# NOTES ON THE KNAPP-ZUCKERMAN THEORY

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The point of these notes is to redefine some of their concepts in terms of the $L$-group. I observe, however, that it is best and indeed essential for further applications that their results be formulated for reductive groups rather than just for simply-connected semi-simple groups. I use the notation of *CIRRAG* (*On the classification of irreducible representations of real algebraic groups*) modified sometimes according to Borel's suggestions.

Since we are dealing with tempered representations we start from $\varphi : W_{\mathbf{C}/\mathbf{R}} \to {}^L G$ with image which is essentially compact. We suppose $\varphi$ defines an element of $\Phi(G)$. Choose a parabolic ${}^L P$ in ${}^L G$ which is minimal with respect to the property that $\varphi(W_{\mathbf{C}/\mathbf{R}}) \subseteq {}^L P$. The group ${}^L P$ defines $P$ and $M$. Let $\rho$ (with character $\Theta$) be one of the representations of $M$ associated to $\varphi$. Thus $\rho \in \Pi_\varphi$, if $\varphi$ is regarded as taking $W_{\mathbf{C}/\mathbf{R}}$ to ${}^L M$. It is

$$\operatorname{Ind}(G, P, \rho)$$

that Knapp-Zuckerman study.

They define $W$ on p. 3, formula [2] of their paper *Classification of irreducible tempered representations of semisimple Lie groups.* We want another definition. For this we observe that $\Omega_{\mathbf{C}}(T, G)$ is isomorphic to $\Omega({}^L T^0, {}^L G^0)$. Here $T$ is a CSG (Cartan subgroup) of $M$. We want to regard $W$ as a subgroup **[2]** of the latter group. We may assume, along the lines of *CIRRAG* that $\varphi(\mathbf{C}^\times) \subseteq {}^L T$, that $\varphi(W_{\mathbf{C}/\mathbf{R}})$ normalizes ${}^L T$, and that ${}^L T \subseteq {}^L M$, a chosen Levi factor of ${}^L P$.

**Lemma 1.** *$W$ is the quotient* $\operatorname{Norm}({}^L T) \cap \operatorname{Cent} \varphi(W_{\mathbf{C}/\mathbf{R}}) / {}^L T^0 \cap \operatorname{Cent} \varphi(W_{\mathbf{C}/\mathbf{R}})$*, the normalizer and centralizer being taken in ${}^L G^0$.*

Let $\{1, \sigma\}$ be $\mathfrak{G}(\mathbf{C}/\mathbf{R})$ so that $W_{\mathbf{C}/\mathbf{R}}$ is generated by $\mathbf{C}^\times$ and $\sigma$ with $\sigma^2 = -1$. As on pages 48 and 49 of *CIRRAG* with $M$ replacing $G$ the homomorphism $\varphi$ is defined by $\mu$, $\nu$ with $\nu = \varphi(\sigma)\mu$ and by $\lambda_0$. If $\omega$ in $\Omega_{\mathbf{R}}(T, G)$ normalizes $M$ then

$$\omega \in W \iff \omega\rho \sim \rho \iff \omega\mu = \omega_1\mu,\ \omega\lambda_0 \equiv \omega_1\lambda_0 \mod \Big({}^L X_* + \big(1 - \varphi(\sigma)\big)({}^L X_* \otimes \mathbf{C})\Big)$$

with $\omega_1 \in \Omega_{\mathbf{R}}(T, M)$ and ${}^L X_* = \operatorname{Hom}\big(GL(1), {}^L T\big)$. Replace $\omega$ by $\omega_1^{-1}\omega$. Since $\omega$ normalizes $M$,

$$\varphi(\sigma)\omega = \omega\varphi(\sigma)$$

on ${}^L X_*$ and

$$\omega\mu = \mu \iff \omega\mu = \mu, \omega\nu = \nu \iff w\varphi(z) = \varphi(z)w \text{ for } z \in \mathbf{C}^\times$$

if $w \in {}^L G^0$ represents $\omega$. We write

$${}^L M = {}^L M^0 \rtimes W_{\mathbf{C}/\mathbf{R}}[\mathbf{3}]$$

and let

$$\varphi(\sigma) = a \rtimes \sigma$$

*Date*: 1977.

