

# ON UNITARY REPRESENTATIONS OF THE VIRASORO ALGEBRA

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The Virasoro algebra  $\mathfrak{v}$  is an infinite-dimensional Lie algebra with basis  $L_m$ ,  $m \in \mathbf{Z}$ , and  $Z$  and defining relations:

- (i)  $[L_m, L_n] = (m - n)L_{m+n} + \frac{m(m^2-1)}{12}\delta_{m,-n}Z$ ;
- (ii)  $[L_m, Z] = 0$ .

Some representations  $\pi$  of  $\mathfrak{v}$  of particular interest [2] are the Verma modules  $(V, \pi) = (V^{h,c}, \pi^{h,c})$ ,  $h, c \in \mathbf{R}$ . They are characterized by the following conditions.

- (i) There is a vector  $v = v_\phi \neq 0$  in  $V$  such that  $L_nv = 0$ ,  $n > 0$ ,  $L_0v = hv$ ,  $Zv = cv$ .
- (ii) Let  $A$  be the set of sequences of integers  $k_1 \geq k_2 \geq \dots \geq k_r > 0$  of arbitrary length, and if  $\alpha \in A$  let  $v_\alpha = \pi(L_{-k_1}) \dots \pi(L_{-k_r})v_\phi$ . Then  $\{v_\alpha \mid \alpha \in A\}$  is a basis of  $V$ .

Observe that  $V$  is just the free vector space with basis  $\{v_\alpha\}$  and is thus independent of  $h$  and  $c$ . It is easy to see [1] that there is a unique sesquilinear form  $\langle u, v \rangle = \langle u, v \rangle^{h,c}$  on  $V$  with the properties:

- (i)  $\langle v_\phi, v_\phi \rangle = 1$ ;
- (ii)  $\langle u, v \rangle = \overline{\langle v, u \rangle}$ ;
- (iii)  $\langle \pi(L_m)u, v \rangle = \langle u, \pi(L_{-m})v \rangle$ ,  $m \in \mathbf{Z}$ .

If this form is non-negative then the representation  $\rho$  of  $\mathfrak{v}$  on the quotient of  $V$  by the space of null vectors is unitary, in the sense that

$$\rho(L_m)^* = \rho(L_{-m}).$$

**Theorem FQS.** *The form  $\langle \cdot, \cdot \rangle_{h,c}$  is non-negative only if either  $c \geq 1$ ,  $h \geq 0$  or there exists an integer  $m \geq 2$  and two integers  $p, q$ ,  $1 \leq p < m$ ,  $1 \leq q \leq p$ , such that*

$$c = 1 - \frac{6}{m(m+1)}, \quad h = \frac{((m+1)p - mq)^2 - 1}{4m(m+1)}.$$

This theorem has been proven by Friedan-Qiu-Shenker [1]. The sketch of the proof that they provided was unconscionably brief, and has evoked some scepticism among mathematicians. In this note, which grew out of a series of lectures at the Centre de recherches mathématiques that overlapped the workshop, details are worked out. In the meantime, Friedan, Qiu and Shenker have themselves provided them [3], but the present account, which turns out to diverge from theirs in some respects, may still be a useful supplement to it. Several other authors have proven that the conditions of the theorem are not only necessary but also sufficient for non-negativity, but that is not the concern here.

The proof proceeds by lemmas. I write  $Lv$  rather than  $\pi(L)v$ ,  $L \in \mathfrak{v}$ ,  $v \in V$ .

**Lemma 1.** *If  $\langle \cdot, \cdot \rangle$  is non-negative then  $h \geq 0$ ,  $c \geq 0$ .*

*Proof.* Since  $L_n L_{-n} v_\phi = L_{-n} L_n v_\phi + 2nhv_\phi + \frac{n(n^2-1)}{12}cv_\phi$ , we have  $\langle L_{-n}v_\phi, L_{-n}v_\phi \rangle = 2nh + \frac{n(n^2-1)}{12}c$ . Taking  $n$  first equal to 1 and then very large we obtain the lemma.  $\square$

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For arbitrary  $m$  we set  $c = c(m) = 1 - \frac{6}{m(m+1)}$ ,  $h_{p,q} = h_{p,q}(m) = \frac{((m+1)p-mq)^2 - 1}{4m(m+1)}$ ,  $p, q \in \mathbf{N}$ . Observe that  $c(-1-m) = c(m)$  and that  $h_{p,q}(-1-m) = h_{q,p}(m)$ .

**Lemma 2.**

- (a) For  $1 < c < 25$ ,  $m$  is not real and neither is  $h_{p,q}(m)$  unless  $p = q$ .
- (b) As  $m$  runs from 2 to  $\infty$ ,  $c$  increases monotonically from 0 to 1.
- (c) For  $c > 1$ ,  $-1 < m < 0$ .
- (d) If  $-1 < m < 0$  then  $h_{p,q}(m) < 0$  unless  $p = q = 1$  when  $h_{p,q}(m) = 0$ .
- (e) If  $p = q$  then  $h_{p,q}(m) = \frac{p^2-1}{24}(1-c)$ .
- (f) If  $p \neq q$  then  $h_{p,q} + h_{q,p} = \frac{p^2+q^2-2}{24}(1-c) + \frac{(p-q)^2}{2}$ . In addition  $h_{p,q}h_{q,p}$  is equal to

$$\begin{aligned} & \frac{(p^2q^2 - p^2 - q^2 + 1)}{16 \cdot 36} (1-c)^2 \\ & + \frac{(2p^2q^2 - pq(p^2 + q^2) - (p-q)^2)}{48} (1-c) \\ & + \frac{(p^4 + q^4 - 4p^3q - 4pq^3 + 6p^2q^2)}{16}. \end{aligned}$$

*Proof.* The first four parts of the lemma are clear, and the last two are straightforward calculations.  $\square$

There is a second sesquilinear form on  $V$  defined by  $\{v_\alpha, v_\beta\} = \delta_{\alpha\beta}$ . If  $V_n$  is the subspace of  $V$  with basis  $\{v_\alpha \mid \alpha = (k_1, \dots, k_r) \mid \sum_{i=1}^r k_i = n\}$ , then  $V = \bigoplus_{n \geq 0} V_n$  and the spaces  $V_n$  are mutually orthogonal with respect to both forms. The first form is defined on  $V_n$  with respect to the second by a hermitian linear transformation  $H_n = H_n(h, c) : \langle u, v \rangle_n = \{H_n u, v\}_n$ . Let  $P(n)$  be the dimension of  $V_n$ . It is the number of partitions of  $n$ . The Kac determinant formula (cf. [1]) is the key to the proof of Theorem FQS.

