ON UNITARY REPRESENTATIONS OF THE VIRASORO ALGEBRA

ROBERT P. LANGLANDS

The Virasoro algebra $\mathfrak{v}$ is an infinite-dimensional Lie algebra with basis L_m, $m \in \mathbf{Z}$, and Z and defining relations:

(i) $[L_m, L_n] = (m-n)L_{m+n} + \frac{m(m^2-1)}{12}\delta_{m,-n}Z$;
(ii) $[L_m, Z] = 0$.

Some representations π of $\mathfrak{v}$ of particular interest [2] are the Verma modules $(V, \pi) = (V^{h,c}, \pi^{h,c})$, h, $c \in \mathbf{R}$. They are characterized by the following conditions.

(i) There is a vector $v = v_\phi \neq 0$ in V such that $L_n v = 0$, $n > 0$, $L_0 v = hv$, $Zv = cv$.
(ii) Let A be the set of sequences of integers $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_r > 0$ of arbitrary length, and if $\alpha \in A$ let $v_\alpha = \pi(L_{-k_1}) \cdots \pi(L_{-k_r})v_\phi$. Then $\{\, v_\alpha \mid \alpha \in A \,\}$ is a basis of V.

Observe that V is just the free vector space with basis $\{v_\alpha\}$ and is thus independent of h and c. It is easy to see [1] that there is a unique sesquilinear form $\langle u, v\rangle = \langle u, v\rangle^{h,c}$ on V with the properties:

(i) $\langle v_\phi, v_\phi\rangle = 1$;
(ii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$;
(iii) $\bigl\langle \pi(L_m)u, v\bigr\rangle = \bigl\langle u, \pi(L_{-m})v\bigr\rangle$, $m \in \mathbf{Z}$.

If this form is non-negative then the representation ρ of $\mathfrak{v}$ on the quotient of V by the space of null vectors is unitary, in the sense that

$$\rho(L_m)^* = \rho(L_{-m}).$$

Theorem FQS. *The form $\langle\cdot,\cdot\rangle_{h,c}$ is non-negative only if either $c \geqslant 1$, $h \geqslant 0$ or there exists an integer $m \geqslant 2$ and two integers p, q, $1 \leqslant p < m$, $1 \leqslant q \leqslant p$, such that*

$$c = 1 - \frac{6}{m(m+1)}, \quad h = \frac{\bigl((m+1)p - mq\bigr)^2 - 1}{4m(m+1)}.$$

This theorem has been proven by Friedan-Qiu-Shenker [1]. The sketch of the proof that they provided was unconscionably brief, and has evoked some scepticism among mathematicians. In this note, which grew out of a series of lectures at the Centre de recherches mathématiques that overlapped the workshop, details are worked out. In the meantime, Friedan, Qiu and Shenker have themselves provided them [3], but the present account, which turns out to diverge from theirs in some respects, may still be a useful supplement to it. Several other authors have proven that the conditions of the theorem are not only necessary but also sufficient for non-negativity, but that is not the concern here.

The proof proceeds by lemmas. I write Lv rather than $\pi(L)v$, $L \in \mathfrak{v}$, $v \in V$.

Lemma 1. *If $\langle\cdot,\cdot\rangle$ is non-negative then $h \geqslant 0$, $c \geqslant 0$.*

Proof. Since $L_n L_{-n} v_\phi = L_{-n} L_n v_\phi + 2nhv_\phi + \frac{n(n^2-1)}{12}cv_\phi$, we have $\langle L_{-n}v_\phi, L_{-n}v_\phi\rangle = 2nh + \frac{n(n^2-1)}{12}c$. Taking n first equal to 1 and then very large we obtain the lemma. □

Appeared in *Infinite-dimensional Lie algebras and their applications,* World Scientific (1988).

For arbitrary m we set $c = c(m) = 1 - \frac{6}{m(m+1)}$, $h_{p,q} = h_{p,q}(m) = \frac{\left((m+1)p - mq\right)^2 - 1}{4m(m+1)}$, $p,\, q \in \mathbf{N}$. Observe that $c(-1-m) = c(m)$ and that $h_{p,q}(-1-m) = h_{q,p}(m)$.

Lemma 2.

(*a*) *For* $1 < c < 25$, m *is not real and neither is* $h_{p,q}(m)$ *unless* $p = q$.
(*b*) *As* m *runs from* 2 *to* ∞, c *increases monotonically from* 0 *to* 1.
(*c*) *For* $c > 1$, $-1 < m < 0$.
(*d*) *If* $-1 < m < 0$ *then* $h_{p,q}(m) < 0$ *unless* $p = q = 1$ *when* $h_{p,q}(m) = 0$.
(*e*) *If* $p = q$ *then* $h_{p,q}(m) = \frac{p^2-1}{24}(1-c)$.
(*f*) *If* $p \neq q$ *then* $h_{p,q} + h_{q,p} = \frac{p^2+q^2-2}{24}(1-c) + \frac{(p-q)^2}{2}$. *In addition* $h_{p,q}h_{q,p}$ *is equal to*

$$\frac{(p^2q^2 - p^2 - q^2 + 1)}{16 \cdot 36}(1-c)^2 + \frac{\left(2p^2q^2 - pq(p^2+q^2) - (p-q)^2\right)}{48}(1-c) + \frac{(p^4 + q^4 - 4p^3q - 4pq^3 + 6p^2q^2)}{16}.$$

Proof. The first four parts of the lemma are clear, and the last two are straightforward calculations. □

There is a second sesquilinear form on V defined by $\{v_\alpha, v_\beta\} = \delta_{\alpha\beta}$. If V_n is the subspace of V with basis $\left\{ v_\alpha \;\middle|\; \alpha = (k_1, \dots, k_r),\ \sum_{i=1}^r k_i = n \right\}$, then $V = \bigoplus_{n \geqslant 0} V_n$ and the spaces V_n are mutually orthogonal with respect to both forms. The first form is defined on V_n with respect to the second by a hermitian linear transformation $H_n = H_n(h,c) : \langle u, v\rangle_n = \{H_n u, v\}_n$. Let $P(n)$ be the dimension of V_n. It is the number of partitions of n. The Kac determinant formula (cf. [1]) is the key to the proof of Theorem FQS.

