

# NOTES ON THE KNAPP-ZUCKERMAN THEORY

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The point of these notes is to redefine some of their concepts in terms of the  $L$ -group. I observe, however, that it is best and indeed essential for further applications that their results be formulated for reductive groups rather than just for simply-connected semi-simple groups. I use the notation of *CIRRA*G (*On the classification of irreducible representations of real algebraic groups*) modified sometimes according to Borel's suggestions.

Since we are dealing with tempered representations we start from  $\varphi : W_{\mathbf{C}/\mathbf{R}} \rightarrow {}^L G$  with image which is essentially compact. We suppose  $\varphi$  defines an element of  $\Phi(G)$ . Choose a parabolic  ${}^L P$  in  ${}^L G$  which is minimal with respect to the property that  $\varphi(W_{\mathbf{C}/\mathbf{R}}) \subseteq {}^L P$ . The group  ${}^L P$  defines  $P$  and  $M$ . Let  $\rho$  (with character  $\Theta$ ) be one of the representations of  $M$  associated to  $\varphi$ . Thus  $\rho \in \Pi_\varphi$ , if  $\varphi$  is regarded as taking  $W_{\mathbf{C}/\mathbf{R}}$  to  ${}^L M$ . It is

$$\text{Ind}(G, P, \rho)$$

that Knapp-Zuckerman study.

They define  $W$  on p. 3, formula [2] of their paper *Classification of irreducible tempered representations of semisimple Lie groups*. We want another definition. For this we observe that  $\Omega_{\mathbf{C}}(T, G)$  is isomorphic to  $\Omega({}^L T^0, {}^L G^0)$ . Here  $T$  is a CSG (Cartan subgroup) of  $M$ . We want to regard  $W$  as a subgroup [2] of the latter group. We may assume, along the lines of *CIRRA*G that  $\varphi(\mathbf{C}^\times) \subseteq {}^L T$ , that  $\varphi(W_{\mathbf{C}/\mathbf{R}})$  normalizes  ${}^L T$ , and that  ${}^L T \subseteq {}^L M$ , a chosen Levi factor of  ${}^L P$ .

**Lemma 1.**  *$W$  is the quotient  $\text{Norm}({}^L T) \cap \text{Cent } \varphi(W_{\mathbf{C}/\mathbf{R}}) / {}^L T^0 \cap \text{Cent } \varphi(W_{\mathbf{C}/\mathbf{R}})$ , the normalizer and centralizer being taken in  ${}^L G^0$ .*

Let  $\{1, \sigma\}$  be  $\mathfrak{G}(\mathbf{C}/\mathbf{R})$  so that  $W_{\mathbf{C}/\mathbf{R}}$  is generated by  $\mathbf{C}^\times$  and  $\sigma$  with  $\sigma^2 = -1$ . As on pages 48 and 49 of *CIRRA*G with  $M$  replacing  $G$  the homomorphism  $\varphi$  is defined by  $\mu, \nu$  with  $\nu = \varphi(\sigma)\mu$  and by  $\lambda_0$ . If  $\omega$  in  $\Omega_{\mathbf{R}}(T, G)$  normalizes  $M$  then

$$\omega \in W \iff \omega\rho \sim \rho \iff \omega\mu = \omega_1\mu, \omega\lambda_0 \equiv \omega_1\lambda_0 \pmod{({}^L X_* + (1 - \varphi(\sigma))({}^L X_* \otimes \mathbf{C}))}$$

with  $\omega_1 \in \Omega_{\mathbf{R}}(T, M)$  and  ${}^L X_* = \text{Hom}(GL(1), {}^L T)$ . Replace  $\omega$  by  $\omega_1^{-1}\omega$ . Since  $\omega$  normalizes  $M$ ,

$$\varphi(\sigma)\omega = \omega\varphi(\sigma)$$

on  ${}^L X_*$  and

$$\omega\mu = \mu \iff \omega\mu = \mu, \omega\nu = \nu \iff w\varphi(z) = \varphi(z)w \text{ for } z \in \mathbf{C}^\times$$

if  $w \in {}^L G^0$  represents  $\omega$ . We write

$${}^L M = {}^L M^0 \rtimes W_{\mathbf{C}/\mathbf{R}}[\mathbf{3}]$$

and let

$$\varphi(\sigma) = a \rtimes \sigma$$

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with

$$\lambda^\vee(a) = e^{2\pi i \langle \lambda_0, \lambda^\vee \rangle}.$$

By the first paragraph on p. 37 of *Problems in the theory of automorphic forms* we may choose  $w$  so that  $w\sigma = \sigma w$ . But this is the wrong choice. We should choose  $\omega(a) = \sigma(b)b^{-1}a$ . Replace  $w$  by  $bw$  then

$$w\varphi(\sigma)w^{-1} = \sigma(b)b^{-1}ab\sigma(b)^{-1} \rtimes \sigma = a \rtimes \sigma = \varphi(\sigma).$$