with

$$\lambda^\vee(a) = e^{2\pi i \langle \lambda_0, \lambda^\vee \rangle}.$$

By the first paragraph on p. 37 of *Problems in the theory of automorphic forms* we may choose $w$ so that $w\sigma = \sigma w$. But this is the wrong choice. We should choose $\omega(a) = \sigma(b)b^{-1}a$. Replace $w$ by $bw$ then

$$w\varphi(\sigma)w^{-1} = \sigma(b)b^{-1}ab\sigma(b)^{-1} \rtimes \sigma = a \rtimes \sigma = \varphi(\sigma).$$

In other words this new choice of $w$ satisfies

$$w\varphi(v)w^{-1} = \varphi(v) \quad \forall v \in W_{\mathbf{C}/\mathbf{R}}.$$

Since $\omega \in \Omega_{\mathbf{R}}(T, M)$ and $\omega\mu = \mu$ imply that $\omega = 1$ we have found

$$W \hookrightarrow \mathrm{Norm}({}^LT^0) \cap \mathrm{Cent}\,\varphi(W_{\mathbf{C}/\mathbf{R}})/{}^LT^0 \cap \mathrm{Cent}\,\varphi(W_{\mathbf{C}/\mathbf{R}}).$$

To obtain the full lemma we have only to show that if $w$ lies in $\mathrm{Norm}({}^LT^0) \cap \mathrm{Cent}\,\varphi(W_{\mathbf{C}/\mathbf{R}})$ then the corresponding element of the Weyl group stabilizes $M$ and lies in $\Omega_{\mathbf{R}}(T, G)$. It stabilizes ${}^LM$ because [**4**] $\alpha^\vee$ is a root of ${}^LM$ if and only if $\varphi(\sigma)\alpha^\vee = -\alpha^\vee$. Hence it stabilizes $M$. By Lemma 5.2 of Shelstad's thesis

$$\omega = \omega_1\omega_2$$

with $\omega_1 \in \Omega_{\mathbf{C}}(T, M)$, $\omega_2 \in \Omega_{\mathbf{R}}(T, G)$. Then

$$w\varphi = \varphi \implies \omega_1^{-1}\mu = \omega_2\mu,\ \omega_1^{-1}\nu = \omega_2\nu,\ \omega_1^{-1}\lambda_0 \equiv \omega_2\lambda_0.$$

Another lemma of Shelstad implies that $\omega_1 \in \Omega_{\mathbf{R}}(T, M)$. Hence

$$\omega \in \Omega_{\mathbf{R}}(T, G).$$

The advantage of introducing the $L$-group appears immediately when Knapp's $R$-group is discussed. Let $S$ be the centralizer of $\varphi(W_{\mathbf{C}/\mathbf{R}})$ in ${}^LG^0$ and $S^0$ the connected component.

**Lemma 2.** *If $G$ is semi-simple and simply-connected then the $R$-group is $S/S^0$.*

Let ${}^L\mathfrak{t}$ be the Lie algebra of ${}^LT$ and set

$${}^L\mathfrak{t} = {}^L\mathfrak{t}_+ + {}^L\mathfrak{t}_-.$$

where ${}^L\mathfrak{t}_+$ and ${}^L\mathfrak{t}_-$ are the $+1$ and $-1$ eigenspaces for $\varphi(\sigma)$. I claim that ${}^L\mathfrak{t}_+$ which certainly lies in $\mathfrak{s}$, the Lie algebra of $S^0$, is in fact a Cartan subalgebra of $S^0$. Indeed [**5**]

$$\mathfrak{s} \subseteq {}^L\mathfrak{t}_+ + \sum_{\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0} \mathbf{C}X_{\alpha^\vee}.$$

If $\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0$ then $\alpha^\vee$ cannot be a root of ${}^LT$ in ${}^LM$. Hence

$$\varphi(\sigma)\alpha^\vee \neq -\alpha^\vee$$

and $\alpha^\vee$ is not 0 on ${}^L\mathfrak{t}_+$. The assertion follows.