Kac determinant formula. *If* $c = c(m)$ *then*

$$\det H_n(h,c) = A_n \prod_{k \leqslant n} \prod_{pq = k} (h - h_{p,q})^{P(n-k)},$$

where A_n *is a positive constant.*

Lemma 3. *The form* $\langle \cdot, \cdot \rangle_n$ *is non-negative for* $h \geqslant 0$, $c \geqslant 1$.

Proof. By continuity it suffices to treat pairs for which $h > 0$, $c > 1$. Since the previous lemma implies that $\det H_n(h,c)$ is nowhere zero in this region, it suffices to prove that the form is positive for one pair (h,c). If $\alpha = (k_1, \dots, k_r)$, $r = r(\alpha)$, $n(\alpha) = k_1 + \cdots + k_r$, set $v'_\alpha = L_{-k_r} \cdots L_{-k_1} v_\phi$. It is generally different than v_α. It clearly suffices to show that for a given c and h large,

$$\langle v'_\alpha, v'_\alpha \rangle = c_\alpha h^{r(\alpha)}\bigl(1 + o(1)\bigr), \qquad c_\alpha > 0 \tag{3.1}$$

$$\langle v'_\alpha, v'_\beta \rangle = o\Bigl(h^{\left(r(\alpha)+r(\beta)\right)/2}\Bigr), \qquad \alpha \neq \beta. \tag{3.2}$$

This is proved by induction on $n(\alpha) + n(\beta)$. First of all $L_k^a L_{-k}^a$ is equal to

$$L_k^{a-1}(bL_0 + d)L_{-k}^{a-1} + L_k^{a-1} L_{-k} L_k L_{-k}^{a-1}, \qquad b > 0.$$

Moving the single L_k in the second term ever further to the right, we obtain finally

$$L_k^a L_{-k}^a = L_k^{a-1}(bL_0 + d)L_{-k}^{a-1} + L_k^{a-1}L_{-k}^a L_k, \qquad b > 0.$$

Take $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_r > k$. If $\alpha = (k_1, \ldots, k_r, k, \ldots, k)$, then

$$\begin{aligned}\langle v'_\alpha, v'_\alpha\rangle &= \langle L_{k_1} \cdots L_{k_r} L_k^a L_{-k}^a L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi\rangle \\ &= c_{k,a} h\big(1 + o(h)\big)\langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a-1} L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi\rangle \\ &\quad + \langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^a L_k L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi\rangle\end{aligned}$$

with $c_{k,a} > 0$. In the second term we move the L_k further and further to the right obtaining the sum of

$$(k + k_r)\langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^a L_{-k_r} \cdots L_{-k_{j+1}} L_{-(k_j - k)} L_{-k_{j-1}} \cdots L_{-k_1} v_\phi, v_\phi\rangle.$$

The induction assumption together with the defining relations for $\mathfrak{v}$ implies readily that each of these terms is $o(h^{r(\alpha)})$ and that

$$\langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a-1} L_{-k_r} \cdots L_{-k_1} v_\phi, v_\phi\rangle = \langle v'_\gamma, v'_\gamma\rangle = c_\gamma h^{r(\gamma)}\big(1 + o(1)\big),$$

if $\gamma = (k_1, \ldots, k_r, k, \ldots, k)$, with k repeated $a - 1$ times, so that $r(\alpha) = 1 + r(\gamma)$.

On the other hand, if $\beta = (\ell_1, \ldots, \ell_s, k, \ldots, k)$, with k repeated $a' \leqslant a$ times, $a > 0$, $a' \geqslant 0$, $\ell_s \geqslant k$ even if $a' = 0$, then

$$\langle v'_\beta, v'_\alpha\rangle = \langle L_{k_1} \cdots L_{k_r} L_k^a L_{-k}^{a'} L_{-\ell_s} \cdots L_{-\ell_1} v_\phi, v_\phi\rangle$$

is equal to the sum of

$$c_{k,a'} h\big(1 + o(1)\big)\langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a'-1} L_{-\ell_s} \cdots L_{-\ell_1} v_\phi, v_\phi\rangle$$

and

$$\sum_j (k + \ell_j)\langle L_{k_1} \cdots L_{k_r} L_k^{a-1} L_{-k}^{a'} L_{-\ell_s} \cdots L_{-\ell_{j+1}} L_{-(\ell_j - k)} L_{-\ell_{j-1}} \cdots L_{-\ell_1} v_\phi, v_\phi\rangle.$$

We take $c_{k,0} = 0$ if $a' = 0$. So induction yields (3.2). □

Observe that if $m > 0$ and $p > q$ then $h_{p,q} > h_{q,p}$. If $h \geqslant 0$ and $m > 0$ define $M > 0$ by $M^2 = 1 + 4m(m+1)h$. Then $M \geqslant 1$. Let D be the closed shaded region in the diagram I. It is bounded by the lines