**Kac determinant formula.** If  $c = c(m)$  then

$$\det H_n(h, c) = A_n \prod_{k \leq n} \prod_{pq=k} (h - h_{p,q})^{P(n-k)},$$

where  $A_n$  is a positive constant.

**Lemma 3.** The form  $\langle \cdot, \cdot \rangle_n$  is non-negative for  $h \geq 0$ ,  $c \geq 1$ .

*Proof.* By continuity it suffices to treat pairs for which  $h > 0$ ,  $c > 1$ . Since the previous lemma implies that  $\det H_n(h, c)$  is nowhere zero in this region, it suffices to prove that the form is positive for one pair  $(h, c)$ . If  $\alpha = (k_1, \dots, k_r)$ ,  $r = r(\alpha)$ ,  $n(\alpha) = k_1 + \dots + k_r$ , set  $v'_\alpha = L_{-k_r} \cdots L_{-k_1} v_\phi$ . It is generally different than  $v_\alpha$ . It clearly suffices to show that for a given  $c$  and  $h$  large,

$$(3.1) \quad \langle v'_\alpha, v'_\alpha \rangle = c_\alpha h^{r(\alpha)} (1 + o(1)), \quad c_\alpha > 0$$

$$(3.2) \quad \langle v'_\alpha, v'_\beta \rangle = o\left(h^{(r(\alpha)+r(\beta))/2}\right), \quad \alpha \neq \beta.$$

This is proved by induction on  $n(\alpha) + n(\beta)$ . First of all  $L_k^a L_{-k}^a$  is equal to

$$L_k^{a-1} (bL_0 + d) L_{-k}^{a-1} + L_k^{a-1} L_{-k} L_k L_{-k}^{a-1}, \quad b > 0.$$

Moving the single  $L_k$  in the second term ever further to the right, we obtain finally

$$L_k^a L_{-k}^a = L_k^{a-1} (bL_0 + d) L_{-k}^{a-1} + L_k^{a-1} L_{-k}^a L_k, \quad b > 0.$$

Take  $k_1 \geq k_2 \geq \dots \geq k_r > k$ . If  $\alpha = (k_1, \dots, k_r, k, \dots, k)$ , then

$$\begin{aligned} \langle v'_\alpha, v'_\alpha \rangle &= \langle L_{k_1} \cdots L_{k_r} L_k^a L_{-k}^a L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi \rangle \\ &= c_{k,a} h (1 + o(h)) \langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a-1} L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi \rangle \\ &\quad + \langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^a L_k L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi \rangle \end{aligned}$$

with  $c_{k,a} > 0$ . In the second term we move the  $L_k$  further and further to the right obtaining the sum of

$$(k + k_r) \langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^a L_{-k_r} \cdots L_{-k_{j+1}} L_{-(k_j-k)} L_{-k_{j-1}} \cdots L_{-k_1} v_\phi, v_\phi \rangle.$$

The induction assumption together with the defining relations for  $\mathfrak{v}$  implies readily that each of these terms is  $o(h^{r(\alpha)})$  and that

$$\langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a-1} L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi \rangle = \langle v'_\gamma, v'_\gamma \rangle = c_\gamma h^{r(\gamma)} (1 + o(1)),$$

if  $\gamma = (k_1, \dots, k_r, k, \dots, k)$ , with  $k$  repeated  $a - 1$  times, so that  $r(\alpha) = 1 + r(\gamma)$ .

On the other hand, if  $\beta = (\ell_1, \dots, \ell_s, k, \dots, k)$ , with  $k$  repeated  $a' \leq a$  times,  $a > 0$ ,  $a' \geq 0$ ,  $\ell_s \geq k$  even if  $a' = 0$ , then

$$\langle v'_\beta, v'_\alpha \rangle = \langle L_{k_1} \cdots L_{k_r} L_k^a L_{-k}^{a'} L_{-\ell_s} \cdots L_{-\ell_1} v_\phi, v_\phi \rangle$$

is equal to the sum of

$$c_{k,a'} h (1 + o(1)) \langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a'-1} L_{-\ell_s} \cdots L_{-\ell_1} v_\phi, v_\phi \rangle$$

and

$$\sum_j (k + \ell_j) \langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a'} L_{-\ell_s} \cdots L_{-\ell_{j+1}} L_{-(\ell_j-k)} L_{-\ell_{j-1}} \cdots L_{-\ell_1} v_\phi, v_\phi \rangle.$$

We take  $c_{k,0} = 0$  if  $a' = 0$ . So induction yields (3.2).  $\square$

Observe that if  $m > 0$  and  $p > q$  then  $h_{p,q} > h_{q,p}$ . If  $h \geq 0$  and  $m > 0$  define  $M > 0$  by  $M^2 = 1 + 4m(m+1)h$ . Then  $M \geq 1$ . Let  $D$  be the closed shaded region in the diagram I. It is bounded by the lines

