho_2$: $\frac{1}{\xi(2)}\, a(z-1/2)\, a(z+1/2) = \frac{1}{\xi(2)}\, \frac{\xi(z-1/2)}{\xi(z+3/2)}$	$\sigma\rho z^{-1} = \sigma_{4\pi/3}$: $\frac{1}{\xi(2)}\, a(z+1/2) = \frac{1}{\xi(2)}\, \frac{\xi(z+1/2)}{\xi(z+3/2)}$	$z\rho z^{-1} = \rho_3$: $\frac{1}{\xi(2)}$

The matrix $M(H)$ is written as $M(z)$ where z is the coordinate on $\mathfrak{s}_1$. In the table below the element $M(\sigma\rho z^{-1}, z\rho H)$, where $H = H(z)$, is put in the row labelled σ and the column labelled z.

	ρ	σ	z
ρ	$\frac{1}{\xi(2)}$	$\frac{1}{\xi(2)}\, \frac{\xi(\frac{1}{2}-z)}{\xi(\frac{3}{2}-z)}$	$\frac{1}{\xi(2)}\, \frac{\xi(-z-1/2)}{\xi(-z+3/2)}$
σ	$\frac{1}{\xi(2)}\, \frac{\xi(z+1/2)}{\xi(z+3/2)}$	$\frac{1}{\xi(2)}\, \frac{\xi(\frac{1}{2}-z)}{\xi(\frac{3}{2}-z)}\, \frac{\xi(\frac{1}{2}+z)}{\xi(\frac{3}{2}+z)}$	$\frac{1}{\xi(2)}\, \frac{\xi(-z+1/2)}{\xi(-z+3/2)}$
z	$\frac{1}{\xi(2)}\, \frac{\xi(z-1/2)}{\xi(z+3/2)}$	$\frac{1}{\xi(2)}\, \frac{\xi(1/2+z)}{\xi(3/2+z)}$	$\frac{1}{\xi(2)}$

The matrix is of rank one. The remainder is of course

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \sum_{\sigma \in \Omega(\mathfrak{s}_i, \mathfrak{s}_j)} \frac{1}{2\pi} \int_{\operatorname{Re}\lambda = \lambda_i} n(\sigma, \lambda)\, \Phi(\lambda)\, \overline{\Phi(-\sigma\bar{\lambda})}\, d\lambda$$

As before we replace these integrals by integrals over the line $\operatorname{Re}\lambda = \lambda_i'$ if λ_i' is that point in $\mathfrak{s}_i$ whose coordinate is zero. Each of the six expressions

$$R(\sigma) = \frac{1}{2\pi} \int_{\mathrm{Re}\,\lambda = \lambda_1} \eta(\sigma,\lambda)\,\Phi(\lambda)\,\bar{\bar{\Psi}}(-\sigma\tilde{\lambda})\,d\lambda - \frac{1}{2\pi} \int_{\mathrm{Re}\,\lambda = \lambda_1'} \eta(\sigma,\lambda)\,\Phi(\lambda)\,\bar{\bar{\Psi}}(-\sigma\tilde{\lambda})\,d\lambda$$

Let us calculate this difference.

(i) The coordinate of λ_1 satisfies the inequality $\tfrac{1}{2} < z < 1$

(a)

$$R(\rho) = 0$$

(b) $$R(\sigma) = \frac{1}{\xi^2(2)}\,\bar{\Phi}(\gamma_2)\,\bar{\Psi}(\gamma_2)$$

(c) $$R(z) = -\frac{1}{\xi^2(2)}\,\bar{\Phi}(\gamma_1)\,\bar{\Psi}(\gamma_2)$$

Thus the total contribution to the remainder from s_1 is zero.

(ii) The total contribution to the remainder from s_2 is clearly zero

(iii) The coordinate of λ_3 is greater than $3/2$. If ρ is one half the sum of the positive roots.

(a) $$R(\rho\rho^{z-1}) = \frac{1}{\xi(2)\,\xi(3)}\,\bar{\Phi}(\rho)\,\bar{\Phi}(\rho) - \frac{1}{\xi^2(2)}\,\bar{\Phi}(\gamma_2)\,\bar{\Psi}(\gamma_1)$$

(b) $$R(\sigma\rho z^{-1}) = \frac{1}{\xi^2(2)}\,\bar{\Phi}(\gamma_2)\,\bar{\Psi}(\gamma_1)$$

(c) $$R(z\rho z^{-1}) = 0$$

The total contribution to the remainder from s_3 is

$$\frac{1}{\xi(2)\,\xi(3)}\,\bar{\Phi}(\rho)\,\bar{\Psi}(\rho)$$

Thus the remainder is

$$\frac{1}{\xi(2)\,\xi(3)}\,\bar{\Phi}(\rho)\,\bar{\Psi}(\rho)$$

3. The group A_1 over the field of rational numbers.

The situation here is very simple and I include it only for completeness. Let α be the single positive root. We work with the simply connected group so that if $z = \lambda(H_\alpha)$ then $d\lambda = dz$. The inner product (ϕ', ψ') is equal to

$$\frac{1}{2\pi} \int_{\operatorname{Re} z > 1} \Phi(z)\overline{\Psi}(-\bar{z}) + \frac{\xi(z)}{\xi(z+1)} \Phi(z)\overline{\Psi}(\bar{z}) \, dz$$

$$= \frac{1}{2\pi} \int_{\operatorname{Re} z = 0} \left\{ \Phi(z)\overline{\Psi}(-\bar{z}) + \frac{\xi(z)}{\xi(z+1)} \Phi(z)\overline{\Psi}(\bar{z}) \right\} dz \quad + \frac{1}{\xi(2)} \Phi(1)\overline{\Psi}(1)$$

AUTOMORPHIC FORMS FOR A UNITARY GROUP

1. $V = \mathbb{Q}^2$ the space of column vectors of length 2 over $\mathbb{Q}$

   L: lattice of integral vectors in V.

   $K = \mathbb{Q}(\sqrt{-1})$, O integers in K

   $W = V \otimes_{\mathbb{Q}} K \simeq K^2$, a four dimensional vector space over $\mathbb{Q}$

   σ: non-trivial element in $G_{K/\mathbb{Q}}$. σ of course acts on W.

2. If $x = (x_1, x_2)$ and $y = (y_1, y_2)$ lie in W set $(x, y) = x_1y_1 + x_2y_2$.

3. $U_{\mathbb{Q}}$ set of all linear transformations A of W over $\mathbb{Q}$ such that

   (i) $(Aw)a = A(wa)$ $\quad \forall a \in K$

   (ii) $(Ax, \sigma Ay) = (x, \sigma y)$ $\quad \forall x, y \in W$.

   (iii) $\det A$ (as a linear transformation of W as a vector space over K) is 1.

4. If p is a finite or infinite prime set.

   $$W_p = W \otimes_{\mathbb{Q}} \mathbb{Q}_p = V \otimes_{\mathbb{Q}} (K \otimes_{\mathbb{Q}} \mathbb{Q}_p)$$

   Let $U_{\mathbb{Q}_p}$ be the set of all linear transformations, the group defined in (3) with entries from $\mathbb{Q}_p$ and let $U_{\mathbb{Z}_p}$ be the set of all A in $U_{\mathbb{Q}_p}$ which take $L \otimes_{\mathbb{Z}} O \otimes_{\mathbb{Z}} \mathbb{Z}_p$ onto itself

5. The elements of $U_{\mathbb{Q}}$ may be represented as matrices

   $$\begin{bmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{bmatrix}$$

   with $\alpha = a + bi$, $\beta = c + di$ and $a^2 + b^2 + c^2 + d^2 = 1$. As linear transformations acting on W they are represented by matrices

   $$\begin{bmatrix} a & -b & c & -d \\ b & a & d & c \\ -c & -d & a & b \\ d & -c & -b & a \end{bmatrix}$$

   Thus the elements of $U_{\mathbb{Q}_p}$ are the matrices of this form in $\mathbb{Q}_p$ with $a^2+b^2+c^2+d^2 = 1$. The elements of $U_{\mathbb{Z}_p}$ are the matrices with $a, b, c, d \in \mathbb{Z}_p$

6. (a) If p splits in K let j be a square root of -1 in $\mathbb{Z}_p$. Then

$$\frac{1}{2}\begin{bmatrix} 1 & -j & 0 & 0 \\ 1 & j & 0 & 0 \\ 0 & 0 & 1 & -j \\ 0 & 0 & 1 & j \end{bmatrix}\begin{bmatrix} a & -b & c & -d \\ b & a & d & c \\ -c & -d & a & b \\ d & -c & -b & a \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ j & -j & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & j & -j \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -j & 0 & 0 \\ 1 & j & 0 & 0 \\ 0 & 0 & 1 & -j \\ 0 & 0 & 1 & j \end{bmatrix}\begin{bmatrix} a-bj & a+bj & c-dj & c+dj \\ b+aj & b-aj & d+cj & d-cj \\ -c-dj & -c+dj & a+bj & a-bj \\ d-cj & d+cj & -b+aj & -b-aj \end{bmatrix}$$

$$= \begin{bmatrix} a-bj & 0 & c-dj & 0 \\ 0 & a+bj & 0 & c+dj \\ -c-dj & 0 & a+bj & 0 \\ 0 & -c+dj & 0 & a-bj \end{bmatrix}$$

Thus the map of the matrix of s to

$$\begin{bmatrix} a+bj & c+dj \\ -c+dj & a-bj \end{bmatrix}$$

is an isomorphism of U_{Q_q} with $SL(2,Q_q)$. Since $q \neq (2)$ the image of U_{Z_q} is $SL(2, Z_q)$

(b) Suppose $q \neq (2)$, and does not split in K. Let $K_q = K \otimes_Q Q_q$, $O_q = O \otimes_Z \mathbb{Z}_q$.
Choose γ in O_q so that $\gamma\gamma^\sigma = -1$. The map

$$\begin{bmatrix} \alpha & \beta \\ -\beta^\sigma & \alpha^\sigma \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^\sigma \end{bmatrix}\begin{bmatrix} \alpha & \beta \\ -\beta^\sigma & \alpha^\sigma \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \gamma \end{bmatrix} = \begin{bmatrix} \alpha & \beta\gamma \\ (\beta\gamma)^\sigma & \alpha^\sigma \end{bmatrix} = \begin{bmatrix} \alpha & \beta\gamma \\ \overline{\beta\gamma} & \bar\alpha \end{bmatrix}$$

is an isomorphism of U_{Q_q} with the unitary group of the form $|x_1|^2 - |x_2|^2$. The image of U_{Z_q} is V_{Z_q}.
Then

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} \alpha & \beta\gamma \\ \overline{\beta\gamma} & \bar\alpha \end{bmatrix}\begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\begin{bmatrix} \alpha+\beta\gamma & \alpha i - \beta\gamma i \\ \overline{\beta\gamma} + \bar\alpha & \overline{\beta\gamma} i - \bar\alpha i \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\alpha+\bar\alpha}{2} + \frac{\beta\gamma + \overline{\beta\gamma}}{2} & \frac{\bar\alpha - \alpha}{2i} + \frac{\beta\gamma - \overline{\beta\gamma}}{2i} \\ \frac{\alpha-\bar\alpha}{2i} + \frac{\beta\gamma - \overline{\beta\gamma}}{2i} & \frac{\alpha+\bar\alpha}{2} - \frac{\beta\gamma + \overline{\beta\gamma}}{2} \end{bmatrix}$$

Thus we have a map of U_{Q_q} onto $SL(2, Q_q)$ which takes $U_{\mathbb{Z}_q}$ to $SL(2,\mathbb{Z}_q)$

7. Let $U_\mathbb{Z}$ be the group of integral matrices in U_Q, that is, the group of matrices which take $L \otimes_\mathbb{Z} O$ onto itself.

$$U_\mathbb{Z} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \,\middle|\, a, b \in O,\ |a|^2 + |b|^2 = 1 \right\}$$

Then $a = \pm 1, \pm i$, $b = 0$ or $b = \pm 1, \pm i$, $a = 0$ and $U_\mathbb{Z}$ is a group of order 16 (8?)

8. Every element of $U_{\mathbb{Z}_2}$ is a product of an element of $U_\mathbb{Z}$ and an element of $\overset{0}{U}_{\mathbb{Z}_2} = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar\alpha & \bar\beta \end{pmatrix} \in \mathbb{Z}_2 \,\middle|\, \alpha \equiv 1 \pmod 2,\ \beta \equiv 0 \pmod 2 \right\}$. Indeed if $\alpha = a + ib$, $\beta = b + ic$ and $a^2 + b^2 + c^2 + d^2 = 1$ then exactly one of a, b, c, d is odd and thus $\begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix}$ is congruent to an element of $U_\mathbb{Z}$ mod 2. Observe

that $U_{\mathbb{Z}} \cap U^{\circ}_{\mathbb{Z}_2} = \{\pm I\}$

9. Lemma: If U_A is the idele group of U and if $U_0 = U_{\mathbb{R}} \times U^{\circ}_{\mathbb{Z}_2} \times \prod_{p \neq 2} U_{\mathbb{Z}_p}$ then $U_A = U_{\mathbb{Q}} \cdot U_0$.

By the preceding section it is enough to show that if $U_1 = U_{\mathbb{R}} \times U_{\mathbb{Z}_2} \times \prod_{p\neq 2} U_{\mathbb{Z}_p}$ then $U_A = U_{\mathbb{Q}} \cdot U_1$. Let $G_{\mathbb{Q}}$ be $SL(2, K)$. Define G_A and G_1 in the same way as U_A and U_1. $U_A \subseteq G_A$. Since the class number of K is one every element g of U_A is a product $g_1 g_2^{-1}$ with $g_1 \in G_{\mathbb{Q}}$, $g_2 \in G_1$. The form $(g_2 x, g_2 x)$ is integral and of determinant one. Since by the Kenigs thesis of Victor Katz there is only one class of integral unitary forms of determinant one $g_2 = g_3 g_4$ with $g_3 \in U_{\mathbb{Q}}$, $g_4 \in G_{\mathbb{Z}}$. Then $g = g_3 (g_4 g_2^{-1})$ and $g_4 g_2^{-1} \in U_A \cap G_1 = U_1$.

10 Rather than U_A we shall consider U'_A the adele group of the group obtained from U by factoring out the centre. Then $U'_{\mathbb{Q}}$ is the set of matrices of the form

that is by letting it act on the 4×4 matrices under inner automorphisms.

$$\frac{1}{\sqrt{a^2+b^2+c^2+d^2}}\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi\end{pmatrix}$$

11. (a). $U'_{\mathbb{R}} = U_{\mathbb{R}} / \pm 1$.

(b) If p splits then $U'_{\mathbb{Q}_p}$ corresponds to the group of matrices in $SL(2, \mathbb{Q}_p)$ of the form

$$\frac{1}{\sqrt{ad-bc}}\begin{pmatrix} a & b \\ c & d\end{pmatrix} \qquad a, b, c, d \in O_p$$

The matrix corresponds to an element of $U'_{\mathbb{Z}_p}$ if $ad-bc$ is a unit.

(c) If $p \neq 2$ does not split then $U'_{\mathbb{Q}_p}$ corresponds to the group of matrices in $SL(2, \mathbb{Q}_p)$ of the form

$$\frac{1}{\sqrt{ad-bc}}\begin{pmatrix} a & b \\ c & d\end{pmatrix} \qquad a, b, c, d \in O_p$$

The matrix corresponds to an element of $U'_{\mathbb{Z}_p}$ if $ad-bc$ is a unit.