$$mx - (m+1)y = \pm M \quad \text{and} \quad (m+1)x - my = M.$$

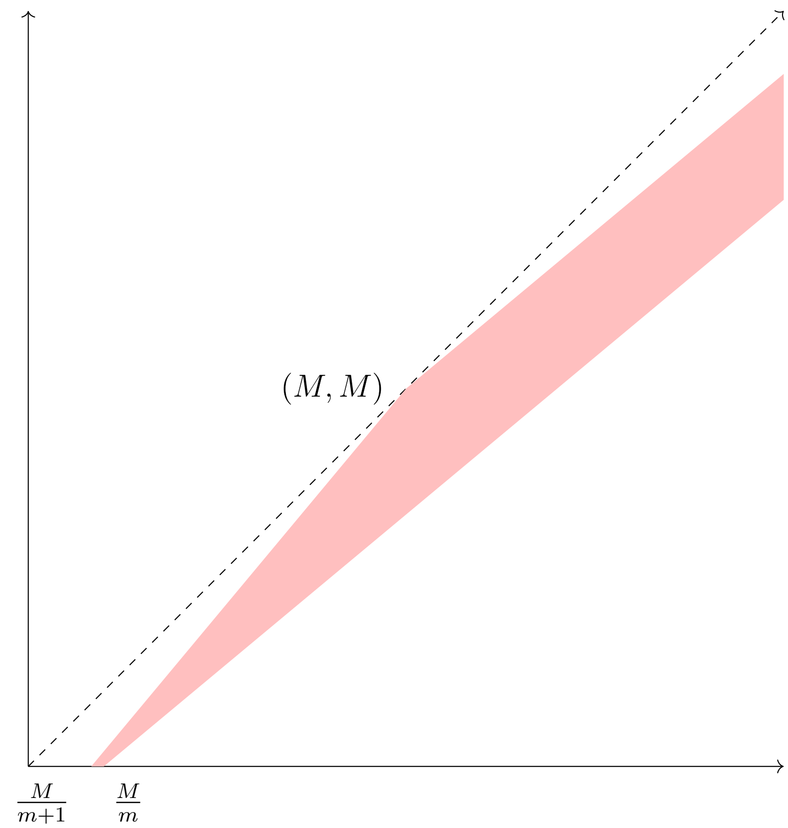

Diagram I

Lemma 4.

(*a*) $h_{p,q} \geqslant h \geqslant h_{q,p}$ *if and only if* $(p, q) \in D$.

(*b*) D *contains an integral point* (p, q) *with* $q > 0$.

Proof. Since $h_{p,q} \geqslant h$ if and only if $\big((m+1)p - mq\big)^2 \geqslant M^2$ and $h \geqslant h_{q,p}$ if and only if $\big((m+1)q - mp\big)^2 \leqslant M^2$, the first statement of the lemma is clear. For the second choose a large integer p and let $a = \frac{mp-M}{m+1}$. Then the points (p, q) with $a \leqslant q \leqslant a + \frac{2M}{m+1}$ lie in D. So do the points $(p+1, q)$, $a + \frac{m}{m+1} \leqslant q \leqslant a + \frac{m+2M}{m+1}$ and so on. So we need only show that one of the intervals $\left[a + \frac{km}{m+1}, a + \frac{km+2M}{m+1}\right]$, $k \in \mathbf{Z}$, $k \geqslant 0$, contains an integer. This is clear if $\frac{m}{m+1}$ is irrational. Otherwise, increasing q if necessary, we may suppose that a is as close to its integral part as any $a + \frac{km}{m+1}$. Then $a + \frac{m}{m+1} < [a] + 1$, but $a + \frac{m+2M}{m+1} \geqslant a + \frac{m+2}{m+1} > [a] + 1$, and the interval $\left[a + \frac{m}{m+1}, a + \frac{m+2M}{m+1}\right]$ contains $[a] + 1$. □

Let $p(h, c) = \min_{(p,q)\in D} p$ and let $q(h, c) = \min_{(p,q)\in D} q$. It is clear that

$$P(h, c) = \big(p(h, c), q(h, c)\big) \in D.$$

In the following geometrical arguments, it is sometimes necessary to recall that $h - h_{p_0,p_0} < 0$ if and only if $p_0 > M$.

Lemma 5. *If* $P(h, c)$ *lies in the interior of* D *then* $\langle v, v\rangle$ *assumes negative values in* V.

Proof. Let $(p,q) = P(h,c)$ and let $n = pq$. If $p_0q_0 \leqslant n$, $p_0 \geqslant q_0$ and $(p_0,q_0) \neq (p,q)$ then either $p_0 < p$ or $q_0 < q$ so that $(p_0,q_0) \notin D$. In general set

$$\begin{aligned} \phi_{p_0,q_0} &= (h - h_{p_0,q_0})(h - h_{q_0,p_0}), && p_0 \neq q_0, \\ &= h - h_{p_0,q_0}, && p_0 = q_0. \end{aligned}$$

If $(p_0,q_0) \notin D$ and $p_0 \neq q_0$ then $\phi_{p_0,q_0} > 0$.

Suppose that for some p_0 with $p_0^2 \leqslant pq$ we had $h - h_{p_0,p_0} < 0$. Then there would be a minimum such p_0 and if $n_0 = p_0^2$ then

$$\det H_{n_0} = A_{n_0} \prod_{\substack{p_1 \geqslant q_1 \\ n_1 = p_1q_1 \leqslant n_0}} \phi_{p_1,q_1}^{P(n_0-n_1)}$$

Since $P(h,c)$ lies in the interior of D, $p \neq q$ and none of the pairs (p_1,q_1) that intervene here lie in D. Moreover, all terms of the products are positive save $\phi_{p_0,p_0}^{P(0)} = \phi_{p_0,p_0}$. Since this is negative, $\langle\cdot,\cdot\rangle$ assumes negative values on V_{n_0}.

If, however, $\phi_{p_0,p_0} > 0$ for all $p_0 \leqslant q$ then the same argument shows that $\det H_n < 0$, so that $\langle\cdot,\cdot\rangle$ assumes negative values on V_n. □

The treatment of those points (h,c) for which $P(h,c)$ lies on the boundary of D is more delicate. There are at first three possibilities for $(p,q) = P(h,c)$:

(A) $mp - (m+1)q = M$;
(B) $(m+1)p - mq = M$;
(C) $mp - (m+1)q = -M$, $p \neq q$;

Lemma 6. *The case* (C) *above does not occur.*

Proof. It is clear from the diagram defining D that in case (C), $p \geqslant M$, $q \geqslant M$. If $q = 1$ then $M = 1$ and $p = 1$, so that we have rather case (B). If $q > 1$ then $p > 1$ and $(m+1)(q-1) - m(p-1) = (m+1)q - mp - 1$, so that $M > (m+1)(q-1) - m(p-1) > -M$. Moreover, $(m+1)(p-1) - m(q-1) - M = (m+1)(p-1-q) - m(q-1-p) = (2M+1)(p-q) - 1$. Since $m \geqslant 2$ this is positive if $p \neq q$. Consequently $(p-1, q-1) \in D$, and this is a contradiction. □

Fix (p,q). In case (A) we have $h = h_{q,p}(m)$, $c = c(m)$. In case (B) we have $h = h_{p,q}(m)$, $c = c(m)$.

