ON ABEL'S DIFFERENTIAL EQUATIONS.*

By Phillip A. Griffiths.**

1. **Introduction.** In this talk I would like to describe two related variations on the classical *Abel theorem* concerning the abelian integrals attached to an algebraic curve. Both results have as common theme that the presence of *addition theorems* (loosely interpreted) or *functional equations* may have rather striking consequences—both local and global—on a geometric configuration.

My interest in these affairs first arose in thinking about threefolds-i.e., algebraic varieties of dimension three. Here the theory of divisors and linear systems which served so beautifully in the study of curves and surfaces does not seem to suffice. Rather, the deeper analysis of specific threefolds appears to frequently require understanding of the curves on the variety. One explanation for this is cohomological: Due to the Lefschetz theorems the "new" cohomology on a smooth projective variety V of dimension n is in $H^n(V)$,[1] and this cohomology relates to the subvarieties of dimension $[n/2]$. For $n=1$ and 2 we have divisors, but for $n \geqslant 3$ one is faced with higher codimensional cycles. For this—or whatever reason—it seemed to me that general methods, probably of a less linear nature, were required for understanding specific geometric questions in higher codimension. The possibility of using "addition theorems" to study the integral varieties of Abel's differential equations provides the motivation for the results discussed below.

Before getting more into specifics, it might be worthwhile to try and cast some perspective on what will be discussed. Birationally attached to an algebraic variety V is the algebra $\Omega^*(V) = \bigoplus_{q>0} \Omega^q(V)$ of regular differentials. These are extremely important invariants of the variety, ones

*Talk presented at the "J. J. Sylvester Symposium on Algebraic Geometry" held at Johns Hopkins University, April 21–23, 1976.

**Miller Institute for Basic Research in Science, Berkeley and Harvard University. Research partially supported by NSF Grant MPS 72-05154.

[1]The so-called hard Lefschetz theorem reduces the study of $H^*(V)$ to $H^k(V)$ for $k<n$, and then the Lefschetz hyperplane theorem tells us that for $k \leqslant n-1$ $H^k(V)$ lies on an $(n-1)$—dimensional variety.

which may be said to have three faces:

(i) *Cohomology classes*. The regular differentials define the holomorphic part $H^{*,0}(V) = \bigoplus_{q>0} H^{q,0}(\tilde{V})$ of the *Hodge decomposition* for any non-singular model $\tilde{V}$ of V.

(ii) *Homogeneous coordinates*. Taking for simplicity the n-forms $\Omega^n(V)$ where $n=\dim V$, we may use any basis $\omega_0, \dots, \omega_N$ to define a rational map

$$[\omega_0, \dots, \omega_N] : V \to \mathbf{P}^N,$$

the so-called *canonical mapping*, which frequently gives an intrinsic projective model of V; and

(iii) *Differential equations*. For a variable 0-cycle $\Gamma = P_1 + \cdots + P_d$ the *abelian differential equations*[2]

$$\omega(P_1) + \cdots + \omega(P_d) \equiv 0, \qquad \omega \in \Omega^*(V) \tag{1.1}$$

define birationally invariant differential systems on the various symmetric products $V^{(d)}$ of V. For curves we have an essentially linear theory which amounts to the classical Abel theorem, but for higher dimensional varieties the differential systems are generally non-linear. It is probably for this reason that, although the first two facets of the regular differentials have been the subject of considerable study, not much attention has been directed toward the integrals of these abelian differential equations (1.1). It is the possibility of using general theorems to understand at least the maximal ones among these integral varieties which is at the back of this work.

It is a pleasure to thank many colleagues, especially Bernard Saint-Donat, Joe Harris, and S. S. Chern for helpful discussions on the subject of Abel's theorem.

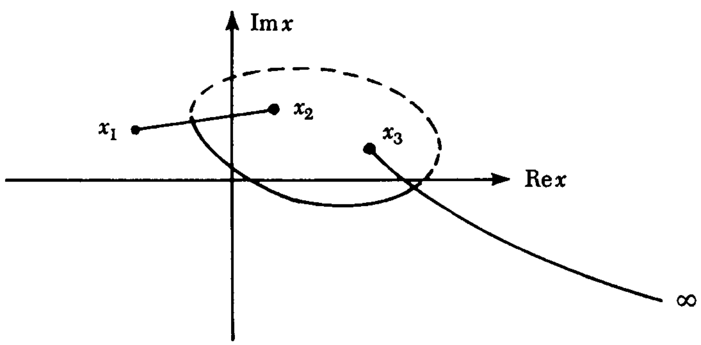
2. **The Classical Elliptic Integral.** A good place to begin is with the inversion of the classical elliptic integral following the original method of Abel, which is now just 150 years old. Suppose that we consider a non-singular cubic curve C in the complex projective plane. For example,

[2]The notation is explained in Section 2.

we may assume that C has affine equation

$$f(x,y)=y^2-p(x)=0$$

where $p(x)=(x-x_1)(x-x_2)(x-x_3)$ is a cubic with distinct roots. The Riemann surface associated to C is a two-sheeted covering of the x-plane branched at the x_ν and ∞



and consequently has genus one. We consider the *elliptic integral*

$$u(P)=\int_{P_0}^{P}\omega \tag{2.1}$$

where

$$\omega=dx/y=dx/\sqrt{p(x)}\ .$$

It is easily verified that ω is everywhere regular and non-zero on the Riemann surface, and consequently $u(P)$ is locally holomorphic and globally well-defined modulo the lattice Λ in $\mathbf{C}$ generated by the two periods of ω. Integrals such as (2.1) arose frequently in problems in geometry and mechanics in the 18th century, but they caused a lot of trouble since they were not expressible in terms of elementary functions. The key to understanding general integrals of rational differentials on algebraic curves was provided by Abel's general addition theorem, and we shall apply his method to study (2.1).