In other words this new choice of  $w$  satisfies

$$w\varphi(v)w^{-1} = \varphi(v) \quad \forall v \in W_{\mathbf{C}/\mathbf{R}}.$$

Since  $\omega \in \Omega_{\mathbf{R}}(T, M)$  and  $\omega\mu = \mu$  imply that  $\omega = 1$  we have found

$$W \hookrightarrow \text{Norm}({}^L T^0) \cap \text{Cent } \varphi(W_{\mathbf{C}/\mathbf{R}}) / {}^L T^0 \cap \text{Cent } \varphi(W_{\mathbf{C}/\mathbf{R}}).$$

To obtain the full lemma we have only to show that if  $w$  lies in  $\text{Norm}({}^L T^0) \cap \text{Cent } \varphi(W_{\mathbf{C}/\mathbf{R}})$  then the corresponding element of the Weyl group stabilizes  $M$  and lies in  $\Omega_{\mathbf{R}}(T, G)$ . It stabilizes  ${}^L M$  because [4]  $\alpha^\vee$  is a root of  ${}^L M$  if and only if  $\varphi(\sigma)\alpha^\vee = -\alpha^\vee$ . Hence it stabilizes  $M$ . By Lemma 5.2 of Shelstad's thesis

$$\omega = \omega_1 \omega_2$$

with  $\omega_1 \in \Omega_{\mathbf{C}}(T, M)$ ,  $\omega_2 \in \Omega_{\mathbf{R}}(T, G)$ . Then

$$w\varphi = \varphi \implies \omega_1^{-1}\mu = \omega_2\mu, \quad \omega_1^{-1}\nu = \omega_2\nu, \quad \omega_1^{-1}\lambda_0 \equiv \omega_2\lambda_0.$$

Another lemma of Shelstad implies that  $\omega_1 \in \Omega_{\mathbf{R}}(T, M)$ . Hence

$$\omega \in \Omega_{\mathbf{R}}(T, G).$$

The advantage of introducing the  $L$ -group appears immediately when Knapp's  $R$ -group is discussed. Let  $S$  be the centralizer of  $\varphi(W_{\mathbf{C}/\mathbf{R}})$  in  ${}^L G^0$  and  $S^0$  the connected component.

**Lemma 2.** *If  $G$  is semi-simple and simply-connected then the  $R$ -group is  $S/S^0$ .*

Let  ${}^L \mathfrak{t}$  be the Lie algebra of  ${}^L T$  and set

$${}^L \mathfrak{t} = {}^L \mathfrak{t}_+ + {}^L \mathfrak{t}_-.$$

where  ${}^L \mathfrak{t}_+$  and  ${}^L \mathfrak{t}_-$  are the  $+1$  and  $-1$  eigenspaces for  $\varphi(\sigma)$ . I claim that  ${}^L \mathfrak{t}_+$  which certainly lies in  $\mathfrak{s}$ , the Lie algebra of  $S^0$ , is in fact a Cartan subalgebra of  $S^0$ . Indeed [5]

$$\mathfrak{s} \subseteq {}^L \mathfrak{t}_+ + \sum_{\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0} \mathbf{C} X_{\alpha^\vee}.$$

If  $\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0$  then  $\alpha^\vee$  cannot be a root of  ${}^L T$  in  ${}^L M$ . Hence

$$\varphi(\sigma)\alpha^\vee \neq -\alpha^\vee$$

and  $\alpha^\vee$  is not 0 on  ${}^L \mathfrak{t}_+$ . The assertion follows.

