

NOTES ON THE KNAPP-ZUCKERMAN THEORY

ROBERT P. LANGLANDS

The point of these notes is to redefine some of their concepts in terms of the L -group. I observe, however, that it is best and indeed essential for further applications that their results be formulated for reductive groups rather than just for simply-connected semi-simple groups. I use the notation of *CIRRA*G (*On the classification of irreducible representations of real algebraic groups*) modified sometimes according to Borel's suggestions.

Since we are dealing with tempered representations we start from $\varphi : W_{\mathbf{C}/\mathbf{R}} \rightarrow {}^L G$ with image which is essentially compact. We suppose φ defines an element of $\Phi(G)$. Choose a parabolic ${}^L P$ in ${}^L G$ which is minimal with respect to the property that $\varphi(W_{\mathbf{C}/\mathbf{R}}) \subseteq {}^L P$. The group ${}^L P$ defines P and M . Let ρ (with character Θ) be one of the representations of M associated to φ . Thus $\rho \in \Pi_\varphi$, if φ is regarded as taking $W_{\mathbf{C}/\mathbf{R}}$ to ${}^L M$. It is

$$\text{Ind}(G, P, \rho)$$

that Knapp-Zuckerman study.

They define W on p. 3, formula [2] of their paper *Classification of irreducible tempered representations of semisimple Lie groups*. We want another definition. For this we observe that $\Omega_{\mathbf{C}}(T, G)$ is isomorphic to $\Omega({}^L T^0, {}^L G^0)$. Here T is a CSG (Cartan subgroup) of M . We want to regard W as a subgroup [2] of the latter group. We may assume, along the lines of *CIRRA*G that $\varphi(\mathbf{C}^\times) \subseteq {}^L T$, that $\varphi(W_{\mathbf{C}/\mathbf{R}})$ normalizes ${}^L T$, and that ${}^L T \subseteq {}^L M$, a chosen Levi factor of ${}^L P$.

Lemma 1. *W is the quotient $\text{Norm}({}^L T) \cap \text{Cent } \varphi(W_{\mathbf{C}/\mathbf{R}})/{}^L T^0 \cap \text{Cent } \varphi(W_{\mathbf{C}/\mathbf{R}})$, the normalizer and centralizer being taken in ${}^L G^0$.*

Let $\{1, \sigma\}$ be $\mathfrak{G}(\mathbf{C}/\mathbf{R})$ so that $W_{\mathbf{C}/\mathbf{R}}$ is generated by \mathbf{C}^\times and σ with $\sigma^2 = -1$. As on pages 48 and 49 of *CIRRA*G with M replacing G the homomorphism φ is defined by μ, ν with $\nu = \varphi(\sigma)\mu$ and by λ_0 . If ω in $\Omega_{\mathbf{R}}(T, G)$ normalizes M then

$$\omega \in W \iff \omega\rho \sim \rho \iff \omega\mu = \omega_1\mu, \omega\lambda_0 \equiv \omega_1\lambda_0 \pmod{{}^L X_* + (1 - \varphi(\sigma))({}^L X_* \otimes \mathbf{C})}$$

with $\omega_1 \in \Omega_{\mathbf{R}}(T, M)$ and ${}^L X_* = \text{Hom}(GL(1), {}^L T)$. Replace ω by $\omega_1^{-1}\omega$. Since ω normalizes M ,

$$\varphi(\sigma)\omega = \omega\varphi(\sigma)$$

on ${}^L X_*$ and

$$\omega\mu = \mu \iff \omega\mu = \mu, \omega\nu = \nu \iff w\varphi(z) = \varphi(z)w \text{ for } z \in \mathbf{C}^\times$$

if $w \in {}^L G^0$ represents ω . We write

$${}^L M = {}^L M^0 \rtimes W_{\mathbf{C}/\mathbf{R}}[3]$$

and let

$$\varphi(\sigma) = a \rtimes \sigma$$

Date: 1977.