(d) If $p=2$ then $U'_{\mathbb{Q}_p}$ is the image group of matrices of the form

$$\frac{1}{\sqrt[2]{\pm(a^2+b^2+c^2+d^2)}}\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi\end{pmatrix} \qquad \text{(Remember } i \text{ is a symbol)} \quad a, b, c, d \in O_p$$

The element is in $U'_{\mathbb{Z}_p}$ if $ad-bc$ is a unit. The index of the image of $U_{\mathbb{Q}_p}$ in $U'_{\mathbb{Q}_p}$ is the index of the group of squares in $\mathbb{Q}_p$ in the set of numbers $a^2+b^2+c^2+d^2$. The index of $U_{\mathbb{Z}_p}$ in $U'_{\mathbb{Z}_p}$ is the index of the group of squares in $O_p \cap O_p^{-1}$ in the set of numbers $a^2+b^2+c^2+d^2$ in O_p. Since $p=2$ this is 4

12. From the above it follows that $\mathfrak{U}'_A = \mathfrak{U}'_Q \cdot \mathfrak{U}'_1$ and that $\mathfrak{U}'_Z$ is the image of $\mathfrak{U}_Z$. Now

$$\tilde{\tilde{\mathfrak{U}}}'_{Z_2} = \left\{ \frac{1}{\sqrt[2]{(a^2+b^2+c^2+d^2)}} \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \middle| \ a-1 \equiv b \equiv c \equiv d \equiv 0 \pmod 2 \right\}$$ is a normal subgroup of $\mathfrak{U}'_{Z_2}$}

The quotient $\tilde{\tilde{\mathfrak{U}}}'_{Z_2} \backslash \mathfrak{U}'_{Z_2}$ is a group of order 8. It contains the image of $\mathfrak{U}'_Z$ which is a group of order 4. Since

$$\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix}\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} = \begin{pmatrix} a^2-b^2-c^2-d^2+2abi & 2ac+2adi \\ -2ac+2adi & a^2-b^2-c^2-d^2-2abi \end{pmatrix}$$

$$\equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2$$

every element in the group is of order 2. The group is thus commutative and is the direct product of the image of $\mathfrak{U}'_Z$ and the group generated by

$$\frac{1}{\sqrt[2]{3}} \begin{pmatrix} 1 & 1+i \\ -1+i & 1 \end{pmatrix}$$

Let $\tilde{\mathfrak{U}}'_{Z_2}$ be the group generated by $\tilde{\tilde{\mathfrak{U}}}'_{Z_2}$ and this matrix. If $\mathfrak{U}'_0 = \mathfrak{U}'_{\mathbb{R}} \times \tilde{\mathfrak{U}}'_{Z_2} \times \prod_{p \neq 2} \mathfrak{U}'_{Z_p}$ then every element of $\mathfrak{U}'_A$ can be written uniquely as a product of an element of $\mathfrak{U}'_Q$ and an element of $\mathfrak{U}'_0$.

13. We will have a theory of Hecke operators for this group. According to the final assertion above if $p \neq 2$ the projection on $\mathfrak{U}'_{Q_p}$ of the images in $\mathfrak{U}'_Q$ of the matrices

$$\frac{1}{\sqrt{p}} \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \qquad \begin{matrix} a^2+b^2+c^2+d^2 = p \\ a \equiv 1 \pmod 2 \quad c \equiv d \pmod 2 \\ b \equiv 0 \pmod 2 \quad a > 0 \end{matrix}$$

form a set of right cosets with respect to $\mathfrak{U}'_{Z_p}$ of the double coset which generates the Hecke ring for $\mathfrak{U}'_{Q_p}$. We shall tabulate these matrices and then use the table to compute the eigenvalues of some of the Hecke operators on various automorphic forms for $\mathfrak{U}'_Q \backslash \mathfrak{U}'_A$. In the tables the factor $\frac{1}{\sqrt{p}}$ in the front is omitted.

$p=3$

$\begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1-i \\ -1-i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -1+i \\ 1+i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix}$

$p=5$

$\begin{bmatrix} 1+2i & 0 \\ 0 & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1-2i & 0 \\ 0 & 1+2i \end{bmatrix}$ $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 2i \\ 2i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -2i \\ -2i & 1 \end{bmatrix}$

$p=7$

$\begin{bmatrix} 1+2i & 1+i \\ -1+i & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1+2i & 1-i \\ -1-i & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1+2i & -1+i \\ 1+i & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1+2i & -1-i \\ 1-i & 1-2i \end{bmatrix}$

$\begin{bmatrix} 1-2i & 1+i \\ -1+i & 1+2i \end{bmatrix}$ $\begin{bmatrix} 1-2i & 1-i \\ -1-i & 1+2i \end{bmatrix}$ $\begin{bmatrix} 1-2i & -1+i \\ 1+i & 1+2i \end{bmatrix}$ $\begin{bmatrix} 1-2i & -1-i \\ 1-i & 1+2i \end{bmatrix}$

$p=11$

$\begin{bmatrix} 3 & 1+i \\ -1+i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 1-i \\ -1-i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -1+i \\ 1+i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -1-i \\ 1-i & 3 \end{bmatrix}$

$\begin{bmatrix} 1 & 3+i \\ -3+i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 3-i \\ -3-i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -3+i \\ 3+i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -3-i \\ 3-i & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1+3i \\ -1+3i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1-3i \\ -1-3i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -1+3i \\ 1+3i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -1-3i \\ 1-3i & 1 \end{bmatrix}$

$p=13$

$\begin{bmatrix} 1+2i & 2+2i \\ -2+2i & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1+2i & 2-2i \\ -2-2i & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1+2i & -2+2i \\ 2+2i & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1+2i & -2-2i \\ 2-2i & 1-2i \end{bmatrix}$

$\begin{bmatrix} 1-2i & 2+2i \\ -2+2i & 1+2i \end{bmatrix}$ $\begin{bmatrix} 1-2i & 2-2i \\ -2-2i & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1-2i & -2+2i \\ 2+2i & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1-2i & -2-2i \\ 2-2i & 1+2i \end{bmatrix}$

$\begin{bmatrix} 3+2i & 0 \\ 0 & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & 0 \\ 0 & 3+2i \end{bmatrix}$ $\begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 2i \\ 2i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -2i \\ -2i & 3 \end{bmatrix}$

$p=17$

$\begin{bmatrix} 3 & 2+2i \\ -2+2i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 2-2i \\ -2-2i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -2+2i \\ 2+2i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -2-2i \\ 2-2i & 3 \end{bmatrix}$

$\begin{bmatrix} 3+2i & 2 \\ -2 & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3+2i & -2 \\ 2 & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & 2 \\ -2 & 3+2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & -2 \\ 2 & 3+2i \end{bmatrix}$

$\begin{bmatrix} 3+2i & 2i \\ 2i & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3+2i & -2i \\ -2i & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & 2i \\ 2i & 3+2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & -2i \\ -2i & 3+2i \end{bmatrix}$

$\begin{bmatrix} 1+4i & 0 \\ 0 & 1-4i \end{bmatrix}$ $\begin{bmatrix} 1-4i & 0 \\ 0 & 1+4i \end{bmatrix}$ $\begin{bmatrix} 1 & 4 \\ -4 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -4 \\ 4 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 4i \\ 4i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -4i \\ -4i & 1 \end{bmatrix}$

p=19

$\begin{bmatrix} 1 & 3+3i \\ -3+3i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 3-3i \\ -3-3i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -3+3i \\ 3+3i & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -3-3i \\ 3-3i & 1 \end{bmatrix}$

$\begin{bmatrix} 3 & 1+3i \\ -1+3i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 1-3i \\ -1-3i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -1+3i \\ 1+3i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -1-3i \\ 1-3i & 3 \end{bmatrix}$

$\begin{bmatrix} 3 & 3+i \\ -3+i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 3-i \\ -3-i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -3+i \\ 3+i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -3-i \\ 3-i & 3 \end{bmatrix}$

$\begin{bmatrix} 1+4i & 1+i \\ -1+i & 1-4i \end{bmatrix}$ $\begin{bmatrix} 1+4i & 1-i \\ -1-i & 1-4i \end{bmatrix}$ $\begin{bmatrix} 1+4i & -1+i \\ 1+i & 1-4i \end{bmatrix}$ $\begin{bmatrix} 1+4i & -1-i \\ 1-i & 1-4i \end{bmatrix}$

$\begin{bmatrix} 1-4i & 1+i \\ -1+i & 1+4i \end{bmatrix}$ $\begin{bmatrix} 1-4i & 1-i \\ -1-i & 1+4i \end{bmatrix}$ $\begin{bmatrix} 1-4i & -1+i \\ 1+i & 1+4i \end{bmatrix}$ $\begin{bmatrix} 1-4i & -1-i \\ 1-i & 1+4i \end{bmatrix}$

p=23

$\begin{bmatrix} 1+2i & -3+3i \\ -3+3i & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1+2i & 3-3i \\ -3-3i & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1+2i & -3+3i \\ 3+3i & 1-2i \end{bmatrix}$ $\begin{bmatrix} 1+2i & -3-3i \\ 3-3i & 1-2i \end{bmatrix}$

$\begin{bmatrix} 1-2i & 3+3i \\ -3+3i & 1+2i \end{bmatrix}$ $\begin{bmatrix} 1-2i & 3-3i \\ -3-3i & 1+2i \end{bmatrix}$ $\begin{bmatrix} 1-2i & -3+3i \\ 3+3i & 1+2i \end{bmatrix}$ $\begin{bmatrix} 1-2i & -3-3i \\ 3-3i & 1+2i \end{bmatrix}$

$\begin{bmatrix} 3+2i & 1+3i \\ -1+3i & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3+2i & 1-3i \\ -1-3i & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3+2i & -1+3i \\ 1+3i & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3+2i & -1-3i \\ 1-3i & 3-2i \end{bmatrix}$

$\begin{bmatrix} 3-2i & 1+3i \\ -1+3i & 3+2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & 1-3i \\ -1-3i & 3+2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & -1+3i \\ 1+3i & 3+2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & -1-3i \\ 1-3i & 3+2i \end{bmatrix}$

$\begin{bmatrix} 3+2i & 3+i \\ -3+i & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3+2i & 3-i \\ -3-i & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3+2i & -3+i \\ 3+i & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3+2i & -3-i \\ 3-i & 3-2i \end{bmatrix}$

$\begin{bmatrix} 3-2i & 3+i \\ -3+i & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & 3-i \\ -3-i & 3+2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & -3+i \\ 3+i & 3+2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & -3-i \\ 3-i & 3+2i \end{bmatrix}$

p=29

1 1 1 1

$\begin{bmatrix} 3+2i & 4 \\ -4 & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3+2i & -4 \\ 4 & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & 4 \\ -4 & 3+2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & -4 \\ 4 & 3+2i \end{bmatrix}$ $\begin{bmatrix} 3+2i & 4i \\ 4i & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3+2i & -4i \\ -4i & 3-2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & 4i \\ 4i & 3+2i \end{bmatrix}$ $\begin{bmatrix} 3-2i & -4i \\ -4i & 3+2i \end{bmatrix}$

$\begin{bmatrix} 3+4i & 2 \\ -2 & 3-4i \end{bmatrix}$ $\begin{bmatrix} 3+4i & -2 \\ 2 & 3-4i \end{bmatrix}$ $\begin{bmatrix} 3-4i & 2 \\ -2 & 3+4i \end{bmatrix}$ $\begin{bmatrix} 3-4i & -2 \\ 2 & 3+4i \end{bmatrix}$ $\begin{bmatrix} 3+4i & 2i \\ 2i & 3-4i \end{bmatrix}$ $\begin{bmatrix} 3+4i & -2i \\ -2i & 3-4i \end{bmatrix}$ $\begin{bmatrix} 3-4i & 2i \\ -2i & 3+4i \end{bmatrix}$ $\begin{bmatrix} 3-4i & -2i \\ 2i & 3+4i \end{bmatrix}$

$\begin{bmatrix} 3 & 2+4i \\ -2+4i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 2-4i \\ -2-4i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -2+4i \\ 2-4i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -2-4i \\ 2-4i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 4+2i \\ -4+2i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & 4-2i \\ -4-2i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -4+2i \\ 4+2i & 3 \end{bmatrix}$ $\begin{bmatrix} 3 & -4-2i \\ 4-2i & 3 \end{bmatrix}$

$\begin{bmatrix} 5+2i & 0 \\ 0 & 5-2i \end{bmatrix}$ $\begin{bmatrix} 5-2i & 0 \\ 0 & 5+2i \end{bmatrix}$ $\begin{bmatrix} 5 & 2 \\ -2 & 5 \end{bmatrix}$ $\begin{bmatrix} 5 & -2 \\ 2 & 5 \end{bmatrix}$ $\begin{bmatrix} 5 & 2i \\ 2i & 5 \end{bmatrix}$ $\begin{bmatrix} 5 & -2i \\ -2i & 5 \end{bmatrix}$

p = 31

14. Now we want to consider some automorphic forms. Functions on $U_{\mathbb{Q}}'\backslash U_A'$ may be identified with fcns on U_0'. We shall be interested in the action of the Hecke operators on the function on U_0'/H where H is an open subgroup of U_0'.

15. H is the product of $U_{\mathbb{R}}' \times \prod_{p \neq 2} U_{\mathbb{Z}_p}'$ and

$$\{g \in \tilde{U}_{\mathbb{Z}_2}' \mid g \equiv \pm I \pmod 8\} = H_0$$

U_0'/H may be identified with $\tilde{U}_{\mathbb{Z}_2}'/H_0$. $\tilde{\mathfrak{A}}_{\mathbb{Z}_2}'/H_0$ is a normal subgroup of order 64 and of index 2 in this group. It is an abelian subgroup because

$$\left(I + 2\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}\right)\left(I + 2\begin{bmatrix} a'+b'i & c'+d'i \\ -c'+d'i & a'-b'i \end{bmatrix}\right)$$

$$= I + 2\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} + 2\begin{bmatrix} a'+b'i & c'+d'i \\ -c'+d'i & a'-b'i \end{bmatrix} + 4\begin{bmatrix} aa'-bb'+cc'-dd'+i(ab'+a'b+cd'-c'd) & '' \\ -ac'-a'c-b'd+bd'+i(a'd+ad'-b'c-bc') & '' \end{bmatrix}$$

There is one non-trivial inner automorphism σ of this subgroup and it is given by

$$1+2\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} \rightarrow 1+6\begin{bmatrix} 1 & 1+i \\ -1+i & 1 \end{bmatrix}\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix}\begin{bmatrix} 1 & -1-i \\ 1-i & 1 \end{bmatrix}$$

$$= 1+6\begin{bmatrix} 3a+(-3+2(d-c))i & '' \\ -3c+2(c-d-b)i+(3d+2(c-d-b))i & '' \end{bmatrix}$$

$$\equiv I + 2\begin{bmatrix} a+bi+2(d-c)i & '' \\ (-c+di)+2(c-d-b)(1+i) & '' \end{bmatrix}$$

Thus $\{u\sigma u^{-1}\}$ is the image of the set of matrices of the form

$$\left\{ I + 4\begin{bmatrix} (a)+(d-c)i & (c-d-b)(-1+i) \\ 2(c-d-b)(1+i) & a-(d-c)i \end{bmatrix} + 4\begin{bmatrix} a^2+b^2+c^2+d^2 & 0 \\ 0 & a^2+b^2+c^2+d^2 \end{bmatrix}\right\} \frac{1}{(1+2a)^2+4b^2+4c^2+4d^2}$$

$$= \begin{bmatrix} 1+4(d-c)i & 4(c-d-b)(-1+i) \\ (c-d-b)1+i & 1-4(d-c)i \end{bmatrix}$$

$$\equiv \begin{Bmatrix} 1+4\beta i & 4\gamma+4\gamma i \\ -4\gamma+4\gamma i & 1-4\beta i \end{Bmatrix}$$