We may identify  $\text{Hom}({}^L \mathfrak{t}, \mathbf{C})$  with  $\mathfrak{t} \otimes \mathbf{C}$  as a  $\mathfrak{G}(\mathbf{C}/\mathbf{R})$ -module if  $\mathfrak{t}$  is the Lie algebra of  $T$ . If  $\alpha^\vee$  is a root of  ${}^L T^0$  in  ${}^L G^0$  with  $\varphi(\sigma)\alpha^\vee \neq -\alpha^\vee$  set

$$\mathfrak{a}_{\alpha^\vee} = ({}^L \mathfrak{t}_- + \mathbf{C}\alpha^\vee)^\perp.$$

Then  $G_{\alpha^\vee}$  the centralizer of  $\mathfrak{a}_{\alpha^\vee}$  in  $G$  is defined over  $\mathbf{R}$  and  $M$  is the Levi factor of a maximal  $PSG$  of  $G_{\alpha^\vee}$ . Let  $\mu(\rho, \alpha^\vee)$  be the value of the Plancherel measure for

$$\text{Ind}(G_{\alpha^\vee}(\mathbf{R}), M(\mathbf{R}), \rho).$$

Let

$$\mathfrak{X}_{\alpha^\vee} = \{\beta^\vee \mid \varphi(\sigma)\beta^\vee \neq -\beta^\vee, G_{\beta^\vee} = G_{\alpha^\vee}\}.$$

The centralizer of  ${}^L\mathfrak{t}_+$  is

$${}^L\mathfrak{t}_+ + \sum_{\varphi(\sigma)\alpha^\vee = -\alpha^\vee} \mathbf{C}X_{\alpha^\vee} [\mathbf{6}]$$

and this is the Lie algebra of  ${}^LM$ . Moreover

$$S/S^0 \simeq \text{Norm}_S({}^L\mathfrak{t}_+)/\text{Norm}_{S^0}({}^L\mathfrak{t}_+).$$

If  $w \in \text{Norm}_S({}^L\mathfrak{t}_+)$  then  $w$  normalizes  ${}^LM^0$  and centralizes  $\varphi(\mathbf{C}^\times)$ . Consequently it normalizes  ${}^L\mathfrak{t}$  and we have

$$\text{Norm}_S({}^L\mathfrak{t}_+)/{}^LT_+ \simeq W.$$

The lemma and indeed more will be established once the following facts are proved. They will be proved for any  $G$ .

- (i)  $\dim \mathfrak{s}_{\alpha^\vee} = \dim \left( \left( \sum_{\beta^\vee \in \mathfrak{X}_{\alpha^\vee}} \mathbf{C}X_{\beta^\vee} \right) \cap \mathfrak{s} \right) \leq 1$ .
- (ii) It is equal to 1 if and only if  $\mu(\rho, \alpha^\vee) = 0$ .
- (iii) If it is one then  $\mathfrak{s}_{\alpha^\vee}$  defines a root space of  ${}^L\mathfrak{t}_+$  in  $\mathfrak{t}$ . The corresponding reflection in  ${}^L\mathfrak{t}_+$  is the same as that defined by the real root of  $T$  in  $G_{\alpha^\vee}$ .

There are a number of possibilities to consider.

- (a)  $\mathfrak{X}_{\alpha^\vee}$  consists of a single element. Then  $\varphi(\sigma)\alpha^\vee = \alpha^\vee$  and  $\alpha$ , the corresponding root of  $T$ , is real. Since  $\sigma\mu = \nu$ ,  $\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle$  and  $\dim \mathfrak{s}_{\alpha^\vee} = 1$  if and only if  $\langle \mu, \alpha^\vee \rangle = 0$  and

$$\varphi(\sigma)X_{\alpha^\vee} = X_{\alpha^\vee}. [\mathbf{7}]$$

Certainly  $T(\mathbf{R})$  is not fundamental. According to the formula on p. 141 of Harish-Chandra's preprint *Harmonic analysis III*,  $\mu(\rho, \alpha^*)$  is 0 if and only if

$$\nu_\alpha = 0 \quad \text{and} \quad \frac{(-1)^{\rho_\alpha}}{2} (\sigma_{a^*}(\gamma) + \sigma_{a^*}(\gamma^{-1})) \neq 1.$$

