

ON THE NOTION OF AN AUTOMORPHIC REPRESENTATION

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The irreducible representations of a reductive group over a local field can be obtained from the square-integrable representations of Levi factors of parabolic subgroups by induction and formation of subquotients [2, 4]. Over a global field F the same process leads from the cuspidal representations, which are analogues of square-integrable representations, to all automorphic representations.

Suppose P is a parabolic subgroup of G with Levi factor M and $\sigma = \bigotimes \sigma_v$ a cuspidal representation of $M(\mathbf{A})$. Then $\text{Ind } \sigma = \bigotimes_v \text{Ind } \sigma_v$ is a representation of $G(\mathbf{A})$ which may not be irreducible, and may not even have a finite composition series. As usual an irreducible subquotient of this representation is said to be a constituent of it.

For almost all v , $\text{Ind } \sigma_v$ has exactly one constituent π_v° containing the trivial representation of $G(\mathbf{O}_v)$. If $\text{Ind } \sigma_v$ acts on X_v then π_v° can be obtained by taking the smallest $G(F_v)$ -invariant subspace V_v of X_v containing nonzero vectors fixed by $G(\mathbf{O}_v)$ together with the largest $G(F_v)$ -invariant subspace U_v of V_v containing no such vectors and then letting $G(F_v)$ act on V_v/U_v .

Lemma 1. *The constituents of $\text{Ind } \sigma$ are the representations $\pi = \bigotimes \pi_v$ where π_v is a constituent of $\text{Ind } \sigma_v$ and $\pi_v = \pi_v^\circ$ for almost all v .*

That any such representation is a constituent is clear. Conversely let the constituent π act on V/U with $0 \subseteq U \subseteq V \subseteq X = \bigotimes X_v$. Recall that to construct the tensor product one chooses a finite set of places S_0 and for each v not in S_0 a vector x_v° which is not zero and is fixed by $G(\mathbf{O}_v)$. We can find a finite set of places S , containing S_0 , and a vector x_S in $X_S = \bigotimes_{v \in S} X_v$ which are such that $x = x_S \otimes (\bigotimes_{v \notin S} x_v^\circ)$ lies in V but not in U .

Let V_S be the smallest subspace of X_S containing x_S and invariant under $G_S = \prod_{v \in S} G(F_v)$. There is clearly a surjective map

$$V_S \otimes \left(\bigotimes_{v \notin S} V_v \right) \rightarrow V/U,$$

and if $v_0 \notin S$ the kernel contains $V_S \otimes U_{v_0} \otimes (\bigotimes_{v \notin S \cup \{v_0\}} V_v)$. We obtain a surjection $V_S \otimes (\bigotimes_{v \in S} V_v/U_v) \rightarrow V/U$ with a kernel of the form $U_S \otimes (\bigotimes_{v \in S} V_v/U_v)$, U_S lying in V_S . The representation of G_S on V_S/U_S is irreducible and, since $\text{Ind } \sigma_v$ has a finite composition series, of the form $\bigotimes_{v \in S} \pi_v$, π_v being a constituent of $\text{Ind } \sigma_v$. Thus $\pi = \bigotimes \pi_v$ with $\pi_v = \pi_v^\circ$ when $v \notin S$.

The purpose of this note is to establish the following proposition.¹

Proposition 2. *A representation π of $G(\mathbf{A})$ is an automorphic representation if and only if π is a constituent of $\text{Ind } \sigma$ for some P and some σ .*

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¹The definition of an automorphic representation is given in the paper [1] by A. Borel and H. Jacquet to which this paper was a supplement.

The proof that every constituent of $\text{Ind } \sigma$ is an automorphic representation will invoke the theory of Eisenstein series, which has been fully developed only when the global field F has characteristic zero [3]. One can expect however that the analogous theory for global fields of positive characteristic will appear shortly; so there is little point in making the restriction to characteristic zero explicit in the proposition. Besides, the proof that every automorphic representation is a constituent of some $\text{Ind } \sigma$ does not involve the theory of Eisenstein series in any serious way.

We begin by remarking some simple lemmas.

Lemma 3. *Let Z be the centre of G . Then an automorphic form is $Z(\mathbf{A})$ -finite.*

This is verified in [1].

Lemma 4. *Suppose K is a maximal compact subgroup of $G(\mathbf{A})$ and φ an automorphic form with respect to K . Let P be a parabolic subgroup of G . Choose $g \in G(\mathbf{A})$ and let K' be a maximal compact subgroup of $M(\mathbf{A})$ containing the projection of $gKg^{-1} \cap P(\mathbf{A})$ on $M(\mathbf{A})$. Then*

$$\varphi_P(m; g) = \int_{N(F) \backslash N(\mathbf{A})} \varphi(nmg) \, dn$$

is an automorphic form on $M(\mathbf{A})$ with respect to K' .