$$mx - (m+1)y = \pm M \text{ and } (m+1)x - my = M.$$

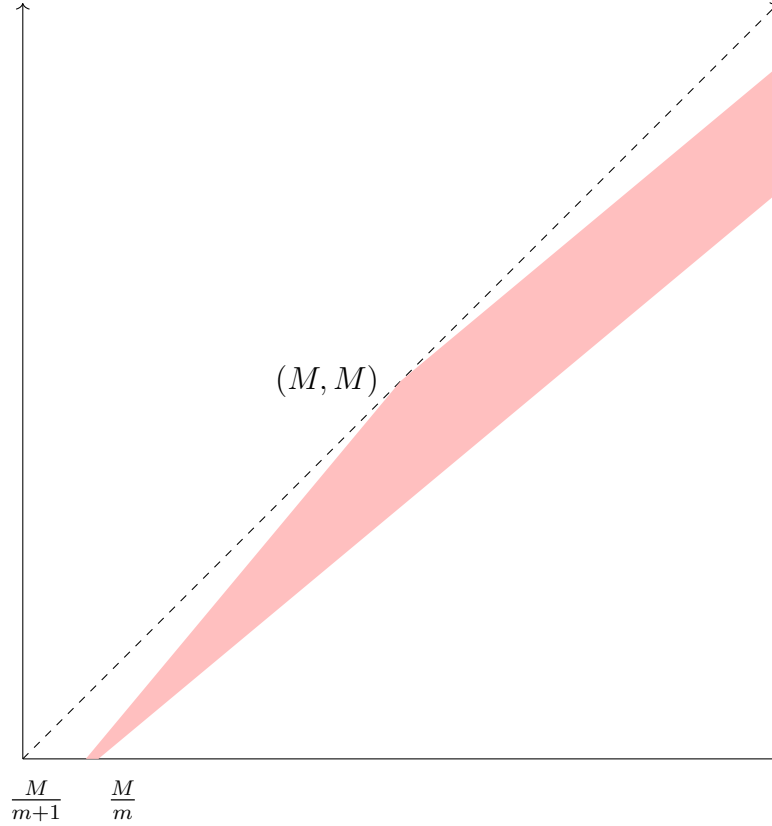


Diagram I

**Lemma 4.**

- (a)  $h_{p,q} \geq h \geq h_{q,p}$  if and only if  $(p, q) \in D$ .  
 (b)  $D$  contains an integral point  $(p, q)$  with  $q > 0$ .

*Proof.* Since  $h_{p,q} \geq h$  if and only if  $((m+1)p - mq)^2 \geq M^2$  and  $h \geq h_{q,p}$  if and only if  $((m+1)q - mp)^2 \leq M^2$ , the first statement of the lemma is clear. For the second choose a large integer  $p$  and let  $a = \frac{mp-M}{m+1}$ . Then the points  $(p, q)$  with  $a \leq q \leq a + \frac{2M}{m+1}$  lie in  $D$ . So do the points  $(p+1, q)$ ,  $a + \frac{m}{m+1} \leq q \leq a + \frac{m+2M}{m+1}$  and so on. So we need only show that one of the intervals  $\left[ a + \frac{km}{m+1}, a + \frac{km+2M}{m+1} \right]$ ,  $k \in \mathbf{Z}$ ,  $k \geq 0$ , contains an integer. This is clear if  $\frac{m}{m+1}$  is irrational. Otherwise, increasing  $q$  if necessary, we may suppose that  $a$  is as close to its integral part as any  $a + \frac{km}{m+1}$ . Then  $a + \frac{m}{m+1} < [a] + 1$ , but  $a + \frac{m+2M}{m+1} \geq a + \frac{m+2}{m+1} > [a] + 1$ , and the interval  $\left[ a + \frac{m}{m+1}, a + \frac{m+2M}{m+1} \right]$  contains  $[a] + 1$ .  $\square$

Let  $p(h, c) = \min_{(p,q) \in D} p$  and let  $q(h, c) = \min_{(p,q) \in D} q$ . It is clear that

$$P(h, c) = (p(h, c), q(h, c)) \in D.$$

In the following geometrical arguments, it is sometimes necessary to recall that  $h - h_{p_0, p_0} < 0$  if and only if  $p_0 > M$ .

**Lemma 5.** *If  $P(h, c)$  lies in the interior of  $D$  then  $\langle v, v \rangle$  assumes negative values in  $V$ .*

*Proof.* Let  $(p, q) = P(h, c)$  and let  $n = pq$ . If  $p_0 q_0 \leq n$ ,  $p_0 \geq q_0$  and  $(p_0, q_0) \neq (p, q)$  then either  $p_0 < p$  or  $q_0 < q$  so that  $(p_0, q_0) \notin D$ . In general set

$$\begin{aligned}\phi_{p_0, q_0} &= (h - h_{p_0, q_0})(h - h_{q_0, p_0}), & p_0 &\neq q_0, \\ &= h - h_{p_0, q_0}, & p_0 &= q_0.\end{aligned}$$

If  $(p_0, q_0) \notin D$  and  $p_0 \neq q_0$  then  $\phi_{p_0, q_0} > 0$ .

Suppose that for some  $p_0$  with  $p_0^2 \leq pq$  we had  $h - h_{p_0, p_0} < 0$ . Then there would be a minimum such  $p_0$  and if  $n_0 = p_0^2$  then

$$\det H_{n_0} = A_{n_0} \prod_{\substack{p_1 \geq q_1 \\ n_1 = p_1 q_1 \leq n_0}} \phi_{p_1, q_1}^{P(n_0 - n_1)}$$

Since  $P(h, c)$  lies in the interior of  $D$ ,  $p \neq q$  and none of the pairs  $(p_1, q_1)$  that intervene here lie in  $D$ . Moreover, all terms of the products are positive save  $\phi_{p_0, p_0}^{P(0)} = \phi_{p_0, p_0}$ . Since this is negative,  $\langle \cdot, \cdot \rangle$  assumes negative values on  $V_{n_0}$ .

If, however,  $\phi_{p_0, p_0} > 0$  for all  $p_0 \leq q$  then the same argument shows that  $\det H_n < 0$ , so that  $\langle \cdot, \cdot \rangle$  assumes negative values on  $V_n$ .  $\square$

The treatment of those points  $(h, c)$  for which  $P(h, c)$  lies on the boundary of  $D$  is more delicate. There are at first three possibilities for  $(p, q) = P(h, c)$ :

- (A)  $mp - (m + 1)q = M$ ;
- (B)  $(m + 1)p - mq = M$ ;
- (C)  $mp - (m + 1)q = -M$ ,  $p \neq q$ ;

**Lemma 6.** *The case (C) above does not occur.*

*Proof.* It is clear from the diagram defining  $D$  that in case (C),  $p \geq M$ ,  $q \geq M$ . If  $q = 1$  then  $M = 1$  and  $p = 1$ , so that we have rather case (B). If  $q > 1$  then  $p > 1$  and  $(m + 1)(q - 1) - m(p - 1) = (m + 1)q - mp - 1$ , so that  $M > (m + 1)(q - 1) - m(p - 1) > -M$ . Moreover,  $(m + 1)(p - 1) - m(q - 1) - M = (m + 1)(p - 1 - q) - m(q - 1 - p) = (2M + 1)(p - q) - 1$ . Since  $m \geq 2$  this is positive if  $p \neq q$ . Consequently  $(p - 1, q - 1) \in D$ , and this is a contradiction.  $\square$

Fix  $(p, q)$ . In case (A) we have  $h = h_{q, p}(m)$ ,  $c = c(m)$ . In case (B) we have  $h = h_{p, q}(m)$ ,  $c = c(m)$ .