Now

$$\nu_\alpha = \langle \mu, \alpha^\vee \rangle.$$

Also  $\mathfrak{s}_{a^*}$  is now of dimension one and

$$\sigma_{a^*}(\gamma) = \sigma_{a^*}(\gamma^{-1}) = \chi(\alpha^\vee(-1)).$$

Here  $\chi$  is associated to  $\varphi : W_{\mathbf{C}/\mathbf{R}} \rightarrow {}^LM$  as on p. 50 of *CIRrag* and if the definition of a coroot is taken into account

$$\gamma = \alpha^\vee(-1).$$

Thus (cf. p. 51 of *CIRrag*)

$$\chi(\alpha^\vee(-1)) = e^{2\pi i \langle \lambda_0, \alpha^\vee \rangle}.$$

Apologies are necessary for this phase of the discussion but the transition from Harish-Chandra's notation to that used in *CIRrag* is clumsy.

On the other hand

$$\varphi(\sigma) = a \rtimes \sigma [\mathbf{8}]$$

and

$$\varphi(\sigma)X_{\alpha^\vee} = e^{2\pi i \langle \lambda_0, \alpha^\vee \rangle} \varphi'(\sigma)(X_{\alpha^\vee})$$

if  $\varphi'(\sigma) = a' \rtimes \sigma$ ,  $a' \in {}^L M_{\text{der}}$ ,  $a^{-1}a' \in {}^L T^0$ . The assertion (ii) will be verified if we show that

$$\varphi'(\sigma)(X_{\alpha^\vee}) = -(-1)^{\rho_\alpha} X_{\alpha^\vee}.$$

Now, by p. 122 of *Harmonic Analysis III*

$$\rho_\alpha = \langle \rho_{\alpha^\vee}, \alpha^\vee \rangle$$

if  $\rho_{\alpha^\vee}$  is one-half the sum of the positive roots of  $G_{\alpha^\vee}$ . But in the present circumstances the derived algebra of  $\mathfrak{g}_{\alpha^\vee}$  is a direct sum because  $\alpha^\vee$  is perpendicular to all roots of  $G_{\alpha^\vee}$  except  $\pm\alpha^\vee$ . Thus

$$\langle \rho_\alpha, \alpha^\vee \rangle = \frac{1}{2} \langle \alpha, \alpha^\vee \rangle = 1.$$

Moreover  $\alpha^\vee$  must be a simple root and so by the definition of  ${}^L M$

$$\varphi'(\sigma)(X_{\alpha^\vee}) = \sigma(X_{\alpha^\vee}) = 1.$$

The assertion (ii) follows. Since the reflections corresponding to  $\alpha$  and  $\alpha^\vee$  are the same, the assertion (iii) does also.

(b) Suppose  $\varphi(\sigma)\alpha^\vee = \alpha^\vee$  and  $\beta^\vee$  different from  $\alpha^\vee$  lies in  $\mathfrak{X}_{\alpha^\vee}$ . [9]

(i) Suppose

$$\langle \mu, \beta^\vee \rangle = \langle \nu, \beta^\vee \rangle = 0.$$

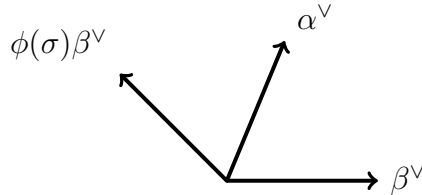
Then

$$\langle \mu, \varphi(\sigma)\beta^\vee \rangle = \langle \nu, \varphi(\sigma)\beta^\vee \rangle = 0.$$

Since  $\varphi(\sigma)\beta^\vee$  lies in the span of  $\{\alpha^\vee, \beta^\vee\}$  and is different from  $\beta^\vee$ , both  $\mu$  and  $\nu$  vanish on this two-dimensional space. As a consequence there are no roots  $\gamma^\vee$  on it orthogonal to  $\alpha^\vee$ . For then  $\varphi(\sigma)\gamma^\vee$  would be  $-\gamma^\vee$  and as a consequence

$$\langle \mu, \gamma^\vee \rangle \neq 0.$$

This leaves only



of type  $A_2$ .