We may identify $\mathrm{Hom}({}^L\mathfrak{t}, \mathbf{C})$ with $\mathfrak{t} \otimes \mathbf{C}$ as a $\mathfrak{G}(\mathbf{C}/\mathbf{R})$-module if $\mathfrak{t}$ is the Lie algebra of $T$. If $\alpha^\vee$ is a root of ${}^LT^0$ in ${}^LG^0$ with $\varphi(\sigma)\alpha^\vee \neq -\alpha^\vee$ set

$$\mathfrak{a}_{\alpha^\vee} = ({}^L\mathfrak{t}_- + \mathbf{C}\alpha^\vee)^\perp.$$

Then $G_{\alpha^\vee}$ the centralizer of $\mathfrak{a}_{\alpha^\vee}$ in $G$ is defined over $\mathbf{R}$ and $M$ is the Levi factor of a maximal *PSG* of $G_{\alpha^\vee}$. Let $\mu(\rho, \alpha^\vee)$ be the value of the Plancherel measure for

$$\mathrm{Ind}\big(G_{\alpha^\vee}(\mathbf{R}), M(\mathbf{R}), \rho\big).$$

Let
$$\mathfrak{X}_{\alpha^\vee} = \{\, \beta^\vee \mid \varphi(\sigma)\beta^\vee \neq -\beta^\vee, G_{\beta^\vee} = G_{\alpha^\vee} \,\}.$$
The centralizer of ${}^L\mathfrak{t}_+$ is
$${}^L\mathfrak{t}_+ + \sum_{\varphi(\sigma)\alpha^\vee = -\alpha^\vee} \mathbf{C}X_{\alpha^\vee} [\mathbf{6}]$$
and this is the Lie algebra of ${}^LM$. Moreover
$$S/S^0 \simeq \mathrm{Norm}_S({}^L\mathfrak{t}_+)/\,\mathrm{Norm}_{S^0}({}^L\mathfrak{t}_+).$$
If $w \in \mathrm{Norm}_S({}^L\mathfrak{t}_+)$ then $w$ normalizes ${}^LM^0$ and centralizes $\varphi(\mathbf{C}^\times)$. Consequently it normalizes ${}^L\mathfrak{t}$ and we have
$$\mathrm{Norm}_S({}^L\mathfrak{t}_+)/{}^LT_+ \simeq W.$$
The lemma and indeed more will be established once the following facts are proved. They will be proved for any $G$.

(i) $\dim \mathfrak{s}_{\alpha^\vee} = \dim\Big(\big(\sum_{\beta^\vee \in \mathfrak{X}_{\alpha^\vee}} \mathbf{C}X_{\beta^\vee}\big) \cap \mathfrak{s}\Big) \leqslant 1.$
(ii) It is equal to 1 if and only if $\mu(\rho, \alpha^\vee) = 0$.
(iii) If it is one then $\mathfrak{s}_{\alpha^\vee}$ defines a root space of ${}^L\mathfrak{t}_+$ in $\mathfrak{t}$. The corresponding reflection in ${}^L\mathfrak{t}_+$ is the same as that defined by the real root of $T$ in $G_{\alpha^\vee}$.

There are a number of possibilities to consider.

(a) $\mathfrak{X}_{\alpha^\vee}$ consists of a single element. Then $\varphi(\sigma)\alpha^\vee = \alpha^\vee$ and $\alpha$, the corresponding root of $T$, is real. Since $\sigma\mu = \nu$, $\langle \mu, \alpha^\vee\rangle = \langle \nu, \alpha^\vee \rangle$ and $\dim \mathfrak{s}_{\alpha^\vee} = 1$ if and only if $\langle \mu, \alpha^\vee \rangle = 0$ and
$$\varphi(\sigma)X_{\alpha^\vee} = X_{\alpha^\vee}.[\mathbf{7}]$$
Certainly $T(\mathbf{R})$ is not fundamental. According to the formula on p. 141 of Harish-Chandra's preprint *Harmonic analysis III,* $\mu(\rho, \alpha^*)$ is 0 if and only if
$$\nu_\alpha = 0 \quad \text{and} \quad \frac{(-1)^{\rho_\alpha}}{2}\big(\sigma_{a^*}(\gamma) + \sigma_{a^*}(\gamma^{-1})\big) \neq 1.$$
Now
$$\nu_\alpha = \langle \mu, \alpha^\vee \rangle.$$
Also $\mathfrak{s}_{a^*}$ is now of dimension one and
$$\sigma_{a^*}(\gamma) = \sigma_{a^*}(\gamma^{-1}) = \chi\big(\alpha^\vee(-1)\big).$$
Here $\chi$ is associated to $\varphi : W_{\mathbf{C}/\mathbf{R}} \to {}^LM$ as on p. 50 of *CIRRAG* and if the definition of a coroot is taken into account
$$\gamma = \alpha^\vee(-1).$$
Thus (cf. p. 51 of *CIRRAG*)
$$\chi\big(\alpha^\vee(-1)\big) = e^{2\pi i \langle \lambda_0, \alpha^\vee \rangle}.$$
Apologies are necessary for this phase of the discussion but the transition from Harish-Chandra's notation to that used in *CIRRAG* is clumsy.