Lemma 7.

(*a*) *The set of all* $m \geqslant 2$ *for which* $h = h_{q,p}(m)$, $c = c(m)$ *yields case* (A) *is the interval* $m > q + p - 1$.
(*b*) *The set of all* $m \geqslant 2$ *for which* $h = h_{p,q}(m)$, $c = c(m)$ *yields case* (B) *is the interval* $m > q + p - 1$ *if* $(p,q) \neq (1,1)$ *and is the interval* $m \geqslant 2$ *if* $(p,q) = (1,1)$.

It will be helpful, when proving this and the following lemmas, to keep the diagrams IIA and IIB in mind.

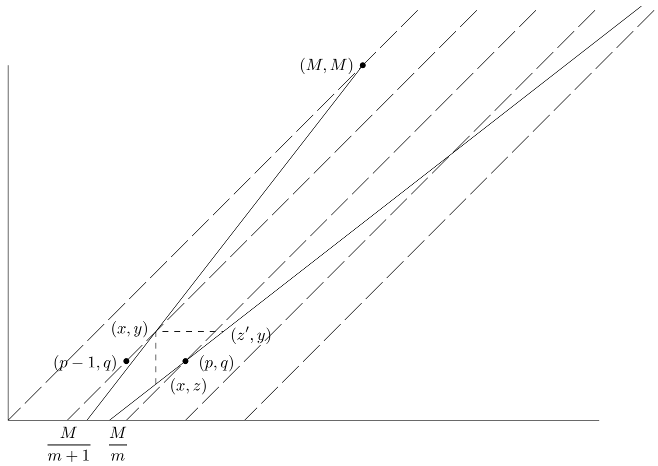

Diagram IIA

Proof. We first show that if $h_{q,p}(m_0)$, $c(m_0)$ yield case (A) then so does $\big(h_{q,p}(m), c(m)\big)$ for $m \geqslant m_0$. It is clear from the diagram that it is sufficient to verify that M, $\frac{M}{m+1}$, and $\frac{M}{m}$ are increasing functions of m. But $M = m(p-q) - q$, $\frac{M}{m} = (p-q) - \frac{q}{m}$, $\frac{M}{m+1} = (p-q) - \frac{p}{m+1}$. It is also clear that

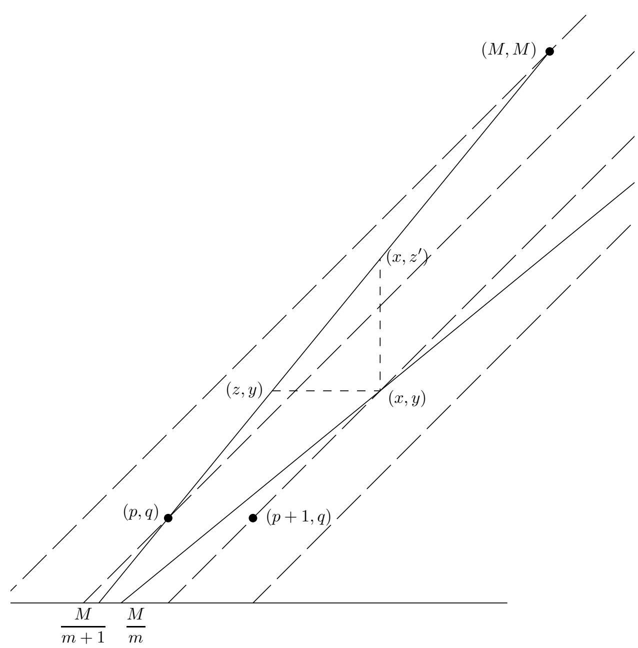

Diagram IIB

we can decrease m without passing out of case (A) so long as $M = m(p-q) - q$ remains greater than or equal to 1 and $(m+1)(p-1) - mq > mp - (m+1)q$. But

$$(m+1)(p-1) - mq = mp - (m+1)q \iff m = p + q - 1.$$

As we decrease to these points, M decreases to

$$(p+q-1)(p-q)-q = p^2-q^2-p = (p-1)^2-q^2+p-1.$$

This number is greater than 1 because $p > q \geqslant 1$.

For case (B), $M = m(p-q)+p$ is a non-decreasing function of m, and $\frac{M}{m} = (p-q)+\frac{p}{m}$, $\frac{M}{m+1} = (p-q)+\frac{q}{m+1}$ are decreasing functions. Since the slope of $mp-(m+1)q = M$ is $1-\frac{1}{m+1}$, it is increasing and the conclusion is the same. The minimal value of m is given by

$$(m+1)p-mq = mp-(m+1)(q-1) \iff m = p+q-1.$$

because

$$(p+q-1)(p-q)+p = p^2-q^2+q \geqslant 1,$$

unless $p=q=1$ when m cannot go below 2. □

In case (A) the intersection of the two lines $(m+1)x-my = M$ and $x-y = p-q-1$ is a point $\big(x(m),y(m)\big)$ with $p' \geqslant x(m) > p'-1$ where p' is an integer, $p' \geqslant p$. If $x(m) = p'$ then $y(m) = q' = p'-p+q+1$, and $m = p'+q$. Thus $m \in \{2,3,\dots\}$, $p' < m$, $q' \leqslant p'$ and $c = c(m)$, $h = h_{p',q'}(m)$.

