

ON UNITARY REPRESENTATIONS OF THE VIRASORO ALGEBRA

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The Virasoro algebra \mathfrak{v} is an infinite-dimensional Lie algebra with basis L_m , $m \in \mathbf{Z}$, and Z and defining relations:

- (i) $[L_m, L_n] = (m - n)L_{m+n} + \frac{m(m^2-1)}{12}\delta_{m,-n}Z$;
- (ii) $[L_m, Z] = 0$.

Some representations π of \mathfrak{v} of particular interest [2] are the Verma modules $(V, \pi) = (V^{h,c}, \pi^{h,c})$, $h, c \in \mathbf{R}$. They are characterized by the following conditions.

- (i) There is a vector $v = v_\phi \neq 0$ in V such that $L_nv = 0$, $n > 0$, $L_0v = hv$, $Zv = cv$.
- (ii) Let A be the set of sequences of integers $k_1 \geq k_2 \geq \dots \geq k_r > 0$ of arbitrary length, and if $\alpha \in A$ let $v_\alpha = \pi(L_{-k_1}) \dots \pi(L_{-k_r})v_\phi$. Then $\{v_\alpha \mid \alpha \in A\}$ is a basis of V .

Observe that V is just the free vector space with basis $\{v_\alpha\}$ and is thus independent of h and c . It is easy to see [1] that there is a unique sesquilinear form $\langle u, v \rangle = \langle u, v \rangle^{h,c}$ on V with the properties:

- (i) $\langle v_\phi, v_\phi \rangle = 1$;
- (ii) $\langle u, v \rangle = \overline{\langle v, u \rangle}$;
- (iii) $\langle \pi(L_m)u, v \rangle = \langle u, \pi(L_{-m})v \rangle$, $m \in \mathbf{Z}$.

If this form is non-negative then the representation ρ of \mathfrak{v} on the quotient of V by the space of null vectors is unitary, in the sense that

$$\rho(L_m)^* = \rho(L_{-m}).$$

Theorem FQS. *The form $\langle \cdot, \cdot \rangle_{h,c}$ is non-negative only if either $c \geq 1, h \geq 0$ or there exists an integer $m \geq 2$ and two integers $p, q, 1 \leq p < m, 1 \leq q \leq p$, such that*

$$c = 1 - \frac{6}{m(m+1)}, \quad h = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}.$$

This theorem has been proven by Friedan-Qiu-Shenker [1]. The sketch of the proof that they provided was unconscionably brief, and has evoked some scepticism among mathematicians. In this note, which grew out of a series of lectures at the Centre de recherches mathématiques that overlapped the workshop, details are worked out. In the meantime, Friedan, Qiu and Shenker have themselves provided them [3], but the present account, which turns out to diverge from theirs in some respects, may still be a useful supplement to it. Several other authors have proven that the conditions of the theorem are not only necessary but also sufficient for non-negativity, but that is not the concern here.

The proof proceeds by lemmas. I write Lv rather than $\pi(L)v$, $L \in \mathfrak{v}$, $v \in V$.

Lemma 1. *If $\langle \cdot, \cdot \rangle$ is non-negative then $h \geq 0, c \geq 0$.*

Proof. Since $L_n L_{-n} v_\phi = L_{-n} L_n v_\phi + 2nhv_\phi + \frac{n(n^2-1)}{12}cv_\phi$, we have $\langle L_{-n}v_\phi, L_{-n}v_\phi \rangle = 2nh + \frac{n(n^2-1)}{12}c$. Taking n first equal to 1 and then very large we obtain the lemma. \square

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For arbitrary m we set $c = c(m) = 1 - \frac{6}{m(m+1)}$, $h_{p,q} = h_{p,q}(m) = \frac{((m+1)p-mq)^2-1}{4m(m+1)}$, $p, q \in \mathbf{N}$. Observe that $c(-1-m) = c(m)$ and that $h_{p,q}(-1-m) = h_{q,p}(m)$.

Lemma 2.

- (a) For $1 < c < 25$, m is not real and neither is $h_{p,q}(m)$ unless $p = q$.
- (b) As m runs from 2 to ∞ , c increases monotonically from 0 to 1.
- (c) For $c > 1$, $-1 < m < 0$.
- (d) If $-1 < m < 0$ then $h_{p,q}(m) < 0$ unless $p = q = 1$ when $h_{p,q}(m) = 0$.
- (e) If $p = q$ then $h_{p,q}(m) = \frac{p^2-1}{24}(1-c)$.
- (f) If $p \neq q$ then $h_{p,q} + h_{q,p} = \frac{p^2+q^2-2}{24}(1-c) + \frac{(p-q)^2}{2}$. In addition $h_{p,q}h_{q,p}$ is equal to

$$\begin{aligned} & \frac{(p^2q^2 - p^2 - q^2 + 1)}{16 \cdot 36} (1-c)^2 \\ & + \frac{(2p^2q^2 - pq(p^2 + q^2) - (p-q)^2)}{48} (1-c) \\ & + \frac{(p^4 + q^4 - 4p^3q - 4pq^3 + 6p^2q^2)}{16}. \end{aligned}$$

Proof. The first four parts of the lemma are clear, and the last two are straightforward calculations. \square

There is a second sesquilinear form on V defined by $\{v_\alpha, v_\beta\} = \delta_{\alpha\beta}$. If V_n is the subspace of V with basis $\{v_\alpha \mid \alpha = (k_1, \dots, k_r) \mid \sum_{i=1}^r k_i = n\}$, then $V = \bigoplus_{n \geq 0} V_n$ and the spaces V_n are mutually orthogonal with respect to both forms. The first form is defined on V_n with respect to the second by a hermitian linear transformation $H_n = H_n(h, c) : \langle u, v \rangle_n = \{H_n u, v\}_n$. Let $P(n)$ be the dimension of V_n . It is the number of partitions of n . The Kac determinant formula (cf. [1]) is the key to the proof of Theorem FQS.