The point is to consider an *abelian sum* $\sum_i u(P_i)$ where the P_i are the points of intersection of C with a variable curve D. For instance, suppose L is a variable line in the plane meeting C in three points $P_i(L)$. Then a

special case of *Abel's theorem* states that the abelian sum

$$\sum_i u(P_i(L))\equiv \text{Constant} \quad (\text{modulo periods}). \tag{2.2}$$

His proof was elementary using only calculus, and in this instance runs as follows: Suppose that $L=L(a,b)$ has equation

$$y=ax+b$$

where (a,b) are affine coordinates in the dual projective plane $\mathbf{P}^{2^*}$ of lines in $\mathbf{P}^2$. Assuming that $L(0,0)$ meets C in three distinct points, for small a and b the points of intersection are of the form

$$P_i(a,b)=(x_i(a,b),y_i(a,b))$$

where the x_i are the roots of the equation

$$F(x,a,b)=(ax+b)^2-p(x)=0$$

and $y_i=ax_i+b$. Differentiation of the abelian sum (2.2) gives

$$\begin{aligned}\frac{\partial}{\partial a}\Big(\sum_i u(P_i(a,b))\Big)&=\frac{\partial}{\partial a}\Big(\sum_i\int_{(x_0,y_0)}^{(x_i,y_i)}dx/y\Big)\\ &=\sum_i\frac{\partial x_i(a,b)}{\partial a}\Big/y_i(a,b).\end{aligned}$$

Now then

$$F(x_i(a,b),a,b)\equiv 0,$$

and so

$$F'\frac{\partial x_i}{\partial a}+\frac{\partial F}{\partial a}=0$$

where $F'=\dfrac{\partial F}{\partial x}$. Since $\dfrac{\partial F}{\partial a}=2(ax+b)x$

$$\begin{aligned}\frac{\partial x_i}{\partial a}\Big/y_i&=-\frac{\partial F}{\partial a}\Big/F'y_i\\ &=-2x_i/F'(x_i)\end{aligned}$$

and consequently

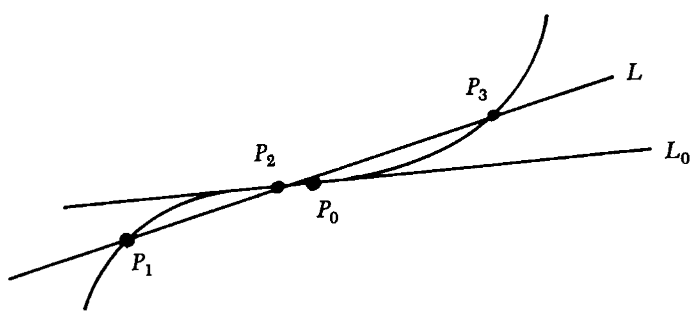
$$\frac{\partial}{\partial a}\Big(\sum u(P_i(a,b))\Big) = -2\bigg(\sum_i \frac{x_i}{F'(x_i)}\bigg).$$

At this juncture we recall the *Lagrange interpolation formula*

$$\sum_i \frac{g(x_i)}{h'(x_i)} = 0 \tag{2.4}$$

where $g(x), h(x)$ are polynomials with $\deg g \leqslant \deg h - 2$ and the x_i are the roots of $h(x)$. Comparing (2.3) and (2.4) gives that $\partial/\partial a$ of the abelian sum (2.2) is zero, and similarly $\partial/\partial b$ of it also vanishes. Q.E.D.

We may use Abel's theorem to construct the elliptic functions as follows: Let L_0 be a *flex tangent*—i.e., a line with third order contact—to C at some point P_0, so that $L_0 \cdot C = 3P_0$. Elementary algebra gives that there are nine of these.



Lines L close to L_0 will meet C in three points P_i close to P_0, and by (2.2)

$$u(P_1) + u(P_2) + u(P_3) = 0.$$

Using the ordinary inverse function theorem and $\omega(P_0) \neq 0$ we may define a local parametrization

$$P(u) = (x(u), y(u))$$

of C near P_0 by inverting the elliptic integral (2.1) according to the relation

$$u = \int_{P_0}^{P(u)} \omega.$$

Here u varies in a disc $\Delta_\epsilon = \{u \in \mathbf{C} | |u| < \epsilon\}$. Now, for u_1 and u_2 in say $\Delta_{\epsilon/2}$ the line the line $\overrightarrow{P(u_1)P(u_2)}$ will meet C in a third point $P(u_3)$ whose coordinates are visibly *rational* functions of those of $P(u_1)$ and $P(u_2)$. On the other hand by Abel's theorem $u_1 + u_2 + u_3 = 0$. Consequently, we obtain a functional equation