On the other hand
$$\varphi(\sigma) = a \rtimes \sigma [\mathbf{8}]$$
and
$$\varphi(\sigma)X_{\alpha^\vee} = e^{2\pi i\langle \lambda_0, \alpha^\vee\rangle}\varphi'(\sigma)(X_{\alpha^\vee})$$

if $\varphi'(\sigma) = a' \rtimes \sigma$, $a' \in {}^L M_{\mathrm{der}}$, $a^{-1}a' \in {}^L T^0$. The assertion (ii) will be verified if we show that

$$\varphi'(\sigma)(X_{\alpha^\vee}) = -(-1)^{\rho_\alpha} X_{\alpha^\vee}.$$

Now, by p. 122 of *Harmonic Analysis III*

$$\rho_\alpha = \langle \rho_{\alpha^\vee}, \alpha^\vee \rangle$$

if $\rho_{\alpha^\vee}$ is one-half the sum of the positive roots of $G_{\alpha^\vee}$. But in the present circumstances the derived algebra of $\mathfrak{g}_{\alpha^\vee}$ is a direct sum because $\alpha^\vee$ is perpendicular to all roots of $G_{\alpha^\vee}$ except $\pm\alpha^\vee$. Thus

$$\langle \rho_\alpha, \alpha^\vee \rangle = \frac{1}{2}\langle \alpha, \alpha^\vee \rangle = 1.$$

Moreover $\alpha^\vee$ must be a simple root and so by the definition of ${}^L M$

$$\varphi'(\sigma)(X_{\alpha^\vee}) = \sigma(X_{\alpha^\vee}) = 1.$$

The assertion (ii) follows. Since the reflections corresponding to $\alpha$ and $\alpha^\vee$ are the same, the assertion (iii) does also.

(b) Suppose $\varphi(\sigma)\alpha^\vee = \alpha^\vee$ and $\beta^\vee$ different from $\alpha^\vee$ lies in $\mathfrak{X}_{\alpha^\vee}$. **[9]**

(i) Suppose

$$\langle \mu, \beta^\vee \rangle = \langle \nu, \beta^\vee \rangle = 0.$$

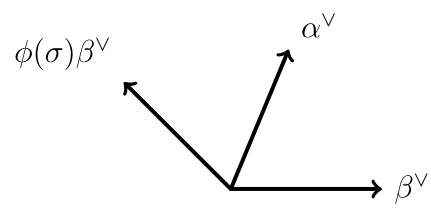
Then

$$\big\langle \mu, \varphi(\sigma)\beta^\vee \big\rangle = \big\langle \nu, \varphi(\sigma)\beta^\vee \big\rangle = 0.$$

Since $\varphi(\sigma)\beta^\vee$ lies in the span of $\{\alpha^\vee, \beta^\vee\}$ and is different from $\beta^\vee$, both $\mu$ and $\nu$ vanish on this two-dimensional space. As a consequence there are no roots $\gamma^\vee$ on it orthogonal to $\alpha^\vee$. For then $\varphi(\sigma)\gamma^\vee$ would be $-\gamma^\vee$ and as a consequence

$$\langle \mu, \gamma^\vee \rangle \neq 0.$$

This leaves only




of type $A_2$.