**Lemma 7.**

- (a) *The set of all  $m \geq 2$  for which  $h = h_{q, p}(m)$ ,  $c = c(m)$  yields case (A) is the interval  $m > q + p - 1$ .*
- (b) *The set of all  $m \geq 2$  for which  $h = h_{p, q}(m)$ ,  $c = c(m)$  yields case (B) is the interval  $m > q + p - 1$  if  $(p, q) \neq (1, 1)$  and is the interval  $m \geq 2$  if  $(p, q) = (1, 1)$ .*

It will be helpful, when proving this and the following lemmas, to keep the diagrams IIA and IIB in mind.

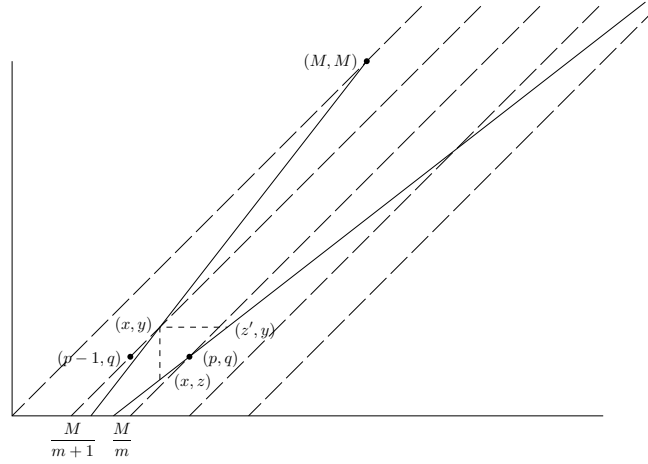


Diagram IIA

*Proof.* We first show that if  $h_{q,p}(m_0), c(m_0)$  yield case (A) then so does  $(h_{q,p}(m), c(m))$  for  $m \geq m_0$ . It is clear from the diagram that it is sufficient to verify that  $M, \frac{M}{m+1}$ , and  $\frac{M}{m}$  are increasing functions of  $m$ . But  $M = m(p - q) - q$ ,  $\frac{M}{m} = (p - q) - \frac{q}{m}$ ,  $\frac{M}{m+1} = (p - q) - \frac{p}{m+1}$ . It is also clear that

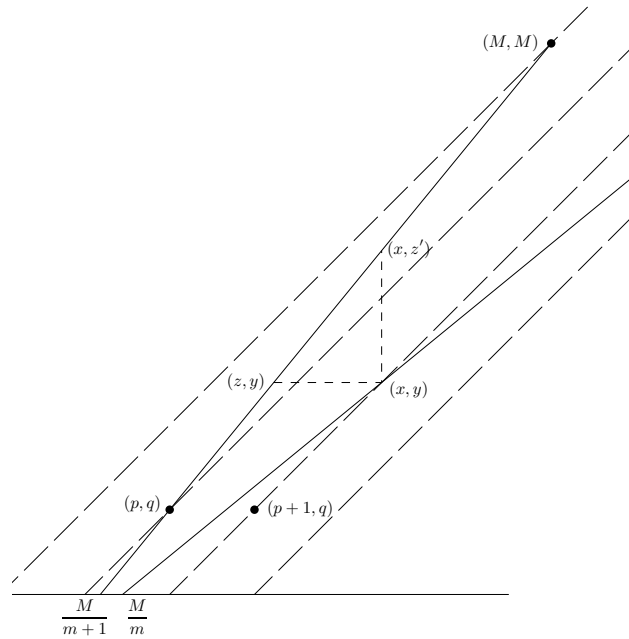


Diagram IIB

we can decrease  $m$  without passing out of case (A) so long as  $M = m(p - q) - q$  remains greater than or equal to 1 and  $(m + 1)(p - 1) - mq > mp - (m + 1)q$ . But

$$(m + 1)(p - 1) - mq = mp - (m + 1)q \iff m = p + q - 1.$$

As we decrease to these points,  $M$  decreases to

$$(p + q - 1)(p - q) - q = p^2 - q^2 - p = (p - 1)^2 - q^2 + p - 1.$$

This number is greater than 1 because  $p > q \geq 1$ .

For case (B),  $M = m(p - q) + p$  is a non-decreasing function of  $m$ , and  $\frac{M}{m} = (p - q) + \frac{p}{m}$ ,  $\frac{M}{m+1} = (p - q) + \frac{q}{m+1}$  are decreasing functions. Since the slope of  $mp - (m + 1)q = M$  is  $1 - \frac{1}{m+1}$ , it is increasing and the conclusion is the same. The minimal value of  $m$  is given by

$$(m + 1)p - mq = mp - (m + 1)(q - 1) \iff m = p + q - 1.$$

because

$$(p + q - 1)(p - q) + p = p^2 - q^2 + q \geq 1,$$

unless  $p = q = 1$  when  $m$  cannot go below 2.  $\square$

In case (A) the intersection of the two lines  $(m + 1)x - my = M$  and  $x - y = p - q - 1$  is a point  $(x(m), y(m))$  with  $p' \geq x(m) > p' - 1$  where  $p'$  is an integer,  $p' \geq p$ . If  $x(m) = p'$  then  $y(m) = q' = p' - p + q + 1$ , and  $m = p' + q$ . Thus  $m \in \{2, 3, \dots\}$ ,  $p' < m$ ,  $q' \leq p'$  and  $c = c(m)$ ,  $h = h_{p', q'}(m)$ .

In case (B) the intersection of the lines  $x - y = p - q + 1$  and  $mx - (m + 1)y = M$  is a point  $(x(m), y(m))$  with  $p' \geq x(m) > p' - 1$ ,  $p' - 1 \geq p$ . If  $x(m) = p'$  then  $m = p + q'$  lies in  $\{2, 3, \dots\}$ ,  $q \leq p$ ,  $p < m$  and  $c = c(m)$ ,  $h = h_{p, q}(m)$ .

Thus to prove the theorem it suffices to establish the following proposition.

**Proposition.** *If case (A) or (B) obtains and  $p' > x(m) > p' - 1$  then the form  $\langle \cdot, \cdot \rangle$  assumes negative values in  $V$ .*

We assume the contrary and derive a contradiction. We occasionally abbreviate  $c(m)$  to  $c$  and  $h_{q, p}(m)$  or  $h_{p, q}(m)$  to  $h(m)$  or to  $h$ .