4 (4)

32 Y

	u	u^σ	X			X																			
×	1 0 / 0 1	1 0 / 0 -1	1	1	1	1	1	1	1	1	1 0 / 0 1	1 0 / 0 1	1 0 / 0 -1	1 0 / 0 1	1 0 / 0 1	1 0 / 0 1	1 0 / 0 1	1 0 / 0 1	1 0 / 0 1	1 0 / 0 1	1 0 / 0 1	1 0 / 0 1	1
	1+i 4i / 4i 1	1+i 4i / 4i 1	-1	1	1	1	-1	-1	-1	-1	1 0 / 0 1	-1 0 / 0 -1	1 0 / 0 1	-1 0 / 0 1	-1 0 / 0 -1	-1 0 / 0 -1							2
	1+4i 4 / -4 1-4i	1+4i 4 / -4 1-4i	-1	1	1	1	-1	-1	-1	-1	1 0 / 0 1	-1 0 / 0 -1	-1 0 / 0 -1	-1 0 / 0 -1	1 0 / 0 1	1 0 / 0 1							3
×	1+4i 4+4i / -4+4i 1-4i	1+4i 4+4i / -4+4i 1-4i	1	1	1	1	1	1	1	1	1 0 / 0 1	1 0 / 0 1	-1 0 / 0 -1	-1 0 / 0 -1	-1 0 / 0 -1	-1 0 / 0 -1							4
×	1+4i 0 / 0 1-4i	1+4i 0 / 0 1-4i	1	1	1	1	1	1	1	1	-1 0 / 0 -1	-1 0 / 0 -1	1 0 / 0 1	-1 0 / 0 -1	1 0 / 0 1	-1 0 / 0 -1							5
	1+4i 4i / 4i 1-4i	1+4i 4i / 4i 1-4i	-1	1	1	1	-1	-1	-1	-1	-1 0 / 0 -1	1 0 / 0 1	1 0 / 0 1	-1 0 / 0 -1	-1 0 / 0 -1	1 0 / 0 1							6
	1 4 / -4 1	1 4 / -4 1	-1	1	1	1	-1	-1	-1	-1	-1 0 / 0 -1	1 0 / 0 1	-1 0 / 0 -1	1 0 / 0 1	1 0 / 0 1	-1 0 / 0 -1							7
×	1 4+4i / -4+4i 1	1 4+4i / -4+4i 1	1	1	1	1	1	1	1	1	-1 0 / 0 -1	-1 0 / 0 -1	-1 0 / 0 -1	1 0 / 0 1	-1 0 / 0 -1	1 0 / 0 1							8
	1+2i 0 / 0 1-2i	1+2i 4+4i / -4+4i 1-2i	1	1	1	1	1	1	1	1	i 0 / 0 -i	i 0 / 0 -i	1 0 / 0 -1	i 0 / 0 i	1 0 / 0 -1	-i 0 / 0 i							9
	1+2i 4i / 4i 1-2i	1+2i 4 / -4 1-2i	1	1	1	1	-1	-1	-1	-1	i 0 / 0 -i	-i 0 / 0 i	1 0 / 0 -1	i 0 / 0 i	-1 0 / 0 1	-i 0 / 0 -i							10
	1+6i 4 / -4 1-6i	1+6i 4i / 4i 1-6i	1	1	1	1	-1	-1	-1	-1	i 0 / 0 -i	-i 0 / 0 i	-1 0 / 0 1	-i 0 / 0 -i	1 0 / 0 -1	i 0 / 0 i							11
	1+6i 4+4i / -4+4i 1-6i	1+6i 0 / 0 1-6i	1	1	1	1	1	1	1	1	i 0 / 0 -i	i 0 / 0 -i	-1 0 / 0 1	-i 0 / 0 -i	-1 0 / 0 1	-i 0 / 0 -i							12
	1+6i 0 / 0 1-6i	1+6i 4+4i / -4+4i 1-6i	1	1	1	1	1	1	1	1	-i 0 / 0 i	-i 0 / 0 i	1 0 / 0 -1	-i 0 / 0 -i	1 0 / 0 -1	-i 0 / 0 -i							13
	1+6i 4i / 4i 1-6i	1+6i 4 / -4 1-6i	1	1	1	1	-1	-1	-1	-1	-i 0 / 0 i	i 0 / 0 -i	1 0 / 0 -1	-i 0 / 0 -i	-1 0 / 0 1	i 0 / 0 i							14
	1+2i 4 / -4 1-2i	1+2i 4i / 4i 1-2i	1	1	1	1	-1	-1	-1	-1	-i 0 / 0 i	i 0 / 0 -i	-1 0 / 0 1	i 0 / 0 i	1 0 / 0 -1	-i 0 / 0 -i							15
	1+2i 4+4i / -4+4i 1-2i	1+2i 0 / 0 1-2i	1	1	1	1	1	1	1	1	-i 0 / 0 i	-i 0 / 0 i	-1 0 / 0 1	i 0 / 0 i	-1 0 / 0 1	i 0 / 0 i							16
	1+2i 2 / -2 1-2i	1+6i 2 / -2 1-6i	1	1	-1	-1	i	i	-i	-i	1 0 / 0 -1	i 0 / 0 -i	i 0 / 0 i	i 0 / 0 -i	1 0 / 0 1	1 0 / 0 -1							17
	1+2i 2+4i / -2+4i 1-2i	1+6i 2+4i / -2+4i 1-2i	1	1	-1	-1	-i	-i	i	i	1 0 / 0 -1	-i 0 / 0 i	i 0 / 0 i	i 0 / 0 -i	-1 0 / 0 -1	-1 0 / 0 [illegible]							18
	1+6i 6 / -6 1-6i	1+2i 6 / -6 1-2i	1	1	-1	-1	-i	-i	i	i	1 0 / 0 -1	-i 0 / 0 i	-i 0 / 0 -i	-i 0 / 0 i	1 0 / 0 1	1 0 / 0 -1							19
	1+6i 6+4i / -6+4i 1-6i	1+2i 6+4i / -6+4i 1-2i	1	1	-1	-1	i	i	-i	-i	1 0 / 0 -1	i 0 / 0 -i	-i 0 / 0 -i	-i 0 / 0 i	-1 0 / 0 -1	-1 0 / 0 1							20
	1+6i 2 / -2 1-6i	1+2i 2 / -2 1-2i	1	1	-1	-1	i	i	-i	-i	-1 0 / 0 1	-i 0 / 0 i	i 0 / 0 i	-i 0 / 0 i	1 0 / 0 1	-1 0 / 0 1							21
	1+6i 2+4i / -2+4i 1-6i	1+2i 2+4i / -2+4i 1-2i	1	1	-1	-1	-i	-i	i	i	-1 0 / 0 1	i 0 / 0 -i	i 0 / 0 i	-i 0 / 0 i	-1 0 / 0 -1	1 0 / 0 -1							22
	1+2i 6 / -6 1-2i	1+6i 6 / -6 1-6i	1	1	-1	-1	-i	-i	i	i	-1 0 / 0 1	i 0 / 0 -i	-i 0 / 0 -i	i 0 / 0 -i	1 0 / 0 1	-1 0 / 0 1							23
	1+2i 6+4i / -6+4i 1-2i	1+6i 6+4i / -6+4i 1-6i	1	1	-1	-1	i	i	-i	-i	-1 0 / 0 1	-i 0 / 0 i	-i 0 / 0 -i	i 0 / 0 -i	-1 0 / 0 -1	1 0 / 0 -1							24
	1+4i 2+4i / -2+4i 1-4i	1 6 / -6 1	1	1	-1	-1	i	i	-i	-i	i 0 / 0 i	-1 0 / 0 -1	i 0 / 0 -i	-1 0 / 0 1	1 0 / 0 -1	i 0 / 0 -i							25
	1+4i 2 / -2 1-4i	1 6+4i / -6+4i 1	1	1	-1	-1	-i	-i	i	i	i 0 / 0 i	1 0 / 0 1	i 0 / 0 -i	-1 0 / 0 1	-1 0 / 0 1	-i 0 / 0 i							26
	1 6+4i / -6+4i 1	1+4i 2 / -2 1-4i	1	1	-1	-1	-i	-i	i	i	i 0 / 0 i	1 0 / 0 1	-i 0 / 0 i	1 0 / 0 -1	1 0 / 0 -1	i 0 / 0 -i							27
	1 6 / -6 1	1+4i 2+4i / -2+4i 1-4i	1	1	-1	-1	i	i	-i	-i	i 0 / 0 i	-1 0 / 0 -1	-i 0 / 0 i	1 0 / 0 -1	-1 0 / 0 1	-i 0 / 0 i							28
	1 2+4i / -2+4i 1	1+4i 6 / -6 1-4i	1	1	-1	-1	i	i	-i	-i	-i 0 / 0 -i	1 0 / 0 1	i 0 / 0 -i	1 0 / 0 -1	1 0 / 0 -1	-i 0 / 0 i							29
	1 2 / -2 1	1+4i 6+4i / -6+4i 1+4i	1	1	-1	-1	-i	-i	i	i	-i 0 / 0 -i	-1 0 / 0 -1	i 0 / 0 -i	1 0 / 0 -1	-1 0 / 0 1	i 0 / 0 -i							30
	1+4i 6+4i / -6+4i 1-4i	1 2i / -2 1	1	1	-1	-1	-i	-i	i	i	-i 0 / 0 -i	-1 0 / 0 -1	-i 0 / 0 i	-1 0 / 0 1	1 0 / 0 -1	-i 0 / 0 i							31
	1+4i 6 / -6 1-4i	1 2+4i / -2+4i 1	1	1	-1	-1	i	i	-i	-i	-i 0 / 0 -i	1 0 / 0 1	-i 0 / 0 i	-1 0 / 0 1	-1 0 / 0 1	i 0 / 0 -i							32
	1+2i 2i / 2i 1-2i	1+6i 2i / 2i 1-6i	1	-1	1	-1	i	-i	i	-i	i 0 / 0 -i	1 0 / 0 1	1 0 / 0 1	i 0 / 0 i	i 0 / 0 i	1 0 / 0 1							33
	1+2i 6i / 6i 1-2i	1+6i 6i / 6i 1-6i	1	-1	1	-1	-i	i	-i	i	i 0 / 0 -i	-1 0 / 0 -1	1 0 / 0 1	i 0 / 0 i	-i 0 / 0 -i	-1 0 / 0 -1							34
	1+6i 4+2i / -4+2i 1-6i	1+2i 4+2i / -4+2i 1-6i	1	-1	1	-1	-i	i	-i	i	i 0 / 0 -i	-1 0 / 0 -1	-1 0 / 0 -1	-i 0 / 0 -i	i 0 / 0 i	1 0 / 0 1							35
	1+6i 4+6i / -4+6i 1-6i	1+2i 4+6i / -4+6i 1-2i	1	-1	1	-1	i	-i	i	-i	i 0 / 0 -i	1 0 / 0 1	-1 0 / 0 -1	-i 0 / 0 -i	-i 0 / 0 -i	-1 0 / 0 -1							36

1+6i 2i 2i 1-6i	1+2i 2i 2i 1-2i	1	-1	1	-1	i	-i	i	-i	-i 0	0 i	1 0	0 1	1 0	0 1	i 0	0 i	i 0	0 i	1 0	0 1	37
1+6i 6i 6i 1-6i	1+2i 6i 6i 1-2i	1	-1	-1	-1	-i	i	-i	i	-i 0	0 i	-1 0	0 -1	1 0	0 1	i 0	0 i	-i 0	0 -i	-1 0	0 -1	38
1+2i 4+2i -4+2i 1-2i	1+6i 4+2i -4+2i 1-6i	1	-1	1	-1	-i	i	-i	i	-i 0	0 i	-1 0	0 -1	-1 0	0 -1	-i 0	0 -i	i 0	0 i	1 0	0 1	39
1+2i 4+6i -4+6i 1-2i	1+6i 4+6i -4+6i 1-6i	1	-1	1	-1	i	-i	i	-i	-i 0	0 i	1 0	0 1	-1 0	0 -1	-i 0	0 -i	-i 0	0 -i	-1 0	0 -1	40
1+4i 4+2i -4+2i 1-4i		1	-1	1	-1	i	-i	i	-i													41
1+4i 4+6i -4+6i 1-4i		1	-1	1	-1	-i	i	-i	i													42
1 2i 2i 1		1	-1	1	-1	-i	i	-i	i													43
1 6i 6i 1		1	-1	1	-1	i	-i	i	-i													44
1 4+2i -4+2i 1		1	-1	1	-1	i	-i	i	-i													45
1 4+6i -4+6i 1		1	-1	1	-1	-i	i	-i	i													46
1+4i 2i 2i 1-4i		1	-1	1	-1	-i	i	-i	i													47
1+4i 6i 6i 1-4i		1	-1	1	-1	i	-i	i	-i													48
1 6+6i -6+6i 1		1	-1	-1	1	-1	1	1	-1 *													49
1 6+2i -6+2i 1		1	-1	-1	1	1	-1	-1	1													50
1+4i 2+6i -2+6i 1-4i		1	-1	-1	1	1	-1	-1	1													51
1+4i 2+2i -2+2i 1-4i		1	-1	-1	1	-1	1	1	-1													52
1+4i 6+6i -6+6i 1-4i		1	-1	-1	1	-1	1	1	-1													53
1+4i 6+2i -6+2i 1-4i		1	-1	-1	1	1	-1	-1	1													54
1 2+6i -2+6i 1		1	-1	-1	1	1	-1	-1	1													55
1 2+2i -2+2i 1		1	-1	-1	1	-1	1	1	-1													56
1+2i 2+2i -2+2i 1-2i		1	-1	-1	1	-1	1	1	-1													57
1+2i 2+6i -2+6i 1-2i		1	-1	-1	1	1	-1	-1	1													58
1+6i 6+2i -6+6i 1-6i		1	-1	-1	1	1	-1	-1	1													59
1+6i 6+6i -6+2i 1-6i		1	-1	-1	1	-1	1	1	-1 *													60
1+6i 2+2i -2+2i 1-6i		1	-1	-1	1	-1	1	1	-1													61
1+6i 2+6i -2+6i 1-6i		1	-1	-1	1	1	-1	-1	1													62
1+2i 6+2i -6+2i 1-2i		1	-1	-1	1	1	-1	-1	1													63
1+2i 6+6i -6+6i 1-2i		1	-1	-1	1	-1	1	1	-1 ?													64
1 1+i -1+i 1		1	i	i	1	i	1	1	i													65
1+4i 5+i -5+i 1-4i		1	i	i	1	-i	-1	-1	-i													66
1 5+i -5+i 1		1	i	i	1	-i	-1	-1	-i													67
1+4i 1+i -1+i 1		1	i	i	1	i	1	1	i													68
1+4i 5+5i -5+5i 1+4i		1	i	i	1	i	1	-1	i													69
1 1+5i -1+5i 1		1	i	i	1	-i	-1	-1	-i													70
1+4i 1+5i -1+5i 1-4i		1	i	i	1	-i	-1	-1	-i													71
1 5+5i -5+5i 1		1	i	i	1	i	1	1	i													72