with

$$\lambda^\vee(a) = e^{2\pi i \langle \lambda_0, \lambda^\vee \rangle}.$$

By the first paragraph on p. 37 of *Problems in the theory of automorphic forms* we may choose w so that $w\sigma = \sigma w$. But this is the wrong choice. We should choose $\omega(a) = \sigma(b)b^{-1}a$. Replace w by bw then

$$w\varphi(\sigma)w^{-1} = \sigma(b)b^{-1}ab\sigma(b)^{-1} \rtimes \sigma = a \rtimes \sigma = \varphi(\sigma).$$

In other words this new choice of w satisfies

$$w\varphi(v)w^{-1} = \varphi(v) \quad \forall v \in W_{\mathbf{C}/\mathbf{R}}.$$

Since $\omega \in \Omega_{\mathbf{R}}(T, M)$ and $\omega\mu = \mu$ imply that $\omega = 1$ we have found

$$W \hookrightarrow \text{Norm}({}^L T^0) \cap \text{Cent } \varphi(W_{\mathbf{C}/\mathbf{R}})/{}^L T^0 \cap \text{Cent } \varphi(W_{\mathbf{C}/\mathbf{R}}).$$

To obtain the full lemma we have only to show that if w lies in $\text{Norm}({}^L T^0) \cap \text{Cent } \varphi(W_{\mathbf{C}/\mathbf{R}})$ then the corresponding element of the Weyl group stabilizes M and lies in $\Omega_{\mathbf{R}}(T, G)$. It stabilizes ${}^L M$ because [4] α^\vee is a root of ${}^L M$ if and only if $\varphi(\sigma)\alpha^\vee = -\alpha^\vee$. Hence it stabilizes M . By Lemma 5.2 of Shelstad's thesis

$$\omega = \omega_1 \omega_2$$

with $\omega_1 \in \Omega_{\mathbf{C}}(T, M)$, $\omega_2 \in \Omega_{\mathbf{R}}(T, G)$. Then

$$w\varphi = \varphi \implies \omega_1^{-1}\mu = \omega_2\mu, \quad \omega_1^{-1}\nu = \omega_2\nu, \quad \omega_1^{-1}\lambda_0 \equiv \omega_2\lambda_0.$$

Another lemma of Shelstad implies that $\omega_1 \in \Omega_{\mathbf{R}}(T, M)$. Hence

$$\omega \in \Omega_{\mathbf{R}}(T, G).$$

The advantage of introducing the L -group appears immediately when Knapp's R -group is discussed. Let S be the centralizer of $\varphi(W_{\mathbf{C}/\mathbf{R}})$ in ${}^L G^0$ and S^0 the connected component.

Lemma 2. *If G is semi-simple and simply-connected then the R -group is S/S^0 .*

Let ${}^L \mathfrak{t}$ be the Lie algebra of ${}^L T$ and set

$${}^L \mathfrak{t} = {}^L \mathfrak{t}_+ + {}^L \mathfrak{t}_-.$$

where ${}^L \mathfrak{t}_+$ and ${}^L \mathfrak{t}_-$ are the $+1$ and -1 eigenspaces for $\varphi(\sigma)$. I claim that ${}^L \mathfrak{t}_+$ which certainly lies in \mathfrak{s} , the Lie algebra of S^0 , is in fact a Cartan subalgebra of S^0 . Indeed [5]

$$\mathfrak{s} \subseteq {}^L \mathfrak{t}_+ + \sum_{\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0} \mathbf{C} X_{\alpha^\vee}.$$

If $\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0$ then α^\vee cannot be a root of ${}^L T$ in ${}^L M$. Hence

$$\varphi(\sigma)\alpha^\vee \neq -\alpha^\vee$$

and α^\vee is not 0 on ${}^L \mathfrak{t}_+$. The assertion follows.