I claim next that if  $\gamma^\vee$  lies in  $X_{\alpha^\vee}$  and is different from  $\alpha^\vee$ ,  $\beta^\vee$ , and  $\varphi(\sigma)\beta^\vee$  then either  $\langle \mu, \gamma^\vee \rangle \neq 0$  or  $\langle \nu, \gamma^\vee \rangle \neq 0$ . If not, consider all roots in the span of  $\{\alpha^\vee, \beta^\vee, \gamma^\vee\}$ . They form a root system of rank 3 on which  $\varphi(\sigma)$  acts. If  $\delta^\vee$  lies in this system then  $\langle \mu, \delta^\vee \rangle = \langle \nu, \delta^\vee \rangle = 0$  so  $\varphi(\sigma)\delta^\vee \neq -\delta^\vee$ . As a consequence [10]

$$\delta^\vee + \varphi(\sigma)\delta^\vee = a\alpha^\vee \quad a \neq 0$$

and

$$\{\delta^\vee \mid \langle \alpha, \delta^\vee \rangle \geq 0\}$$

defines a system of positive roots stable under  $\varphi(\sigma)$ . Let  $\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee$  be the simple roots. They are permuted amongst themselves by  $\varphi(\sigma)$ . Thus by a suitable numbering

$$a_1^\vee = \alpha^\vee, \quad a_3^\vee = \varphi(\sigma)a_2^\vee.$$

Then

$$a\alpha^\vee = \alpha_2^\vee + a_3^\vee.$$

This is a contradiction.

Also we may take

$$X_{\alpha^\vee} = [X_{\beta^\vee}, \varphi(\sigma)X_{\beta^\vee}]$$

and

$$\varphi(\sigma)X_{\alpha^\vee} = -X_{\alpha^\vee}.$$

Thus

$$\mathfrak{s}_{\alpha^\vee} = \mathbf{C}(X_{\beta^\vee} + \varphi(\sigma)X_{\beta^\vee})$$

has dimension 1. Since [11]

$$\langle \mu, \beta' \rangle = (\lambda + i\nu)(H_\beta),$$

the right side conforming to Harish-Chandra's notation, the measure  $\mu(\rho, \alpha^\vee)$  is certainly zero. The reflection defined by  $\mathfrak{s}_{\alpha^\vee}$  is clearly correct on  ${}^L\mathfrak{t}_+$ .

(ii) Suppose that for every  $\beta^\vee$  different from  $\alpha^\vee$  in  $\mathfrak{X}_{\alpha^\vee}$

$$\langle \mu, \beta^\vee \rangle \neq 0 \text{ or } \langle \nu, \beta^\vee \rangle \neq 0.$$

Then  $\dim \mathfrak{s}_{\alpha^\vee} = 1$  if and only if

$$\langle \mu, \alpha^\vee \rangle = 0, \quad \varphi(\sigma)X_{\alpha^\vee} = X_{\alpha^\vee}.$$

Again the first condition is equivalent to  $\nu_\alpha = 0$ . We have to show that when this is so then the second is equivalent to

$$\frac{(-1)^{\rho_\alpha}}{2} (\sigma_{a^*}(\gamma) + \sigma_{a^*}(\gamma^{-1})) \neq 1.$$

Let

$$\varphi(\sigma)X_{\alpha^\vee} = \lambda X_{\alpha^\vee}.$$

We show that

$$\left( \frac{(-1)^{\rho_\alpha}}{2} \right) (\sigma_{a^*}(\gamma) + \sigma_{a^*}(\gamma^{-1})) = -\lambda.$$

This is enough, for  $\lambda = \pm 1$ . As before [12]

$$\sigma_{a^*}(\gamma) = \sigma_{a^*}(\gamma^{-1}) = e^{2\pi i \langle \lambda_0, \alpha^\vee \rangle}$$

and

$$\varphi(\sigma)X_{\alpha^\vee} = e^{2\pi i \langle \lambda_0, \alpha^\vee \rangle} \varphi'(\sigma)(X_{\alpha^\vee}).$$

if  $\varphi'(\sigma)$  is defined as before. What we must do is show that

$$\varphi'(\sigma)(X_{\alpha^\vee}) = -(-1)^{\langle \rho_{\alpha^\vee}, \alpha^\vee \rangle} X_{\alpha^\vee}.$$