Kac determinant formula. If $c = c(m)$ then

$$\det H_n(h, c) = A_n \prod_{k \leq n} \prod_{pq=k} (h - h_{p,q})^{P(n-k)},$$

where A_n is a positive constant.

Lemma 3. The form $\langle \cdot, \cdot \rangle_n$ is non-negative for $h \geq 0, c \geq 1$.

Proof. By continuity it suffices to treat pairs for which $h > 0, c > 1$. Since the previous lemma implies that $\det H_n(h, c)$ is nowhere zero in this region, it suffices to prove that the form is positive for one pair (h, c) . If $\alpha = (k_1, \dots, k_r), r = r(\alpha), n(\alpha) = k_1 + \dots + k_r$, set $v'_\alpha = L_{-k_r} \cdots L_{-k_1} v_\phi$. It is generally different than v_α . It clearly suffices to show that for a given c and h large,

$$(3.1) \quad \langle v'_\alpha, v'_\alpha \rangle = c_\alpha h^{r(\alpha)} (1 + o(1)), \quad c_\alpha > 0$$

$$(3.2) \quad \langle v'_\alpha, v'_\beta \rangle = o\left(h^{(r(\alpha)+r(\beta))/2}\right), \quad \alpha \neq \beta.$$

This is proved by induction on $n(\alpha) + n(\beta)$. First of all $L_k^a L_{-k}^a$ is equal to

$$L_k^{a-1} (bL_0 + d) L_{-k}^{a-1} + L_k^{a-1} L_{-k} L_k L_{-k}^{a-1}, \quad b > 0.$$

Moving the single L_k in the second term ever further to the right, we obtain finally

$$L_k^a L_{-k}^a = L_k^{a-1} (bL_0 + d) L_{-k}^{a-1} + L_k^{a-1} L_{-k}^a L_k, \quad b > 0.$$

Take $k_1 \geq k_2 \geq \dots \geq k_r > k$. If $\alpha = (k_1, \dots, k_r, k, \dots, k)$, then

$$\begin{aligned} \langle v'_\alpha, v'_\alpha \rangle &= \langle L_{k_1} \cdots L_{k_r} L_k^a L_{-k}^a L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi \rangle \\ &= c_{k,a} h (1 + o(h)) \langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a-1} L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi \rangle \\ &\quad + \langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^a L_k L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi \rangle \end{aligned}$$

with $c_{k,a} > 0$. In the second term we move the L_k further and further to the right obtaining the sum of

$$(k + k_r) \left\langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^a L_{-k_r} \cdots L_{-k_{j+1}} L_{-(k_j-k)} L_{-k_{j-1}} \cdots L_{-k_1} v_\phi, v_\phi \right\rangle.$$

The induction assumption together with the defining relations for \mathfrak{v} implies readily that each of these terms is $o(h^{r(\alpha)})$ and that

$$\langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a-1} L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi \rangle = \langle v'_\gamma, v'_\gamma \rangle = c_\gamma h^{r(\gamma)} (1 + o(1)),$$

if $\gamma = (k_1, \dots, k_r, k, \dots, k)$, with k repeated $a - 1$ times, so that $r(\alpha) = 1 + r(\gamma)$.

On the other hand, if $\beta = (\ell_1, \dots, \ell_s, k, \dots, k)$, with k repeated $a' \leq a$ times, $a > 0$, $a' \geq 0$, $\ell_s \geq k$ even if $a' = 0$, then

$$\langle v'_\beta, v'_\alpha \rangle = \langle L_{k_1} \cdots L_{k_r} L_k^a L_{-k}^{a'} L_{-\ell_s} \cdots L_{-\ell_1} v_\phi, v_\phi \rangle$$

is equal to the sum of

$$c_{k,a'} h (1 + o(1)) \langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a'-1} L_{-\ell_s} \cdots L_{-\ell_1} v_\phi, v_\phi \rangle$$

and

$$\sum_j (k + \ell_j) \left\langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a'} L_{-\ell_s} \cdots L_{-\ell_{j+1}} L_{-(\ell_j-k)} L_{-\ell_{j-1}} \cdots L_{-\ell_1} v_\phi, v_\phi \right\rangle.$$

We take $c_{k,0} = 0$ if $a' = 0$. So induction yields (3.2). \square

Observe that if $m > 0$ and $p > q$ then $h_{p,q} > h_{q,p}$. If $h \geq 0$ and $m > 0$ define $M > 0$ by $M^2 = 1 + 4m(m+1)h$. Then $M \geq 1$. Let D be the closed shaded region in the diagram I. It is bounded by the lines