$1+2i$ $7+3i$ $-7+3i$ $1-2i$	1	i	i	1	i	1	1	i	73
$1+6i$ $3+3i$ $-3+3i$ $1-6i$	1	i	i	1	$-i$	-1	-1	$-i$	74
$1+2i$ $3+3i$ $-3+3i$ $1-2i$	1	i	i	1	$-i$	-1	-1	$-i$	75
$1+6i$ $7+3i$ $-7+3i$ $1-6i$	1	i	i	1	i	1	1	i	76
$1+6i$ $3+7i$ $-3+7i$ $1-6i$	1	i	i	1	i	1	1	i	77
$1+2i$ $7+7i$ $-7+7i$ $1-2i$	1	i	i	1	$-i$	-1	-1	$-i$	78
$1+6i$ $7+7i$ $-7+7i$ $1-6i$	1	i	i	1	$-i$	-1	-1	$-i$	79
$1+2i$ $3+7i$ $-3+7i$ $1-2i$	1	i	i	1	i	1	1	i	80
$1+4i$ $7+5i$ $-7+5i$ $1-4i$	1	i	$-i$	-1	-1	i	$-i$	1	81
1 $3+5i$ $-3+5i$ 1	1	i	$-i$	-1	1	$-i$	i	-1	82
$1+4i$ $3+5i$ $-3+5i$ $1-4i$	1	i	$-i$	-1	1	$-i$	i	-1	83
1 $7+5i$ $-7+5i$ 1	1	i	$-i$	-1	-1	i	$-i$	1	84
1 $3+i$ $-3+i$ 1	1	i	$-i$	-1	-1	i	$-i$	1	85
$1+4i$ $7+i$ $-7+i$ $1-4i$	1	i	$-i$	-1	1	$-i$	i	-1	86
1 $7+i$ $-7+i$ 1	1	i	$-i$	-1	1	$-i$	i	-1	87
$1+4i$ $3+i$ $-3+i$ $1-4i$	1	i	$-i$	-1	-1	i	$-i$	1	88
$1+6i$ $5+3i$ $-5+3i$ $1-6i$	1	i	$-i$	-1	-1	i	$-i$	1	89
$1+2i$ $1+3i$ $-1+3i$ $1-2i$	1	i	$-i$	-1	1	$-i$	i	-1	90
$1+6i$ $1+3i$ $-1+3i$ $1-6i$	1	i	$-i$	-1	1	$-i$	i	-1	91
$1+2i$ $5+3i$ $-5+3i$ $1-2i$	1	i	$-i$	-1	-1	i	$-i$	1	92
$1+2i$ $1+7i$ $-1+7i$ $1-2i$	1	i	$-i$	-1	-1	i	$-i$	1	93
$1+6i$ $5+7i$ $-5+7i$ $1-6i$	1	i	$-i$	-1	1	$-i$	i	-1	94
$1+2i$ $5+7i$ $-5+7i$ $1-2i$	1	i	$-i$	-1	1	$-i$	i	-1	95
$1+6i$ $1+7i$ $-1+7i$ $1-6i$	1	i	$-i$	-1	-1	i	$-i$	1	96
1 $1+3i$ $-1+3i$ 1	1	$-i$	i	-1	-1	$-i$	i	1	97
$1+4i$ $5+3i$ $-5+3i$ $1-4i$	1	$-i$	i	-1	1	i	$-i$	-1	98
1 $5+3i$ $-5+3i$ 1	1	$-i$	i	-1	1	i	$-i$	-1	99
$1+4i$ $1+3i$ $-1+3i$ $1-4i$	1	$-i$	i	-1	-1	$-i$	i	1	100
$1+4i$ $5+7i$ $-5+7i$ $1-4i$	1	$-i$	i	-1	-1	$-i$	i	1	101
1 $1+7i$ $-1+7i$ 1	1	$-i$	i	-1	1	i	$-i$	-1	102
1 $5+3i$ $-5+3i$ 1	1	$-i$	i	-1	1	i	$-i$	-1	103
1 $5+7i$ $-5+7i$ 1	1	$-i$	i	-1	-1	$-i$	i	1	104
$1+2i$ $3+5i$ $-3+5i$ $1-2i$	1	$-i$	i	-1	-1	$-i$	i	1	105
$1+6i$ $7+5i$ $-7+5i$ $1-6i$	1	$-i$	i	-1	1	i	$-i$	-1	106
$1+2i$ $7+5i$ $-7+5i$ $1-2i$	1	$-i$	i	-1	1	i	$-i$	-1	107
$1+6i$ $3+5i$ $-3+5i$ $1-6i$	1	$-i$	i	-1	-1	$-i$	i	1	108

1+6i 7+i -7+i 1-6i	1	-i	i	-1	-1	-i	i	1	109
1+2i 3+i -3+i 1-2i	1	-i	i	-1	1	i	-i	-1	110
1+6i 3+i -3+i 1-6i	1	-i	i	-1	1	i	-i	-1	111
1+2i 7+i -7+i 1-2i	1	-i	i	-1	-1	-i	i	1	112
1 3+3i -3+3i 1	1	-i	-i	1	-i	1	1	-i	113
1+4i 7+3i -7+3i 1-4i	1	-i	-i	1	-i	-1	-1	i	114
1 7+3i -7+3i 1	1	-i	-i	1	-i	-1	-1	i	115
1+4i 3+3i -3+3i 1-4i	1	-i	-i	1	-i	1	1	-i	116
1+4i 7+7i -7+7i 1-4i	1	-i	-i	1	-i	1	1	-i	117
1 3+7i -3+7i 1	1	-i	-i	1	-i	-1	-1	-i	118
1+4i 3+7i -3+7i 1-4i	1	-i	-i	1	-i	1	-1	-i	119
1 7+7i -7+7i 1	1	-i	-i	1	-i	1	1	-i	120
1+2i 5+i -5+i 1-2i	1	-i	-i	1	-i	1	1	-i	121
1+6i 1+i -1+i 1-6i	1	-i	-i	1	i	-1	-1	i	122
1+2i 1+i -5+i 1-6i	1	-i	-i	1	i	-1	-1	i	123
1+6i 5+i -1+i 1-2i	1	-i	-i	1	-i	1	1	-i	124
1+6i 1+5i -1+5i 1-6i	1	-i	-i	1	-i	1	1	-i	125
1+2i 5+5i -5+5i 1-2i	1	-i	-i	1	i	-1	-1	i	126
1+6i 5+5i -5+5i 1-6i	1	-i	-i	1	i	-1	-1	i	127
1+2i 1+5i -1+5i 1-2i	1	-i	-i	1	-i	1	1	-i	128
p=3	4	0	0	0	2	2	2	-2	
p=5	6	2	2	-2	2	2	2	2?	
p=7	8	0	0	0	-4	-4	-4	4	
p=11	12	0	0	0	-2	-2	-2	2	
p=13	14	-6	-6	6	2	2	2	2	
p=17	18								
p=19	20								
p=23	24								
p=29	30								

$J \equiv 2 \pmod 5$

1+2i	0	15	8	5	0									
0	1-2i	0	13	0	1	1	1	1	1	1	1	1	1	
1-2i	0	-3	0	1	0									
0	1+2i	0	5	0	5	1	1	1	1	1	1	1	1	
1	2	1	2	5	2									
-2	1	-2	1	0	1	1	1	-1	-1	-i	-i	i	i	
1	-2	1	-2	5	3									
2	1	2	1	0	1	1	1	-1	-1	i	i	-i	-i	
1	2i	1	4	5	4									
2i	1	4	1	0	1	1	-1	1	-1	-i	i	-i	i	
1	-2i	1	-4	5	1									
-2i	1	-4	1	0	1	1	-1	1	-1	i	-i	i	-i	
3+4i	0	31	0	1	0									
0	3-4i	0	-25	0	25	1	1	1	1	1	1	1	1	
3-4i	0	-25	0	25	0									
0	3+4i	0	31	0	1	1	1	1	1	1	1	1	1	X
3	4	3	4	25	18									
-4	3	-4	3	0	1	1	1	1	1	-1	-1	-1	-1	x
3	-4	3	-4	25	7									
4	3	4	3	0	1	1	1	1	1	-1	-1	-1	-1	
3	4i	3	28	25	1									
4i	3	28	3	0	1	1	1	1	1	-1	-1	-1	-1	
3	-4i	3	-28	25	24									
-4i	3	-28	3	0	1	1	1	1	1	-1	-1	-1	-1	
1+4i	2+2i	29	16	25	17									
-2+2i	1-4i	12	-27	0	1	1	-1	-1	1	-1	1	1	-1	
1+4i	-2-2i	29	-16	25	8									
2-2i	1-4i	-12	-27	0	1	1	-1	-1	1	-1	1	1	-1	
1+4i	2-2i	29	-12	25	6									
-2-2i	1-4i	-16	-27	0	1	1	-1	-1	1	1	-1	-1	1	
1+4i	-2+2i	29	12	25	19									
2+2i	1-4i	16	-27	0	1	1	-1	-1	1	1	-1	-1	1	
1-4i	2+2i	-27	16	25	4									
-2+2i	1+4i	12	29	0	1	1	-1	-1	1	-1	1	1	-1	
1-4i	-2-2i	-27	-16	25	21									
2-2i	1+4i	-12	29	0	1	1	-1	-1	1	-1	1	1	-1	
1-4i	2-2i	-27	-12	25	22									
-2-2i	1+4i	-16	29	0	1	1	-1	-1	1	1	-1	-1	1	
1-4i	-2+2i	-27	12	25	3									
2+2i	1+4i	16	29	0	1	1	-1	-1	1	1	-1	-1	1	
1+2i	4+2i	115	118	25	14									
-4+2i	1-2i	110	-113	0	1	1	-1	1	-1	-i	i	-i	i	
1+2i	-4-2i	115	-118	25	11									
4-2i	1-2i	-110	-113	0	1	1	-1	1	-1	i	-i	i	-i	
1+2i	4-2i	115	-110	25	20									
-4-2i	1-2i	-118	-113	0	1	1	-1	1	-1	i	-i	i	-i	x
1+2i	-4+2i	115	110	25	5									
4+2i	1-2i	118	-113	0	1	1	-1	1	-1	-i	i	-i	i	x
1-2i	4+2i	-113	118	5	1									
-4+2i	1+2i	110	115	0	5	1	-1	1	-1	-i	i	-i	i	←
1-2i	-4-2i	-113	-118	5	4									
4-2i	1+2i	-110	115	0	5	1	-1	1	-1	i	-i	i	-i	←
1-2i	-4+2i	-113	110	25	9									
4+2i	1+2i	118	115	0	1	1	-1	1	-1	-i	i	-i	i	
1-2i	4-2i	-113	-110	25	16									
-4-2i	1+2i	-118	115	0	1	1	-1	1	-1	i	-i	i	-i	
1+2i	2+4i	115	230	25	15									
-2+4i	1-2i	226	-113	0	1	1	1	-1	-1	-i	-i	i	i	✓
1+2i	-2-4i	115	-230	25	10									
2-4i	1-2i	-226	-113	0	1	1	1	-1	-1	i	i	-i	-i	✓
1+2i	2-4i	115	-226	25	2									
-2-4i	1-2i	-230	-113	0	1	1	1	-1	-1	-i	-i	i	i	
1+2i	-2+4i	115	226	25	23									
2+4i	1-2i	230	-113	0	1	1	1	-1	-1	i	i	-i	-i	
1-2i	2+4i	-113	230	25	12									
-2+4i	1+2i	226	115	0	1	1	1	-1	-1	-i	-i	i	i	
1-2i	-2-4i	-113	-230	25	13									
2-4i	1+2i	-226	115	0	1	1	1	-1	-1	i	i	-i	-i	
1-2i	2-4i	-113	-226	5	3									
-2-4i	1+2i	-230	115	0	5	1	1	-1	-1	-i	-i	i	i	←
1-2i	-2+4i	-113	226	5	2									
2+4i	1+2i	230	115	0	5	1	1	-1	-1	i	i	-i	-i	←

Representatives (modulo 4) — [illegible]

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix} \quad \begin{pmatrix} 1+2i & 2 \\ -2 & 1-2i \end{pmatrix} \quad \begin{pmatrix} 1 & 6 \\ -6 & 1 \end{pmatrix}$$

1 1 −1 −1 Fourth column

$$\begin{pmatrix} 1+2i & 6i \\ 6i & 1-2i \end{pmatrix} \quad \begin{pmatrix} 1 & 2i \\ 2i & 1 \end{pmatrix} \quad \begin{pmatrix} 1+i & 2+2i \\ -2+2i & 1-i \end{pmatrix} \quad \begin{pmatrix} 1+2i & 2+2i \\ -2+2i & 1-2i \end{pmatrix}$$

−1 −1 1 1

$p = 17$

1	1	1	1		
−1	−1	−1	−1		
−1	−1	−1	−1		
1	1	1	1	1	1

p=29 (Column 4)

1 1 1 1 1 1 1 1

-1 -1 -1 -1 -1 -1 -1 -1

-1 -1 -1 -1 -1 -1 -1 -1

1 1 -1 -1 -1 -1

(-10)

as required

p=37

$\begin{pmatrix} 1+6i & 0 \\ 0 & 1-6i \end{pmatrix}$	$\begin{pmatrix} 1-6i & 0 \\ 0 & 1+6i \end{pmatrix}$	$\begin{pmatrix} 1 & 6 \\ -6 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -6 \\ 6 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 6i \\ 6i & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -6i \\ -6i & 1 \end{pmatrix}$
1	1	−1	−1	−1	−1

6

$\begin{pmatrix} 5+2i & 2+2i \\ -2+2i & 5-2i \end{pmatrix}$: 8 possibilities with sign changes — 8

$\begin{pmatrix} 1+2i & 4+4i \\ -4+4i & 1-2i \end{pmatrix}$: " — 8

$\begin{pmatrix} 1+4i & 2+4i \\ -2+4i & 1-4i \end{pmatrix}$ " — −8

$\begin{pmatrix} 1+4i & 4+2i \\ -4+2i & 1-4i \end{pmatrix}$ " — −8

$\Big\}$ −2 as required

1 0 0 1	1 0 0 1	1 0 0 1	1 0 0 1

Matrix		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1 0 0 1		1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1 4i 4i 1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1+4i 4 -4 1-4i	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1+4i 4+4i -4+4i 1-4i	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1+4i 0 0 1-4i	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1+4i 4i 4i 1-4i	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1 4i 4i 1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1 4+4i -4+4i 1-i	1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1+2i 0 0 1-2i		i	i	i	i	i	i	i	i	-i	-i	-i	-i	-i	-i	-i	-i
1+2i 4i 4i 1-2i		i	i	i	i	i	i	i	i	-i	-i	-i	-i	-i	-i	-i	-i
1+6i 4 -4 1-6i		i	i	i	i	i	i	i	i	-i	-i	-i	-i	-i	-i	-i	-i
1+6i 4+4i -4+4i 1-6i		i	i	i	i	i	i	i	i	-i	-i	-i	-i	-i	-i	-i	-i
1+6i 0 0 1-6i		-i	-i	-i	-i	-i	-i	-i	-i	i	i	i	i	i	i	i	i
1+6i 4i 4i 1-6i		-i	-i	-i	-i	-i	-i	-i	-i	i	i	i	i	i	i	i	i
1+2i 4 -4 1-2i		-i	-i	-i	-i	-i	-i	-i	-i	i	i	i	i	i	i	i	i
1+2i 4+4i -4+4i 1-2i		-i	-i	-i	-i	-i	-i	-i	-i	i	i	i	i	i	i	i	i
1+2i 2 -2 1-2i		1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
1+3i 2+4i -2+4i 1-2i		1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
1+6i 6 -6 1-6i		1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
1+6i 6+4i -6+4i 1-6i		1	1	1	1	-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1
1+6i 2 -2 1-6i		-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1
1+6i 2+4i -2+4i 1-6i		-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1
1+2i 6 -6 1-2i		-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1
1+2i 6+4i -6+4i 1-2i		-1	-1	-1	-1	1	1	1	1	-1	-1	-1	-1	1	1	1	1
1+4i 2+4i -2+4i 1-4i		i	i	i	i	-i	-i	-i	-i	-i	-i	-i	-i	i	i	i	i
1+4i 2 -2 1-4i		i	i	i	i	-i	-i	-i	-i	-i	-i	-i	-i	i	i	i	i
1 6+4i -6+4i 1		i	i	i	i	-i	-i	-i	-i	-i	-i	-i	-i	i	i	i	i
1 6 -6 1		i	i	i	i	-i	-i	-i	-i	-i	-i	-i	-i	i	i	i	i
1 2+4i -2+4i 1		-i	-i	-i	-i	i	i	i	i	i	i	i	i	-i	-i	-i	-i
1 2 -2 1		-i	-i	-i	-i	i	i	i	i	i	i	i	i	-i	-i	-i	-i
1+4i 6+4i -6+4i 1-4i		-i	-i	-i	-i	i	i	i	i	i	i	i	i	-i	-i	-i	-i
1+4i 6 -6 1-4i		-i	-i	-i	-i	i	i	i	i	i	i	i	i	-i	-i	-i	-i
1+2i 2i 2i 1-2i		i	i	-i	-i	i	i	-i	-i	i	i	-i	-i	i	i	-i	-i
1+2i 6i -6i 1-2i		i	i	-i	-i	i	i	-i	-i	i	i	-i	-i	i	i	-i	-i
1+6i 4+2i -4+2i 1-6i		i	i	-i	-i	i	i	-i	-i	i	i	-i	-i	i	i	-i	-i
1+6i 4+6i -4+6i 1-6i		i	i	-i	-i	i	i	-i	-i	i	i	-i	-i	i	i	-i	-i