It is clear that the growth conditions are hereditary and that $\varphi_P(\cdot; g)$ is smooth and K' -finite. That it transforms under admissible representations of the local Hecke algebras of M is a consequence of theorems in [2] and [4].

We say that φ is orthogonal to cusp forms if $\int_{\Omega_{G_{\text{der}}(\mathbf{A})}} \varphi(g) \psi(g) \, dg = 0$ whenever ψ is a cusp form and Ω is a compact set in $G(\mathbf{A})$. If P is a parabolic subgroup we write $\psi \perp P$ if $\varphi_P(\cdot; g)$ is orthogonal to cusp forms on $M(\mathbf{A})$ for all g . We recall a simple lemma [3].

Lemma 5. *If $\varphi \perp P$ for all P then φ is zero.*

We now set about proving that any automorphic representation π is a constituent of some $\text{Ind } \sigma$. We may realize π on V/U , where U, V are subspaces of the space \mathcal{A} of automorphic forms and V is generated by a single automorphic form φ . Totally order the conjugacy classes of parabolic subgroups in such a way that $\{P\} < \{P'\}$ implies $\text{rank } P \leq \text{rank } P'$ and $\text{rank } P < \text{rank } P'$ implies $\{P\} < \{P'\}$. Given φ let $\{P_\varphi\}$ be the minimum $\{P\}$ for which $\{P\} < \{P'\}$ implies $\varphi \perp P'$. Amongst all the φ serving to generate π choose one for which $\{P\} = \{P_\varphi\}$ is minimal. If $\psi \in \mathcal{A}$ let $\psi_P(g) = \psi_P(1, g)$ and consider the map $\psi \rightarrow \psi_P$ on V . If U and V had the same image we could realize π as a constituent of the kernel of the map. But this is incompatible with our choice of φ and hence if U_P and V_P are the images of U and V we can realize π in the quotient V_P/U_P .

Let \mathcal{A}_P° be the space of smooth functions ψ on $N(\mathbf{A})P(F) \backslash G(\mathbf{A})$ satisfying the following conditions.

- (a) ψ is K -finite.
- (b) For each g the function $m \rightarrow \psi(m, g) = \psi(mg)$ is automorphic and cuspidal.

Then $V_P \subseteq \mathcal{A}_P^\circ$. Since there is no point in dragging the subscript P about, we change notation, letting π be realized on V/U with $U \subseteq V \subseteq \mathcal{A}_P^\circ$. We suppose that V is generated by a single function φ .

Lemma 6. *Let A be the centre of M . We may so choose φ and V that there is a character χ of $A(\mathbf{A})$ satisfying $\varphi(ag) = \chi(a)\varphi(g)$ for all $g \in G(\mathbf{A})$ and all $a \in A(\mathbf{A})$.*

Since $P(\mathbf{A}) \backslash G(\mathbf{A})/K$ is finite, Lemma 3 implies that any φ in \mathcal{A}_P° is $A(\mathbf{A})$ -finite. Choose V and the φ generating it to be such that the dimension of the span Y of $\{l(a)\varphi \mid a \in A(\mathbf{A})\}$ is minimal. Here $l(a)$ is left translation by a . If this dimension is one the lemma is valid. Otherwise there is an $a \in A(\mathbf{A})$ and $\alpha \in \mathbf{C}$ such that $0 < \dim(l(a) - \alpha)Y < \dim Y$.

There are two possibilities. Either

$$(l(a) - \alpha)U = (l(a) - \alpha)V \quad \text{or} \quad (l(a) - \alpha)U \neq (l(a) - \alpha)V.$$

In the second case we may replace φ by $(l(a) - \alpha)\varphi$, contradicting our choice. In the first we can realize π as a subquotient of the kernel of $l(a) - \alpha$ in V .

What we do then is to choose a lattice B in $A(\mathbf{A})$ such that $BA(F)$ is closed and $BA(F) \backslash A(\mathbf{A})$ is compact. Amongst all those φ and V for which Y has the minimal possible dimension we choose one φ for which the subgroup of B defined as $\{b \in B \mid l(b)\varphi = \beta\varphi, \beta \in \mathbf{C}\}$ has maximal rank. What we conclude from the previous paragraph is that this rank must be that of B . Since φ is invariant under $A(F)$ and $BA(F) \backslash A(\mathbf{A})$ is compact, we conclude that Y must now have dimension one. The lemma follows.