We may identify $\text{Hom}({}^L \mathfrak{t}, \mathbf{C})$ with $\mathfrak{t} \otimes \mathbf{C}$ as a $\mathfrak{G}(\mathbf{C}/\mathbf{R})$ -module if \mathfrak{t} is the Lie algebra of T . If α^\vee is a root of ${}^L T^0$ in ${}^L G^0$ with $\varphi(\sigma)\alpha^\vee \neq -\alpha^\vee$ set

$$\mathfrak{a}_{\alpha^\vee} = ({}^L \mathfrak{t}_- + \mathbf{C}\alpha^\vee)^\perp.$$

Then G_{α^\vee} the centralizer of $\mathfrak{a}_{\alpha^\vee}$ in G is defined over \mathbf{R} and M is the Levi factor of a maximal PSG of G_{α^\vee} . Let $\mu(\rho, \alpha^\vee)$ be the value of the Plancherel measure for

$$\text{Ind}(G_{\alpha^\vee}(\mathbf{R}), M(\mathbf{R}), \rho).$$

Let

$$\mathfrak{X}_{\alpha^\vee} = \{ \beta^\vee \mid \varphi(\sigma)\beta^\vee \neq -\beta^\vee, G_{\beta^\vee} = G_{\alpha^\vee} \}.$$

The centralizer of ${}^L\mathfrak{t}_+$ is

$${}^L\mathfrak{t}_+ + \sum_{\varphi(\sigma)\alpha^\vee = -\alpha^\vee} \mathbb{C}X_{\alpha^\vee}[\mathbf{6}]$$

and this is the Lie algebra of LM . Moreover

$$S/S^0 \simeq \text{Norm}_S({}^L\mathfrak{t}_+)/\text{Norm}_{S^0}({}^L\mathfrak{t}_+).$$

If $w \in \text{Norm}_S({}^L\mathfrak{t}_+)$ then w normalizes ${}^LM^0$ and centralizes $\varphi(\mathbb{C}^\times)$. Consequently it normalizes ${}^L\mathfrak{t}$ and we have

$$\text{Norm}_S({}^L\mathfrak{t}_+)/{}^LT_+ \simeq W.$$

The lemma and indeed more will be established once the following facts are proved. They will be proved for any G .

- (i) $\dim \mathfrak{s}_{\alpha^\vee} = \dim \left(\left(\sum_{\beta^\vee \in \mathfrak{X}_{\alpha^\vee}} \mathbb{C}X_{\beta^\vee} \right) \cap \mathfrak{s} \right) \leq 1.$
- (ii) It is equal to 1 if and only if $\mu(\rho, \alpha^\vee) = 0.$
- (iii) If it is one then $\mathfrak{s}_{\alpha^\vee}$ defines a root space of ${}^L\mathfrak{t}_+$ in \mathfrak{t} . The corresponding reflection in ${}^L\mathfrak{t}_+$ is the same as that defined by the real root of T in $G_{\alpha^\vee}.$

There are a number of possibilities to consider.

- (a) $\mathfrak{X}_{\alpha^\vee}$ consists of a single element. Then $\varphi(\sigma)\alpha^\vee = \alpha^\vee$ and α , the corresponding root of T , is real. Since $\sigma\mu = \nu$, $\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle$ and $\dim \mathfrak{s}_{\alpha^\vee} = 1$ if and only if $\langle \mu, \alpha^\vee \rangle = 0$ and

$$\varphi(\sigma)X_{\alpha^\vee} = X_{\alpha^\vee}[\mathbf{7}]$$

Certainly $T(\mathbf{R})$ is not fundamental. According to the formula on p. 141 of Harish-Chandra's preprint *Harmonic analysis III*, $\mu(\rho, \alpha^*)$ is 0 if and only if

$$\nu_\alpha = 0 \quad \text{and} \quad \frac{(-1)^{\rho_\alpha}}{2} (\sigma_{a^*}(\gamma) + \sigma_{a^*}(\gamma^{-1})) \neq 1.$$

Now

$$\nu_\alpha = \langle \mu, \alpha^\vee \rangle.$$

Also \mathfrak{s}_{a^*} is now of dimension one and

$$\sigma_{a^*}(\gamma) = \sigma_{a^*}(\gamma^{-1}) = \chi(\alpha^\vee(-1)).$$

Here χ is associated to $\varphi : W_{\mathbf{C}/\mathbf{R}} \rightarrow {}^LM$ as on p. 50 of *CIRAG* and if the definition of a coroot is taken into account

$$\gamma = \alpha^\vee(-1).$$

Thus (cf. p. 51 of *CIRAG*)

$$\chi(\alpha^\vee(-1)) = e^{2\pi i \langle \lambda_0, \alpha^\vee \rangle}.$$

Apologies are necessary for this phase of the discussion but the transition from Harish-Chandra's notation to that used in *CIRAG* is clumsy.