$$mx - (m+1)y = \pm M \text{ and } (m+1)x - my = M.$$

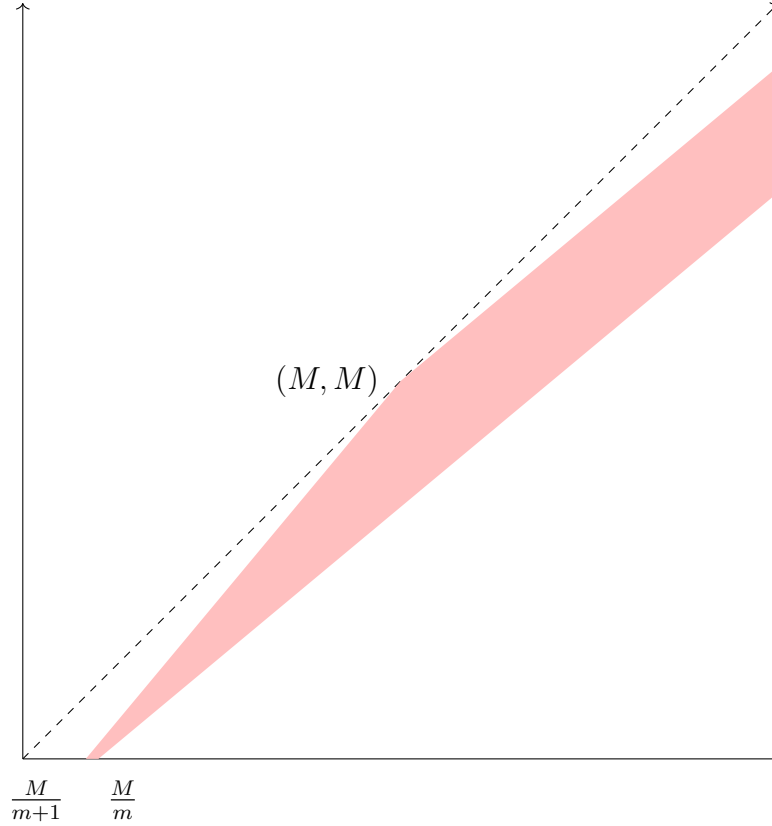


Diagram I

Lemma 4.

- (a) $h_{p,q} \geq h \geq h_{q,p}$ if and only if $(p, q) \in D$.
 (b) D contains an integral point (p, q) with $q > 0$.

Proof. Since $h_{p,q} \geq h$ if and only if $((m+1)p - mq)^2 \geq M^2$ and $h \geq h_{q,p}$ if and only if $((m+1)q - mp)^2 \leq M^2$, the first statement of the lemma is clear. For the second choose a large integer p and let $a = \frac{mp-M}{m+1}$. Then the points (p, q) with $a \leq q \leq a + \frac{2M}{m+1}$ lie in D . So do the points $(p+1, q)$, $a + \frac{m}{m+1} \leq q \leq a + \frac{m+2M}{m+1}$ and so on. So we need only show that one of the intervals $\left[a + \frac{km}{m+1}, a + \frac{km+2M}{m+1} \right]$, $k \in \mathbf{Z}, k \geq 0$, contains an integer. This is clear if $\frac{m}{m+1}$ is irrational. Otherwise, increasing q if necessary, we may suppose that a is as close to its integral part as any $a + \frac{km}{m+1}$. Then $a + \frac{m}{m+1} < [a] + 1$, but $a + \frac{m+2M}{m+1} \geq a + \frac{m+2}{m+1} > [a] + 1$, and the interval $\left[a + \frac{m}{m+1}, a + \frac{m+2M}{m+1} \right]$ contains $[a] + 1$. \square

Let $p(h, c) = \min_{(p,q) \in D} p$ and let $q(h, c) = \min_{(p,q) \in D} q$. It is clear that

$$P(h, c) = (p(h, c), q(h, c)) \in D.$$

In the following geometrical arguments, it is sometimes necessary to recall that $h - h_{p_0, p_0} < 0$ if and only if $p_0 > M$.

Lemma 5. *If $P(h, c)$ lies in the interior of D then $\langle v, v \rangle$ assumes negative values in V .*

Proof. Let $(p, q) = P(h, c)$ and let $n = pq$. If $p_0 q_0 \leq n, p_0 \geq q_0$ and $(p_0, q_0) \neq (p, q)$ then either $p_0 < p$ or $q_0 < q$ so that $(p_0, q_0) \notin D$. In general set

$$\begin{aligned}\phi_{p_0, q_0} &= (h - h_{p_0, q_0})(h - h_{q_0, p_0}), & p_0 \neq q_0, \\ &= h - h_{p_0, q_0}, & p_0 = q_0.\end{aligned}$$

If $(p_0, q_0) \notin D$ and $p_0 \neq q_0$ then $\phi_{p_0, q_0} > 0$.

Suppose that for some p_0 with $p_0^2 \leq pq$ we had $h - h_{p_0, p_0} < 0$. Then there would be a minimum such p_0 and if $n_0 = p_0^2$ then

$$\det H_{n_0} = A_{n_0} \prod_{\substack{p_1 \geq q_1 \\ n_1 = p_1 q_1 \leq n_0}} \phi_{p_1, q_1}^{P(n_0 - n_1)}$$

Since $P(h, c)$ lies in the interior of $D, p \neq q$ and none of the pairs (p_1, q_1) that intervene here lie in D . Moreover, all terms of the products are positive save $\phi_{p_0, p_0}^{P(0)} = \phi_{p_0, p_0}$. Since this is negative, $\langle \cdot, \cdot \rangle$ assumes negative values on V_{n_0} .

If, however, $\phi_{p_0, p_0} > 0$ for all $p_0 \leq q$ then the same argument shows that $\det H_n < 0$, so that $\langle \cdot, \cdot \rangle$ assumes negative values on V_n . \square

The treatment of those points (h, c) for which $P(h, c)$ lies on the boundary of D is more delicate. There are at first three possibilities for $(p, q) = P(h, c)$:

- (A) $mp - (m + 1)q = M$;
- (B) $(m + 1)p - mq = M$;
- (C) $mp - (m + 1)q = -M, p \neq q$;

Lemma 6. *The case (C) above does not occur.*

Proof. It is clear from the diagram defining D that in case (C), $p \geq M, q \geq M$. If $q = 1$ then $M = 1$ and $p = 1$, so that we have rather case (B). If $q > 1$ then $p > 1$ and $(m + 1)(q - 1) - m(p - 1) = (m + 1)q - mp - 1$, so that $M > (m + 1)(q - 1) - m(p - 1) > -M$. Moreover, $(m + 1)(p - 1) - m(q - 1) - M = (m + 1)(p - 1 - q) - m(q - 1 - p) = (2M + 1)(p - q) - 1$. Since $m \geq 2$ this is positive if $p \neq q$. Consequently $(p - 1, q - 1) \in D$, and this is a contradiction. \square

Fix (p, q) . In case (A) we have $h = h_{q, p}(m), c = c(m)$. In case (B) we have $h = h_{p, q}(m), c = c(m)$.