$$\begin{cases} x(-(u_1+u_2)) = R\,(x(u_1), y(u_1), x(u_2), y(u_2)) \\ y(-(u_1+u_2)) = S\,(x(u_1), y(u_1), x(u_2), y(u_2)) \end{cases}$$

valid for u_1, u_2 in $\Delta_{\epsilon/2}$ and where R, S are rational functions. Taking $u_1 = u_2$ we find a *duplication formula* which allows us to express $x(2u)$ and $y(2u)$ rationally in terms of $x(-u)$ and $y(-u)$. Then we may use this relation to analytically continue $x(u), y(u)$ to $\Delta_{2\epsilon}$, then to $\Delta_{4\epsilon}$, etc. In this way we arrive at entire meromorphic functions which are easily seen to be doubly periodic with period lattice Λ and which satisfy

$$u \equiv \int_{(x_0, y_0)}^{(x(u), y(u))} dx/y \quad (\Lambda) \tag{2.5}$$

$$\begin{vmatrix} 1 & x(u_1) & y(u_1) \\ 1 & x(u_2) & y(u_2) \\ 1 & x(u_3) & y(u_3) \end{vmatrix} = 0 \Leftrightarrow u_1 + u_2 + u_2 \equiv 0\,(\Lambda). \tag{2.6}$$

These are of course just the Weierstrass elliptic functions, here constructed by elementary reasoning based on Abel's theorem. The essential properties of these functions are immediate consequences of (2.5) and (2.6): Differentiation of (2.5) gives

$$1 = x'(u)/y(u)$$

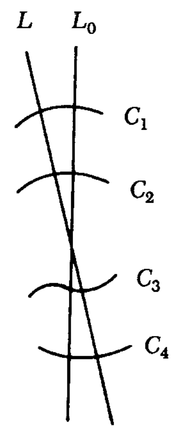
which implies the Weierstrass differential equation

$$x'(u)^2 = p(x(u)).$$

Then substituting $y(u) = x'(u)$ in (2.6) gives the addition theorem for $x(u)$ if we note that $x(-u) = x(u)$.

3. The Generalized Sophus Lie Theorem. The philosophy underlying the preceding discussion of the inversion of the elliptic integral may be summarized as stating that the presence of a suitable functional equation allows one to propagate a locally given analytic object into one which is global. We shall now give a theorem which is a geometric variant of this philosophy. The result had its origins in some work of Sophus Lie in which he wanted to characterize surfaces of double translation type as being the theta divisor of plane quartic curves. The Sophus Lie theorem has recently been deeply studied and extended by Bernard Saint-Donat, and it was he who first got me interested in this whole business. The result I shall describe is proved in my paper "Variations on a theorem of Abel", Inventiones Math., vol. 35 (1976), pp. 321–390.

Lie's theorem was based on the case $n=4$ of the following analytical result: *Suppose we are given n distinct points P_i on a line L_0 in the projective plane $\mathbf{P}^2$, and germs of analytic arcs C_i meeting L_0 transversely at P_i.*



Assume, moreover, that we are given local parameters t_i on C_i such that the addition theorem

$$\sum_{i=1}^{n} t_i(C_i \cdot L) = \text{Constant}$$

is valid for lines L in a neighborhood U of L_0 in the dual projective plane $\mathbf{P}^{2*}$. Then there is a plane algebraic curve C and rational differential ω on this curve such that each C_i is a piece of C, and for P on C_i

$$t_i(P) = \int_{P_0}^{P} \omega + \text{Constant}.$$

It should be recalled that, in case C is non-singular and ω is an everywhere holomorphic differential on the curve, then a general form of Abel's theorem gives

$$\sum_i \int_{P_0}^{P_i(L)} \omega \equiv K \quad \text{(modulo periods)}$$

where K is a constant and

$$L \cdot C = \sum_i P_i(L)$$

are the variable points of intersection of a line L with C. So the above is a kind of converse to Abel's theorem. When Saint-Donat told me the Sophus Lie theorem, I was quite struck by the beauty of the result and in trying to find a proof for myself was led to a generalization which will now be explained.

First we need to recall the notion of the *trace* of a differential form relative to a family of zero-cycles on a variety. Suppose that V is such a variety, say of dimension n and with no assumptions about non-singularity, and that ω is a rational q-form on V. A family of zero-cycles, written

$$\Gamma(t) = P_1(t) + \cdots + P_d(t) \qquad (t \in T),$$

is given by a parameter variety T and subvariety $I \subset T \times V$ such that the projection

$$I \xrightarrow{\pi} T$$

is generically finite. Thus, for generic $t \in T$ the intersection

$$I \cdot (t \times V) = t \times \Gamma(t)$$

where $\Gamma(t)$ is a set of points $P_i(t)$ as above. The trace of ω is the rational q-form ω on T defined by

$$\omega(t) = \omega(P_1(t)) + \cdots + \omega(P_d(t)) \tag{3.1}$$

where $\omega(P_i(t))$ is the pullback to T (defined locally of course) of the map $t \to P_i(t)$. We naturally assume that none of the points $P_i(t)$ varies entirely in the polar divisor of ω, since otherwise (3.1) would not make sense. In fact, if we say that the family of zero-cycles is *non-degenerate* in case each $P_i(t)$ is not constrained to lie in a subvariety of V—i.e., around a generic point it covers a little open set on V—then we may always form the trace. For our purposes one may think of the families of zero-cycles as being non-degenerate.