On the other hand

$$\varphi(\sigma) = a \rtimes \sigma[\mathbf{8}]$$

and

$$\varphi(\sigma)X_{\alpha^\vee} = e^{2\pi i \langle \lambda_0, \alpha^\vee \rangle} \varphi'(\sigma)(X_{\alpha^\vee})$$

if $\varphi'(\sigma) = a' \rtimes \sigma$, $a' \in {}^L M_{\text{der}}$, $a^{-1}a' \in {}^L T^0$. The assertion (ii) will be verified if we show that

$$\varphi'(\sigma)(X_{\alpha^\vee}) = -(-1)^{\rho_\alpha} X_{\alpha^\vee}.$$

Now, by p. 122 of *Harmonic Analysis III*

$$\rho_\alpha = \langle \rho_{\alpha^\vee}, \alpha^\vee \rangle$$

if ρ_{α^\vee} is one-half the sum of the positive roots of G_{α^\vee} . But in the present circumstances the derived algebra of $\mathfrak{g}_{\alpha^\vee}$ is a direct sum because α^\vee is perpendicular to all roots of G_{α^\vee} except $\pm\alpha^\vee$. Thus

$$\langle \rho_\alpha, \alpha^\vee \rangle = \frac{1}{2} \langle \alpha, \alpha^\vee \rangle = 1.$$

Moreover α^\vee must be a simple root and so by the definition of ${}^L M$

$$\varphi'(\sigma)(X_{\alpha^\vee}) = \sigma(X_{\alpha^\vee}) = 1.$$

The assertion (ii) follows. Since the reflections corresponding to α and α^\vee are the same, the assertion (iii) does also.

(b) Suppose $\varphi(\sigma)\alpha^\vee = \alpha^\vee$ and β^\vee different from α^\vee lies in $\mathfrak{X}_{\alpha^\vee}$. [9]

(i) Suppose

$$\langle \mu, \beta^\vee \rangle = \langle \nu, \beta^\vee \rangle = 0.$$

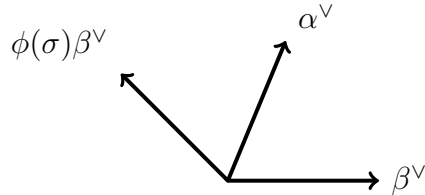
Then

$$\langle \mu, \varphi(\sigma)\beta^\vee \rangle = \langle \nu, \varphi(\sigma)\beta^\vee \rangle = 0.$$

Since $\varphi(\sigma)\beta^\vee$ lies in the span of $\{\alpha^\vee, \beta^\vee\}$ and is different from β^\vee , both μ and ν vanish on this two-dimensional space. As a consequence there are no roots γ^\vee on it orthogonal to α^\vee . For then $\varphi(\sigma)\gamma^\vee$ would be $-\gamma^\vee$ and as a consequence

$$\langle \mu, \gamma^\vee \rangle \neq 0.$$

This leaves only



of type A_2 .

I claim next that if γ^\vee lies in X_{α^\vee} and is different from α^\vee , β^\vee , and $\varphi(\sigma)\beta^\vee$ then either $\langle \mu, \gamma^\vee \rangle \neq 0$ or $\langle \nu, \gamma^\vee \rangle \neq 0$. If not, consider all roots in the span of $\{\alpha^\vee, \beta^\vee, \gamma^\vee\}$. They form a root system of rank 3 on which $\varphi(\sigma)$ acts. If δ^\vee lies in this system then $\langle \mu, \delta^\vee \rangle = \langle \nu, \delta^\vee \rangle = 0$ so $\varphi(\sigma)\delta^\vee \neq -\delta^\vee$. As a consequence [10]