Lemma 7.

- (a) *The set of all $m \geq 2$ for which $h = h_{q, p}(m), c = c(m)$ yields case (A) is the interval $m > q + p - 1$.*
- (b) *The set of all $m \geq 2$ for which $h = h_{p, q}(m), c = c(m)$ yields case (B) is the interval $m > q + p - 1$ if $(p, q) \neq (1, 1)$ and is the interval $m \geq 2$ if $(p, q) = (1, 1)$.*

It will be helpful, when proving this and the following lemmas, to keep the diagrams IIA and IIB in mind.

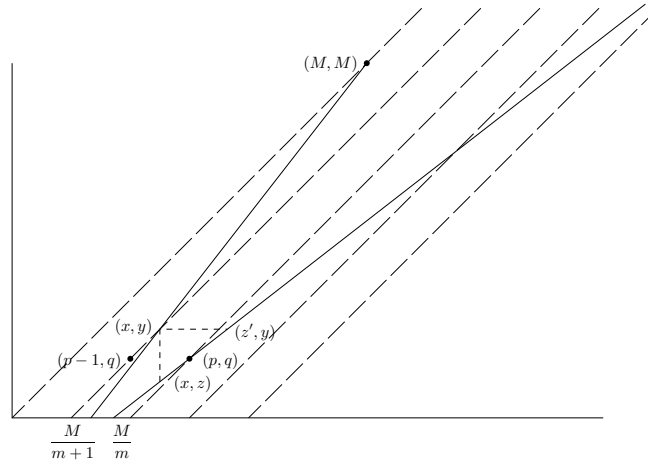


Diagram IIA

Proof. We first show that if $h_{q,p}(m_0), c(m_0)$ yield case (A) then so does $(h_{q,p}(m), c(m))$ for $m \geq m_0$. It is clear from the diagram that it is sufficient to verify that $M, \frac{M}{m+1}$, and $\frac{M}{m}$ are increasing functions of m . But $M = m(p - q) - q, \frac{M}{m} = (p - q) - \frac{q}{m}, \frac{M}{m+1} = (p - q) - \frac{q}{m+1}$. It is also clear that

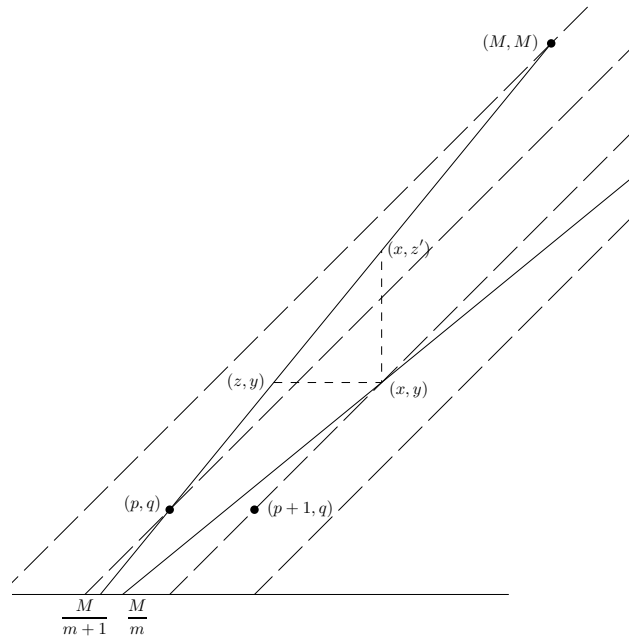


Diagram IIB

we can decrease m without passing out of case (A) so long as $M = m(p - q) - q$ remains greater than or equal to 1 and $(m + 1)(p - 1) - mq > mp - (m + 1)q$. But

$$(m + 1)(p - 1) - mq = mp - (m + 1)q \iff m = p + q - 1.$$

As we decrease to these points, M decreases to

$$(p+q-1)(p-q) - q = p^2 - q^2 - p = (p-1)^2 - q^2 + p - 1.$$

This number is greater than 1 because $p > q \geq 1$.

For case (B), $M = m(p-q)+p$ is a non-decreasing function of m , and $\frac{M}{m} = (p-q) + \frac{p}{m}$, $\frac{M}{m+1} = (p-q) + \frac{q}{m+1}$ are decreasing functions. Since the slope of $mp - (m+1)q = M$ is $1 - \frac{1}{m+1}$, it is increasing and the conclusion is the same. The minimal value of m is given by

$$(m+1)p - mq = mp - (m+1)(q-1) \iff m = p+q-1.$$

because

$$(p+q-1)(p-q) + p = p^2 - q^2 + q \geq 1,$$

unless $p = q = 1$ when m cannot go below 2. \square

In case (A) the intersection of the two lines $(m+1)x - my = M$ and $x - y = p - q - 1$ is a point $(x(m), y(m))$ with $p' \geq x(m) > p' - 1$ where p' is an integer, $p' \geq p$. If $x(m) = p'$ then $y(m) = q' = p' - p + q + 1$, and $m = p' + q$. Thus $m \in \{2, 3, \dots\}$, $p' < m$, $q' \leq p'$ and $c = c(m)$, $h = h_{p', q'}(m)$.

In case (B) the intersection of the lines $x - y = p - q + 1$ and $mx - (m+1)y = M$ is a point $(x(m), y(m))$ with $p' \geq x(m) > p' - 1$, $p' - 1 \geq p$. If $x(m) = p'$ then $m = p + q'$ lies in $\{2, 3, \dots\}$, $q \leq p$, $p < m$ and $c = c(m)$, $h = h_{p, q}(m)$.