Alternatively, a family of zero-cycles is given by a rational map

$$\Gamma : T \to V^{(d)}$$

into the d-fold symmetric product

$$V^{(d)} = \underbrace{V \times \cdots \times V}_{d\text{-times}} / (\text{permutations})$$

whose points $\Gamma = P_1 + \cdots + P_d \in V^{(d)}$ are just the set of *all* zero-cycles of degree d. The form ω on V induces a form ω on the symmetric product by

$$\omega(\Gamma) = \omega(P_1) + \cdots + \omega(P_d),$$

and the trace is just $\omega(\Gamma(t))$.

A basic fact is that holomorphic forms may be defined for general varieties and have the properties (i) that the pullback of a holomorphic form under a rational map is again holomorphic, and (ii) that the induced form ω on the symmetric product of a holomorphic form ω on the original variety is again holomorphic. Consequently, the trace of a holomorphic form is again holomorphic.

For example, suppose that our variety is the cubic plane curve discussed in Section 2 above, that T is the dual projective plane $\mathbf{P}^{2*}$ of lines L in $\mathbf{P}^2$, and that our family of zero-cycles are the residual intersections

$$\Gamma(L) = L \cdot C.$$

Then, in the notation from that section, the trace

$$\begin{aligned}\omega(a,b) &= \sum_i \omega(x_i(a,b), y_i(a,b)) \\ &= \sum_i dx_i(a,b)/y_i(a,b) \\ &= d\left(\sum \int_{(x_0,y_0)}^{(x_i,y_i)} \omega\right) \\ &= d\left(\sum u(P_i(L))\right)\end{aligned}$$

is the differential of the abelian sum. Since ω is holomorphic on C, ω is holomorphic on $\mathbf{P}^{2*}$ and consequently $\omega \equiv 0$. This is Abel's theorem again, proved here from the local properties of the trace and global fact that $H^0(\mathbf{P}^{2*}, \Omega^1) = 0$.

In general, we shall say that ω is *of the first kind relative to the family* $\Gamma(t)$ *of zero-cycles* if the trace (3.1) is holomorphic on T. If this is the case and if $H^0(T, \Omega^q) = 0$, then we obtain a general form

$$\sum_i \omega(P_i(t)) \equiv 0 \tag{3.2}$$

of Abel's theorem. We may view (3.2) as a sort of addition theorem linking together the local behavior of the branches of V around the points $P_i(t)$. The relation has been expressed in differential form since we cannot take the indefinite integral of a q-form for $q > 1$.

As an example suppose that V is embedded in $\mathbf{P}^{n+k}$ and T is the Grassmannian $\mathbf{G}(k, n+k)$ of projective k-planes in $\mathbf{P}^{n+k}$. A general such k-plane A will meet V in $d =$ degree of V points $P_i(A)$, and we write $A \cdot V = \sum_i P_i(A)$. The incidence coorespondence

$$I \subset \mathbf{G}(k, n+k) \times V$$

is defined as $\{(A, P) : P \in V \cap A\}$ and so $\pi : I \to \mathbf{G}(k, n+k)$ is generically finite with fibre $A \cdot V$. If ω is a rational q-form on V which is of the first kind relative to the zero-cycles $A \cdot V$, then

$$\sum_i \omega(P_i(A)) \equiv 0. \tag{3.3}$$

In case V is smooth, ω is of the first kind here exactly when it is holomorphic on the complex manifold V. On the other hand, suppose we consider the situation where $V \subset \mathbf{P}^{n+1}$ is a hypersurface with affine equation

$$f(x_1, \ldots, x_n, y) = f(x, y) = 0.$$

If we write a rational n-form as

$$\omega = \frac{r(x,y)\, dx_1 \wedge \cdots \wedge dx_n}{\partial f / \partial y(x,y)}, \tag{3.4}$$

then ω *is of the first kind relative to the lines in* $\mathbf{P}^{n+1}$ *if and only if* $r(x,y)$ *is a polynomial of degree* $d - n - 2$.

The proof of this last assertion is based on a rather nice formula which we now explain. Suppose that

$$x = ay + b$$

gives the equation of a line $L(a,b)$ in $\mathbf{P}^{n+1}$. Here x, a, and b are vectors and (a,b) are affine coordinates in $\mathbf{G}((1,n+1)$ (recall that $\dim \mathbf{G}(k,n+k) = (k+1)n$). Now then the intersection

$$L(a,b)\cdot V = \sum_i P_i(a,b)$$

where

$$P_i(a,b) = (x_i(a,b), y_i(a,b))$$

satisfies $x_i(a,b) = ay_i(a,b) + b$. For any function $q(x,y)$ we set $Q(y) = q(ay+b,y)$. Then our formula for the trace of ω as given by (3.4) is

$$\omega(a,b) = \sum_J \pm\left(\sum_i \frac{R(y_i)}{F'(y_i)}\, y_i^{|J|}\right) da_J \wedge db_{J^c} \tag{3.5}$$

where $J = (j_1, \ldots, j_q)$ runs over all increasing index sets selected from $(1,\ldots,n)$, $da_J = da_{j_1} \wedge \cdots \wedge da_{j_q}$, J^c is the complementary index set, and $|J| = q$ is the number of elements in J. Our assertion about ω being of the first kind follows from (3.5) and the Lagrange interpolation formula (2.4).