$$\delta^\vee + \varphi(\sigma)\delta^\vee = a\alpha^\vee \quad a \neq 0$$

and

$$\{ \delta^\vee \mid \langle \alpha, \delta^\vee \rangle \geq 0 \}$$

defines a system of positive roots stable under $\varphi(\sigma)$. Let $\alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee$ be the simple roots. They are permuted amongst themselves by $\varphi(\sigma)$. Thus by a suitable numbering

$$a_1^\vee = \alpha^\vee, \quad a_3^\vee = \varphi(\sigma)a_2^\vee.$$

Then

$$a\alpha^\vee = \alpha_2^\vee + a_3^\vee.$$

This is a contradiction.

Also we may take

$$X_{\alpha^\vee} = [X_{\beta^\vee}, \varphi(\sigma)X_{\beta^\vee}]$$

and

$$\varphi(\sigma)X_{\alpha^\vee} = -X_{\alpha^\vee}.$$

Thus

$$\mathfrak{s}_{\alpha^\vee} = \mathbf{C}(X_{\beta^\vee} + \varphi(\sigma)X_{\beta^\vee})$$

has dimension 1. Since [11]

$$\langle \mu, \beta' \rangle = (\lambda + i\nu)(H_\beta),$$

the right side conforming to Harish-Chandra's notation, the measure $\mu(\rho, \alpha^\vee)$ is certainly zero. The reflection defined by $\mathfrak{s}_{\alpha^\vee}$ is clearly correct on ${}^L\mathfrak{t}_+$.

(ii) Suppose that for every β^\vee different from α^\vee in $\mathfrak{X}_{\alpha^\vee}$

$$\langle \mu, \beta^\vee \rangle \neq 0 \text{ or } \langle \nu, \beta^\vee \rangle \neq 0.$$

Then $\dim \mathfrak{s}_{\alpha^\vee} = 1$ if and only if

$$\langle \mu, \alpha^\vee \rangle = 0, \quad \varphi(\sigma)X_{\alpha^\vee} = X_{\alpha^\vee}.$$

Again the first condition is equivalent to $\nu_\alpha = 0$. We have to show that when this is so then the second is equivalent to

$$\frac{(-1)^{\rho_\alpha}}{2}(\sigma_{a^*}(\gamma) + \sigma_{a^*}(\gamma^{-1})) \neq 1.$$

Let

$$\varphi(\sigma)X_{\alpha^\vee} = \lambda X_{\alpha^\vee}.$$

We show that

$$\left(\frac{(-1)^{\rho_\alpha}}{2} \right) (\sigma_{a^*}(\gamma) + \sigma_{a^*}(\gamma^{-1})) = -\lambda.$$

This is enough, for $\lambda = \pm 1$. As before [12]

$$\sigma_{a^*}(\gamma) = \sigma_{a^*}(\gamma^{-1}) = e^{2\pi i \langle \lambda_0, \alpha^\vee \rangle}$$

and

$$\varphi(\sigma)X_{\alpha^\vee} = e^{2\pi i \langle \lambda_0, \alpha^\vee \rangle} \varphi'(\sigma)(X_{\alpha^\vee}).$$

if $\varphi'(\sigma)$ is defined as before. What we must do is show that

$$\varphi'(\sigma)(X_{\alpha^\vee}) = -(-1)^{\langle \rho_{\alpha^\vee}, \alpha^\vee \rangle} X_{\alpha^\vee}.$$

This is a statement about a reductive group G_{α^\vee} and a Levi factor M of a maximal parabolic, M and G both having compact CSGs. It is not bound to the present situation and may be proved by induction on the rank of G_{α^\vee} . Let β^\vee be the largest root of one of the simple factors of ${}^L M_{\text{der}}$ and introduce a_2, a_1 as on p. 46 of *CIRRA*G. We may take $a' = a_2 a_1$. If ρ' is the analogue of ρ_{α^\vee} for the roots perpendicular to β^\vee then by induction

$$a_1 \rtimes \sigma(X_{\alpha^\vee}) = -(-1)^{\langle \rho', \alpha^\vee \rangle} X_{\alpha^\vee}.$$