1+6i 2i / 2i 1-6i		-i	-i	i	i	-i	-i	i	i	-i	-i	i	i	-i	-i	i	i	
1+6i 6i / 6i 1-6i		-i	-i	i	i	-i	-i	-i	i	-i	-i	i	i	-i	-i	i	i	
1+6i 4+2i / -4+2i 1-6i		-i	-i	i	i	-i	-i	i	i	-i	-i	i	i	-i	-i	i	i	
1+2i 4+6i / -4+6i 1-2i		-i	-i	i	i	-i	-i	i	i	-i	-i	i	i	-i	-i	i	i	
1+4i 4+2i / -4+2i 1-4i	x	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	
1+4i 4+6i / -4+2i 1-4i		-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	
1 2i / -2i 1		-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	
1 6i / 6i 1		-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	
1 -4+2i / -4+2i 1		1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	
1 4+6i / -4+6i 1		1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	
1+4i 2i / -2i 1-4i		1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	
1+4i 6i / 6i 1+4i		1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	
1 6+6i / -6+6i 1	x	i	i	-i	-i	-i	-i	i	i	i	i	-i	-i	-i	-i	i	i	
1 6+2i / -6+2i 1		i	i	-i	-i	-i	-i	i	i	i	i	-i	-i	-i	-i	i	i	
1+4i 2+6i / -2+6i 1-4i		i	i	-i	-i	-i	-i	i	i	i	i	-i	-i	-i	-i	i	i	
1+4i 2+2i / -2+2i 1-4i		i	i	-i	-i	-i	-i	i	i	i	i	-i	-i	-i	-i	i	i	
1+4i 6+6i / -6+6i 1-4i		-i	-i	i	i	i	i	-i	-i	-i	-i	i	i	i	i	-i	-i	
1+4i 6+2i / -6+2i 1-4i		-i	-i	i	i	i	i	-i	-i	-i	-i	i	i	i	i	-i	-i	
1 2+6i / -2+6i 1		-i	-i	i	i	i	i	-i	-i	-i	-i	i	i	i	i	-i	-i	
1 2+2i / -2+2i 1		-i	-i	i	i	i	i	-i	-i	-i	-i	i	i	i	i	-i	-i	
1+2i 2+2i / -2+2i 1-2i	1 x	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	
1+2i 2+6i / -2+6i 1-2i	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	
1+6i 6+6i / -6+6i 1-6i	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	
1+6i 6+2i / -6+2i 1-6i	1	-1	-1	1	1	1	1	-1	-1	1	1	-1	-1	-1	-1	1	1	
1+6i 2+2i / -2+2i 1-6i	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	
1+6i 2+6i / -2+6i 1-6i	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	
1+2i 6+2i / -6+2i 1-2i	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	
1+2i 6+6i / -6+6i 1-2i	1	1	1	-1	-1	-1	-1	1	1	-1	-1	1	1	1	1	-1	-1	
1 1+i / -1+i 1		$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	0 0 0 0 / 1 0 1 0 1 0 1 0
1+4i 5+i / -5+i 1-4i		$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{2}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	
1 5+i / -5+i 1		$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	
1+4i 1+i / -1+i 1		$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	
1+4i 5+5i / -5+5i 1+4i		$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	
1 1+5i / -1+5i 1		$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	
1+4i 1+5i / -1+5i 1		$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	
1 5+5i / -5+5i 1		$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{2}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	

1+2i 7+3i -7+3i 1-2i	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$
1+6i 3+3i -3+3i 1-6i	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$
1+2i 3+3i -3+3i 1-2i	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$
1+6i 7+3i -7+3i 1-6i	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$
1+6i 3+7i -3+7i 1-6i	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1+2i 7+7i -7+7i 1-2i	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1+6i 7+7i -7+7i 1-6i	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1+2i 3+7i -3+7i 1-2i	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1+4i 7+5i -7+5i 1-4i	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$
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1+4i 3+5i -3+5i 1-4i	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$
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1 3+i -3+i 1	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$
1+4i 7+i -7+i 1-4i	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$
1 7+i -7+i 1	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$
1+4i 3+i -3+i 1-4i	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$
1+6i 5+3i -5+3i 1-6i	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1+2i 1+3i -1+3i 1-2i	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1+6i 1+3i -1+3i 1-6i	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1+2i 5+3i -5+3i 1-2i	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1+2i 1+7i -1+7i 1-2i	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$
1+6i 5+7i -5+7i 1-6i	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$
1+2i 5+7i -5+7i 1-2i	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$
1+6i 1+7i -1+7i 1-6i	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$
1 1+3i -1+3i 1	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$
1+4i 5+3i -5+3i 1-4i	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$
1 5+3i -5+3i 1	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$
1+4i 1+3i -1+3i 1-4i	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$
1+4i 5+7i -5+7i 1-4i	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1 1+7i -1+7i 1	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1 5+3i -5+3i 1	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1 5+7i -5+7i 1	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$
1+2i 3+5i -3+5i 1-2i	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$
1+6i 7+5i -7+5i 1-6i	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$
1+2i 7+5i -7+5i 1-2i	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$
1+6i 3+5i -3+5i 1-6i	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{1-i}{\sqrt2}$	$\frac{-1+i}{\sqrt2}$	$\frac{-1-i}{\sqrt2}$	$\frac{1+i}{\sqrt2}$

1+6i 7+i -7+i 1-6i	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$
1+2i 3+i -3+i 1-2i	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$
1+6i 3+i -3+i 1-6i	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$
1+2i 7+i -7+i 1-2i	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$
1 3+3i -3+3i 1	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$
1+4i 7+3i -7+3i 1-4i	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$
1 7+3i -7+3i 1	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$
1+4i 3+3i -3+3i 1-4i	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$
1+4i 7+7i -7+7i 1-4i	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$
1 3+7i -3+7i 1	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$
1+4i 3+7i -3+7i 1-4i	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$
1 7+7i -7+7i 1	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$
1+2i 5+i -5+i 1-2i	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$
1+6i 1+i -1+i 1-6i	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$
1+6i 5+i -5+i 1-6i	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$
1+2i 1+i -1+i 1-2i	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$
1+6i 1+5i -1+5i 1-6i	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$
1+2i 5+5i -5+5i 1-2i	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$
1+6i 5+5i -5+5i 1-6i	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$
1+2i 1+5i -1+5i 1-2i (20)	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{1-i}{\sqrt{2}}$	$\frac{-1+i}{\sqrt{2}}$	$\frac{-1-i}{\sqrt{2}}$	$\frac{1+i}{\sqrt{2}}$
p = 3	[illegible]	[illegible]	[illegible]	[illegible]	-1+2i	-1+2i	1+2i	1+2i	1+2i	1+2i	[illegible]					
p = 5	-1±2i	-1±2i	1±2i	1±2i	-1±2i	-1±2i	1±2i	1±2i	1±2i	1±2i	-1±2i	-1±2i	1±2i	1±2i	-1±2i	1±2i
p = 7																
p = 11																
p = 13	$-1\pm2\sqrt{3}i$	$-1\pm2\sqrt{3}i$	$1\pm2\sqrt{3}i$	$1\pm2\sqrt{3}i$	$-1\pm2\sqrt{3}i$	$-1\pm2\sqrt{3}i$	$1\pm2\sqrt{3}i$	$1\pm2\sqrt{3}i$	$1\pm2\sqrt{3}i$	$1\pm2\sqrt{3}i$	$-1\pm2\sqrt{3}i$	$-1\pm2\sqrt{3}i$	$1\pm2\sqrt{3}i$	$1\pm2\sqrt{3}i$	$-1\pm2\sqrt{3}i$	$-1\pm2\sqrt{3}i$
p = 17	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i	-1±4i
p = 19																
p = 23																
p = 29	$3\pm2\sqrt{5}i$	$3\pm2\sqrt{5}i$	$-3\pm2\sqrt{5}i$	$-3\pm2\sqrt{5}i$	$3\pm2\sqrt{5}i$	$3\pm2\sqrt{5}i$	$-3\pm2\sqrt{5}i$	$-3\pm2\sqrt{5}i$	$-3\pm2\sqrt{5}i$	$-3\pm2\sqrt{5}i$	$3\pm2\sqrt{5}i$	$3\pm2\sqrt{5}i$	$-3\pm2\sqrt{5}i$	$-3\pm2\sqrt{5}i$	$3\pm2\sqrt{5}i$	$3\pm2\sqrt{5}i$

There are only 8 one-dimensional representations listed although the group admits 16. However all the elements have ~~character~~ norms which are congruent to 1 modulo 4 ~~and~~. These norms are determined modulo 16. ~~Comparing~~ Thus are four characters factoring through the norm. Thus of the form $\chi \circ Nm$. Take χ non-trivial on 5. ~~Then~~ Multiplying the first eight ~~rows~~ columns by these yields the remaining 8 characters

	$(-\frac14\ \frac14\ -\frac14\ \frac14) + (xxyy)$	$(-\frac14\ \frac14\ -\frac14\ \frac14) + (yyxx)$	$(-\frac14\ -\frac14\ \frac14\ \frac14) + (xyyx)$	$(-\frac14\ -\frac14\ \frac14\ \frac14) + (yxxy)$	$(-\frac14\ -\frac14\ \frac14\ \frac14) + (xyxy)$	$(-\frac14\ -\frac14\ \frac14\ \frac14) + (yxyx$
$(\frac14\ -\frac14\ \frac14\ -\frac14)$ $(xxyy)$	c'	$c'(13)(14)(23)(24)$	$c'(13)(14)$	$c'(14)(24)$	$c'(14)$	$c'(14)(13)(24)$
$\frac14\ -\frac14\ \frac14\ -\frac14$ $(yyxx)$	$C'(13)(14)(23)(24)$	c'	$C'(14)(24)$	$C'(13)(14)$	$C'(14)(13)(24)$	$C'(14)$
$(\frac14\ \frac14\ -\frac14\ -\frac14)$ $(xyyx)$	$C'(24)(34)$	$C'(12)(13)$	$C'(12)(13)(24)(34)$	$C'(13)(24)$	$C'(13)(24)(34)$	$C'(13)(12)(24)$
$(\frac14\ \frac14\ -\frac14\ -\frac14)$ $(yxxy)$	$C'(12)(13)$	$C'(24)(34)$	$C'(13)(24)$	$C'(12)(13)(24)(34)$	$C'(13)(12)(24)$	$C'(13)(24)(34)$
$(\frac14\ \frac14\ -\frac14\ -\frac14)$ $(xyxy$	$C'(23)$	$C'(12)(14)(34)$	$C'(12)(14)(23)$	$C'(14)(23)(34)$	$C'(14)(23)$	$C'(12)(14)(23)(34)$
$(\frac14\ \frac14\ -\frac14\ -\frac14)$ $(yxyx)$						

	$(\frac{1}{2}0 -\frac{1}{2}0) + (xxxy)$	$(\frac{1}{2}00 -\frac{1}{2}) + (xxyx)$	$(\frac{1}{2}00 -\frac{1}{2}) + (xyxx)$	$(0\ \frac{1}{2}0 -\frac{1}{2}) + (yxxx)$
$(\frac{1}{2}0 -\frac{1}{2}0) + (xxxy)$	$(-\frac{1}{2}0 +\frac{1}{2}0) + (xxxy)$ C''	$(-\frac{1}{2}00\frac{1}{2}) + (xxyx)$ $C''(14)$	$(-\frac{1}{2}00\frac{1}{2}) + (xyxx)$	$(0 -\frac{1}{2}0\frac{1}{2}) + (yxxx)$
$(\frac{1}{2}00 -\frac{1}{2}) + (xxyx)$	$(-\frac{1}{2}0\frac{1}{2}0) + (xxxy)$ $C''(34)$	$(-\frac{1}{2}00\frac{1}{2}) + (xxyx)$ $C''(13)(34)$	$(-\frac{1}{2}00\frac{1}{2}) + (xyxx)$ $C''(13)(23)(34)$	$(0 -\frac{1}{2}0\frac{1}{2}) + (yxxx)$ $C''(13)(23)$
$(\frac{1}{2}00 -\frac{1}{2}) + (xyxx)$	$(-\frac{1}{2}0\frac{1}{2}0) + (xxxy)$ $C''(23)(24)$	$(-\frac{1}{2}00\frac{1}{2}) + (xxyx)$ $C''(12)(23)(24)$	$(-\frac{1}{2}00\frac{1}{2}) + (xyxx)$ $C''(12)(24)$	$(0 -\frac{1}{2}0\frac{1}{2}) + (yxxx)$ $C''(12)$
$(0\ \frac{1}{2}0 -\frac{1}{2}) + (yxxx)$	$(-\frac{1}{2}0\frac{1}{2}0) + (xxxy)$ $C''(13)(14)(12)$	$(-\frac{1}{2}00\frac{1}{2}) + (xxyx)$ $C''(13)(14)$	$(-\frac{1}{2}00\frac{1}{2}) + (xyxx)$ $C''(14)$	$(0 -\frac{1}{2}0\frac{1}{2}) + (yxxx)$ C''

that $a(t) = \pi^{-t}\Gamma(t)\zeta(2t)$. It follows from 'Dirichlet series associated with quadratic forms' that $N(\sigma, s) = \dfrac{\phi(\sigma s)}{\phi(s)}$ if $\phi(s) = \prod_{i<j} a(1/2 + s_i - s_j)$

Let $\sigma(s_1, \ldots, s_n) = s_{\sigma^{-1}(1)} \ldots s_{\sigma^{-1}(n)}$ (σ acts on $(1, \ldots, n)$ from left). Then

$$\frac{\phi(\sigma s)}{\phi(s)} = \prod_{\substack{i<j \\ \sigma(i) > \sigma(j)}} \frac{a(1/2 - (s_i - s_j))}{a(\frac{1}{2} + (s_i - s_j))} = \prod_{\substack{i<j \\ \sigma(i) > \sigma(j)}} \frac{a(s_i - s_j)}{a(\frac{1}{2} + s_i - s_j)}$$

$$= \pi^{1/2} \frac{\Gamma(s_i - s_j)}{\Gamma(\frac{1}{2} + s_i - s_j)} \frac{\zeta(2(s_i - s_j))}{\zeta(1 + 2(s_i - s_j))}$$

The denominator has no zeros for $Re(s_i - s_j) > 0$ and the numerator has a pole at $s_i - s_j = 1/2$ or 0.