The result we set out to describe is a converse to (3.3). Here is the precise statement:

THEOREM 1. *Suppose that A_0 is a k-plane in $\mathbf{P}^{n+k}$ and $V_1, \ldots, V_d$ are germs of n-dimensional analytic varieties each meeting A_0 in a single point. Suppose moreover that we are given holomorphic n-forms $\omega_i \neq 0$ on V_i such that the addition theorem*

$$\sum_i \omega_i(A\cdot V_i) \equiv 0 \tag{3.6}$$

is valid for k-planes A varying in a neighborhood of A_0. Then there is an algebraic variety V in $\mathbf{P}^{n+k}$ and rational n-form ω on V, of the first kind relative to the zero-cycles $A\cdot V$ ($A \in \mathbf{G}(k,n+k)$), and with $V_i \subset V$ and $\omega | V_i = \omega_i$.

It is perhaps of interest to outline the proof of this theorem. For simplicity of notation we take the case $k = n = 1$ of plane curves.

The crucial observation is that what is being sought is the *pair* $\{V,\omega\}$, and moreover this pair is equivalent to giving the rational 2-form

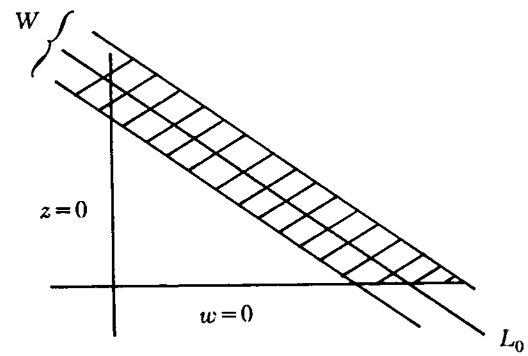
$$\Omega = \frac{p(x,y)\,dx\wedge dy}{f(x,y)} \tag{3.7}$$

on $\mathbf{P}^2$ whose polar variety is V and whose *Poincaré residue*

$$\mathrm{Res}_V\,\Omega = \frac{p(x,y)\,dx}{\partial f/\partial y(x,y)}\ {}^{3}$$

is ω. Here the right-hand side is restricted to V.

Conversely, suppose that we can find a neighborhood W of L_0 in $\mathbf{P}^2$ and meromorphic 2-form Ω in W with polar divisor $\sum_i V_i$ and $\mathrm{Res}_{V_i}(\Omega) = \omega_i$. Then, taking L_0 to be the line at infinity in $\mathbf{P}^2$ and shrinking W so that $\mathbf{P}^2 - W$ is the complement of a ball in $\mathbf{C}^2$,



a variant of the Levi-Hartogs theorem gives a meromorphic extension of Ω to all of $\mathbf{P}^2$. This will be our desired form (3.7), and so the problem is to construct Ω on W.

Next, we change the picture slightly and assume that L_0 is now the y-axis $\{x=0\}$ and that the lines $L(a,b)$ given by

$$x = ay + b, \qquad |a|,|b| < \epsilon$$

[3]Poincaré residues are discussed in the paper "Variations on a theorem of Abel" mentioned at the beginning of this section. Since $(\partial f/\partial x)dx + (\partial f/\partial y)dy \equiv 0$ on V, we note that on V

$$\frac{p\,dx}{\partial f/\partial y} = -\frac{p\,dy}{\partial f/\partial x}.$$

fill out W. We write the intersection

$$L(a,b)\cdot V_i=(x_i(a,b),y_i(a,b))=P_i(a,b).$$

Now consider the disjoint union

$$I_W=\bigcup_{|a|,|b|<\epsilon} L(a,b)$$

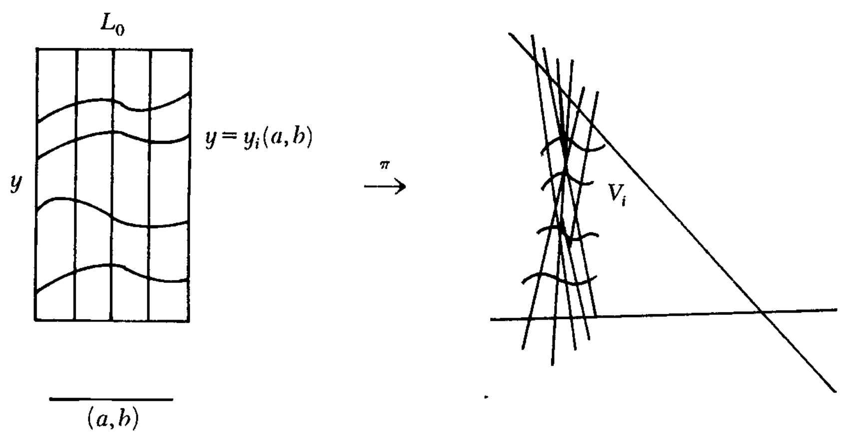
of the lines in W. This is just the restriction of the incidence correspondence $I\subset \mathbf{P}^{2*}\times\mathbf{P}^2$ to W, and there is a fibering