What we have to do is show that

$$a_2(X_{\alpha^\vee}) = (-1)^\ell X_{\alpha^\vee}, \quad \ell = \left(\frac{1}{2}\right) \sum_{\substack{\langle \gamma, \beta^\vee \rangle \neq 0 \\ \gamma > 0}} \langle \gamma, \alpha^\vee \rangle. \quad [\mathbf{13}]$$

Suppose $\gamma > 0$, $\langle \gamma, \beta^\vee \rangle \neq 0$, $\langle \gamma, \alpha^\vee \rangle \neq 0$ and γ^\vee is not in the plane spanned by α^\vee, β^\vee . Then:

- 1) $\gamma^\vee = a_2 \gamma^\vee \implies \gamma = a_2 \gamma \implies \langle \gamma, \beta^\vee \rangle = 0$ —impossible
- 2) $\gamma^\vee = \varphi(\sigma) \gamma^\vee \implies \gamma^\vee = \pm \alpha^\vee$ —impossible
- 3) $\gamma^\vee = a_2 \varphi(\sigma) \gamma^\vee \implies \gamma^\vee$ in plane of α^\vee, β^\vee because $(\alpha^\vee, \beta^\vee) = 0$. Thus $\gamma, a_2 \gamma, \varphi(\sigma) \gamma, a_2 \varphi(\sigma) \gamma$ are distinct and positive. Since

$$\langle \gamma, \alpha^\vee \rangle = \langle a_2 \gamma, \alpha^\vee \rangle = \langle \varphi(\sigma) \gamma, \alpha^\vee \rangle = \langle a_2 \varphi(\sigma) \gamma, \alpha^\vee \rangle$$

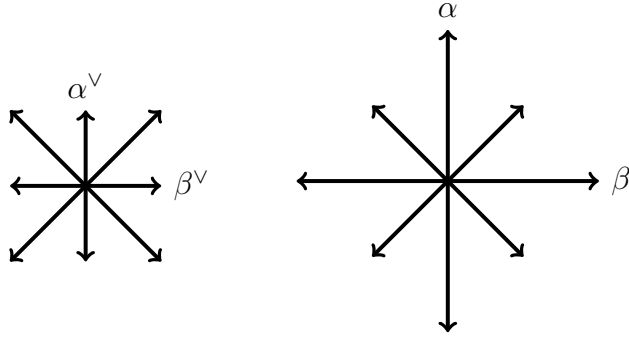
the sum of the four of them even after division by 2 is even and may be dropped from the exponent. So may those $\langle \gamma, \alpha^\vee \rangle$ which are 0. We confine ourselves to γ with γ^\vee in the plane of α^\vee, β^\vee .

The possibilities are:

- A) No roots except $\pm \alpha^\vee, \pm \beta^\vee$ in the plane. Then the exponent is 0 and

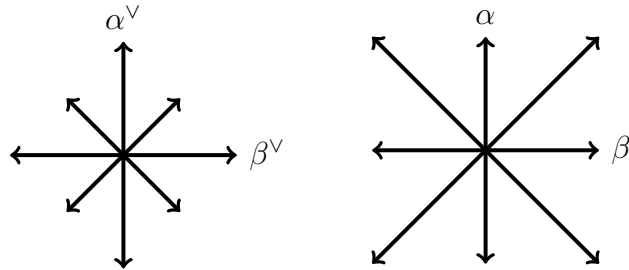
$$a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}.$$

B)