This is a statement about a reductive group  $G_{\alpha^\vee}$  and a Levi factor  $M$  of a maximal parabolic,  $M$  and  $G$  both having compact CSGs. It is not bound to the present situation and may be proved by induction on the rank of  $G_{\alpha^\vee}$ . Let  $\beta^\vee$  be the largest root of one of the simple factors of  ${}^L M_{\text{der}}$  and introduce  $a_2, a_1$  as on p. 46 of *CIRRAG*. We may take  $a' = a_2 a_1$ . If  $\rho'$  is the analogue of  $\rho_{\alpha^\vee}$  for the roots perpendicular to  $\beta^\vee$  then by induction

$$a_1 \times \sigma(X_{\alpha^\vee}) = -(-1)^{\langle \rho', \alpha^\vee \rangle} X_{\alpha^\vee}.$$

What we have to do is show that

$$a_2(X_{\alpha^\vee}) = (-1)^\ell X_{\alpha^\vee}, \quad \ell = \left(\frac{1}{2}\right) \sum_{\substack{\langle \gamma, \beta^\vee \rangle \neq 0 \\ \gamma > 0}} \langle \gamma, \alpha^\vee \rangle. \quad [13]$$

Suppose  $\gamma > 0$ ,  $\langle \gamma, \beta^\vee \rangle \neq 0$ ,  $\langle \gamma, \alpha^\vee \rangle \neq 0$  and  $\gamma^\vee$  is not in the plane spanned by  $\alpha^\vee, \beta^\vee$ . Then:

- 1)  $\gamma^\vee = a_2 \gamma^\vee \implies \gamma = a_2 \gamma \implies \langle \gamma, \beta^\vee \rangle = 0$ —impossible
- 2)  $\gamma^\vee = \varphi(\sigma) \gamma^\vee \implies \gamma^\vee = \pm \alpha^\vee$ —impossible
- 3)  $\gamma^\vee = a_2 \varphi(\sigma) \gamma^\vee \implies \gamma^\vee$  in plane of  $\alpha^\vee, \beta^\vee$  because  $(\alpha^\vee, \beta^\vee) = 0$ . Thus  $\gamma, a_2 \gamma, \varphi(\sigma) \gamma, a_2 \varphi(\sigma) \gamma$  are distinct and positive. Since

$$\langle \gamma, \alpha^\vee \rangle = \langle a_2 \gamma, \alpha^\vee \rangle = \langle \varphi(\sigma) \gamma, \alpha^\vee \rangle = \langle a_2 \varphi(\sigma) \gamma, \alpha^\vee \rangle$$

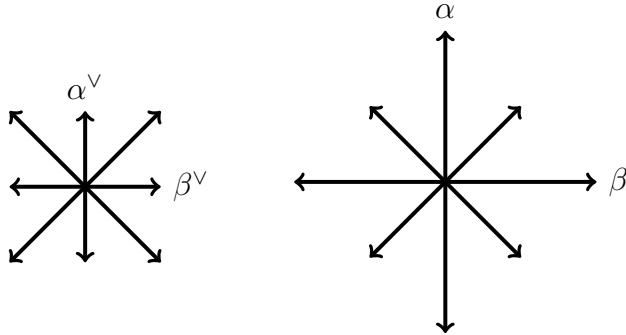
the sum of the four of them even after division by 2 is even and may be dropped from the exponent. So may those  $\langle \gamma, \alpha^\vee \rangle$  which are 0. We confine ourselves to  $\gamma$  with  $\gamma^\vee$  in the plane of  $\alpha^\vee, \beta^\vee$ .

The possibilities are:

- A) No roots except  $\pm \alpha^\vee, \pm \beta^\vee$  in the plane. Then the exponent is 0 and

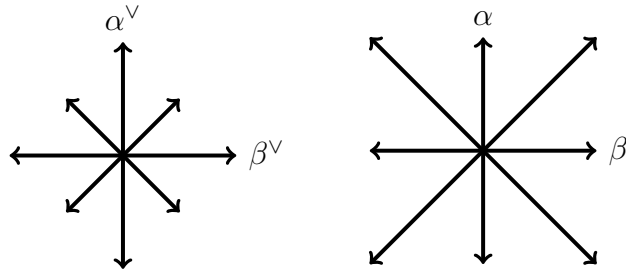
$$a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}.$$

- B)