$$I_W\xrightarrow{\pi} W$$

with one dimensional fibers. The surfaces

$$y=y_i(a,b)$$

in I_W project onto V_i, so that the picture is something like



Taking (y,a,b) as product coordinates in I_W, since

$$\pi^*dx\wedge dy=(y\,da+db)\wedge dy$$

if our form Ω exists then it will be uniquely given on I_W by an expression

$$\pi^*\Omega=\frac{h(y,a,b)(y\,da+db)\wedge dy}{\prod_i(y-y_i(a,b))} \tag{3.8}$$

where $h(y,a,b)$ is holomorphic in a,b and rational in y. We may determine the function h by noting that along the line $L(a,b)$

$$\frac{h(y,a,b)\,dy}{\prod_i(y-y_i(a,b))}=\sum_i\rho_i(a,b)\frac{dy}{y-y_i(a,b)} \tag{3.9}$$

where the $\rho_i(a,b)$ are given by

$$\rho_i(a,b)=\operatorname{Res}_{P_i(a,b)}\left(\frac{\omega_i}{x-ay-b}\right). \tag{3.10}$$

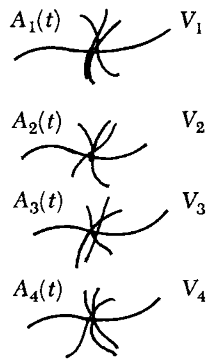
Combining (3.8)–(3.10) gives a unique candidate Ω_W for $\pi^*\Omega$, and it is not difficult to see that Ω_W is of the form $\pi^*\Omega$ if and only if the exterior derivative

$$d\Omega_W\equiv 0. \tag{3.11}$$

The last step, which is not difficult but is a little miraculous in how elegantly it works out, is proved by a computation based on (3.5) which shows that (3.11) is equivalent to the addition theorem (3.6).

4. Abel's Theorem and Webs. Now even though Theorem 1 does give a converse to an Abel type theorem, it seems pretty reasonable to expect deeper results when one deals with *all* the differentials of the first kind and not just one among them. Indeed, such considerations will be essential in order to hope to come to grips with the integral varieties of Abel's differential equations (1.1). One recalls that a proper understanding of inversion of the abelian integrals $\int r(x,y)\,dx$ of the first kind on an algebraic curve $f(x,y)=0$ is achieved only by simultaneously inverting all of them. As a first approximation, what is needed here is a Sophus Lie type theorem where one is given the local pieces of analytic variety V_i as before, but where the global linear spaces are replaced by a family $A_i(t)$

($t \in T$) of pieces of analytic variety in correspondence



and where a number of abelian equations

$$\sum \omega_{\alpha,i}(A_i(t)\cdot V_i)=0 \qquad (\alpha=1,\ldots,r)$$

are satisfied.

Now it so happens that related questions, although in quite a different guise, were extensively discussed in the 1930's by Blaschke and his school in Hamburg. They were interested in the subject of *webs*, and there is a book by Blaschke-Bol entitled "Geometrie die Gewebe" published by Springer-Verlag in which they present their theory. Chern was a student in Hamburg toward the end of this period, and he wrote a thesis on webs which appeared in two papers in the Abhandlung Math Seminar[4] of Hamburg. The first of these deals with the subject of Abel's theorem and webs, which occupied the last third of the Blaschke-Bol book concluding with a rather beautiful theorem due to Bol, one which in fact used the Sophus Lie addition theorem. Since I was spending the year at Berkeley it was quite natural that Chern and I should begin discussing these matters, and we were able to clarify and find a generalization of Bol's theorem which I would like to describe now.[5] The method of proof of this result is of a rather general geometric character, and it is hoped that the techniques will prove useful in dealing with the abelian differential equations (1.1); indeed this has already happened in a few special cases.

---

[4]"Abzahlungen für Gewebe" Abh. Math. Sem. Hamburg (1935), pp. 163–170 and "Eine Invariantentheorie der Dreigewebe aur r-dimensionalen Mannigfaltigkeiten im R^{2r}", *loc. cit* (1936), pp. 333–358.

[5]This result will appear in a joint paper with Chern entitled "Abel's theorem and webs" to appear in Deutsche Math. Ver.

The questions to be discussed are local in nature, and so will take place in a sufficiently small open set U in $\mathbf{R}^n$ or in $\mathbf{C}^n$. In the first case we will work with C^∞ functions, and with holomorphic functions in the second—in either case we shall typically write such functions as $u(x)=u(x_1,\ldots,x_n)$ where $(x_1,\ldots,x_n)$ is an appropriate coordinate system in U. In the complex analytic case no use will be made of the conjugate variables $\bar{x}_\alpha$ or Cauchy-Riemann equations $\dfrac{\partial u}{\alpha \bar{x}_\alpha}=0$, so that it won't really be necessary to specify which situation we are discussing.