I claim next that if $\gamma^\vee$ lies in $X_{\alpha^\vee}$ and is different from $\alpha^\vee$, $\beta^\vee$, and $\varphi(\sigma)\beta^\vee$ then either $\langle \mu, \gamma^\vee \rangle \neq 0$ or $\langle \nu, \gamma^\vee \rangle \neq 0$. If not, consider all roots in the span of $\{\alpha^\vee, \beta^\vee, \gamma^\vee\}$. They form a root system of rank 3 on which $\varphi(\sigma)$ acts. If $\delta^\vee$ lies in this system then $\langle \mu, \delta^\vee \rangle = \langle \nu, \delta^\vee \rangle = 0$ so $\varphi(\sigma)\delta^\vee \neq -\delta^\vee$. As a consequence **[10]**

$$\delta^\vee + \varphi(\sigma)\delta^\vee = a\alpha^\vee \quad a \neq 0$$

and

$$\big\{\, \delta^\vee \;\big|\; \langle \alpha, \delta^\vee \rangle \geqslant 0 \,\big\}$$

defines a system of positive roots stable under $\varphi(\sigma)$. Let $\alpha_1^\vee$, $\alpha_2^\vee$, $\alpha_3^\vee$ be the simple roots. They are permuted amongst themselves by $\varphi(\sigma)$. Thus by a suitable numbering

$$a_1^\vee = \alpha^\vee, \quad a_3^\vee = \varphi(\sigma)a_2^\vee.$$

Then

$$a\alpha^\vee = \alpha_2^\vee + a_3^\vee.$$

This is a contradiction.

Also we may take

$$X_{\alpha^\vee} = \big[X_{\beta^\vee}, \varphi(\sigma)X_{\beta^\vee}\big]$$

and

$$\varphi(\sigma)X_{\alpha^\vee} = -X_{\alpha^\vee}.$$

Thus

$$\mathfrak{s}_{\alpha^\vee} = \mathbf{C}\big(X_{\beta^\vee} + \varphi(\sigma)X_{\beta^\vee}\big)$$

has dimension 1. Since **[11]**

$$\langle \mu, \beta' \rangle = (\lambda + i\nu)(H_\beta),$$

the right side conforming to Harish-Chandra's notation, the measure $\mu(\rho, \alpha^\vee)$ is certainly zero. The reflection defined by $\mathfrak{s}_{\alpha^\vee}$ is clearly correct on ${}^L\mathfrak{t}_+$.

(ii) Suppose that for every $\beta^\vee$ different from $\alpha^\vee$ in $\mathfrak{X}_{\alpha^\vee}$

$$\langle \mu, \beta^\vee \rangle \neq 0 \text{ or } \langle \nu, \beta^\vee \rangle \neq 0.$$

Then $\dim \mathfrak{s}_{\alpha^\vee} = 1$ if and only if

$$\langle \mu, \alpha^\vee \rangle = 0, \qquad \varphi(\sigma)X_{\alpha^\vee} = X_{\alpha^\vee}.$$

Again the first condition is equivalent to $\nu_\alpha = 0$. We have to show that when this is so then the second is equivalent to

$$\frac{(-1)^{\rho_\alpha}}{2}\big(\sigma_{a^*}(\gamma) + \sigma_{a^*}(\gamma^{-1})\big) \neq 1.$$

Let

$$\varphi(\sigma)X_{\alpha^\vee} = \lambda X_{\alpha^\vee}.$$

We show that

$$\left(\frac{(-1)^{\rho_\alpha}}{2}\right)\big(\sigma_{a^*}(\gamma) + \sigma_{a^*}(\gamma^{-1})\big) = -\lambda.$$

This is enough, for $\lambda = \pm 1$. As before **[12]**

$$\sigma_{a^*}(\gamma) = \sigma_{a^*}(\gamma^{-1}) = e^{2\pi i \langle \lambda_0, \alpha^\vee \rangle}$$

and

$$\varphi(\sigma)X_{\alpha^\vee} = e^{2\pi i \langle \lambda_0, \alpha^\vee \rangle}\varphi'(\sigma)(X_{\alpha^\vee}).$$

if $\varphi'(\sigma)$ is defined as before. What we must do is show that

$$\varphi'(\sigma)(X_{\alpha^\vee}) = -(-1)^{\langle \rho_{\alpha^\vee}, \alpha^\vee \rangle}X_{\alpha^\vee}.$$

This is a statement about a reductive group $G_{\alpha^\vee}$ and a Levi factor $M$ of a maximal parabolic, $M$ and $G$ both having compact CSGs. It is not bound to the present situation and may be proved by induction on the rank of $G_{\alpha^\vee}$. Let $\beta^\vee$ be the largest root of one of the simple factors of ${}^LM_{\mathrm{der}}$ and introduce $a_2$, $a_1$ as on p. 46 of *CIRRAG*. We may take $a' = a_2 a_1$. If $\rho'$ is the analogue of $\rho_{\alpha^\vee}$ for the roots perpendicular to $\beta^\vee$ then by induction

$$a_1 \rtimes \sigma(X_{\alpha^\vee}) = -(-1)^{\langle \rho', \alpha^\vee \rangle}X_{\alpha^\vee}.$$