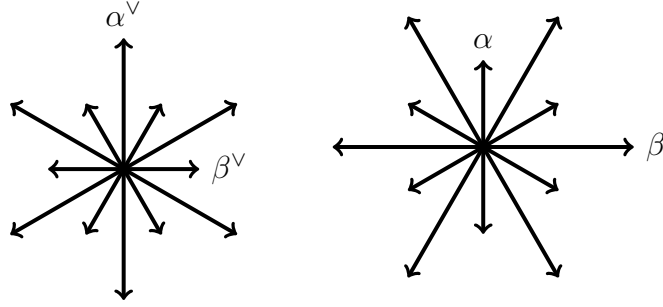
$$\left(\frac{1}{2}\right) \sum \langle \gamma, \alpha^\vee \rangle = \frac{1}{2} \langle \alpha, \alpha^\vee \rangle = 1, \quad a_2(X_{\alpha^\vee}) = -X_{\alpha^\vee} \quad [14]$$

- C)



$$\left(\frac{1}{2}\right) \sum \langle \gamma, \alpha^\vee \rangle = \langle \alpha, \alpha^\vee \rangle = 2, \quad a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}$$

- D)



$$\left(\frac{1}{2}\right) \sum \langle \gamma, \alpha^\vee \rangle = 2 \langle \alpha, \alpha^\vee \rangle = 4, \quad a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}$$

E) The roles of  $\alpha$ ,  $\alpha^\vee$  and  $\beta$ ,  $\beta^\vee$  are reversed

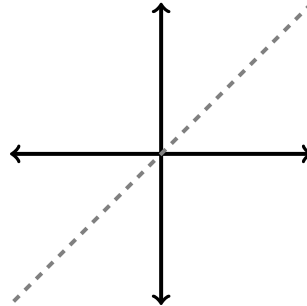
$$\left(\frac{1}{2}\right) \sum \langle \gamma, \alpha^\vee \rangle = \langle \alpha, \alpha^\vee \rangle = 2, \quad a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}.$$

All that is claimed in A) through E) is easy to check. Finally it is clear that the reflection defined by  $\mathfrak{s}_{\alpha^\vee}$  is that defined by  $\alpha$  or  $\alpha^\vee$ .

(i) Suppose that  $\varphi(\sigma)\beta^\vee \neq \beta^\vee$  for all  $\beta^\vee$  in  $\mathfrak{X}_{\alpha^\vee}$ . Then  $\beta^\vee + \varphi(\sigma)\beta^\vee$  is not a root, nor is [15]

$$\frac{\beta^\vee + \varphi(\sigma)\beta^\vee}{2}.$$

(ii) Suppose that  $\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0$ . Then  $\alpha^\vee - \varphi(\sigma)\alpha^\vee$  is not a root and  $\langle \alpha^\vee, \varphi(\sigma)\alpha^\vee \rangle = 0$ . Since  $\alpha^\vee$  and  $\varphi(\sigma)\alpha^\vee$  have the same length, the root diagram of the plane spanned by  $\alpha^\vee, \varphi(\sigma)\alpha^\vee$  is



I claim that if  $\beta^\vee$  lies in  $\mathfrak{X}_{\alpha^\vee}$  but not in this plane then either  $\langle \mu, \beta^\vee \rangle = 0$  or  $\langle \nu, \beta^\vee \rangle = 0$ . Otherwise in the three-dimensional plane spanned by  $\alpha^\vee, \varphi(\sigma)\alpha^\vee, \beta^\vee, \varphi(\sigma)\beta^\vee$  we have a root system and

$$\left\{ \gamma^\vee \mid \langle \gamma, \alpha^\vee + \varphi(\sigma)\alpha^\vee \rangle \geq 0 \right\}$$

is a set of positive roots, for

$$\langle \gamma, \alpha^\vee + \varphi(\sigma)\alpha^\vee \rangle$$

is never 0, because if it were then  $\varphi(\sigma)\gamma^\vee = -\gamma^\vee$ . Since  $\langle \mu, \gamma^\vee \rangle = \langle \nu, \gamma^\vee \rangle = 0$  this is impossible. Then  $\varphi(\sigma)$  permutes the three simple roots amongst themselves, and leaves one fixed. This is a contradiction. Thus [16]

$$\mathfrak{s}_\alpha = \mathbf{C}(X_{\alpha^\vee} + \varphi(\sigma)X_{\alpha^\vee})$$

has dimension one. Since  $T$  is fundamental in  $G_{\alpha^\vee}$ , the formula on p. 97 of *Harmonic analysis III* shows that  $\mu(\rho, \alpha^\vee) = 0$ . The three assertions follow again.