Recall that a *foliation* of U by codimension k submanifolds is according to the Frobenius theorem given by a collection of everywhere linearly independent 1-forms

$$\omega_1(x),\ldots,\omega_k(x)$$

which satisfy the complete integrability condition

$$d\omega_\nu \equiv \{\omega_1,\ldots,\omega_k\}$$

where $\{\omega_1,\ldots,\omega_k\}$ is the ideal generated by the ω_ν's. The linear independence is expressed by the non-vanishing of the differential form

$$\Omega(x)=\omega_1(x)\wedge\cdots\wedge\omega_k(x),$$

and the complete integrability by

$$d\omega_\nu\wedge\Omega\equiv 0.$$

We may think of $\Omega(x)$ as being the *normal* to the leaves of the foliation. The tangent $(n-k)$-planes are defined either by the equations

$$\langle\omega_\nu,\xi\rangle=0$$

or

$$\xi \mathbin{\overline{\wedge}} \Omega=0$$

where ξ is a tangent vector and $\overline{\wedge}$ is the contraction operator. Sometimes we shall simply write

$$\omega_\nu(x)=0$$

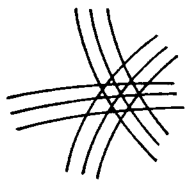
as defining the equations of the leaves in the foliation.

Here is the basic

Definition. *A d-web of codimension k is given by d foliations of U by codimension k submanifolds.*

It is always assumed that the web is *non-degenerate* in the sense that the tangent $(n-k)$–planes to the d leaves of the foliation passing through any point of U are planes in general position.

For example, when $k=1$ and $n=2$ a d-web in an open set U in the plane is given by d families of curves meeting pairwise transversely



$d=3$

In general, when we are discussing the codimension one case we shall only speak of a *d-web.*

In a few cases a web may be put in standard local form. For example, when $d=1$ the web is equivalent to the usual linear foliation

$$\{x_1=\text{const},\ldots,x_k=\text{const}\}.$$

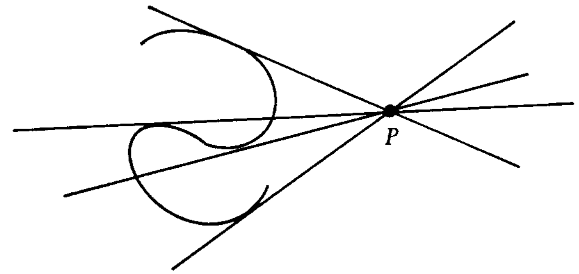
On the other hand, in the case $k=1$ of hypersurface webs the leaves may be assumed to be given by the level sets

$$u_i(x)=\text{constant} \qquad (i=1,\ldots,d)$$

of suitable functions. Since the web is non-degenerate, when $d \leqslant n$ we may take the $u_i(x)$ as part of a coordinate system and thereby put the web in the standard form consisting of d families of parallel hyperplanes.

In general, however, there will be local invariants of a web and it was the study of these which interested Blaschke and his colleagues. A basic question concerns the linearization of a web. Suppose that we say that a web is *linear* in case the leaves of all the foliations are (pieces of) linear subspaces of $\mathbf{R}^n$, and that a web is *linearizable* in case it is diffeomorphic to a linear web. A natural question is to find conditions when a web is linearizable. As an example of an interesting linear web, let C be an algebraic plane curve and recall that through a general point P

there will pass $d=$ *class* of C distinct tangent lines to the curve



As P varies in an open set we have found a linear d-web. Note that the various families of lines are not all parallel. There are some beautiful pictures of such webs when $d=3$ in the book by Blaschke-Bol.

As another type of a web coming from algebraic geometry, we consider a k-dimensional variety V and family

$$\Gamma(t)=P_1(t)+\cdots+P_d(t) \qquad (t\in T)$$

of zero-cycles as discussed in the previous section. Here T may be either a global algebraic variety or may be an open set in $\mathbf{C}^n$. We do make the non-degeneracy assumption that for a general point of T the $P_i(t)$ are distinct and the mapping

$$t\rightarrow P_i(t)$$

has rank k, so that the points each vary in an open set on V. Now then, fixing a point $P_i(t_0)$ defines a codimension-k submanifold passing through the point t_0 in T. In this way the d equations

$$P_i(t)=P_i(t_0)$$

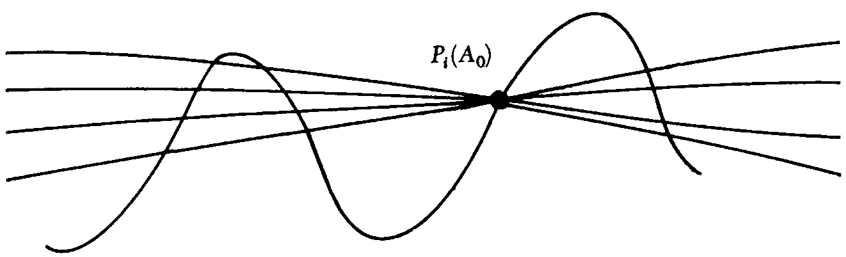
define a d-web of codimension k in the neighborhood of a general point in T. We shall call this the web defined by the family $\{\Gamma(t)\}_{t\in T}$ of zero-cycles, and emphasize that *the codimension of the leaves is the same as the dimension of the variety* V.