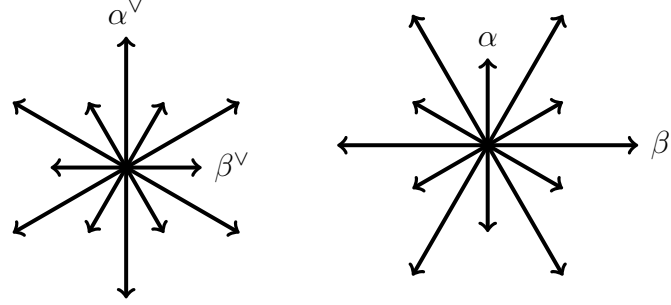
$$\left(\frac{1}{2}\right) \sum \langle \gamma, \alpha^\vee \rangle = \frac{1}{2} \langle \alpha, \alpha^\vee \rangle = 1, \quad a_2(X_{\alpha^\vee}) = -X_{\alpha^\vee} \quad [\mathbf{14}]$$

C)



$$\left(\frac{1}{2}\right) \sum \langle \gamma, \alpha^\vee \rangle = \langle \alpha, \alpha^\vee \rangle = 2, \quad a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}$$

D)



$$\left(\frac{1}{2}\right) \sum \langle \gamma, \alpha^\vee \rangle = 2 \langle \alpha, \alpha^\vee \rangle = 4, \quad a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}$$

E) The roles of α, α^\vee and β, β^\vee are reversed

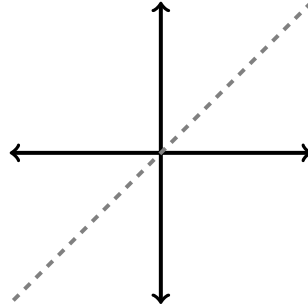
$$\left(\frac{1}{2}\right) \sum \langle \gamma, \alpha^\vee \rangle = \langle \alpha, \alpha^\vee \rangle = 2, \quad a_2(X_{\alpha^\vee}) = X_{\alpha^\vee}.$$

All that is claimed in A) through E) is easy to check. Finally it is clear that the reflection defined by $\mathfrak{s}_{\alpha^\vee}$ is that defined by α or α^\vee .

(i) Suppose that $\varphi(\sigma)\beta^\vee \neq \beta^\vee$ for all β^\vee in $\mathfrak{X}_{\alpha^\vee}$. Then $\beta^\vee + \varphi(\sigma)\beta^\vee$ is not a root, nor is [15]

$$\frac{\beta^\vee + \varphi(\sigma)\beta^\vee}{2}.$$

(ii) Suppose that $\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0$. Then $\alpha^\vee - \varphi(\sigma)\alpha^\vee$ is not a root and $\langle \alpha^\vee, \varphi(\sigma)\alpha^\vee \rangle = 0$. Since α^\vee and $\varphi(\sigma)\alpha^\vee$ have the same length, the root diagram of the plane spanned by $\alpha^\vee, \varphi(\sigma)\alpha^\vee$ is



I claim that if β^\vee lies in $\mathfrak{X}_{\alpha^\vee}$ but not in this plane then either $\langle \mu, \beta^\vee \rangle = 0$ or $\langle \nu, \beta^\vee \rangle = 0$. Otherwise in the three-dimensional plane spanned by $\alpha^\vee, \varphi(\sigma)\alpha^\vee, \beta^\vee, \varphi(\sigma)\beta^\vee$ we have a root system and

$$\left\{ \gamma^\vee \mid \langle \gamma, \alpha^\vee + \varphi(\sigma)\alpha^\vee \rangle \geq 0 \right\}$$

is a set of positive roots, for

$$\langle \gamma, \alpha^\vee + \varphi(\sigma)\alpha^\vee \rangle$$

is never 0, because if it were then $\varphi(\sigma)\gamma^\vee = -\gamma^\vee$. Since $\langle \mu, \gamma^\vee \rangle = \langle \nu, \gamma^\vee \rangle = 0$ this is impossible. Then $\varphi(\sigma)$ permutes the three simple roots amongst themselves, and leaves one fixed. This is a contradiction. Thus [16]

$$\mathfrak{s}_\alpha = \mathbf{C}(X_{\alpha^\vee} + \varphi(\sigma)X_{\alpha^\vee})$$

has dimension one. Since T is fundamental in G_{α^\vee} , the formula on p. 97 of *Harmonic analysis III* shows that $\mu(\rho, \alpha^\vee) = 0$. The three assertions follow again.