- (iii) Suppose that for any  $\beta^\vee$  in  $\mathfrak{X}_{\alpha^\vee}$  either  $\langle \mu, \beta^\vee \rangle \neq 0$  or  $\langle \nu, \beta^\vee \rangle \neq 0$ . Then  $\mathfrak{s}_{\alpha^\vee} = 0$ . By the same formula in *Harmonic analysis III*,

$$\mu(\rho, \alpha^\vee) \neq 0.$$

Lemma 2 is now completely proved. I should observe, for it will remove a confusion that could otherwise arise, that

$$-\overline{\langle \mu, \alpha^\vee \rangle} = \langle \nu, \alpha^\vee \rangle$$

for any  $\alpha^\vee$ .

It is also possible to give Zuckerman's proof that the  $R$ -group is a sum of  $Z_2$ 's in the above context. Let  $\text{Norm}_S^+({}^L\mathfrak{t}_+)$  be the set of elements of  $\text{Norm}_S({}^L\mathfrak{t}_+)$  that take positive roots of  $\mathfrak{s}_0$  to positive roots. Then

$$R = S/S^0 \simeq \text{Norm}_S^+({}^L\mathfrak{t}_+)/{}^L\mathfrak{t}_+.$$

Let

$$\mathfrak{s}_1 = {}^L\mathfrak{t}_+ + \sum_{\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0} \mathbf{C}X_{\alpha^\vee} [\mathbf{17}]$$

The elements of  $\text{Norm}_S^+({}^L\mathfrak{t}_+)$  take  $\mathfrak{s}_1$  to itself. Let  $Q$  be the operator

$$\frac{1}{|R|} \sum_R r$$

on  $\mathfrak{t} \otimes \mathbf{C}$ . Since the centralizer of  $\varphi(\mathbf{C}^\times)$  is connected,  $S$  lies in the connected group  $S_1$  with Lie algebra  $\mathfrak{s}_1$ . Thus by Chevalley's theorem  $R$  is contained in the group generated by the reflections associated to the roots  $\alpha^\vee$  of  $\mathfrak{s}_1$  for which  $Q\alpha^\vee = 0$ .

If  $\alpha^\vee$  is a root of  $\mathfrak{s}_1$  then  $\varphi(\sigma)\alpha^\vee \neq -\alpha^\vee$ . Suppose  $\varphi(\sigma)\alpha^\vee \neq \alpha^\vee$ . Then

$$X_{\alpha^\vee} + \varphi(\sigma)X_{\alpha^\vee} \neq 0$$

and lies in  $\mathfrak{s}$ . Thus  $\alpha^\vee$  restricted to  ${}^L\mathfrak{t}_+$  defines a root of  $\mathfrak{s}$ . Since the elements of  $r$  stabilize  ${}^L\mathfrak{t}_+$  and each  $r$  takes positive roots of  ${}^L\mathfrak{t}_+$  in  $\mathfrak{s}$  to positive roots,

$$Q\alpha^\vee \neq 0.$$

Thus if  $\alpha^\vee$  is a root of  $\mathfrak{s}_1$  then

$$Q\alpha^\vee = 0 \implies \varphi(\sigma)\alpha^\vee = \alpha^\vee.$$

Moreover  $\alpha^\vee$  cannot be a root of  $\mathfrak{s}$  and therefore

$$\varphi(\sigma)X_{\alpha^\vee} = -X_{\alpha^\vee}. [\mathbf{18}]$$

Finally if  $Q\alpha^\vee = 0$ ,  $Q\beta^\vee = 0$  then  $\alpha^\vee \pm \beta^\vee$  is not a root because  $\varphi(\sigma)X_{\alpha^\vee + \beta^\vee} = \varphi(\sigma)[X_{\alpha^\vee}, X_{\beta^\vee}] = [-X_{\alpha^\vee}, -X_{\beta^\vee}] = X_{\alpha^\vee + \beta^\vee}$  and  $\alpha^\vee + \beta^\vee$  would have to be a root of  $\mathfrak{s}$ . This is inconsistent with

$$Q(\alpha^\vee + \beta^\vee) = 0.$$

The set of positive  $\alpha^\vee$  for which  $\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0$  and  $Q\alpha^\vee = 0$  is the strongly orthogonal system needed for Zuckerman's argument.



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