Choose a point in $\{s_i - s_j > 1/2\}$ such that $s_i - s_{i+1} > s_{i+1} - s_n$, $i = 1, \ldots, n-1$.

Residue at $s_i - s_j = 1/2$ $i<j$ is a constant times the following terms.

1 Say σ takes $(\ldots, i, \ldots, j, \ldots) \to (\ldots j, \ldots, i, \ldots)$ then the residue of $N(\sigma, s)$ is the product of

$\dfrac{a(s_k - s_\ell)}{a(\frac{1}{2} + s_k - s_\ell)}$ $k<\ell$ $\sigma(k) > \sigma(\ell)$

$\dfrac{a(s_k - s_j)}{a(\frac{1}{2} + s_k - s_j)}$ $i<k<j$ $\sigma(k) > \sigma(j)$ No other complication

$i<k$ or $i<k$

$\dfrac{a(s_i - s_k)}{a(\frac{1}{2} + s_i - s_k)}$ $i<k$ $\sigma(i) > \sigma(k)$ No other complication

$k<j$

$\dfrac{a(s_i - s_k)\, a(s_j - s_k)}{a(\frac{1}{2} + s_i - s_k)\, a(\frac{1}{2} + s_j - s_k)} = \dfrac{a(\cancel{s_i - s_k})}{a(\cancel{s_i} - s_k)} \dfrac{a(s_j - s_k)}{a(\frac{1}{2} + s_i - s_k)}$ $k > j$ $\sigma(k) < \sigma(j) < \sigma(i)$

$$\frac{a(s_k-s_i)\,a(s_k-s_j)}{a(\tfrac12+s_k-s_i)\,a(\tfrac12+s_k-s_j)} = \frac{a(s_k-s_i)}{a(\tfrac12+s_k-s_j)}$$

$k<i \quad \sigma(k)>\sigma(i)>\sigma(j)$

Now we have

$(s_1, \ldots, s'_j, s_{j+1}, \ldots, s'_j, s_{k+1}, \ldots, s_n)$

$1/4 \quad -1/4$

$s_k - s_\ell > s_{k'} - s_{\ell'}$ if $k<k'$

$(0, \ldots, 0\ 1/4, 0\ 0, -1/4 \ldots)$

$s_k - s_\ell > s_{k'} - s_{\ell'}$ if $k<k'$ & $k' \neq i$

exceptional term change

none = i or j.

$s_{k'} - s_j > s_k - a$

$S_i - S_\ell < 1/2 \quad k > i$

$s_i - s_\ell$ -

$s_k - s_\ell < 1/2 \quad k>i$

$s_j - s_\ell < 1/2 \qquad s_i - s_\ell < 1/2 \quad$ ~~k~~ $i \neq \ell < j$

Don't interest — $s_i = 1/4$ ⟸ $s_i - s_\ell > 1/2 \quad j > \ell$

unnatural

Don't interest — 0, $n \cdot 1/4$ ⟵ $s_k - s_\ell > 1/2 \quad k < i$

Singularities (a) $s_k - s_\ell = 1/2$ $\qquad \{s_k - s_\ell < 1/2, k > i$; ~~so~~ $s_k - s_\ell > 1/2$, $k < i$?

$k<$ $\qquad$ so $k < i$

$k \neq i$ or j

$\ell \neq i$ or j

(b) $s_k - s_j$ $\to 1/4$ $\quad k<j,\ k\ i<k \Rightarrow s_k - s_j < 1/2$ no singularity

$k < i \qquad \sigma(k) < \sigma(i) \quad s_k - s_j > 1/2$

$s_k - s_j$

$= s_k - s_j$

$s_k\ s_j$

(c) $s_i - s_k$ $\to 1/4$ $\quad i<k \quad \sigma(i) > \sigma(k) \qquad k<j \Rightarrow s_i - s_k < 1/2$

$k > j \quad \sigma(k) > \sigma(j) \qquad s_k - s_k > 1/2$

(d) $k > j \quad \sigma(k) < \sigma(j) \quad s_i - s_k > 1/2$ | $k < i\ \sigma(k) > \sigma(i)$

$s_j - s_k \to -1/4$ singularity at $\tfrac12$ so. | ▬ $s_k - s_i$

1 $i=1, j=4$ Then $\mathrm{Re}\, s_k - s_l < 1/2$ unless $k=1, l=4$. moreover $s_k - s_l \to 0$

if $k=2, l=3$ $\to 1/4$ if $k=1, l=2$ or 3 $\to 1/4$ if $k=2,3, l=4$. Then

are no $k \geq j$ and no $k < i$ so there are no singularities.

(a) $(1\,2\,3\,4) \to (4\,1\,2\,3)$; $(14)(24)(34)$; $\left|\left(\frac{1}{4}, 0, 0, -1/4\right) + (x, y\, z, x)\right|$

$\sigma(4)=1$

$\sigma(2)=3$

$\sigma(3)=4$

$\sigma(4)=1$

$(-\frac{1}{4}, +1/4, 0, 0)$ (x, x, y, z)

$$c\, \frac{a(s_2 - s_4)}{a(\frac{1}{2}+s_2-s_4)} \frac{a(s_3 - s_4)}{a(\frac{1}{2}+s_3-s_4)}$$

(b) $(1\,2\,3\,4) \to (4\,1\,3\,2)$ $(14)(24)(34)(23)$

$(-\frac{1}{4}, +1/4, 0, 0)$ $(x, x, z\, y)$

$$c\, \frac{a(s_2 - s_4)}{a(\frac{1}{2}+s_2-s_4)} \frac{a(s_3 - s_4)}{a(\frac{1}{2}+s_3-s_4)} \frac{a(s_2 - s_3)}{a(\frac{1}{2}+s_2-s_3)}$$

(c) $(1\,2\,3\,4) \to (4\,2\,1\,3)$ $(12)(14)(24)(34)$ $(-\frac{1}{4}, 1/4, 0)(x\, y\, x\, z)$

$\to (4\,3\,1\,2)$ $(13)(14)(24)(23)(34)$ $(-\frac{1}{4}, 0, -1/4, 0))$ $(x\, z\, x\, y)$

$$\left\{c\, \frac{a(s_1 - s_2)}{a(\frac{1}{2}+s_1-s_2)} \quad \frac{a(s_1 - s_3)}{a(\frac{1}{2}+s_1-s_3)}\right\} \left\{\frac{a(s_2 - s_1)}{a(\frac{1}{2}+s_2-s_1)} \frac{a(s_2 - s_4)}{a(\frac{1}{2}+s_2-s_4)}\right\} \left\{1 \quad \frac{a(\frac{1}{2}+s_2-s_3)}{a(\frac{1}{2}+s_2-s_3)}\right\}$$

"

(d) $(1\,2\,3\,4) \to (4\,2\,3\,1)$ $(14)(12)(13)(24)(34)$ $(-\frac{1}{4}\ 0\ 0\ +1/4) + (x\, y\, z\, x)$

$(4\,3\,2\,1)$ $(14)(12)(13)(24)(34)(23)$ $(-\frac{1}{4}, 0, 0, +1/4)$ $(x\, z\, y\, x)$

(e) $(1\,2\,3\,4) \to (2\,4\,3\,1)$ $(14)(12)(13)(34)$ $(\ -1/4\ \ 1/4)(y\, x\, z\, x)$

$(3\,4\,2\,1)$ $(14)(12)(13)(24)(23)$ $(0\ -1/4\ 0\ 1/4)$ $(z\, x\, y\, x)$

f) $(1\,2\,3\,4) \to (2\,3\,4\,1)$ (14) : $(12)(13)$ $(\ 0\ 0\ -1/4\ 1/4)$ $(y\, z\, x\, y)$

$(3\,2\,4\,1)$ (14) $(12)(13)$ (23) $(\ 0\ 0\ -1/4\ 1/4)$ $(z\, y\, x\, x)$

(3) $i=1, j=3$ $\quad s_i - s_j < 1/2$ except $s_1 - s_4) > 1/2$

Singularities $s_1 - s_4$, $\sigma(k) < \sigma(1)$ | $k=4$ or $\sigma(k) < \sigma(3)$ $\quad a(s_3 - s_4)$ $\cancel{<1/2} \to -1/4$, >0

(a) $(1234) \to (3124)$ No singularity
(3142)

(b) $(1234) \to (3214)$ No singularity
(3412)

(3412) $\quad (13)(14)(23)(24)$ $\quad s_1 - s_4 \to 1/4$ Singularity at $1/2$

Singular line $(1/3, 0, -1/6, -1/6) + (x y x x) \to (-1/6, -1/6, 1/3, 0) + (x x x y)$

Residue $\quad c'\ (23)(24)$ $\quad$ (ii)

(c) $(1234) \to (3241)$ $\quad s_1 - s_4 \to 1/4$ Singularity
(3421) — $s_1 - s_4 \to 1/4$ Singularity

$(12)(13)(14)(23)$ | $c'\ (12)(23)$ | $(1/3\ 0, -1/6, -1/6) + (x y x x) \to$
$(12)(13)(14)(23)(24)$ | $c'\ (12)((23)(24)$ | $(1/3, 0, -1/6, -1/6) + (x y x x) \to$

$(-1/6, 0, -1/6\ \ 1/3) + (x y x x)$
$(-1/6 - 1/6, 0, 1/3) + (x x y x)$ $\quad$ (i)

(d) $(1234) \to (2341)$ $\quad$ Singularity $s_1 - s_4 = 1/2$ $\quad (12)(13)(14)$
(4321) $\quad$ Singularity $s_3 - s_4 = 0$ $\quad (12)(13)(14)(34)(23)(24)$

$c'\ (12)$ $\quad (1/3\ 0, -1/6, -1/6) \to (0, -1/6\ -1/6\ 1/3) + (y x x x)$

? $(12)(23)(24)$ $\quad (1/3\ 0\ -1/6)(-1/6) \to (-1/6\ -1/6\ 0\ \ 1/3) + (x x y x)$ $\quad$ (i)

(e) $(1234) \to (2314)$ No singularity $\quad (12)(13)$
(4312) $\quad s_1 - \cancel{s_4} = 1/2 \quad s_3 - s_4 = 0$ $\quad (13)(14)(34)$ $\quad (23)(24)$
$s_3 - s_4 < 1/2$

? $(23)(24)$ $\quad (\tfrac{1}{3}\ 0, -1/6, -1/6) \to (-1/6\ -1/6\ \tfrac{1}{3}\ 0) + (x x x y)$ $\quad$ (ii)

(f) $(1234) \to (24\,31)$ $s_3 - s_4 = 0$ $(12)(13)(14)(34)$

$(42\,31)$ $s_3 - s_4 = 0$ $(12)(13)(14)(34)(24)$

$(24) = (23)$

$(1/3\; 0, -1/6, -1/6) \to (0\; -1/6\; -1/6\; 1/3)$

$(1/3\; 0, -1/6\; -1/6)$ $(-1/6\; 0\; -1/6\; 1/3)$

3 $i = 1$ $j = 2$

(a) $(1234) \to (2134)$ $\cancel{(12)}$

(2143) $\cancel{(12)}(34)$ $s_3 - s_4 < 1/2$ No singularity.

(b) $(1234) \to (2314)$ $(\cancel{12})(13)$ $s_1 - s_3 > 1/2$ singularity at $s_1 - s_3 = 1/2$

(2413) $(\cancel{12})(14)(34)$ $s_1 - s_4) > 1/2$ singularity at $s_1 - s_4 = 1/2$

$(1/3\; 1/6\; -1/6\; 0) + (x\,x\,x\,y) \to (-1/6\; -1/6\; 1/3\; 0) + (x\,x\,x\,y)$ | c' (i)

$(1/3\; -1/6\; 0\; -1/6) + (x\,x\,y\,x) \to (-1/6\; -1/6\; 1/3\; 0) + (x\,x\,x\,y)$ $c'(34)$ (ii)

(c) $(1234) \to (2341)$ $(12)(13)(14)$ $s_1 - s_3 = 1/2$ $s_1 - s_4 = 1/2$

(2431) $(12)(13)(14)(34)$ $s_1 - s_3 = 1/2$ $s_1 - s_4 = 1/2$

(iii)

$(1/3\; -1/6\; -1/6\; 0) \to (-1/6\; -1/6\; 0, 1/3)$ $c'(1,4)$ | $(1/3, -1/6, -1/6\; 0) \to (-1/6, 0\; -1/6\; 1/3)$ $c'(13)$ (iii)

(v)

$(1/3\; -1/6\; -1/6\; 0) \to (-1/6\; 0\; -1/6\; 1/3)$ $c'(14)(34)$ | $(1/3, -1/6, 0, -1/6) \to (-1/6\; -1/6\; 0\; 1/3)$ $c'(13)(34)$ (iv)

(d) $(1234) \to (3214)$ $(12)(13)(23)$ $s_2 - s_3 = 1/2 \to -1/2$ Singularity at $s_2 - s_3 = 0$

(4213) $(12)(14)(24)(34)$ $s_2 - s_4 = 1/2 \to -1/2$ Singularity $s_2 - s_3 = 0$

$(1/3\; -1/6\; -1/6\; 0) \to (-1/6\; -1/6\; 1/3\; 0)$? (i)

$(1/3\; -1/6\; 0\; -1/6) \to (-1/6\; -1/6\; 1/3\; 0)$?(34) (ii)

(e) $(1234) \to (3241)$ $(\cancel{12})(13)(14)(23)$ $s_1 - s_4 > 1/2$ $s_2 - s_3 < 1/2$

(4231) $(\cancel{12})(13)(14)(34)(24)$ $s_1 - s_3 > 1/2$ $s_3 - s_4 < 1/2$ $s_2 - s_4 < 1/2$

Singularity $s_1 - s_4 = 1/2$ $\quad c'\,(13)(23)$: $\left(\tfrac{1}{3}\ -1/6\ 0\ -1/6\right) \to \left(0\ -1/6\ -1/6\ 1/3\right)$ (vii)

" $\quad s_1 - s_3 = 1/2$ $\quad c'\,(14)(24)(34)$: $\left(1/3\ -1/6\ -1/6\ 0\right) \to \left(0\ -1/6\ -1/6\ 1/3\right)$ (vi)

Singularity $s_2 - s_3 = 0$ $\quad ?\ (14)$: $\left(1/3\ -1/6\ -1/6\ 0\right) \to \left(-1/6\ -1/6\ 0\ 1/3\right)$ (iii)

Singularity at $s_2 - s_4 = 0$ $\quad ?\ (13)(34)$ $\left(1/3\ -1/6\ 0\ -1/6\right) \to \left(-1/6\ -1/6\ 0\ 1/3\right)$ (iv)

f) $(1234) \to (3421)$ $\quad (12)(13)(14)(23)(24)$ $\quad$ Singularity $s_2 = s_3$ $\quad s_2 = s_4$

(4321) $\quad (12)(13)(14)(23)(24)(34)$ $\quad s_2 = s_3$ $\quad s_2 = s_4$

Singularity at $s_2 - s_3 = 0$: $\quad ?\ (14)(24)$ $\left(1/3\ -1/6\ -1/6\ 0\right) \to \left(-1/6\ 0\ -1/6\ 1/3\right)$ (v)

34

" $\quad$ " $\quad$ " $=0$: $\quad ?\ (14)(24)(34)$ $\left(1/3\ -1/6\ -1/6\ 0\right) \to \left(-1/6\ 0\ -1/6\ 1/3\right)$ (vi)

" $\quad$ " $\quad s_2 - s_4 = 0$ $\quad ?\ (13)(23)$ $\left(1/3\ -1/6\ 0\ -1/6\right) \to \left(0\ -1/6\ -1/6\ 1/3\right)$ (vii)

" $\quad$ " $\quad s_2 - s_4 = 0$ $\quad ?\ (13)(23)(34)$ $\left(\tfrac{1}{3}\ -1/6\ 0\ -1/6\right) \to \left(-1/6\ 0\ -1/6\ 1/3\right)$

"1 = (23)(32)

Apparently then these three terms give no further contribution

That is they contribute only to the two dimensional spectrum.