What we have to do is show that

$$a_2(X_{\alpha^\vee}) = (-1)^\ell X_{\alpha^\vee}, \quad \ell = \left(\frac{1}{2}\right) \sum_{\substack{\langle\gamma,\beta^\vee\rangle\neq 0\\ \gamma>0}} \langle\gamma,\alpha^\vee\rangle.\mathbf{[13]}$$

Suppose $\gamma > 0$, $\langle\gamma,\beta^\vee\rangle \neq 0$, $\langle\gamma,\alpha^\vee\rangle \neq 0$ and $\gamma^\vee$ is not in the plane spanned by $\alpha^\vee$, $\beta^\vee$. Then:

1) $\gamma^\vee = a_2\gamma^\vee \implies \gamma = a_2\gamma \implies \langle\gamma,\beta^\vee\rangle = 0$—impossible
2) $\gamma^\vee = \varphi(\sigma)\gamma^\vee \implies \gamma^\vee = \pm\alpha^\vee$—impossible
3) $\gamma^\vee = a_2\varphi(\sigma)\gamma^\vee \implies \gamma^\vee$ in plane of $\alpha^\vee$, $\beta^\vee$ because $(\alpha^\vee,\beta^\vee) = 0$. Thus $\gamma$, $a_2\gamma$, $\varphi(\sigma)\gamma$, $a_2\varphi(\sigma)\gamma$ are distinct and positive. Since

   $$\langle\gamma,\alpha^\vee\rangle = \langle a_2\gamma,\alpha^\vee\rangle = \big\langle\varphi(\sigma)\gamma,\alpha^\vee\big\rangle = \big\langle a_2\varphi(\sigma)\gamma,\alpha^\vee\big\rangle$$

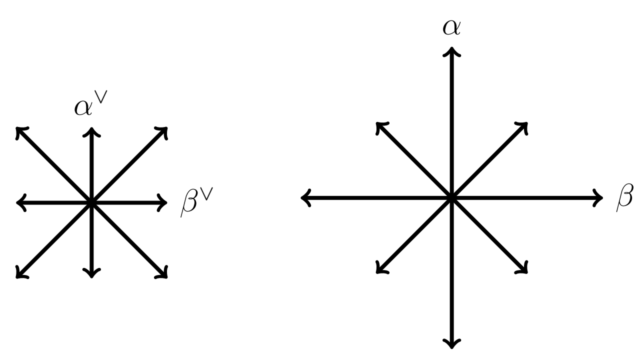
   the sum of the four of them even after division by 2 is even and may be dropped from the exponent. So may those $\langle\gamma,\alpha^\vee\rangle$ which are 0. We confine ourselves to $\gamma$ with $\gamma^\vee$ in the plane of $\alpha^\vee$, $\beta^\vee$.

The possibilities are:

A) No roots except $\pm\alpha^\vee$, $\pm\beta^\vee$ in the plane. Then the exponent is 0 and

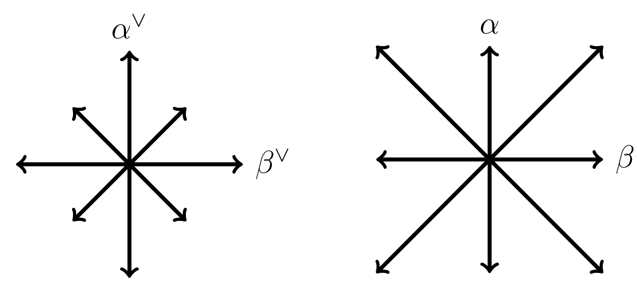
$$a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}.$$

B)



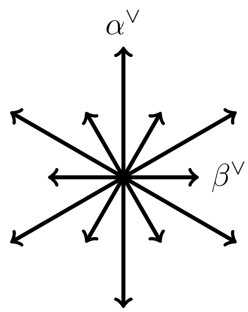

$$\left(\frac{1}{2}\right)\sum\langle\gamma,\alpha^\vee\rangle = \frac{1}{2}\langle\alpha,\alpha^\vee\rangle = 1, \quad a_2(X_{\alpha^\vee}) = -X_{\alpha^\vee}\mathbf{[14]}$$