Thus to prove the theorem it suffices to establish the following proposition.

Proposition. *If case (A) or (B) obtains and $p' > x(m) > p' - 1$ then the form $\langle \cdot, \cdot \rangle$ assumes negative values in V .*

We assume the contrary and derive a contradiction. We occasionally abbreviate $c(m)$ to c and $h_{q,p}(m)$ or $h_{p,q}(m)$ to $h(m)$ or to h .

Lemma 8.

- (a) *Suppose $p' > x(m) > p' - 1$. If (p_1, q_1) lies on the boundary of $D(h, c)$ and $p_1 q_1 \leq p' q'$ then $(p_1, q_1) = (p, q)$.*
- (b) *Define m' by $p' = x(m')$ and set $c' = c(m')$, $h' = h_{q,p}(m')$ or $h_{p,q}(m')$. If (p_1, q_1) lies on the boundary of $D(h', c')$ and $p_1 q_1 \leq p' q'$ then (p_1, q_1) is (p, q) or (p', q') .*

Proof. Set $(x, y) = (x(m), y(m))$ and define z, z' as indicated by the diagrams. It clearly suffices to show that in case (A) $y - z < 2$, $z' - x < 2$, and that in case (B), $x - z < 2$, $z' - y < 2$. In case (A) elementary algebra yields $m = x + q$, $y - z = \frac{x+z}{m} = 1 + \frac{z-q}{x+q}$ and $\frac{z-q}{x+q} = \frac{x-p}{x+q} \cdot \frac{z-q}{x-p} < 1$. On the other hand $z' - x = \frac{x+y}{m} = 1 + \frac{y-q}{p+y-1} < 2$. A similar argument works for case (B). \square

Since p, q and p' are fixed it will be useful to let C denote the curve $c = c(m)$, $h = h_{q,p}(m)$ (A) or $h = h_{p,q}(m)$ (B), $m > p' - 1$.

Lemma 9.

- (a) *If $x(m) > p' - 1$, $x(m) \neq p'$, and $n_1 \leq n'$, then the dimension of the space of null vectors in V_{n_1} is $P(n_1 - n)$.*
- (b) *If $x(m) = p'$ and $n_1 < n'$ then the dimension of the space of null vectors in V_{n_1} is $P(n_1 - n)$, but if $n_1 = n'$ it is $P(n_1 - n) + 1$.*

Proof. Observe that $P(n_1 - n) = 0$ if $n_1 < n$ and that when this is so the lemma is clear. So take $n_1 \geq n$ and denote the pertinent dimension by $d_{n_1}^0$. We begin by showing that $d_{n_1}^0 > 0$ and that $d_{n_1}^0 \leq P(n_1 - n)$ unless $x(m) = p'$ and $n_1 = n'$ when $d_{n_1}^0 \leq P(n_1 - n) + 1$.

For $0 \leq c < 1$, m is locally an analytic function of c and we may write $h_{p,q}(m) = h_{p,q}(c) = h(c)$ or $h_{q,p}(m) = h_{q,p}(c) = h(c)$. Fix c and consider $H_{n_1}(h, c)$ as a function of h near $h(c)$. Its eigenvalues are the roots of a polynomial equation with real analytic, indeed polynomial, coefficients and they are all real for h real. It is easily seen that this implies that there is no ramification at $h = h(c)$ and that in a neighborhood of this point there are expansions

$$\alpha_i(h) = \alpha_{i0} + \alpha_{i1}(h - h(c)) + \alpha_{i2}(h - h(c))^2 + \cdots, \quad 1 \leq i \leq P(n_1)$$

for the eigenvalues of H_{n_1} . Thus

$$\det H_{n_1}(h, c) = \prod_{i=1}^{P(n_1)} \left(\alpha_{i0} + \alpha_{i1}(h - h(c)) + \cdots \right),$$

and the power of $h - h(c)$ that divides it is greater than or equal to the number of zero eigenvalues of $H_{n_1}(h(c), c)$. On the other hand, the left side is equal to

$$A_n \prod_{k \leq n_1} \prod_{p_1 q_1 = k} (h - h_{p_1, q_1}(c))^{P(n_1 - k)},$$

and $h_{p_1, q_1}(c) = h(c)$ only if (p_1, q_1) or (q_1, p_1) lies in the boundary of D . Thus the assertion follows from Lemma 8.

Choosing $n_1 = n$, we see in particular that the dimension of the null space of V_n is 1. Thus if $m > p' - 1$ then in a neighborhood of $(h(m), c(m))$ we can find an analytic function $v(h, c)$ with values in V_n such that $v(h, c)$ has length 1, is an eigenvector of $H_n(h, c)$, and corresponds to the eigenvalue 0 when (h, c) falls on the curve C .

Since

$$\begin{aligned} L_0 v(h(m), c(m)) &= (h(m) + n)v(h(m), c(m)), \\ L_k v(h(m), c(m)) &= 0, \quad k > 0, \end{aligned}$$

there is a homomorphism of \mathfrak{v} -modules, $\phi: V^{h(m)+n, c(m)} \rightarrow V^{h(m), c(m)}$, taking $v_\phi^{h(m)+n, c(m)}$ to $v(h(m), c(m))$. If it is injective on $V_{n_1-n}^{h(m)+n, c(m)}$ then $d_{n_1}^0 \geq P(n_1 - n)$ because the image consists of null vectors. Since $d_{n_1}^0$ is lower semicontinuous, $d_{n_1}^0$ will be greater than or equal to $P(n_1 - n)$ everywhere on C if it is so on a dense set. The homomorphism ϕ will be injective if $\det H_{n_1-n}^{h(m)+n, c(m)} \neq 0$ because the kernel consists of null vectors. So it is enough to show that this determinant does not vanish identically on C . However, if $h(m) + n = h_{p_1, q_1}(m)$ then

$$((m+1)p + mq)^2 = ((m+1)p_1 - mq_1)^2$$

or

$$(mp + (m+1)q)^2 = ((m+1)p_1 - mq_1)^2.$$

This can occur for at most two values of m . □

It remains to show that at m' the dimension of the space of null vectors in $V_{n'}$ is $P(n' - n) + 1$. For this we need further lemmas.