4. (

4. $i=2$ $j=3$.

(a) $(1234) \to (3214)$ $\quad$ (23) $\quad$ $(12)(13)$ $\quad\quad$ $s_1-s_2 > 1/2 \to -1/4$

(3241) $\quad$ $\cancel{(23)}$ $\quad$ $(12)(13)(14)$ $\quad\quad$ $s_1-s_2 > 1/2 \to -1/4$.

$s_1-s_4 > 1/2$ ✓

Singularities at $s_1-s_2=1/2$, $s_1-s_2=0$

Singularity at $s_1-s_2=1/2$ $\quad$ c'' $\quad$ $(1/2\ 0\ -1/2\ 0)+(xxxy) \to (-1/2\ 0\ 1/2\ 0)+(xxxy)$

" " $s_1-s_2=1/2$ $\quad$ $c''(14)$ $\quad$ $(1/2\ 0\ -1/2\ 0)+(xxxy$ $\quad$ $(-1/2\ 0\ 0\ 1/2)+(xxyx)$

Singularity at $s_1-s_2=0$ $\quad$? $\quad$ $(1/6\ 1/6\ -1/3\ 0)+(xxxy) \to (-1/3\ 1/6\ 1/6\ 0)+(xxxy)$ (i)

" " $s_1-s_2=0$ $\quad$? (14) $\quad$ $(1/6\ 1/6\ -1/3\ 0)+(xxxy) \to (-1/3\ 1/6\ 0\ 1/6)+(xxyx)$ (ii)

Singularity at $s_1-s_4=1/2$ $\quad$ $c'(12)(13)$ $\quad$ $(\frac14\ \frac14\ -\frac14\ -\frac14)+(xyy\overset{x}{\underset{1/4}{}}) \to (-\frac14\ \frac14\ -\frac14\ \frac14)+(yyxx)$

(b) $(1234) \to (3124)$ $\quad$ $(13)\cancel{(23)}$ $\quad\quad$ $s_1-s_3>1/2$ $\quad$ Singularity at $1/2$

(3421) $\quad$ $(13)(12)$ $\quad$ $(14)\cancel{(23)}(24)$ $\quad\quad$ $s_1-s_2 > 1/2$ $\quad$ " $\quad$ " 0 $1/2$

$s_2-s_4>1/2$ $\quad$ $s_1-s_4>1/2$ $\quad$ s

$\searrow 1/4$ $\quad\quad$ $\searrow 0$

singularity at $1/2$ $\quad$ singularity at $1/2$

Singularity at $s_1-s_3=1/2$ n c' $\quad$ $(1/6\ 1/6\ -1/3\ 0) \to (-1/3, 1/6\ 1/6, 0)$ $\quad\quad$ (i)

Singularity at $s_1-s_2=1/2$ $\quad$ $c''(14)(24)$ $\quad$ $(1/2\ 0\ -1/2\ 0)+(xxxy) \to (-1/2\ 0\ 0\ 1/2)+(xyxx)$

Singularity at $s_1-s_2=0$ $\quad$? $(14)(24)$ $\quad$ $(\frac16\ \frac16\ -1/3\ 0)+(xxxy) \to (-1/3\ 0\ 1/6\ 1/6)+(xyxx)$ (iii)

Singularity at $s_2-s_4=1/2$ $\quad$ c' $(14)(13)(12)$ $\quad$ $(0\ 1/3\ -1/6\ -1/6)+(yxxx) \to (-1/6\ -1/6\ 1/3\ 0)+(xxyx)$ (vii)

$c'(13)(12)(24)$

Singularity at $s_1-s_4=1/2$ $\quad$ $(1/4\ 1/4\ -1/4\ -1/4)+(xyyx) \to$ $\quad$ $(-1/4\ -1/4\ 1/4\ 1/4)+(yxyx)$

(c) $(1234) \to (3142)$ $\quad$ $(13)(23)(24)$ $\quad\quad$ $s_1-s_3>1/2$ $\quad$ $s_2-s_4>1/2$

(3412) $\quad$ $(13)(14)\cancel{(23)}(24)$ $\quad\quad$ $s_1-s_3>1/2$ $s_1-s_4>1/2$ $s_2-s_4>1/2$

Singularity at $s_1-s_3=1/2$ $\quad$ $c'(24)$ $\quad$ $(1/6\ 1/6\ -1/3\ 0) \to (-1/3\ 1/6\ 0\ 1/6)$ $\quad$ 4i)

| Singularity at $s_1 - s_3 = 1/2$ c' (14)(24) $(\frac{1}{6}\ \frac{1}{6}\ -1/3\ 0) \to (-1/3\ 0\ 1/6\ 1/6)$ (iii)

| Singularity at $s_2 - s_4 = 1/2$ c' (13) $(0\ \frac{1}{3}\ -1/6\ -1/6) \to (-1/6\ 0\ -1/6\ 1/3)$ (vi)

| Singularity at $s_2 - s_4 = 1/2$ c' (13)(24) $(0\ 1/3\ -1/6\ -1/6) \to (-1/6\ -1/6\ 0\ 1/3)$ (iv)

| Singularity at $s_1 - s_4 = 1/2$ c' (13)(24) $(1/4\ \frac{1}{4}\ -1/4\ -1/4) + (xyyx) \to (-1/4\ -1/4\ \frac{1}{4}\ \frac{1}{4}) + (yxxy)$

(d) (1234) → (1324)   ~~(23)~~   No singularity

(4321)   (12)(13)(14)~~(23)~~(24)(34) $s_1 - s_2 > 1/2$ ~~$s_3 - s_4 > 1/2$~~ X

$s_1 - s_4 > 1/2$

| Singularity at $s_1 - s_2 = 1/2$ c'' (14)(24)(34) $(1/2\ 0\ -1/2\ 0) + (xxxy) \to (0\ -1/2\ 0\ 1/2) + (yxxx)$

| Singularity at $s_1 - s_2 = 0$?   (14)(24)(34) $(1/6\ 1/6\ -1/3\ 0) + (xxxy) \to (0\ -1/3\ 1/6\ 1/6) + (yxxx)$ (v)

~~Singularity at $s_3 - s_4 = 1/2$ c'' (14)(12)(13) $(0\ 1/2\ 0\ -1/2) + (xyyy) \to (-1/2\ 0\ 1/2\ 0) + (yyyx)$~~

| Singularity at $s_3 - s_4 = 0$? (12)(13)(14) $(0\ 1/3\ -1/6\ -1/6) + (xyyy) \to (-1/6\ -1/6\ 1/3\ 0) + (yyyx)$ (vii)

| Singularity at $s_3 - s_4 = 1/2$ c' (12)(13)(24)(34) $(\frac{1}{4}\ \frac{1}{4}\ -1/4\ -1/4) + (xyyx) \to (-1/4\ -1/4\ 1/4\ 1/4) + (xyyx)$

(e) (1234) → (1342)   ~~(23)~~ (24) $s_2 - s_4 > 1/2 \to 1/4$

(4312)   (13)(14)~~(23)~~(24)(34) $s_3 - s_4 < 1/2$ $s_1 - s_3 > 1/2$

$s_1 - s_4 > 1/2$

| Singularity at $s_3 - s_4 = 0$?   (13)(14) $(0\ 1/3\ -1/6\ -1/6) + (xyyy) \to (-1/6\ -1/6\ 0\ 1/3) + (yyxy)$ (ii

| Singularity at $s_2 - s_4 = 1/2$ c' $(0\ 1/3\ -1/6\ -1/6) + (xyyy) \to (0\ -1/6\ -1/6\ 1/3) + (xyyy)$ (vii

| Singularity at $s_1 - s_3 = 1/2$ c' (14)(24)(34) $(1/6\ 1/6\ -1/3\ 0) + (xxxy) \to (0\ -1/3\ 1/6\ 1/6)(yxxx)$ (v)

| Singularity at $s_1 - s_4 = 1/2$ c' (13)(24)(34) $(\frac{1}{4}\ \frac{1}{4}\ -1/4\ -1/4) + (xyyx) \to (-1/4\ -1/4\ \frac{1}{4}\ \frac{1}{4}) + (xyxy)$

(f) (1234) → (1432)   ~~(23)~~ (24)(34) $s_3 - s_4 < 1/2$

(4132)   ~~(23)~~ (24)(34)(14) $s_3 - s_4 < 1/2$ $s_1 - s_4 > 1/2$

Singularity at $s_3 - s_4 = 0$ " ? $(0\ {}^1/_3\ {}^{-1}/_6\ {}^{-1}/_6) + (x\,y\,y\,y) \to (0\ {}^{-1}/_6\ {}^{-1}/_6\ {}^1/_3) + (x\,y\,y\,y)$ (viii)

Singularity at $s_3 - s_4 = 0$? (14) $(0\ {}^1/_3\ {}^{-1}/_6\ {}^{-1}/_6) + (x\,y\,y\,y) \to ({}^{-1}/_6\ 0\ {}^{-1}/_6\ {}^1/_3) + (y\,x\,y\,y)$

${}^{-1}/_3$

Singularity at $s_1 - s_4 = {}^1/_2$ c' (24)(34) $({}^1/_4\ {}^1/_4\ {}^{-1}/_4\ {}^{-1}/_4)\ (x\,y\,y\,x) \to (-{}^1/_4\ {}^1/_4\ {}^{-1}/_4\ {}^1/_4)\ (x\,x\,y\,y)$

5 $i=2$ $j=4$

(a) (1234) $\to$ (4213)
(4231)

(12)(14) (24) (34)

(12)(14)(13) (24) (34)

$s_3 - s_4 < {}^1/_2$ $s_1 - s_2 > {}^1/_2$

Singularity $s_1 - s_2 = {}^1/_2$, $s_1 - s_2 = 0$

$s_1 - s_3 > {}^1/_2$

Singularity at $s_1 - s_2 = {}^1/_2$ c'' (34) $({}^1/_2\ 0\ 0\ {}^{-1}/_2) + (x\,x\,y\,x) \to (-{}^1/_2\ 0\ {}^1/_2\ 0) + (x\,x\,x\,y)$

" " $s_1 - s_2 = {}^1/_2$ c'' (13)(34) $({}^1/_2\ 0\ 0\ {}^{-1}/_2) + (x\,x\,y\,x) \to ({}^{-1}/_2\ 0\ 0\ {}^1/_2) + (x\,x\,y\,x)$

Singularity at $s_1 - s_2 = 0$? (34) $({}^1/_6\ \frac{1}{6}\ 0\ {}^{-1}/_3) + (x\,x\,y\,x) \to (-{}^1/_3\ \frac{1}{6}\ \frac{1}{6}\ 0) + (x\,x\,x\,y)$ (i)

" " $s_1 - s_2 = 0$? (13)(34) $(\frac{1}{6}\ \frac{1}{6}\ 0\ {}^{-1}/_3) + (x\,x\,y\,x) \to ({}^{-1}/_3\ \frac{1}{6}\ 0\ \frac{1}{6}) + (x\,x\,y\,x)$ (ii)

Singularity at $s_1 - s_3 = {}^1/_2$ c' (12)(14)(34) $(\frac{1}{4}\ \frac{1}{4}\ -\frac{1}{4}\ -\frac{1}{4}) + (x\,y\,x\,y) \to (-\frac{1}{4}\ \frac{1}{4}\ -\frac{1}{4}\ \frac{1}{4}) + (y\,y\,x\,x)$

(b) (1234) $\to$ (4123) (14)(24)(34) $s_1 - s_4 > {}^1/_2$, $s_3 - s_4 < {}^1/_2$

(4321) (12)(13)(14)(23)(24)(34) $s_1 - s_3 > {}^1/_2$, $s_1 - s_2 > {}^1/_2$, $s_2 - s_3 < {}^1/_2$ $s_3 - s_4 < {}^1/_2$

Singularity at $s_1 - s_4 = {}^1/_2$ c' (34) $(\frac{1}{6}\ \frac{1}{6}\ 0\ {}^{-1}/_3) + (x\,x\,y\,x) \to (-{}^1/_3\ \frac{1}{6}\ \frac{1}{6}\ 0) + (x\,x\,x\,y)$ (i)

Singularity at $s_1 - s_3 = {}^1/_2$ c'(12)(14)(23)(34) $(\frac{1}{4}\ \frac{1}{4}\ \frac{-1}{4}\ {}^{-1}/_4) + (x\,y\,x\,y) \to (-\frac{1}{4}\ \frac{-1}{4}\ \frac{1}{4}\ \frac{1}{4}) + (y\,x\,y\,x)$

" $s_1 - s_2 = {}^1/_2$ c''(13)(23)(34) $(\frac{1}{2}\ 0\ 0\ {}^{-1}/_2) + (x\,x\,y\,x) \to (-{}^1/_2\ 0\ 0\ {}^1/_2) + (x\,y\,x\,x)$

" " $s_1 - s_2 = 0$? (13)(23)(34) $(\frac{1}{6}\ \frac{1}{6}\ 0\ {}^{-1}/_3) + (x\,x\,y\,x) \to (-{}^1/_3\ 0\ \frac{1}{6}\ \frac{1}{6}) + (x\,y\,x\,x)$ (ii)

(c) (1234) $\to$ (4132) (14)(~~24~~)(23)(34) $s_1 - s_4 > {}^1/_2$

(4312) (14)(13) (23)(~~24~~)(34) $s_1 - s_4 > {}^1/_2$ $s_1 - s_3 > {}^1/_2$

Singularity at $s_1 - s_4 = {}^1/_2$ c' (23)(34) (13) $(\frac{1}{6}\ \frac{1}{6}\ 0\ {}^{-1}/_3) + (x\,x\,y\,x) \to ({}^{-1}/_3\ \frac{1}{6}\ 0\ \frac{1}{6}) + (x\,x\,y\,x)$ (ii)

Singularity at $s_1 - s_4 = 1/2$ c' (13)(23)(34) $(\frac{1}{6}\ \frac{1}{6}\ 0\ -1/3) + (x x y x) \to (-1/3\ 0\ \frac{1}{6}\ \frac{1}{6}) + (x y x x)$ (iii)

Singularity at $s_1 - s_3 = 1/2$ c' (14)(23)(34) $(\frac{1}{4}\ \frac{1}{4}\ \frac{-1}{4}\ -1/4) + (x y x y) \to (-1/4\ -1/4\ \frac{1}{4}\ \frac{1}{4}) + (y x x y)$

(d) (1234) $\to$ (1423) $\cancel{(24)}$ (34) $s_3 - s_4 < 1/2$ No singularity.