**Lemma 8.**

- (a) *Suppose  $p' > x(m) > p' - 1$ . If  $(p_1, q_1)$  lies on the boundary of  $D(h, c)$  and  $p_1 q_1 \leq p' q'$  then  $(p_1, q_1) = (p, q)$ .*
- (b) *Define  $m'$  by  $p' = x(m')$  and set  $c' = c(m')$ ,  $h' = h_{q, p}(m')$  or  $h_{p, q}(m')$ . If  $(p_1, q_1)$  lies on the boundary of  $D(h', c')$  and  $p_1 q_1 \leq p' q'$  then  $(p_1, q_1)$  is  $(p, q)$  or  $(p', q')$ .*

*Proof.* Set  $(x, y) = (x(m), y(m))$  and define  $z, z'$  as indicated by the diagrams. It clearly suffices to show that in case (A)  $y - z < 2$ ,  $z' - x < 2$ , and that in case (B),  $x - z < 2$ ,  $z' - y < 2$ . In case (A) elementary algebra yields  $m = x + q$ ,  $y - z = \frac{x+z}{m} = 1 + \frac{z-q}{x+q}$  and  $\frac{z-q}{x+q} = \frac{x-p}{x+q} \cdot \frac{z-q}{x-p} < 1$ . On the other hand  $z' - x = \frac{x+y}{m} = 1 + \frac{y-q}{p+y-1} < 2$ . A similar argument works for case (B).  $\square$

Since  $p, q$  and  $p'$  are fixed it will be useful to let  $C$  denote the curve  $c = c(m)$ ,  $h = h_{q, p}(m)$  (A) or  $h = h_{p, q}(m)$  (B),  $m > p' - 1$ .

**Lemma 9.**

- (a) *If  $x(m) > p' - 1$ ,  $x(m) \neq p'$ , and  $n_1 \leq n'$ , then the dimension of the space of null vectors in  $V_{n_1}$  is  $P(n_1 - n)$ .*
- (b) *If  $x(m) = p'$  and  $n_1 < n'$  then the dimension of the space of null vectors in  $V_{n_1}$  is  $P(n_1 - n)$ , but if  $n_1 = n'$  it is  $P(n_1 - n) + 1$ .*

*Proof.* Observe that  $P(n_1 - n) = 0$  if  $n_1 < n$  and that when this is so the lemma is clear. So take  $n_1 \geq n$  and denote the pertinent dimension by  $d_{n_1}^0$ . We begin by showing that  $d_{n_1}^0 > 0$  and that  $d_{n_1}^0 \leq P(n_1 - n)$  unless  $x(m) = p'$  and  $n_1 = n'$  when  $d_{n_1}^0 \leq P(n_1 - n) + 1$ .

For  $0 \leq c < 1$ ,  $m$  is locally an analytic function of  $c$  and we may write  $h_{p,q}(m) = h_{p,q}(c) = h(c)$  or  $h_{q,p}(m) = h_{q,p}(c) = h(c)$ . Fix  $c$  and consider  $H_{n_1}(h, c)$  as a function of  $h$  near  $h(c)$ . Its eigenvalues are the roots of a polynomial equation with real analytic, indeed polynomial, coefficients and they are all real for  $h$  real. It is easily seen that this implies that there is no ramification at  $h = h(c)$  and that in a neighborhood of this point there are expansions

$$\alpha_i(h) = \alpha_{i0} + \alpha_{i1}(h - h(c)) + \alpha_{i2}(h - h(c))^2 + \cdots, \quad 1 \leq i \leq P(n_1)$$

for the eigenvalues of  $H_{n_1}$ . Thus

$$\det H_{n_1}(h, c) = \prod_{i=1}^{P(n_1)} \left( \alpha_{i0} + \alpha_{i1}(h - h(c)) + \cdots \right),$$

and the power of  $h - h(c)$  that divides it is greater than or equal to the number of zero eigenvalues of  $H_{n_1}(h(c), c)$ . On the other hand, the left side is equal to

$$A_n \prod_{k \leq n_1} \prod_{p_1 q_1 = k} (h - h_{p_1, q_1}(c))^{P(n_1 - k)},$$

and  $h_{p_1, q_1}(c) = h(c)$  only if  $(p_1, q_1)$  or  $(q_1, p_1)$  lies in the boundary of  $D$ . Thus the assertion follows from Lemma 8.

Choosing  $n_1 = n$ , we see in particular that the dimension of the null space of  $V_n$  is 1. Thus if  $m > p' - 1$  then in a neighborhood of  $(h(m), c(m))$  we can find an analytic function  $v(h, c)$  with values in  $V_n$  such that  $v(h, c)$  has length 1, is an eigenvector of  $H_n(h, c)$ , and corresponds to the eigenvalue 0 when  $(h, c)$  falls on the curve  $C$ .

Since

$$\begin{aligned} L_0 v(h(m), c(m)) &= (h(m) + n)v(h(m), c(m)), \\ L_k v(h(m), c(m)) &= 0, \quad k > 0, \end{aligned}$$

there is a homomorphism of  $\mathfrak{v}$ -modules,  $\phi: V^{h(m)+n, c(m)} \rightarrow V^{h(m), c(m)}$ , taking  $v_\phi^{h(m)+n, c(m)}$  to  $v(h(m), c(m))$ . If it is injective on  $V_{n_1-n}^{h(m)+n, c(m)}$  then  $d_{n_1}^0 \geq P(n_1 - n)$  because the image consists of null vectors. Since  $d_{n_1}^0$  is lower semicontinuous,  $d_{n_1}^0$  will be greater than or equal to  $P(n_1 - n)$  everywhere on  $C$  if it is so on a dense set. The homomorphism  $\phi$  will be injective if  $\det H_{n_1-n}^{h(m)+n, c(m)} \neq 0$  because the kernel consists of null vectors. So it is enough to show that this determinant does not vanish identically on  $C$ . However, if  $h(m) + n = h_{p_1, q_1}(m)$  then

$$((m+1)p + mq)^2 = ((m+1)p_1 - mq_1)^2$$

or

$$(mp + (m+1)q)^2 = ((m+1)p_1 - mq_1)^2.$$

This can occur for at most two values of  $m$ . □

It remains to show that at  $m'$  the dimension of the space of null vectors in  $V_{n'}$  is  $P(n' - n) + 1$ . For this we need further lemmas.