Choosing such a φ and V we let ν be that positive character of $M(\mathbf{A})$ which satisfies

$$\nu(a) = |\chi(a)|, \quad a \in A(\mathbf{A}),$$

and introduce the Hilbert space $L_2^\circ = L_2^\circ(M(F) \backslash M(\mathbf{A}), \chi)$ of all measurable functions ψ on $M(\mathbf{Q}) \backslash M(\mathbf{A})$ satisfying the following conditions.

- (i) For all $m \in M(\mathbf{A})$ and all $a \in A(\mathbf{A})$, $\psi(ag) = \chi(a)\psi(g)$.
- (ii) $\int_{A(\mathbf{A})M(\mathbf{Q}) \backslash M(\mathbf{A})} \nu^{-2}(m) |\psi(m)|^2 dm < \infty$.

L_2° is a direct sum of irreducible invariant subspaces and if $\psi \in V$ then $m \rightarrow \psi(m, g)$ lies in L_2° for all $g \in G(\mathbf{A})$. Choose some irreducible component H of L_2° on which the projection of $\psi(\cdot, g)$ is not zero for some $g \in G(\mathbf{A})$.

For each ψ in V define $\psi'(\cdot, g)$ to be the projection of $\psi(\cdot, g)$ on H . It is easily seen that, for all $m_1 \in M(\mathbf{A})$, $\psi'(mm_1, g) = \psi'(m, m_1g)$. Thus we may define $\psi'(g)$ by $\psi'(g) = \psi'(1, g)$. If $V' = \{\psi' \mid \psi \in V\}$, then we realize π as a quotient of V' . However if δ^2 is the modular function for $M(\mathbf{A})$ on $N(\mathbf{A})$ and σ the representation of $M(\mathbf{A})$ on H then V' is contained in the space of $\text{Ind } \delta^{-1}\sigma$.

To prove the converse, and thereby complete the proof of the proposition, we exploit the analytic continuation of the Eisenstein series associated to cusp forms. Suppose π is a representation of the global Hecke algebra \mathcal{H} , defined with respect to some maximal compact subgroup K of $G(\mathbf{A})$. Choose an irreducible representation κ of K which is contained in π . If E_κ is the idempotent defined by κ let $\mathcal{H}_\kappa = E_\kappa \mathcal{H} E_\kappa$ and let π_κ be the irreducible representation of \mathcal{H}_κ on the κ -isotypical subspace of π . To show that π is an automorphic representation, it is sufficient to show that π_κ is a constituent of the representation of \mathcal{H}_κ on the space of automorphic forms of type κ . To lighten the burden on the notation, we henceforth denote π_κ by π and \mathcal{H}_κ by \mathcal{H} .

Suppose P and the cuspidal representation σ of $M(\mathbf{A})$ are given. Let L be the lattice of rational characters of M defined over F and let $L_\mathbf{C} = L \otimes \mathbf{C}$. Each element μ of $L_\mathbf{C}$ defines a character ξ_μ of $M(\mathbf{A})$. Let I_μ be the κ -isotypical subspace of $\text{Ind } \xi_\mu \sigma$ and let $I = I_0$. We want to show that if π is a constituent of the representation of I then π is a constituent of the representation of \mathcal{H} on the space of automorphic forms of type κ .

If $\{g_i\}$ is a set of coset representatives for $P(\mathbf{A}) \backslash G(\mathbf{A})/K$ then we may identify I_μ with I by means of the map $\varphi \rightarrow \varphi_\mu$ with

$$\varphi_\mu(nmg_ik) = \xi_\mu(m)\varphi(nmg_ik).$$

In other words we have a trivialisation of the bundle $\{I_\mu\}$ over $L_{\mathbf{C}}$, and we may speak of a holomorphic section or of a section vanishing at $\mu = 0$ to a certain order. These notions do not depend on the choice of the g_i , although the trivialisation does.

There is a neighborhood U of $\mu = 0$ and a finite set of hyperplanes passing through U such that for μ in the complement of these hyperplanes in U the Eisenstein series $E(\varphi)$ is defined for φ in I_μ . To make things simpler we may even multiply E by a product of linear functions and assume that it is defined on all of U . Since it is only the modified function that we shall use, we may denote it by E , although it is no longer the true Eisenstein series. It takes values in the space of automorphic forms and thus $E(\varphi)$ is a function $g \rightarrow E(g, \varphi)$ on $G(\mathbf{A})$. It satisfies

$$E(\rho_\mu(h)\varphi) = r(h)E(\varphi)$$

if $h \in \mathcal{H}$ and ρ_μ is $\text{Ind } \xi_\mu \sigma$. In addition, if φ_μ is a holomorphic section of $\{I_\mu\}$ in a neighborhood of 0 then $E(g, \varphi_\mu)$ is a holomorphic in μ for each g , and the derivatives of $E(\varphi_\mu)$, taken pointwise, continue to be in \mathcal{A} .