(3421) (12)(13)(14)(23)$\cancel{(24)}$ $s_1 - s_2 > 1/2$ $s_1 - s_3 > 1/2$ $s_2 - s_3 < 1/2$

Singularity at $s_1 - s_2 = 1/2$ c'' (13)(23) $(\frac{1}{2}\ 0\ 0\ -1/2) + (x x y x) \to (0\ -1/2\ 0\ 1/2) + (y x x x)$

| " " $s_1 - s_2 = 0$? (13)(23) $(\frac{1}{6}\ \frac{1}{6}\ 0\ -1/3) + (x x y x) \to (0\ -1/3\ \frac{1}{6}\ \frac{1}{6}) + (y x x x)$ (iv)

+ " " $s_1 - s_3 = 0$ c' (12)(14)(23) $(\frac{1}{4}\ \frac{1}{4}\ -1/4\ \frac{-1}{4}) + (x y x y) \to (-\frac{1}{4}\ -\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}) + (x y y x)$

(e) (1234) $\to$ (1432) (24)(23)(34) $s_2 - s_3 < 1/2$ $s_3 - s_4 < 1/2$ No singularity

(3412) (13)(14)(23)$\cancel{(24)}$ $s_1 - s_3 > 1/2$ $s_1 - s_4 > 1/2$

Singularity at $s_1 - s_3 = 1/2$ c' (14)(23) $(\frac{1}{4}\ \frac{1}{4}\ -1/4\ \frac{-1}{4}) + (x y x y) \to (\frac{-1}{4}\ \frac{-1}{4}\ \frac{1}{4}\ \frac{1}{4})$ $(x y x y)$

Singularity at $s_1 - s_4 = 1/2$ c' (13)(23) $(\frac{1}{6}\ \frac{1}{6}\ 0\ -1/3) + (x x y x) \to (0\ -1/3\ \frac{1}{6}\ \frac{1}{6}) + (y x x x)$ (iv)

(f) (1234) $\to$ (1342) (23)$\cancel{(24)}$ $s_2 - s_3 < 1/2$ No singularity

(3142) (13)(23)$\cancel{(24)}$ $s_2 - s_3 < 1/2$ $s_1 - s_3 > 1/2$

Singularity at $s_1 - s_3 = 1/2$ c'(23) + $(\frac{1}{4}\ \frac{1}{4}\ -1/4\ -\frac{1}{4}) + (x y x y) \to (-1/4\ \frac{1}{4}\ \frac{-1}{4}\ \frac{1}{4}) + (x x y y)$

6 $i = 3$ $j = 4$.

(a) (1234) $\to$ (4312) (13)(14)(23)(24)$\cancel{(34)}$ $s_1 - s_3 > 1/2$ $s_2 - s_4 > 1/2$

(4321) (12)(13)(14)(23)(24)$\cancel{(34)}$ $s_1 - s_2 > 1/2$

Singularity at $s_1 - s_3 = 1/2$ c'' (23)(24) : $(\frac{1}{2}\ 0\ 0\ -1/2) + (x y x x) \to (-1/2\ 0\ 1/2\ 0) + (x x x y)$

|| " " $s_1 - s_3 = 1/2$ c'' (12)(23)(24) : $(\frac{1}{2}\ 0\ 0\ -1/2) + (x y x x) \to (-1/2\ 0\ 0\ 1/2) + (x x y x)$

Singularity at $s_1 - s_3 = 0$? (23)(24) $(\frac{1}{6}\ 0\ \frac{1}{6}\ -1/3) + (xyxx) \to (-\frac{1}{3}\ \frac{1}{6}\ \frac{1}{6}\ 0) + (xxxy)$ (i)

" " $s_1 - s_3 = 0$? (12)(23)(24) $(\frac{1}{6}\ 0\ \frac{1}{6}\ -1/3) + (xyxx) \to (-1/3\ \frac{1}{6}\ 0\ \frac{1}{6}) + (xxyx)$ (viii)
= 1

Singularity at $s_2 - s_3 = 1/2$ c'' (13)(14) : $(0\ \frac{1}{2}\ 0\ -1/2) + (xyyy) \to (-1/2\ 0\ 0\ 1/2) + (yyxy)$

" " " $s_2 - s_3 = 1/2$ c'' (13)(14)(12) : $(0\ 1/2\ 0\ -1/2) + (xyyy) \to (-1/2\ 0\ 1/2\ 0) + (yyyx)$

Singularity at $s_2 - s_3 = 0$? (13)(14) $(0\ \frac{1}{6}\ \frac{1}{6}\ -1/3) + (xyyy) \to (-1/3\ \frac{1}{6}\ 0\ \frac{1}{6}) + (yyxy)$ (v)

" " $s_2 - s_3 = 0$? (13)(14)(12) $(0\ \frac{1}{6}\ \frac{1}{6}\ -1/3) + (xyyy) \to (-1/3\ \frac{1}{6}\ \frac{1}{6}\ 0) + (yyyx)$ (ii)

Singularity at $s_1 - s_2 = 1/2$ c' (13)(14)(23)(24) $(\frac{1}{4}\ -\frac{1}{4}\ \frac{1}{4}\ -1/4) + (xxyy) \to (-1/4\ \frac{1}{4}\ -1/4\ \frac{1}{4}) + (yyxx)$

(b) (1234) $\to$ (4132) (14)(23)(24)($\cancel{34}$) $s_1 - s_4 > 1/2$ $\to 1/4$ $s_2 - s_3 > 1/2$
(4231) (14)(12)(13)(24)($\cancel{34}$) $s_2 - s_4 > 1/2 \to \frac{1}{4}$ $s_1 - s_3 > 1/2 \to -1/4$ $s_1 - s_2 > 1/2 \to 0$

Singularity at $s_1 - s_4 = 1/2$ c'(23)(24) $(\frac{1}{6}\ 0\ \frac{1}{6}\ -1/3) + (xyxx) \to (-\frac{1}{3}\ \frac{1}{6}\ \frac{1}{6}\ 0) + (xxxy)$ (i)

" " $s_2 - s_3 = 1/2$ c'' (14) $(0\ \frac{1}{2}\ 0\ -1/2) + (xyyy) \to (-1/2\ 0\ 0\ \frac{1}{2}) + (yxyy)$

" " $s_2 - s_3 = 0$? (14) $(0\ 1/6\ \frac{1}{6}\ -1/3) + (xyyy) \to (-1/3\ 0\ \frac{1}{6}\ \frac{1}{6}) + (yxyy)$ (iii)

Singularity at $s_2 - s_4 = 1/2$ c' (12)(13)(14) $(0\ \frac{1}{6}\ \frac{1}{6}\ -1/3) + (xyyy) \to (-1/3\ \frac{1}{6}\ \frac{1}{6}\ 0) + (yyyx)$ (ii)

Singularity at $s_1 - s_3 = 1/2$ c'' (12)(24) $(\frac{1}{2}\ 0\ 0\ -1/2) + (xyxx) \to (-1/2\ 0\ 0\ 1/2) + (xyxx)$

Singularity at $s_1 - s_3 = 0$? (12)(24) $(\frac{1}{6}\ 0\ \frac{1}{6}\ -1/3) + (xyxx) \to (-1/3\ 0\ \frac{1}{6}\ \frac{1}{6}) + (xyxx)$ (iv)

Singularity at $s_1 - s_2 = 1/2$ c'(14)(13)(24) $(\frac{1}{4}\ -\frac{1}{4}\ \frac{1}{4}\ -1/4) + (xxyy) \to (-1/4\ -\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}) + (yxyx)$

(c) (1234) $\to$ (4123) (14)(24)($\cancel{34}$) $s_1 - s_4 > 1/2$ $s_2 - s_4 > 1/2$
(4213) (14)(12)(24) 34) $s_1 - s_2 > 1/2$ $s_2 - s_4 > 1/2$ $s_1 - s_4 > 1/2$

Singularity at $s_1 - s_4 = 1/2$ c' (24) $(\frac{1}{6}\ 0\ \frac{1}{6}\ -1/3) + (xyxx) \to (-1/3\ \frac{1}{6}\ 0\ \frac{1}{6}) + (xxyx)$ (viii)

Singularity at $s_2 - s_4 = 1/2$ $\quad c'(14)$ $\quad (0\ \frac{1}{6}\ \frac{1}{6}\ -1/3) + (xyyy) \to (-1/3\ 0\ \frac{1}{6}\ \frac{1}{6}) + (yxyy)$ (iii)

Singularity at $s_1 - s_4 = 1/2$ $\quad c'(12)(24)$ $\quad (1/6\ 0\ \frac{1}{6}\ -1/3) + (xyxx) \to (-1/3\ 0\ \frac{1}{6}\ \frac{1}{6}) + (xyxx)$ (iv)

Singularity at $s_2 - s_4 = 1/2$ $\quad c'(12)(14) \underset{(13)}{=}$ $\quad (0\ \frac{1}{6}\ \frac{1}{6}\ -1/3) + (xyyy) \to (-1/3\ \frac{1}{6}\ 0\ \frac{1}{6}) + (yyxy)$ (v)

Singularity at $s_1 - s_2 = 1/2$ $\quad c'(14)(24)$ $\quad (\frac{1}{4}\ -1/4\ \frac{1}{4}\ -1/4) + (xxyy) \to (-\frac{1}{4}\ -\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}) + (yxxy)$

(d) $(1234) \to (1432)$ $\quad (23)(24)(\cancel{34})$ $\quad s_2 - s_3 > 1/2$

(2431) $\quad (12)(13)(14)(\cancel{34})$ $\quad s_1 - s_3 > 1/2 \quad s_1 - s_2 > 1/2$

Singularity at $s_2 - s_3 = 1/2$ $\quad c''$ $\quad (0\ \frac{1}{2}\ 0\ -1/2) + (xyyy) \to (0\ -1/2\ 0\ 1/2) + (xyyy)$

Singularity at $s_2 - s_3 = 0$ $\quad ?$ $\quad (0\ \frac{1}{6}\ \frac{1}{6}\ -1/3) + (xyyy) \to (0\ -1/3\ \frac{1}{6}\ \frac{1}{6}) + (xyyy)$ (vi)

Singularity at $s_1 - s_3 = 1/2$ $\quad c''(12)$ $\quad (\frac{1}{2}\ 0\ 0\ -1/2) + (xyxx) \to (0\ -1/2\ 0\ 1/2) + (yxxx)$

Singularity at $s_1 - s_3 = 0$ $\quad ?\ (12)$ $\quad (\frac{1}{6}\ 0\ \frac{1}{6}\ -1/3) + (xyxx) \to (0\ -1/3\ \frac{1}{6}\ \frac{1}{6}) + (yxxx)$ (vii)

Singularity at $s_1 - s_2 = 0$ $\quad c'(13)(14)$ $\quad (\frac{1}{4}\ -1/4\ \frac{1}{4}\ -1/4) + (xxyy) \to (-1/4\ -\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}) + (xyyx)$

(e) $(1234) \to (1423)$ $\quad (24)(\cancel{34})$ $\quad s_2 - s_4 > 1/2$

(2413) $\quad (12)(14)(\cancel{34})$ $\quad s_1 - s_2 > 1/2 \quad s_1 - s_4 > 1/2$

Singularity at $s_2 - s_4 = 1/2$ $\quad c'$ $\quad (0\ \frac{1}{6}\ \frac{1}{6}\ -1/3) + (xyyy) \to (0\ -1/3\ \frac{1}{6}\ \frac{1}{6}) + (xyyy)$ (vi)

Singularity at $s_1 - s_4 = 1/2$ $\quad c'(12)$ $\quad (\frac{1}{6}\ 0\ \frac{1}{6}\ -1/3) + (xyxx) \to (0\ -1/3\ \frac{1}{6}\ \frac{1}{6}) + (yxxx)$ (vii)

Singularity at $s_1 - s_2 = 1/2$ $\quad c'(14)$ $\quad (\frac{1}{4}\ -\frac{1}{4}\ \frac{1}{4}\ -\frac{1}{4}) + \cancel{(xxyy)}$ xxyy $\to$ ~~$(-\frac{1}{4}\ -\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}) + (yyxx)$~~

$(-\frac{1}{4}\ -\frac{1}{4}\ \frac{1}{4}\ \frac{1}{4}) + (xyxy)$

(f) $(1234) \to (1243)$ $\quad c'(34)$ $\quad$ No singularity

(2143) $\quad (12)(\cancel{34})$ $\quad s_1 - s_2 > 1/2$

Singularity at $s_1 - s_2 = 1/2$ $\quad c'$ $\quad (\frac{1}{4}\ -1/4\ \frac{1}{4}\ -1/4) + (xxyy) \to (-1/4\ \frac{1}{4}\ -1/4\ \frac{1}{4}) + (xxyy)$

$\cancel{8}x_1 + 2x_2$ $\quad$ $2x_1 + x_2$

$\phi($ $\quad$ $\cancel{2x_1 \; 2x_2} \; \cancel{x_1 + x_2} \; \cancel{x_1 - x_2}$

$\phi($ $\quad$ $\leq \; \cancel{x} \; f(\lambda)$ $\quad$ $(2,1)$

$a(\tfrac{1}{2} +$ $\quad$ $f(\lambda$

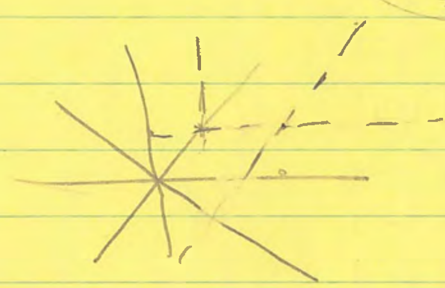
$(2,1)$

$s_1 - s_2 = 1$

$\dfrac{a(\tfrac{1}{2} + \cancel{\otimes}(s)}{a(\tfrac{1}{2} + s)}$ $\quad$ $\cancel{x_1 + x_2} +$ $\quad$ $(2) x_1 - x_2)$

$2^{s_1 + s_2}$ $\quad$ 2^{-1} $\quad$ $s_1 - s_2$ $\quad$ $2s_1 - s_2$

$1 \to 1$



$(s_1 s_2) \to (s_2 s_1)$ $\quad$ $\dfrac{a(s_1 - s_2)}{a(\tfrac{1}{2} + s_1 - s_2)}$

$(s_1 s_2) \to (s_1 - s_2)$ $\quad$ $\dfrac{a(s_2)}{a(\tfrac{1}{2} + s_2)}$ $\quad$ c

$(s_1 s_2) \to (-s_1, s_2)$ $\quad$ $\dfrac{a(2s_1)}{a(\tfrac{1}{2} + s_1)}$ $\quad$ $\dfrac{a(s_1 - s_2)}{a(\tfrac{1}{2} + s_1 - s_2)}$ $\quad$ $\dfrac{a(s_1 + s_2)}{a(\tfrac{1}{2} + s_1 + s_2)}$

$(0,1)$ $\quad$ $(-1,0)$

$(s_1, s_2) \to (s_2, -s_1)$ $\quad$ $\dfrac{a(s_1 - s_2)}{a(\tfrac{1}{2} + s_1 - s_2)}$ $\quad$ $\dfrac{a(s_1)}{a(\tfrac{1}{2} + s_1)}$ ✓

$(s_1, s_2) \to (-s_2, s_1)$ $\quad$ $\dfrac{a(s_2)}{a(\tfrac{1}{2} + s_2)}$ $\quad$ $\dfrac{a(s_1 + s_2)}{a(\tfrac{1}{2} + s_1 + s_2)}$ ✓

$(s_1, s_2) \to (-s_1, -s_2)$ $\quad$ $\dfrac{a(s_1)}{a(\tfrac{1}{2} + s_1)}$ $\quad$ $\dfrac{a(s_1 + s_2)}{a(\tfrac{1}{2} + s_1 + s_2)}$ $\quad$ $\dfrac{a(s_1 - s_2)}{a(\tfrac{1}{2} + s_1 - s_2)}$ $\quad$ $\dfrac{a(s_2)}{a(\tfrac{1}{2} + s_2)}$ ✓

$(s_1, s_2) \to (-s_2, -s_1)$ $\quad$ $a\,\cancel{(s_1 - s_2)}\;\cancel{a(s_1 - s_2)}$

$\dfrac{a(s_1)}{a(\tfrac{1}{2} + s_1)}$ $\quad$ $\dfrac{a(s_1 + s_2)}{a(\tfrac{1}{2} + s_1 + s_2)}$ $\quad$ $\dfrac{a(s_2)}{a(\tfrac{1}{2} + s_2)}$