In case (B) the intersection of the lines $x-y = p-q+1$ and $mx-(m+1)y = M$ is a point $\big(x(m),y(m)\big)$ with $p' \geqslant x(m) > p'-1$, $p'-1 \geqslant p$. If $x(m) = p'$ then $m = p+q'$ lies in $\{2,3,\dots\}$, $q \leqslant p$, $p < m$ and $c = c(m)$, $h = h_{p,q}(m)$.

Thus to prove the theorem it suffices to establish the following proposition.

Proposition. *If case* (A) *or* (B) *obtains and $p' > x(m) > p'-1$ then the form $\langle\cdot,\cdot\rangle$ assumes negative values in V.*

We assume the contrary and derive a contradiction. We occasionally abbreviate $c(m)$ to c and $h_{q,p}(m)$ or $h_{p,q}(m)$ to $h(m)$ or to h.

Lemma 8.

(*a*) *Suppose $p' > x(m) > p'-1$. If (p_1,q_1) lies on the boundary of $D(h,c)$ and $p_1q_1 \leqslant p'q'$ then $(p_1,q_1) = (p,q)$.*

(*b*) *Define m' by $p' = x(m')$ and set $c' = c(m')$, $h' = h_{q,p}(m')$ or $h_{p,q}(m')$. If (p_1,q_1) lies on the boundary of $D(h',c')$ and $p_1q_1 \leqslant p'q'$ then (p_1,q_1) is (p,q) or (p',q').*

Proof. Set $(x,y) = \big(x(m),y(m)\big)$ and define z, z' as indicated by the diagrams. It clearly suffices to show that in case (A) $y-z < 2$, $z'-x < 2$, and that in case (B), $x-z < 2$, $z'-y < 2$. In case (A) elementary algebra yields $m = x+q$, $y-z = \frac{x+z}{m} = 1+\frac{z-q}{x+q}$ and $\frac{z-q}{x+q} = \frac{x-p}{x+q}\cdot\frac{z-q}{x-p} < 1$. On the other hand $z'-x = \frac{x+y}{m} = 1+\frac{y-q}{p+y-1} < 2$. A similar argument works for case (B). □

Since p, q and p' are fixed it will be useful to let C denote the curve $c = c(m)$, $h = h_{q,p}(m)$ (A) or $h = h_{p,q}(m)$ (B), $m > p'-1$.

Lemma 9.

(*a*) *If $x(m) > p'-1$, $x(m) \neq p'$, and $n_1 \leqslant n'$, then the dimension of the space of null vectors in V_{n_1} is $P(n_1-n)$.*

(*b*) *If $x(m) = p'$ and $n_1 < n'$ then the dimension of the space of null vectors in V_{n_1} is $P(n_1-n)$, but if $n_1 = n'$ it is $P(n_1-n)+1$.*

Proof. Observe that $P(n_1-n)=0$ if $n_1<n$ and that when this is so the lemma is clear. So take $n_1\geqslant n$ and denote the pertinent dimension by $d^0_{n_1}$. We begin by showing that $d^0_{n_1}>0$ and that $d^0_{n_1}\leqslant P(n_1-n)$ unless $x(m)=p'$ and $n_1=n'$ when $d^0_{n_1}\leqslant P(n_1-n)+1$.

For $0\leqslant c<1$, m is locally an analytic function of c and we may write $h_{p,q}(m)=h_{p,q}(c)=h(c)$ or $h_{q,p}(m)=h_{q,p}(c)=h(c)$. Fix c and consider $H_{n_1}(h,c)$ as a function of h near $h(c)$. Its eigenvalues are the roots of a polynomial equation with real analytic, indeed polynomial, coefficients and they are all real for h real. It is easily seen that this implies that there is no ramification at $h=h(c)$ and that in a neighborhood of this point there are expansions

$$\alpha_i(h)=\alpha_{i0}+\alpha_{i1}\big(h-h(c)\big)+\alpha_{i2}\big(h-h(c)\big)^2+\cdots,\ 1\leqslant i\leqslant P(n_1)$$

for the eigenvalues of H_{n_1}. Thus

$$\det H_{n_1}(h,c)=\prod_{i=1}^{P(n_1)}\Big(\alpha_{i0}+\alpha_{i1}\big(h-h(c)\big)+\cdots\Big),$$

and the power of $h-h(c)$ that divides it is greater than or equal to the number of zero eigenvalues of $H_{n_1}\big(h(c),c\big)$. On the other hand, the left side is equal to

$$A_n\prod_{k\leqslant n_1}\prod_{p_1q_1=k}\big(h-h_{p_1,q_1}(c)\big)^{P(n_1-k)},$$

and $h_{p_1,q_1}(c)=h(c)$ only if (p_1,q_1) or (q_1,p_1) lies in the boundary of D. Thus the assertion follows from Lemma 8.

Choosing $n_1=n$, we see in particular that the dimension of the null space of V_n is 1. Thus if $m>p'-1$ then in a neighborhood of $\big(h(m),c(m)\big)$ we can find an analytic function $v(h,c)$ with values in V_n such that $v(h,c)$ has length 1, is an eigenvector of $H_n(h,c)$, and corresponds to the eigenvalue 0 when (h,c) falls on the curve C.

Since

$$L_0v\big(h(m),c(m)\big)=\big(h(m)+n\big)v\big(h(m),c(m)\big),$$
$$L_kv\big(h(m),c(m)\big)=0,\quad k>0,$$

there is a homomorphism of $\mathfrak{v}$-modules, ϕ: $V^{h(m)+n,c(m)}\to V^{h(m),c(m)}$, taking $v_\phi^{h(m)+n,c(m)}$ to $v\big(h(m),c(m)\big)$. If it is injective on $V^{h(m)+n,c(m)}_{n_1-n}$ then $d^0_{n_1}\geqslant P(n_1-n)$ because the image consists of null vectors. Since $d^0_{n_1}$ is lower semicontinuous, $d^0_{n_1}$ will be greater than or equal to $P(n_1-n)$ everywhere on C if it is so on a dense set. The homomorphism ϕ will be injective if $\det H^{h(m)+n,c(m)}_{n_1-n}\neq 0$ because the kernel consists of null vectors. So it is enough to show that this determinant does not vanish identically on C. However, if $h(m)+n=h_{p_1,q_1}(m)$ then

$$\big((m+1)p+mq\big)^2=\big((m+1)p_1-mq_1\big)^2$$

or

$$\big(mp+(m+1)q\big)^2=\big((m+1)p_1-mq_1\big)^2.$$

This can occur for at most two values of m. □

It remains to show that at m' the dimension of the space of null vectors in $V_{n'}$ is $P(n'-n)+1$. For this we need further lemmas.