**Lemma 10.**  $\det H_{n'-n}^{h(m') + n, c(m')} \neq 0$ .



*Proof.* It has to be shown that the equality  $h(m') + n = h_{p_1, q_1}(m')$ ,  $p_1 q_1 \leq n' - n$  is impossible. This equality amounts to

$$(A) \quad (m'p + (m' + 1)q)^2 = ((m' + 1)p_1 - m'q_1)^2$$

or

$$(B) \quad ((m' + 1)p + m'q)^2 = ((m' + 1)p_1 - m'q_1)^2.$$

It is not supposed that  $p_1 \geq q_1$ .

The first equation implies that  $m'p + (m' + 1)q = \pm((m' + 1)p_1 - m'q_1)$  or  $m'(p \pm q_1) = (m' + 1)(\pm p_1 - q)$ . Since  $m'$  is an integer this implies  $(p \pm q_1) = a(m' + 1)$ ,  $(\pm p_1 - q) = am'$ . Since  $n' = p'q' = (m' - q)(m' - p + 1)$  the inequality  $n' \geq n + p_1 q_1$  becomes

$$(m' - q)(m' - p + 1) \geq a(m' + 1)q - am'p + a^2 m'(m' + 1)$$

or

$$((1 + a)(m' + 1) - p)((1 - a)m' - q) \geq 0.$$

Since  $m' = p' + q = p + q' - 1$ ,  $m' > q$ ,  $m' + 1 > p$ . So the inequality is possible only for  $a = 0$ , but  $a$  cannot be 0. The case (B) is treated in a similar fashion.  $\square$

For  $n_1 < n'$  or  $m \neq m'$  we let  $U_{n_1} = U_{n_1}(m)$  be the space of null vectors in  $V_{n_1}$ . For  $h, c$  close to  $h(m')$ ,  $c(m')$  we let  $U_{n_1}(h, c)$  be the span of

$$\left\{ L_{-k_1} \cdots L_{-k_r} v(h, c) \mid k_1 \geq \cdots \geq k_r > 0, \sum k_i = n' - n \right\}.$$

We set  $U_{n'}(m) = U_{n'}(h(m), c(m))$ , the two definitions of  $U_{n'}(m)$  coinciding when they both apply. Thus for  $m > p' - 1$ ,  $U_{n_1}(m)$  is defined and analytic as a function of  $m$ . Let  $W_{n_1}$  be its orthogonal complement with respect to the form  $\{\cdot, \cdot\}$ . It follows from that part of Lemma 9 already proved that the restriction  $J_{n_1} = J_{n_1}(m)$  of  $H_{n_1}$  to  $W_{n_1}$  is non-singular unless  $n_1 = n'$ ,  $m = m'$ . In particular, our assumption, which was made for a particular  $m$ , implies that  $J_{n_1}(m)$  is positive for all  $m > p' - 1$  if  $n_1 < n'$ .

**Lemma 11.** *Near  $m'$ ,  $\det J_{n'}(m) = \delta(m)(m - m')$  where  $\frac{1}{\delta} \geq |\delta(m)| \geq \delta > 0$ .*

It will follow from this lemma that the remaining assertion of Lemma 9 is true. In addition the lemma together with our assumption on the non-negativity of  $\langle \cdot, \cdot \rangle$  for a particular  $m$ ,  $p' > x(m) > p' - 1$ , will imply that the form takes negative values for  $m > m'$  because  $\det J_{n'}(m)$  changes sign at  $m'$ .

Let  $v(h, c)$ , defined in a neighborhood of  $(h(m'), c(m'))$ , correspond to the eigenvalue  $\alpha(h, c)$  of  $H_n(h, c)$ . All the other eigenvalues of  $H_n(h, c)$  are bounded above and, if the neighborhood is sufficiently small, away from 0. On the other hand, all factors  $h - h_{p_1, q_1}(c) = h - h_{p_1, q_1}(m)$ ,  $c = c(m)$ , of  $\det H_n(h, c)$  are bounded away from 0 in a neighborhood of  $h(m')$ ,  $c(m')$  except for  $h - h(c)$ , where  $h(c)$  is  $h_{q, p}(c)$  or  $h_{p, q}(c)$  according as we are dealing with case A or case B. Thus we have the following lemma.

**Lemma 12.** *In a neighborhood of  $(h(m'), c(m'))$  we have  $\alpha(h, c) = a(h, c)(h - h(c))$  with  $\frac{1}{a} \geq |a(h, c)| \geq a > 0$ ,  $a$  being a constant.*

Here  $h(c)$  is  $h_{q, p}(m)$  (A) or  $h_{p, q}(m)$  (B),  $c = c(m)$ . More generally we have

**Lemma 13.** *Let  $K_{n'}(h, c)$  be the restriction of  $H_{n'}(h, c)$  to  $U_{n'}(h, c)$ . Then, in a neighborhood of  $(h(m'), c(m'))$ ,  $\det K_{n'}(h, c) = k(h, c)\alpha(h, c)^{P(n' - n)}$ , with  $\frac{1}{k} \geq |k(h, c)| \geq k > 0$ .*

*Proof.* The determinant of  $K_{n'}(h, c)$  is that of the form  $\langle \cdot, \cdot \rangle_{n'}$ , calculated with respect to a basis of  $U_{n'}(h, c)$  orthogonal with respect to the form  $\{ \cdot, \cdot \}_n$ . However the basis

$$\left\{ \phi(v_\alpha) \mid v_\alpha \in V^{h(m)+n, c(m)}, n(\alpha) = n' - n \right\}$$

is related to such a basis by a matrix whose determinant is bounded in absolute value above and below. So it is enough to consider  $\det\left(\{\phi(v_\alpha), \phi(v_\beta)\}\right)$ .