- (iii) Suppose that for any β^\vee in $\mathfrak{X}_{\alpha^\vee}$ either $\langle \mu, \beta^\vee \rangle \neq 0$ or $\langle \nu, \beta^\vee \rangle \neq 0$. Then $\mathfrak{s}_{\alpha^\vee} = 0$. By the same formula in *Harmonic analysis III*,

$$\mu(\rho, \alpha^\vee) \neq 0.$$

Lemma 2 is now completely proved. I should observe, for it will remove a confusion that could otherwise arise, that

$$-\overline{\langle \mu, \alpha^\vee \rangle} = \langle \nu, \alpha^\vee \rangle$$

for any α^\vee .

It is also possible to give Zuckerman's proof that the R -group is a sum of Z_2 's in the above context. Let $\text{Norm}_S^+({}^L\mathfrak{t}_+)$ be the set of elements of $\text{Norm}_S({}^L\mathfrak{t}_+)$ that take positive roots of \mathfrak{s}_0 to positive roots. Then

$$R = S/S^0 \simeq \text{Norm}_S^+({}^L\mathfrak{t}_+)/{}^L\mathfrak{t}_+.$$

Let

$$\mathfrak{s}_1 = {}^L\mathfrak{t}_+ + \sum_{\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0} \mathbf{C}X_{\alpha^\vee} [\mathbf{17}]$$

The elements of $\text{Norm}_S^+({}^L\mathfrak{t}_+)$ take \mathfrak{s}_1 to itself. Let Q be the operator

$$\frac{1}{|R|} \sum_R r$$

on $\mathfrak{t} \otimes \mathbf{C}$. Since the centralizer of $\varphi(\mathbf{C}^\times)$ is connected, S lies in the connected group S_1 with Lie algebra \mathfrak{s}_1 . Thus by Chevalley's theorem R is contained in the group generated by the reflections associated to the roots α^\vee of \mathfrak{s}_1 for which $Q\alpha^\vee = 0$.

If α^\vee is a root of \mathfrak{s}_1 then $\varphi(\sigma)\alpha^\vee \neq -\alpha^\vee$. Suppose $\varphi(\sigma)\alpha^\vee \neq \alpha^\vee$. Then

$$X_{\alpha^\vee} + \varphi(\sigma)X_{\alpha^\vee} \neq 0$$

and lies in \mathfrak{s} . Thus α^\vee restricted to ${}^L\mathfrak{t}_+$ defines a root of \mathfrak{s} . Since the elements of r stabilize ${}^L\mathfrak{t}_+$ and each r takes positive roots of ${}^L\mathfrak{t}_+$ in \mathfrak{s} to positive roots,

$$Q\alpha^\vee \neq 0.$$

Thus if α^\vee is a root of \mathfrak{s}_1 then

$$Q\alpha^\vee = 0 \implies \varphi(\sigma)\alpha^\vee = \alpha^\vee.$$

Moreover α^\vee cannot be a root of \mathfrak{s} and therefore

$$\varphi(\sigma)X_{\alpha^\vee} = -X_{\alpha^\vee}. [\mathbf{18}]$$

Finally if $Q\alpha^\vee = 0$, $Q\beta^\vee = 0$ then $\alpha^\vee \pm \beta^\vee$ is not a root because $\varphi(\sigma)X_{\alpha^\vee + \beta^\vee} = \varphi(\sigma)[X_{\alpha^\vee}, X_{\beta^\vee}] = [-X_{\alpha^\vee}, -X_{\beta^\vee}] = X_{\alpha^\vee + \beta^\vee}$ and $\alpha^\vee + \beta^\vee$ would have to be a root of \mathfrak{s} . This is inconsistent with

$$Q(\alpha^\vee + \beta^\vee) = 0.$$

The set of positive α^\vee for which $\langle \mu, \alpha^\vee \rangle = \langle \nu, \alpha^\vee \rangle = 0$ and $Q\alpha^\vee = 0$ is the strongly orthogonal system needed for Zuckerman's argument.

Compiled on April 18, 2025.