Lemma 10. $\det H_{n'-n}^{h(m')+n, c(m')} \neq 0$.

Proof. It has to be shown that the equality $h(m') + n = h_{p_1, q_1}(m')$, $p_1 q_1 \leq n' - n$ is impossible. This equality amounts to

$$(A) \quad (m'p + (m' + 1)q)^2 = ((m' + 1)p_1 - m'q_1)^2$$

or

$$(B) \quad ((m' + 1)p + m'q)^2 = ((m' + 1)p_1 - m'q_1)^2.$$

It is not supposed that $p_1 \geq q_1$.

The first equation implies that $m'p + (m' + 1)q = \pm((m' + 1)p_1 - m'q_1)$ or $m'(p \pm q_1) = (m' + 1)(\pm p_1 - q)$. Since m' is an integer this implies $(p \pm q_1) = a(m' + 1)$, $(\pm p_1 - q) = am'$. Since $n' = p'q' = (m' - q)(m' - p + 1)$ the inequality $n' \geq n + p_1 q_1$ becomes

$$(m' - q)(m' - p + 1) \geq a(m' + 1)q - am'p + a^2 m'(m' + 1)$$

or

$$((1 + a)(m' + 1) - p)((1 - a)m' - q) \geq 0.$$

Since $m' = p' + q = p + q' - 1$, $m' > q$, $m' + 1 > p$. So the inequality is possible only for $a = 0$, but a cannot be 0. The case (B) is treated in a similar fashion. \square

For $n_1 < n'$ or $m \neq m'$ we let $U_{n_1} = U_{n_1}(m)$ be the space of null vectors in V_{n_1} . For h, c close to $h(m')$, $c(m')$ we let $U_{n_1}(h, c)$ be the span of

$$\left\{ L_{-k_1} \cdots L_{-k_r} v(h, c) \mid k_1 \geq \cdots \geq k_r > 0, \sum k_i = n' - n \right\}.$$

We set $U_{n'}(m) = U_{n'}(h(m), c(m))$, the two definitions of $U_{n'}(m)$ coinciding when they both apply. Thus for $m > p' - 1$, $U_{n_1}(m)$ is defined and analytic as a function of m . Let W_{n_1} be its orthogonal complement with respect to the form $\{\cdot, \cdot\}$. It follows from that part of Lemma 9 already proved that the restriction $J_{n_1} = J_{n_1}(m)$ of H_{n_1} to W_{n_1} is non-singular unless $n_1 = n'$, $m = m'$. In particular, our assumption, which was made for a particular m , implies that $J_{n_1}(m)$ is positive for all $m > p' - 1$ if $n_1 < n'$.

Lemma 11. *Near m' , $\det J_{n'}(m) = \delta(m)(m - m')$ where $\frac{1}{\delta} \geq |\delta(m)| \geq \delta > 0$.*

It will follow from this lemma that the remaining assertion of Lemma 9 is true. In addition the lemma together with our assumption on the non-negativity of $\langle \cdot, \cdot \rangle$ for a particular m , $p' > x(m) > p' - 1$, will imply that the form takes negative values for $m > m'$ because $\det J_{n'}(m)$ changes sign at m' .

Let $v(h, c)$, defined in a neighborhood of $(h(m'), c(m'))$, correspond to the eigenvalue $\alpha(h, c)$ of $H_n(h, c)$. All the other eigenvalues of $H_n(h, c)$ are bounded above and, if the neighborhood is sufficiently small, away from 0. On the other hand, all factors $h - h_{p_1, q_1}(c) = h - h_{p_1, q_1}(m)$, $c = c(m)$, of $\det H_n(h, c)$ are bounded away from 0 in a neighborhood of $h(m')$, $c(m')$ except for $h - h(c)$, where $h(c)$ is $h_{q, p}(c)$ or $h_{p, q}(c)$ according as we are dealing with case A or case B. Thus we have the following lemma.

Lemma 12. *In a neighborhood of $(h(m'), c(m'))$ we have $\alpha(h, c) = a(h, c)(h - h(c))$ with $\frac{1}{a} \geq |a(h, c)| \geq a > 0$, a being a constant.*

Here $h(c)$ is $h_{q, p}(m)$ (A) or $h_{p, q}(m)$ (B), $c = c(m)$. More generally we have

Lemma 13. *Let $K_{n'}(h, c)$ be the restriction of $H_{n'}(h, c)$ to $U_{n'}(h, c)$. Then, in a neighborhood of $(h(m'), c(m'))$, $\det K_{n'}(h, c) = k(h, c)\alpha(h, c)^{P(n' - n)}$, with $\frac{1}{k} \geq |k(h, c)| \geq k > 0$.*

Proof. The determinant of $K_{n'}(h, c)$ is that of the form $\langle \cdot, \cdot \rangle_{n'}$, calculated with respect to a basis of $U_{n'}(h, c)$ orthogonal with respect to the form $\{ \cdot, \cdot \}_n$. However the basis

$$\left\{ \phi(v_\alpha) \mid v_\alpha \in V^{h(m)+n, c(m)}, n(\alpha) = n' - n \right\}$$

is related to such a basis by a matrix whose determinant is bounded in absolute value above and below. So it is enough to consider $\det\left(\{\phi(v_\alpha), \phi(v_\beta)\}\right)$.