We have

$$\begin{aligned} \langle \phi(v_\alpha), \phi(v_\beta) \rangle &= \langle L_{\ell_s} \cdots L_{\ell_1} L_{-k_1} \cdots L_{-k_r} v(h, c), v(h, c) \rangle \\ &= \{ L_{\ell_s} \cdots L_{\ell_1} L_{-k_1} \cdots L_{-k_r} v(h, c), H_{n'}(h, c) v(h, c) \} \\ &= \alpha(h, c) \{ L_{\ell_s} \cdots L_{\ell_1} L_{-k_1} \cdots L_{-k_r} v(h, c), v(h, c) \}. \end{aligned}$$

At  $h(m), c(m)$  the value of  $\det\left(\{ L_{\ell_s} \cdots L_{\ell_1} L_{-k_1} \cdots L_{-k_r} v(h, c), v(h, c) \}\right)$  is

$$\det\left(\langle v_\alpha, v_\beta \rangle_{n'-n}^{h(m')+n, c(m')}\right).$$

By Lemma 10 this is not 0. Lemma 13 follows.  $\square$

In a neighborhood of  $h(m), c(m)$  we decompose  $V_{n'}$  as an orthogonal sum  $U_{n'} \oplus W_{n'}$ . The linear transformation  $H_{n'}(h, c)$ , or its matrix with respect to a compatible basis, then decomposes into blocks. I claim that the entries in the off-diagonal blocks are  $O(h - h_{p,q}(c))$  in a neighborhood of  $h(m), c(m)$ . To verify this it is sufficient, for the pertinent basis can be supposed to depend analytically on  $h, c$ , to verify that they are zero when  $h = h_{p,q}(c)$ , but that is clear by the definition of  $U_{n'}$ .

It follows that

$$(1) \quad \det H_{n'}(h, c) = \det J_{n'}(h, c) \det K_{n'}(h, c) + O\left((h - h_{p,q}(c))^{P(n'-n)+1}\right)$$

if  $J_{n'}(h, c)$  is the matrix in the diagonal block corresponding to  $W_{n'}$ . Since

$$\det H_{n'}(h, c) = A_{n'} \prod_{k \leq n'} \prod_{p_1 q_1 = k} (h - h_{p_1 q_1}(c))^{P(n' - p_1 q_1)}$$

we may divide the relation (1) by  $(h - h_{p_1 q_1}(c))^{P(n'-n)}$  and then set  $h = h_{p,q}(c), c = c(m)$ . The result clearly yields Lemma 11 because  $h(m') = h_{p_1, q_1}(m')$ ,  $p_1, q_1 \leq n'$ , only if  $(p_1, q_1)$  is  $(q, p)$  or  $(p', q')$  (case A) or  $(p, q)$  or  $(q', p')$  (case B).

Our assumption that  $H_{n_1}(h(m), c(m))$  is non-negative for a given  $m, p' > m > p' - 1$ , has led to the conclusion that  $J_{n_1}(m)$  is positive for large  $m$  and  $n_1 < n'$  but that  $J_{n'}(m)$  has negative eigenvalues for large  $m$ . We show not that this is impossible.

As  $m$  approaches infinity, the point  $(h(m), c(m))$  approaches  $(h_0, c_0) = \left(\frac{(p-q)^2}{4}, 1\right)$ . If  $p \neq q$  a suitable coordinate on the curve is  $\mu = \frac{1}{m}$ . If  $p = q$  we may take  $\mu = 1 - c$ . All the matrices  $H_{n_1}(\mu) = H_{n_1}(m) = H_{n_1}(h(m), c(m))$  are analytic functions of  $\mu$ . The eigenvalues of  $H_{n_1}(\mu)$  are given by power series.

$$\alpha_i = \alpha_i(\mu) = \alpha_{i0} + \alpha_{i1}\mu + \alpha_{i2}\mu^2 + \cdots$$

Let  $V_{n_1}^1(\mu)$  be the space spanned by the eigenvectors corresponding to  $\alpha_i$  with  $\alpha_{i0} = 0$ ; let  $V_{n_1}^2(\mu)$  be the space spanned by the eigenvectors corresponding to  $\alpha_i$  with  $\alpha_{i0} = \alpha_{i1} = 0$  and so on. One proves by induction that these spaces are well-defined, depend analytically on  $\mu$  (in

the sense that we have analytic functions  $v_1(\mu), \dots, v_{P(n_1)}(\mu)$ , such that  $\{v_1(\mu), \dots, v_{d_k}(\mu)\}$ ,  $d_k = \dim V_{n_1}^k$  forms a basis of  $V_{n_1}^k(\mu)$  for each  $\mu$ , and that  $\mu^{-k} \{H_{n_1}(\mu)v_i(\mu), v_j(\mu)\}$ ,  $i \leq d_k$ ,  $j \leq P(n_1)$  is analytic for small  $\mu$ . It can even be supposed that  $\{H_{n_1}(\mu)v_i(\mu), v_j(\mu)\} = 0$ ,  $i \leq d_k$ ,  $j > d_k$ .

Let  $V^k = \bigoplus_{n_1} V_{n_1}^k(0)$  and  $X^k = V^k/V^{k+1} = \bigoplus_{n_1} V_{n_1}^k(0)/V_{n_1}^{k+1}(0)$ . If  $u = \sum_{i \leq d_k} a_i v_i(0) \in V_{n_1}^k(0)$  and  $v = \sum_{i \leq d_k} b_i v_i(0) \in V_{n_2}^k(0)$ , define  $\langle u, v \rangle^{(k)}$  to be 0 if  $n_1 \neq n_2$ , and if  $n_1 = n_2$  set

$$\begin{aligned} \langle u, v \rangle^{(k)} &= \langle u, v \rangle_{n_1}^{(k)} = \sum a_i \bar{b}_j \lim_{\mu \rightarrow 0} \mu^{-k} \langle v_i(\mu), v_j(\mu) \rangle \\ &= \sum a_i \bar{b}_j \lim_{\mu \rightarrow 0} \left\{ \mu^{-k} H_{n_1}(\mu) v_i(\mu), v_j(\mu) \right\}. \end{aligned}$$

It is clear that  $H_{n_1}(\mu)$  is non-negative for small  $\mu$  if and only if the forms  $\langle u, v \rangle_{n_1}^{(k)}$  are all positive.

**Lemma 14.**

- (a) The spaces  $V^k$  are all invariant under  $\pi = \pi^{h_0, c_0}$ , so that  $\mathfrak{v}$  operates on  $X^k$ .
- (b) The form  $\langle \cdot, \cdot \rangle^{(k)}$  on  $X^k$  satisfies  $\langle L_m x, y \rangle = \langle x, L_{-m} y \rangle$ ,  $m \in \mathbf{Z}$ .