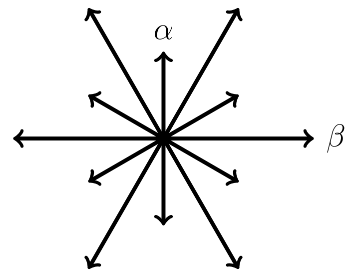
C)




$$\left(\frac{1}{2}\right)\sum\langle\gamma,\alpha^\vee\rangle = \langle\alpha,\alpha^\vee\rangle = 2, \quad a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}$$

D)




$$\left(\frac{1}{2}\right)\sum\langle\gamma,\alpha^\vee\rangle = 2\langle\alpha,\alpha^\vee\rangle = 4, \quad a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}$$

E) The roles of $\alpha$, $\alpha^\vee$ and $\beta$, $\beta^\vee$ are reversed

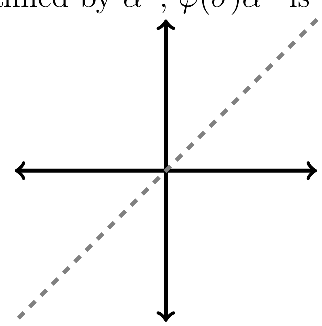
$$\left(\frac{1}{2}\right)\sum\langle\gamma,\alpha^\vee\rangle = \langle\alpha,\alpha^\vee\rangle = 2, \quad a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}.$$

All that is claimed in A) through E) is easy to check. Finally it is clear that the reflection defined by $\mathfrak{s}_{\alpha^\vee}$ is that defined by $\alpha$ or $\alpha^\vee$.

(i) Suppose that $\varphi(\sigma)\beta^\vee \neq \beta^\vee$ for all $\beta^\vee$ in $\mathfrak{X}_{\alpha^\vee}$. Then $\beta^\vee + \varphi(\sigma)\beta^\vee$ is not a root, nor is **[15]**

$$\frac{\beta^\vee + \varphi(\sigma)\beta^\vee}{2}.$$

(ii) Suppose that $\langle\mu,\alpha^\vee\rangle = \langle\nu,\alpha^\vee\rangle = 0$. Then $\alpha^\vee - \varphi(\sigma)\alpha^\vee$ is not a root and $\big\langle\alpha^\vee,\varphi(\sigma)\alpha^\vee\big\rangle = 0$. Since $\alpha^\vee$ and $\varphi(\sigma)\alpha^\vee$ have the same length, the root diagram of the plane spanned by $\alpha^\vee$, $\varphi(\sigma)\alpha^\vee$ is



I claim that if $\beta^\vee$ lies in $\mathfrak{X}_{\alpha^\vee}$ but not in this plane then either $\langle\mu,\beta^\vee\rangle = 0$ or $\langle\nu,\beta^\vee\rangle = 0$. Otherwise in the three-dimensional plane spanned by $\alpha^\vee$, $\varphi(\sigma)\alpha^\vee$, $\beta^\vee$, $\varphi(\sigma)\beta^\vee$ we have a root system and

$$\Big\{\gamma^\vee \;\Big|\; \big\langle\gamma,\alpha^\vee + \varphi(\sigma)\alpha^\vee\big\rangle \geqslant 0\Big\}$$

is a set of positive roots, for

$$\big\langle\gamma,\alpha^\vee + \varphi(\sigma)\alpha^\vee\big\rangle$$

is never 0, because if it were then $\varphi(\sigma)\gamma^\vee = -\gamma^\vee$. Since $\langle\mu,\gamma^\vee\rangle = \langle\nu,\gamma^\vee\rangle = 0$ this is impossible. Then $\varphi(\sigma)$ permutes the three simple roots amongst themselves, and leaves one fixed. This is a contradiction. Thus **[16]**

$$\mathfrak{s}_\alpha = \mathbf{C}\big(X_{\alpha^\vee} + \varphi(\sigma)X_{\alpha^\vee}\big)$$

has dimension one. Since $T$ is fundamental in $G_{\alpha^\vee}$, the formula on p. 97 of *Harmonic analysis III* shows that $\mu(\rho,\alpha^\vee)=0$. The three assertions follow again.