Lemma 10. $\det H^{h(m')+n,c(m')}_{n'-n}\neq 0$.

Proof. It has to be shown that the equality $h(m')+n = h_{p_1,q_1}(m')$, $p_1q_1 \leqslant n'-n$ is impossible. This equality amounts to

$$\text{(A)}\qquad \big(m'p + (m'+1)q\big)^2 = \big((m'+1)p_1 - m'q_1\big)^2$$

or

$$\text{(B)}\qquad \big((m'+1)p + m'q\big)^2 = \big((m'+1)p_1 - m'q_1\big)^2.$$

It is not supposed that $p_1 \geqslant q_1$.

The first equation implies that $m'p + (m'+1)q = \pm\big((m'+1)p_1 - m'q_1\big)$ or $m'(p \pm q_1) = (m'+1)(\pm p_1 - q)$. Since m' is an integer this implies $(p \pm q_1) = a(m'+1)$, $(\pm p_1 - q) = am'$. Since $n' = p'q' = (m'-q)(m'-p+1)$ the inequality $n' \geqslant n + p_1q_1$ becomes

$$(m'-q)(m'-p+1) \geqslant a(m'+1)q - am'p + a^2m'(m'+1)$$

or

$$\big((1+a)(m'+1) - p\big)\big((1-a)m' - q\big) \geqslant 0.$$

Since $m' = p' + q = p + q' - 1$, $m' > q$, $m'+1 > p$. So the inequality is possible only for $a = 0$, but a cannot be 0. The case (B) is treated in a similar fashion. □

For $n_1 < n'$ or $m \neq m'$ we let $U_{n_1} = U_{n_1}(m)$ be the space of null vectors in V_{n_1}. For h, c close to $h(m')$, $c(m')$ we let $U_{n_1}(h,c)$ be the span of

$$\Big\{\, L_{-k_1}\cdots L_{-k_r}v(h,c) \;\Big|\; k_1 \geqslant \cdots \geqslant k_r > 0, \sum k_i = n'-n \,\Big\}.$$

We set $U_{n'}(m) = U_{n'}\big(h(m), c(m)\big)$, the two definitions of $U_{n'}(m)$ coinciding when they both apply. Thus for $m > p'-1$, $U_{n_1}(m)$ is defined and analytic as a function of m. Let W_{n_1} be its orthogonal complement with respect to the form $\{\cdot,\cdot\}$. It follows from that part of Lemma 9 already proved that the restriction $J_{n_1} = J_{n_1}(m)$ of H_{n_1} to W_{n_1} is non-singular unless $n_1 = n'$, $m = m'$. In particular, our assumption, which was made for a particular m, implies that $J_{n_1}(m)$ is positive for all $m > p'-1$ if $n_1 < n'$.

Lemma 11. *Near m', $\det J_{n'}(m) = \delta(m)(m-m')$ where $\frac{1}{\delta} \geqslant |\delta(m)| \geqslant \delta > 0$.*

It will follow from this lemma that the remaining assertion of Lemma 9 is true. In addition the lemma together with our assumption on the non-negativity of $\langle\cdot,\cdot\rangle$ for a particular m, $p' > x(m) > p'-1$, will imply that the form takes negative values for $m > m'$ because $\det J_{n'}(m)$ changes sign at m'.

Let $v(h,c)$, defined in a neighborhood of $\big(h(m'), c(m')\big)$, correspond to the eigenvalue $\alpha(h,c)$ of $H_n(h,c)$. All the other eigenvalues of $H_n(h,c)$ are bounded above and, if the neighborhood is sufficiently small, away from 0. On the other hand, all factors $h - h_{p_1,q_1}(c) = h - h_{p_1,q_1}(m)$, $c = c(m)$, of $\det H_n(h,c)$ are bounded away from 0 in a neighborhood of $h(m')$, $c(m')$ except for $h - h(c)$, where $h(c)$ is $h_{q,p}(c)$ or $h_{p,q}(c)$ according as we are dealing with case A or case B. Thus we have the following lemma.

Lemma 12. *In a neighborhood of $\big(h(m'), c(m')\big)$ we have $\alpha(h,c) = a(h,c)\big(h - h(c)\big)$ with $\frac{1}{a} \geqslant |a(h,c)| \geqslant a > 0$, a being a constant.*

Here $h(c)$ is $h_{q,p}(m)$ (A) or $h_{p,q}(m)$ (B), $c = c(m)$. More generally we have

Lemma 13. *Let $K_{n'}(h,c)$ be the restriction of $H_{n'}(h,c)$ to $U_{n'}(h,c)$. Then, in a neighborhood of $\big(h(m'), c(m')\big)$, $\det K_{n'}(h,c) = k(h,c)\alpha(h,c)^{P(n'-n)}$, with $\frac{1}{k} \geqslant |k(h,c)| \geqslant k > 0$.*

Proof. The determinant of $K_{n'}(h,c)$ is that of the form $\langle\cdot,\cdot\rangle_{n'}$, calculated with respect to a basis of $U_{n'}(h,c)$ orthogonal with respect to the form $\{\cdot,\cdot\}_n$. However the basis

$$\Big\{\,\phi(v_\alpha)\;\Big|\;v_\alpha\in V^{h(m)+n,c(m)},\ n(\alpha)=n'-n\,\Big\}$$

is related to such a basis by a matrix whose determinant is bounded in absolute value above and below. So it is enough to consider $\det\Big(\big\{\phi(v_\alpha),\phi(v_\beta)\big\}\Big)$.