We have

$$\begin{aligned} \langle \phi(v_\alpha), \phi(v_\beta) \rangle &= \langle L_{\ell_s} \cdots L_{\ell_1} L_{-k_1} \cdots L_{-k_r} v(h, c), v(h, c) \rangle \\ &= \{ L_{\ell_s} \cdots L_{\ell_1} L_{-k_1} \cdots L_{-k_r} v(h, c), H_{n'}(h, c) v(h, c) \} \\ &= \alpha(h, c) \{ L_{\ell_s} \cdots L_{\ell_1} L_{-k_1} \cdots L_{-k_r} v(h, c), v(h, c) \}. \end{aligned}$$

At $h(m), c(m)$ the value of $\det\left(\{ L_{\ell_s} \cdots L_{\ell_1} L_{-k_1} \cdots L_{-k_r} v(h, c), v(h, c) \}\right)$ is

$$\det\left(\langle v_\alpha, v_\beta \rangle_{n'-n}^{h(m)+n, c(m)}\right).$$

By Lemma 10 this is not 0. Lemma 13 follows. \square

In a neighborhood of $h(m), c(m)$ we decompose $V_{n'}$ as an orthogonal sum $U_{n'} \oplus W_{n'}$. The linear transformation $H_{n'}(h, c)$, or its matrix with respect to a compatible basis, then decomposes into blocks. I claim that the entries in the off-diagonal blocks are $O(h - h_{p,q}(c))$ in a neighborhood of $h(m), c(m)$. To verify this it is sufficient, for the pertinent basis can be supposed to depend analytically on h, c , to verify that they are zero when $h = h_{p,q}(c)$, but that is clear by the definition of $U_{n'}$.

It follows that

$$(1) \quad \det H_{n'}(h, c) = \det J_{n'}(h, c) \det K_{n'}(h, c) + O\left((h - h_{p,q}(c))^{P(n'-n)+1}\right)$$

if $J_{n'}(h, c)$ is the matrix in the diagonal block corresponding to $W_{n'}$. Since

$$\det H_{n'}(h, c) = A_{n'} \prod_{k \leq n'} \prod_{p_1 q_1 = k} (h - h_{p_1 q_1}(c))^{P(n' - p_1 q_1)}$$

we may divide the relation (1) by $(h - h_{p_1 q_1}(c))^{P(n'-n)}$ and then set $h = h_{p,q}(c), c = c(m)$. The result clearly yields Lemma 11 because $h(m') = h_{p_1, q_1}(m'), p_1, q_1 \leq n'$, only if (p_1, q_1) is (q, p) or (p', q') (case A) or (p, q) or (q', p') (case B).

Our assumption that $H_{n_1}(h(m), c(m))$ is non-negative for a given $m, p' > m > p' - 1$, has led to the conclusion that $J_{n_1}(m)$ is positive for large m and $n_1 < n'$ but that $J_{n'}(m)$ has negative eigenvalues for large m . We show not that this is impossible.

As m approaches infinity, the point $(h(m), c(m))$ approaches $(h_0, c_0) = \left(\frac{(p-q)^2}{4}, 1\right)$. If $p \neq q$ a suitable coordinate on the curve is $\mu = \frac{1}{m}$. If $p = q$ we may take $\mu = 1 - c$. All the matrices $H_{n_1}(\mu) = H_{n_1}(m) = H_{n_1}(h(m), c(m))$ are analytic functions of μ . The eigenvalues of $H_{n_1}(\mu)$ are given by power series.

$$\alpha_i = \alpha_i(\mu) = \alpha_{i0} + \alpha_{i1}\mu + \alpha_{i2}\mu^2 + \cdots$$

Let $V_{n_1}^1(\mu)$ be the space spanned by the eigenvectors corresponding to α_i with $\alpha_{i0} = 0$; let $V_{n_1}^2(\mu)$ be the space spanned by the eigenvectors corresponding to α_i with $\alpha_{i0} = \alpha_{i1} = 0$ and so on. One proves by induction that these spaces are well-defined, depend analytically on μ (in

the sense that we have analytic functions $v_1(\mu), \dots, v_{P(n_1)}(\mu)$, such that $\{v_1(\mu), \dots, v_{d_k}(\mu)\}$, $d_k = \dim V_{n_1}^k$ forms a basis of $V_{n_1}^k(\mu)$ for each μ , and that $\mu^{-k} \{H_{n_1}(\mu)v_i(\mu), v_j(\mu)\}$, $i \leq d_k, j \leq P(n_1)$ is analytic for small μ . It can even be supposed that $\{H_{n_1}(\mu)v_i(\mu), v_j(\mu)\} = 0$, $i \leq d_k, j > d_k$.

Let $V^k = \bigoplus_{n_1} V_{n_1}^k(0)$ and $X^k = V^k/V^{k+1} = \bigoplus_{n_1} V_{n_1}^k(0)/V_{n_1}^{k+1}(0)$. If $u = \sum_{i \leq d_k} a_i v_i(0) \in V_{n_1}^k(0)$ and $v = \sum_{i \leq d_k} b_i v_i(0) \in V_{n_2}^k(0)$, define $\langle u, v \rangle^{(k)}$ to be 0 if $n_1 \neq n_2$, and if $n_1 = n_2$ set

$$\begin{aligned} \langle u, v \rangle^{(k)} &= \langle u, v \rangle_{n_1}^{(k)} = \sum a_i \bar{b}_j \lim_{\mu \rightarrow 0} \mu^{-k} \langle v_i(\mu), v_j(\mu) \rangle \\ &= \sum a_i \bar{b}_j \lim_{\mu \rightarrow 0} \left\{ \mu^{-k} H_{n_1}(\mu) v_i(\mu), v_j(\mu) \right\}. \end{aligned}$$

It is clear that $H_{n_1}(\mu)$ is non-negative for small μ if and only if the forms $\langle u, v \rangle_{n_1}^{(k)}$ are all positive.