(iii) Suppose that for any $\beta^\vee$ in $\mathfrak{X}_{\alpha^\vee}$ either $\langle\mu,\beta^\vee\rangle\neq 0$ or $\langle\nu,\beta^\vee\rangle\neq 0$. Then $\mathfrak{s}_{\alpha^\vee}=0$. By the same formula in *Harmonic analysis III,*

$$\mu(\rho,\alpha^\vee)\neq 0.$$

Lemma 2 is now completely proved. I should observe, for it will remove a confusion that could otherwise arise, that

$$-\overline{\langle\mu,\alpha^\vee\rangle}=\langle\nu,\alpha^\vee\rangle$$

for any $\alpha^\vee$.

It is also possible to give Zuckerman's proof that the $R$-group is a sum of $Z_2$'s in the above context. Let $\mathrm{Norm}_S^+({}^L\mathfrak{t}_+)$ be the set of elements of $\mathrm{Norm}_S({}^L\mathfrak{t}_+)$ that take positive roots of $\mathfrak{s}_0$ to positive roots. Then

$$R=S/S^0\simeq \mathrm{Norm}_S^+({}^L\mathfrak{t}_+)/{}^L\mathfrak{t}_+.$$

Let

$$\mathfrak{s}_1={}^L\mathfrak{t}+\sum_{\langle\mu,\alpha^\vee\rangle=\langle\nu,\alpha^\vee\rangle=0}\mathbf{C}X_{\alpha^\vee}[\mathbf{17}]$$

The elements of $\mathrm{Norm}_S^+({}^L\mathfrak{t}_+)$ take $\mathfrak{s}_1$ to itself. Let $Q$ be the operator

$$\frac{1}{|R|}\sum_R r$$

on $\mathfrak{t}\otimes\mathbf{C}$. Since the centralizer of $\varphi(\mathbf{C}^\times)$ is connected, $S$ lies in the connected group $S_1$ with Lie algebra $\mathfrak{s}_1$. Thus by Chevalley's theorem $R$ is contained in the group generated by the reflections associated to the roots $\alpha^\vee$ of $\mathfrak{s}_1$ for which $Q\alpha^\vee=0$.

If $\alpha^\vee$ is a root of $\mathfrak{s}_1$ then $\varphi(\sigma)\alpha^\vee\neq-\alpha^\vee$. Suppose $\varphi(\sigma)\alpha^\vee\neq\alpha^\vee$. Then

$$X_{\alpha^\vee}+\varphi(\sigma)X_{\alpha^\vee}\neq 0$$

and lies in $\mathfrak{s}$. Thus $\alpha^\vee$ restricted to ${}^L\mathfrak{t}_+$ defines a root of $\mathfrak{s}$. Since the elements of $r$ stabilize ${}^L\mathfrak{t}_+$ and each $r$ takes positive roots of ${}^L\mathfrak{t}_+$ in $\mathfrak{s}$ to positive roots,

$$Q\alpha^\vee\neq 0.$$

Thus if $\alpha^\vee$ is a root of $\mathfrak{s}_1$ then

$$Q\alpha^\vee=0\implies \varphi(\sigma)\alpha^\vee=\alpha^\vee.$$

Moreover $\alpha^\vee$ cannot be a root of $\mathfrak{s}$ and therefore

$$\varphi(\sigma)X_{\alpha^\vee}=-X_{\alpha^\vee}.[\mathbf{18}]$$

Finally if $Q\alpha^\vee=0$, $Q\beta^\vee=0$ then $\alpha^\vee\pm\beta^\vee$ is not a root because $\varphi(\sigma)X_{\alpha^\vee+\beta^\vee}=\varphi(\sigma)[X_{\alpha^\vee},X_{\beta^\vee}]=[-X_{\alpha^\vee},-X_{\beta^\vee}]=X_{\alpha^\vee+\beta^\vee}$ and $\alpha^\vee+\beta^\vee$ would have to be a root of $\mathfrak{s}$. This is inconsistent with

$$Q(\alpha^\vee+\beta^\vee)=0.$$

The set of positive $\alpha^\vee$ for which $\langle\mu,\alpha^\vee\rangle=\langle\nu,\alpha^\vee\rangle=0$ and $Q\alpha^\vee=0$ is the strongly orthogonal system needed for Zuckerman's argument.

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