*Proof.* Set  $L_m(\mu) = \pi^{h(\mu), c(\mu)}(L_m)$  and  $L_m = L_m(0)$ . We have to show for each  $n_1$  that  $L_m v_i \in V^k$  if  $v_i = v_i(0)$  and  $i \leq d_k$ . However

$$L_m v_i = \lim_{\mu \rightarrow 0} L_m(\mu) v_i(\mu) = \lim_{\mu \rightarrow 0} \sum_j c_{ij}(\mu) v'_j(\mu)$$

where the  $c_{ij}$  are analytic functions of  $\mu$ . It is to be shown that  $c_{ij}(0) = 0$  for  $j > d'_k$ . The primes refer to  $n_2 = n_1 - m$  rather than to  $n_1$ . In other words it has to be shown that

$$\{H_{n_2}(\mu) L_m(\mu) v_i(\mu), v'_\ell(\mu)\} = O(\mu^k)$$

for all  $\ell$ . Since  $H_{n_2}(\mu) L_m(\mu) = L_{-m}^*(\mu) H_{n_1}(\mu)$ , the adjoint of  $L_{-m}(\mu)$  being taken with respect to the form  $\{\cdot, \cdot\}$ , this is clear. So is the second assertion of the lemma.  $\square$

For any  $h \geq 0$  the representation  $\pi^{h,1}$  on  $V^{h,1}$  has a unique irreducible quotient  $\rho^{h,1}$  on  $X^{h,1}$ , which by Lemma 3 carries a hermitian form for which  $\rho^{h,1}$  is unitary in the sense that the adjoint  $\rho^{h,1}(L_m)$  is  $\rho^{h,1}(L_{-m})$ . Such a form is unique up to a scalar multiple. Take in particular  $h = \frac{r^2}{4}$ ,  $r \in \mathbf{Z}$ . Then  $h = h_{p_2, q_2}(c)$  if and only if  $(p_2 - q_2)^2 = r^2$ . In particular,  $h = h_{r+1, 1}(c)$ . Thus the lowest weight for a null vector in  $V$  is  $r + 1$  and  $h + r + 1 = \frac{(r+2)^2}{4}$ , so that the kernel of  $V^{h,1} \rightarrow X^{h,1}$  contains a quotient of  $V^{h',1}$ ,  $h' = \frac{(r+2)^2}{4}$ . Thus  $V^{h,1}$  admits a sequence of invariant subspaces  $V^{h,1} = V^{h,1}(0) \supseteq V^{h,1}(1) \supseteq V^{h,1}(2)$  such that the representation on  $V^{h,1}(0)/V^{h,1}(1)$  is  $\rho^{h,1}$  and that on  $V^{h,1}(1)/V^{h,1}(2)$  is  $\rho^{h',1}$ . In general set  $h^{(\ell)} = \frac{1}{4}(r + 2\ell)^2$ .

**Lemma 15.**  $V^{h,1}$  admits an infinite decomposition series  $V^{h,1}(0) \supseteq V^{h,1}(1) \supseteq \dots \supseteq V^{h,1}(\ell) \supseteq \dots$  such that the representation on the quotient  $V^{h,1}(\ell)/V^{h,1}(\ell + 1)$  is  $\rho^{h^{(\ell)},1}$ .

*Proof.* If  $\lambda = h + k$ ,  $k \in \mathbf{Z}$ ,  $k \geq 0$ , let

$$\begin{aligned} d_\lambda &= \dim \left\{ v \in V^{h,1} \mid L_0 v = \lambda v \right\}, \\ d_\lambda(\ell) &= \dim \left\{ v \in X^{h^{(\ell)},1} \mid L_0 v = \lambda v \right\}. \end{aligned}$$

The lemma follows easily from a formula of Kac ([2], Th. 5), according to which  $d_\lambda = \sum_{\ell=0}^{\infty} d_\lambda(\ell)$ . Indeed, suppose we have constructed an initial segment of the series  $V^{h,1}(0) \supset \dots \supset V^{h,1}(\ell)$ . Then  $\frac{1}{4}(r+2\ell)^2$  is a lowest weight in  $V^{h,1}(\ell)$  and

$$\dim \left\{ v \in V^{h,1}(\ell) \mid L_0 v = \frac{1}{4}(r+2\ell)^2 v \right\} = 1.$$

Take  $V^{h,1}(\ell+1)$  to be the sum of all invariant subspaces of  $V^{h,1}(\ell)$  for which the lowest weight is greater than  $\frac{1}{4}(r+2\ell)^2$ .  $\square$

Now take  $r = p - q$ . It follows immediately from the preceding lemma that  $X^k$  is the direct sum of irreducible invariant subspaces  $X_j^k$  carrying distinct representations and that the restriction of  $\langle \cdot, \cdot \rangle^k$  to  $X_j^k$  is either positive or negative. The assumption that we are trying to contradict implies that the form is positive if  $X_j^k$  contains non-zero vectors of weight  $h + n_1$ ,  $n_1 < n'$ , but that for some  $j$  and  $k$  for which  $X_j^k$  contains vectors of weight  $h + n'$ , it is negative.

Thus the following lemma completes the proof of Theorem FQS.

**Lemma 16.** *The equation  $\frac{r^2}{4} + n' = \frac{1}{4}(r+2\ell)^2$  has no solution  $\ell \geq 0$  in  $\mathbf{Z}$ .*

*Proof.* The equation may be written as  $n' = \ell(\ell+r)$ . Recall that  $n'$  is  $(p+a)(q+a+1)$  in case A and  $(p+a+1)(q+a)$  in case B, with  $a \geq 0$ . Since  $r = p - q$ , the equation is  $(p+a+\ell)(q+a+1-\ell) = \ell$  or  $(p+a+1+\ell)(q+a-\ell) = -\ell$ . Both equations are manifestly impossible.  $\square$

## REFERENCES

- [1] Friedan, D., Z. Qiu and S. Shenker, *Conformal invariance, unitarity and two-dimensional critical exponents*, in *Vertex operators in mathematics and physics*, eds. J. Lepowsky, S. Mandelstam and I. M. Singer, Springer (1985).
- [2] Rocha-Caridi, A. *Vacuum-vector representations of the Virasoro algebra*, *ibid.*
- [3] Friedan, D., Z. Qiu and S. Shenker, *Details of the non-unitary proof for highest weight representations of the Virasoro algebra*, *Comm. Math. Phys.* **107** (1986).

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