Lemma 14.

- (a) The spaces V^k are all invariant under $\pi = \pi^{h_0, c_0}$, so that \mathfrak{v} operates on X^k .
- (b) The form $\langle \cdot, \cdot \rangle^{(k)}$ on X^k satisfies $\langle L_m x, y \rangle = \langle x, L_{-m} y \rangle$, $m \in \mathbf{Z}$.

Proof. Set $L_m(\mu) = \pi^{h(\mu), c(\mu)}(L_m)$ and $L_m = L_m(0)$. We have to show for each n_1 that $L_m v_i \in V^k$ if $v_i = v_i(0)$ and $i \leq d_k$. However

$$L_m v_i = \lim_{\mu \rightarrow 0} L_m(\mu) v_i(\mu) = \lim_{\mu \rightarrow 0} \sum_j c_{ij}(\mu) v'_j(\mu)$$

where the c_{ij} are analytic functions of μ . It is to be shown that $c_{ij}(0) = 0$ for $j > d'_k$. The primes refer to $n_2 = n_1 - m$ rather than to n_1 . In other words it has to be shown that

$$\{H_{n_2}(\mu) L_m(\mu) v_i(\mu), v'_\ell(\mu)\} = O(\mu^k)$$

for all ℓ . Since $H_{n_2}(\mu) L_m(\mu) = L_{-m}^*(\mu) H_{n_1}(\mu)$, the adjoint of $L_{-m}(\mu)$ being taken with respect to the form $\{\cdot, \cdot\}$, this is clear. So is the second assertion of the lemma. \square

For any $h \geq 0$ the representation $\pi^{h,1}$ on $V^{h,1}$ has a unique irreducible quotient $\rho^{h,1}$ on $X^{h,1}$, which by Lemma 3 carries a hermitian form for which $\rho^{h,1}$ is unitary in the sense that the adjoint $\rho^{h,1}(L_m)$ is $\rho^{h,1}(L_{-m})$. Such a form is unique up to a scalar multiple. Take in particular $h = \frac{r^2}{4}$, $r \in \mathbf{Z}$. Then $h = h_{p_2, q_2}(c)$ if and only if $(p_2 - q_2)^2 = r^2$. In particular, $h = h_{r+1, 1}(c)$. Thus the lowest weight for a null vector in V is $r + 1$ and $h + r + 1 = \frac{(r+2)^2}{4}$, so that the kernel of $V^{h,1} \rightarrow X^{h,1}$ contains a quotient of $V^{h',1}$, $h' = \frac{(r+2)^2}{4}$. Thus $V^{h,1}$ admits a sequence of invariant subspaces $V^{h,1} = V^{h,1}(0) \supseteq V^{h,1}(1) \supseteq V^{h,1}(2)$ such that the representation on $V^{h,1}(0)/V^{h,1}(1)$ is $\rho^{h,1}$ and that on $V^{h,1}(1)/V^{h,1}(2)$ is $\rho^{h',1}$. In general set $h^{(\ell)} = \frac{1}{4}(r + 2\ell)^2$.

Lemma 15. $V^{h,1}$ admits an infinite decomposition series $V^{h,1}(0) \supseteq V^{h,1}(1) \supseteq \dots \supseteq V^{h,1}(\ell) \supseteq \dots$ such that the representation on the quotient $V^{h,1}(\ell)/V^{h,1}(\ell + 1)$ is $\rho^{h^{(\ell)},1}$.

Proof. If $\lambda = h + k$, $k \in \mathbf{Z}$, $k \geq 0$, let

$$\begin{aligned} d_\lambda &= \dim \left\{ v \in V^{h,1} \mid L_0 v = \lambda v \right\}, \\ d_\lambda(\ell) &= \dim \left\{ v \in X^{h^{(\ell)},1} \mid L_0 v = \lambda v \right\}. \end{aligned}$$

The lemma follows easily from a formula of Kac ([2], Th. 5), according to which $d_\lambda = \sum_{\ell=0}^{\infty} d_\lambda(\ell)$. Indeed, suppose we have constructed an initial segment of the series $V^{h,1}(0) \supset \dots \supset V^{h,1}(\ell)$. Then $\frac{1}{4}(r+2\ell)^2$ is a lowest weight in $V^{h,1}(\ell)$ and

$$\dim \left\{ v \in V^{h,1}(\ell) \mid L_0 v = \frac{1}{4}(r+2\ell)^2 v \right\} = 1.$$

Take $V^{h,1}(\ell+1)$ to be the sum of all invariant subspaces of $V^{h,1}(\ell)$ for which the lowest weight is greater than $\frac{1}{4}(r+2\ell)^2$. \square

Now take $r = p - q$. It follows immediately from the preceding lemma that X^k is the direct sum of irreducible invariant subspaces X_j^k carrying distinct representations and that the restriction of $\langle \cdot, \cdot \rangle^k$ to X_j^k is either positive or negative. The assumption that we are trying to contradict implies that the form is positive if X_j^k contains non-zero vectors of weight $h + n_1, n_1 < n'$, but that for some j and k for which X_j^k contains vectors of weight $h + n'$, it is negative.

Thus the following lemma completes the proof of Theorem FQS.

Lemma 16. *The equation $\frac{r^2}{4} + n' = \frac{1}{4}(r+2\ell)^2$ has no solution $\ell \geq 0$ in \mathbf{Z} .*

Proof. The equation may be written as $n' = \ell(\ell+r)$. Recall that n' is $(p+a)(q+a+1)$ in case A and $(p+a+1)(q+a)$ in case B, with $a \geq 0$. Since $r = p - q$, the equation is $(p+a+\ell)(q+a+1-\ell) = \ell$ or $(p+a+1+\ell)(q+a-\ell) = -\ell$. Both equations are manifestly impossible. \square

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