We have

$$\begin{aligned}\big\langle\phi(v_\alpha),\phi(v_\beta)\big\rangle &= \big\langle L_{\ell_s}\cdots L_{\ell_1}L_{-k_1}\cdots L_{-k_r}v(h,c),v(h,c)\big\rangle\\ &=\big\{L_{\ell_s}\cdots L_{\ell_1}L_{-k_1}\cdots L_{-k_r}v(h,c),H_{n'}(h,c)v(h,c)\big\}\\ &=\alpha(h,c)\big\{L_{\ell_s}\cdots L_{\ell_1}L_{-k_1}\cdots L_{-k_r}v(h,c),v(h,c)\big\}.\end{aligned}$$

At $h(m)$, $c(m)$ the value of $\det\Big(\big\{L_{\ell_s}\cdots L_{\ell_1}L_{-k_1}\cdots L_{-k_r}v(h,c),v(h,c)\big\}\Big)$ is

$$\det\Big(\langle v_\alpha,v_\beta\rangle_{n'-n}^{h(m')+n,c(m')}\Big).$$

By Lemma 10 this is not 0. Lemma 13 follows. □

In a neighborhood of $h(m)$, $c(m)$ we decompose $V_{n'}$ as an orthogonal sum $U_{n'}\oplus W_{n'}$. The linear transformation $H_{n'}(h,c)$, or its matrix with respect to a compatible basis, then decomposes into blocks. I claim that the entries in the off-diagonal blocks are $O\big(h-h_{p,q}(c)\big)$ in a neighborhood of $h(m)$, $c(m)$. To verify this it is sufficient, for the pertinent basis can be supposed to depend analytically on h, c, to verify that they are zero when $h=h_{p,q}(c)$, but that is clear by the definition of $U_{n'}$.

It follows that

$$\det H_{n'}(h,c)=\det J_{n'}(h,c)\det K_{n'}(h,c)+O\Big(\big(h-h_{p,q}(c)\big)^{P(n'-n)+1}\Big)\tag{1}$$

if $J_{n'}(h,c)$ is the matrix in the diagonal block corresponding to $W_{n'}$. Since

$$\det H_{n'}(h,c)=A_{n'}\prod_{k\leqslant n'}\prod_{p_1q_1=k}\big(h-h_{p_1q_1}(c)\big)^{P(n'-p_1q_1)}$$

we may divide the relation (1) by $\big(h-h_{p_1q_1}(c)\big)^{P(n'-n)}$ and then set $h=h_{p,q}(c)$, $c=c(m)$. The result clearly yields Lemma 11 because $h(m')=h_{p_1,q_1}(m')$, p_1, $q_1\leqslant n'$, only if (p_1,q_1) is (q,p) or (p',q') (case A) or (p,q) or (q',p') (case B).

Our assumption that $H_{n_1}\big(h(m),c(m)\big)$ is non-negative for a given m, $p'>m>p'-1$, has led to the conclusion that $J_{n_1}(m)$ is positive for large m and $n_1<n'$ but that $J_{n'}(m)$ has negative eigenvalues for large m. We show not that this is impossible.

As m approaches infinity, the point $\big(h(m),c(m)\big)$ approaches $(h_0,c_0)=\Big(\frac{(p-q)^2}{4},1\Big)$. If $p\neq q$ a suitable coordinate on the curve is $\mu=\frac{1}{m}$. If $p=q$ we may take $\mu=1-c$. All the matrices $H_{n_1}(\mu)=H_{n_1}(m)=H_{n_1}\big(h(m),c(m)\big)$ are analytic functions of μ. The eigenvalues of $H_{n_1}(\mu)$ are given by power series.

$$\alpha_i=\alpha_i(\mu)=\alpha_{i0}+\alpha_{i1}\mu+\alpha_{i2}\mu^2+\cdots$$

Let $V^1_{n_1}(\mu)$ be the space spanned by the eigenvectors corresponding to α_i with $\alpha_{i0}=0$; let $V^2_{n_1}(\mu)$ be the space spanned by the eigenvectors corresponding to α_i with $\alpha_{i0}=\alpha_{i1}=0$ and so on. One proves by induction that these spaces are well-defined, depend analytically on μ (in the sense that we have analytic functions $v_1(\mu), \ldots, v_{P(n_1)}(\mu)$, such that $\{v_1(\mu), \ldots, v_{d_k}(\mu)\}$, $d_k = \dim V^k_{n_1}$ forms a basis of $V^k_{n_1}(\mu)$ for each μ), and that $\mu^{-k}\{H_{n_1}(\mu)v_i(\mu), v_j(\mu)\}$, $i \leqslant d_k$, $j \leqslant P(n_1)$ is analytic for small μ. It can even be supposed that $\{H_{n_1}(\mu)v_i(\mu), v_j(\mu)\} = 0$, $i \leqslant d_k$, $j > d_k$.

Let $V^k = \bigoplus_{n_1} V^k_{n_1}(0)$ and $X^k = V^k/V^{k+1} = \bigoplus_{n_1} V^k_{n_1}(0)/V^{k+1}_{n_1}(0)$. If $u = \sum_{i \leqslant d_k} a_i v_i(0) \in V^k_{n_1}(0)$ and $v = \sum_{i \leqslant d_k} b_i v_i(0) \in V^k_{n_2}(0)$, define $\langle u, v\rangle^{(k)}$ to be 0 if $n_1 \neq n_2$, and if $n_1 = n_2$ set

$$\begin{aligned}\langle u, v\rangle^{(k)} = \langle u, v\rangle^{(k)}_{n_1} &= \sum a_i \bar{b}_j \lim_{\mu\to 0} \mu^{-k}\langle v_i(\mu), v_j(\mu)\rangle \\ &= \sum a_i \bar{b}_j \lim_{\mu\to 0}\Big\{\mu^{-k} H_{n_1}(\mu) v_i(\mu), v_j(\mu)\Big\}.\end{aligned}$$

It is clear that $H_{n_1}(\mu)$ is non-negative for small μ if and only if the forms $\langle u, v\rangle^{(k)}_{n_1}$ are all positive.