Let I_r be the space of germs of degree r at $\mu = 0$ of holomorphic sections of I . Then $\varphi_\mu \rightarrow \rho_\mu(h)\varphi_\mu$ defines an action of \mathcal{H} on I_r . If $s \leq r$ the natural map $I_r \rightarrow I_s$ is an \mathcal{H} -homomorphism. Denote its kernel by I_r^s . Certainly $I_0 = I$. Choosing a basis for the linear forms on $L_{\mathbf{C}}$ we may consider power series with values in the κ -isotypical subspace of \mathcal{A} , $\sum_{|\alpha| \leq r} \mu^\alpha \psi_\alpha$. \mathcal{H} acts by right translation on this space and the representation so obtained is, of course, a direct sum of that on the κ -isotypical automorphic forms. Moreover $\varphi_\mu \rightarrow E(\varphi_\mu)$ defines an \mathcal{H} -homomorphism λ from I_r to this space. To complete the proof of the proposition all one needs is the Jordan-Hölder theorem and the following lemma.

Lemma 7. *For r sufficiently large the kernel of λ is contained in I_r^0 .*

Since we are dealing with Eisenstein series associated to a fixed P and σ we may replace E by the sum of its constant terms for the parabolic associated to P , modifying λ accordingly. All of these constant terms vanish identically if and only if E itself does. If Q_1, \dots, Q_m is a set of representatives for the classes of parabolics associated to P let $E_i(\varphi)$ be the constant term along Q_i . We may suppose that M is a Levi factor of each Q_i . Define $\nu(m)$ for $m \in M(\mathbf{A})$ by $\xi_\mu(m) = e^{\langle \mu, \nu(m) \rangle}$. Thus $\nu(m)$ lies in the dual of $L_{\mathbf{R}}$. If $\varphi \in I_\mu$, the function of $E_i(\varphi)$ has the form

$$E_i(nmg_ik, \varphi) = \sum_{\alpha=1}^a \sum_{\beta=1}^b p_\alpha(\nu(m)) \xi_{\nu_\beta(\mu)}(m) \psi_{\alpha\beta}(m, k).$$

Here $\psi_{\alpha\beta}$ lies in a finite-dimensional space independent of μ and g_j ; ν_β , $1 \leq \beta \leq b$ is a holomorphic function of μ ; and $\{p_\alpha\}$ is a basis for the polynomials of some degree. This representation may not be unique. The next lemma implies that we may shrink the open set U and then find a finite set h_1, \dots, h_n in $G(\mathbf{A})$ such that $E(\varphi_\mu)$ is 0 for $\mu \in U$, $\varphi_\mu \in I_\mu$ if and only if the numbers $E_i(h_j, \varphi_\mu)$, $1 \leq i \leq m$, $1 \leq j \leq n$ are all 0.

Lemma 8. *Let U be a neighborhood of 0 in \mathbf{C}^l , ν_1, \dots, ν_k holomorphic functions on U and p_1, \dots, p_a a basis for the polynomials of some given degree. Then there is a neighborhood V*

of 0 contained in U and a finite set y_1, \dots, y_b in \mathbf{C}^l such that if $\mu \in V$ then

$$(*) \quad \sum p_i(y) e^{\nu_j(\mu) \cdot y} = 0$$

for all y if and only if it is 0 for $y = y_1, \dots, y_b$.

To prove this lemma one has only to observe that the analytic subset of U defined by the equations $(*)$, $y \in \mathbf{C}^l$, is defined in a neighborhood of 0 by a finite number of them.

We may therefore regard E as a function on U with values in the space of linear transformations from the space I , which is finite-dimensional, to the space \mathbf{C}^{mn} . One knows from the theory of Eisenstein series that E_μ is injective for μ in an open subset of U . Then to complete the proof of the proposition, we need only verify the following lemma.

Lemma 9. *Suppose E is a holomorphic function in U , a neighborhood of 0 on \mathbf{C}^l , with values in $\text{Hom}(I, J)$, where I and J are finite-dimensional spaces, and suppose that E_μ is injective on an open subset of U . Then there is an integer r such that if φ_μ is analytic near $\mu = 0$ and the Taylor series of $E_\mu \varphi_\mu$ vanishes to order r then $\varphi_0 = 0$.*

Projecting to a quotient of J , we may assume that $\dim I = \dim J$ and even that $I = J$. Let the first nonzero term of the power-series expansion of $\det E_\mu$ have degree s . Then we will show that r can be taken to be $s + 1$. It is enough to verify this for $l = 1$, for we can restrict to a line on which the leading term of E_μ still has degree s . But then multiplying E fore and aft by nonsingular matrices we may suppose it is diagonal with entries z^α , $0 \leq \alpha \leq s$. Then the assertion is obvious.

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