Lemma 14.

(*a*) *The spaces V^k are all invariant under $\pi = \pi^{h_0,c_0}$, so that $\mathfrak{v}$ operates on X^k.*

(*b*) *The form $\langle\cdot,\cdot\rangle^{(k)}$ on X^k satisfies $\langle L_m x, y\rangle = \langle x, L_{-m} y\rangle$, $m \in \mathbf{Z}$.*

Proof. Set $L_m(\mu) = \pi^{h(\mu),c(\mu)}(L_m)$ and $L_m = L_m(0)$. We have to show for each n_1 that $L_m v_i \in V^k$ if $v_i = v_i(0)$ and $i \leqslant d_k$. However

$$L_m v_i = \lim_{\mu\to 0} L_m(\mu) v_i(\mu) = \lim \sum_j c_{ij}(\mu) v'_j(\mu)$$

where the c_{ij} are analytic functions of μ. It is to be shown that $c_{ij}(0) = 0$ for $j > d'_k$. The primes refer to $n_2 = n_1 - m$ rather than to n_1. In other words it has to be shown that

$$\big\{H_{n_2}(\mu) L_m(\mu) v_i(\mu), v'_\ell(\mu)\big\} = O(u^k)$$

for all ℓ. Since $H_{n_2}(\mu)L_m(\mu) = L^*_{-m}(\mu)H_{n_1}(\mu)$, the adjoint of $L_{-m}(\mu)$ being taken with respect to the form $\{\cdot,\cdot\}$, this is clear. So is the second assertion of the lemma. □

For any $h \geqslant 0$ the representation $\pi^{h,1}$ on $V^{h,1}$ has a unique irreducible quotient $\rho^{h,1}$ on $X^{h,1}$, which by Lemma 3 carries a hermitian form for which $\rho^{h,1}$ is unitary in the sense that the adjoint $\rho^{h,1}(L_m)$ is $\rho^{h,1}(L_{-m})$. Such a form is unique up to a scalar multiple. Take in particular $h = \frac{r^2}{4}$, $r \in \mathbf{Z}$. Then $h = h_{p_2,q_2}(c)$ if and only if $(p_2 - q_2)^2 = r^2$. In particular, $h = h_{r+1,1}(c)$. Thus the lowest weight for a null vector in V is $r+1$ and $h + r + 1 = \frac{(r+2)^2}{4}$, so that the kernel of $V^{h,1} \to X^{h,1}$ contains a quotient of $V^{h',1}$, $h' = \frac{(r+2)^2}{4}$. Thus $V^{h,1}$ admits a sequence of invariant subspaces $V^{h,1} = V^{h,1}(0) \supseteq V^{h,1}(1) \supseteq V^{h,1}(2)$ such that the representation on $V^{h,1}(0)/V^{h,1}(1)$ is $\rho^{h,1}$ and that on $V^{h,1}(1)/V^{h,1}(2)$ is $\rho^{h',1}$. In general set $h^{(\ell)} = \frac{1}{4}(r+2\ell)^2$.

Lemma 15. *$V^{h,1}$ admits an infinite decomposition series $V^{h,1}(0) \supseteq V^{h,1}(1) \supseteq \cdots \supseteq V^{h,1}(\ell) \supseteq \cdots$ such that the representation on the quotient $V^{h,1}(\ell)/V^{h,1}(\ell+1)$ is $\rho^{h(\ell),1}$.*

Proof. If $\lambda = h + k$, $k \in \mathbf{Z}$, $k \geqslant 0$, let

$$\begin{aligned} d_\lambda &= \dim\Big\{ v \in V^{h,1} \;\Big|\; L_0 v = \lambda v \Big\}, \\ d_\lambda(\ell) &= \dim\Big\{ v \in X^{h(\ell),1} \;\Big|\; L_0 v = \lambda v \Big\}. \end{aligned}$$

The lemma follows easily from a formula of Kac ([2], Th. 5), according to which $d_\lambda = \sum_{\ell=0}^{\infty} d_\lambda(\ell)$. Indeed, suppose we have constructed an initial segment of the series $V^{h,1}(0) \supset \cdots \supset V^{h,1}(\ell)$. Then $\frac{1}{4}(r+2\ell)^2$ is a lowest weight in $V^{h,1}(\ell)$ and

$$\dim\left\{ v \in V^{h,1}(\ell) \,\middle|\, L_0 v = \frac{1}{4}(r+2\ell)^2 \right\} = 1.$$

Take $V^{h,1}(\ell+1)$ to be the sum of all invariant subspaces of $V^{h,1}(\ell)$ for which the lowest weight is greater than $\frac{1}{4}(r+2\ell)^2$. □

Now take $r = p - q$. It follows immediately from the preceding lemma that X^k is the direct sum of irreducible invariant subspaces X_j^k carrying distinct representations and that the restriction of $\langle \cdot, \cdot \rangle^k$ to X_j^k is either positive or negative. The assumption that we are trying to contradict implies that the form is positive if X_j^k contains non-zero vectors of weight $h + n_1$, $n_1 < n'$, but that for some j and k for which X_j^k contains vectors of weight $h + n'$, it is negative.

Thus the following lemma completes the proof of Theorem FQS.

Lemma 16. *The equation* $\frac{r^2}{4} + n' = \frac{1}{4}(r+2\ell)^2$ *has no solution* $\ell \geqslant 0$ *in* $\mathbf{Z}$.

Proof. The equation may be written as $n' = \ell(\ell + r)$. Recall that n' is $(p+a)(q+a+1)$ in case A and $(p+a+1)(q+a)$ in case B, with $a \geqslant 0$. Since $r = p - q$, the equation is $(p+a+\ell)(q+a+1-\ell) = \ell$ or $(p+a+1+\ell)(q+a-\ell) = -\ell$. Both equations are manifestly impossible. □

References

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Compiled on February 14, 2025.