For example, in case we have an embedded variety $V_k\subset\mathbf{P}^{k+l}$ of degree d, we may take T to be the Grassmannian $\mathbf{G}(l,k+l)$ of l-planes A in $\mathbf{P}^{k+l}$. The zero-cycles $A\cdot V=\sum_i P_i(A)$ then define a codimension-k d-web in an open set on the Grassmannian (here $n=k(l+1)$). The leaves of this web are the *Schubert cycles* of all l-planes in $\mathbf{P}^{k+l}$ passing through a fixed point in the projective space. In the case $k=1$ when V is a curve in $\mathbf{P}^{l+1}$, the Grassmannian in question is the dual projective space of hyperplanes and the Schubert cycles are themselves linear hyperplanes in this $\mathbf{P}^{l+1*}$. Relabelling, we may say that *a non-degenerate algebraic curve C of degree d in $\mathbf{P}^n$ defines a linear d-web in the dual projective space.* Explicitly we write the intersection

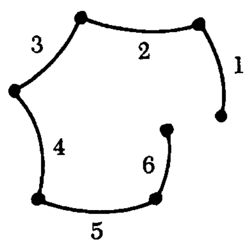
$$A\cdot C=\sum_{i=1}^{d}P_i(A)\qquad (A\in\mathbf{P}^{n*})$$

and then the leaf of the ith foliation passing through A_0 is defined by

$$P_i(A)=P_i(A_0).$$



Returning to the linearization question, the simplest case is that of a 3-web in the plane. Through each point there will pass three curves which we label as 1,2,3. We then attempt to draw an infinitesimal hexagon as indicated by the following figure



The web is said to be *hexagonal* in case this figure closes up, at least modulo higher order terms. One of the first theorems in the subject is that *the web is linearizable exactly when it is hexagonal.* To express the hexagonal condition analytically, we note that if $\omega_i(x)=0$ $(i=1,2,3)$ defines the ith curve of the web, then there is a linear relation

$$F_1(x)\omega_1(x)+F_2(x)\omega_2(x)+F_3(x)\omega_3(x)=0 \tag{4.1}$$

among the three web normals at each point $x=(x_1,x_2)$. This relation is unique up to multiplication by a function, and it turns out that the hexagonal condition is expressed analytically by the equations

$$dF_i(x)\wedge\omega_i(x)=0,$$

or equivalently that $F_i(x)$ should be constant along the ith curve.

Granting this, suppose that ith family of curves is given as the level sets of a function $u_i(x)$ and take $\omega_i(x)=du_i(x)$. Then (4.1) becomes

$$F_1(u_1(x))\,du_1(x)+F_2(u_2(x))\,du_2(x)+F_3(u_3(x))\,du_3(x)\equiv 0.$$

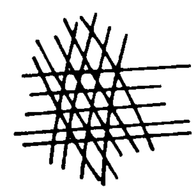
Setting

$$U_i(u)=\int_{u_0}^{u}F_i(t)\,dt$$

and $U_i(x)=U_i(u_i(x))$, the ith family of curves is defined by $U_i(x)=$ constant, and the linear relation (4.1) among the web normals is equivalent to

$$U_1(x)+U_2(x)+U_3(x)=\text{Constant}. \tag{4.2}$$

Taking U_1 and U_2 as local coordinates in the plane we conclude from (4.2) that in this coordinate system our web is the standard linear web



Once the linearization has been achieved, we may think of the web as defining three arcs in the dual projective plane of lines—to each x there are associated 3 lines passing through that point, and these describe the arcs in $\mathbf{P}^{2*}$. When this is done, (4.1) becomes an abelian differential equation of the sort encountered in the Sophus Lie theorem, with the upshot being: *A 3 web in the plane is linearizable exactly when it is equivalent to the web defined by the tangents to a curve of class three.*

Hopefully these remarks at least support a connection between the linearization question for webs and algebraic geometry, where the link is provided by interpreting the linear relations among the web normals as being something very similar to abelian differential equations. The theorem of Blaschke-Bol and subsequent generalization by Chern and myself is along these lines.

Before stating this result we need to give one more definition. Suppose that

$$\omega_{i,1}(x)=\cdots\omega_{i,k}(x)=0 \qquad (i=1,\ldots,d)$$

defines a d-web of codimension k in some open set U.

Definition. *An abelian q-equation is given by a linear relation*

$$\sum_{i,\lambda_1,\ldots,\lambda_q} F_{i,\lambda_1,\ldots,\lambda_q}(x)\omega_{i,\lambda_1}(x)\wedge\cdots\wedge\omega_{i,\lambda_q}(x)\equiv 0 \tag{4.3}$$

where

$$dF_{i,\lambda_1,\ldots,\lambda_q}\equiv 0\{\omega_{i,1},\ldots,\omega_{i,k}\}. \tag{4.4}$$

The number of linearly independent abelian q-equations is called the q-rank *of the web.*

We note that as before the integrability condition (4.4) is equivalent to stating that the coefficient function $F_{i,\lambda_1,\ldots,\lambda_q}(x)$ is constant on the leaves of the ith foliation.

As an example, suppose that we return to a family $\{\Gamma(t)\}_{t\in T}$ of zero-cycles on an algebraic variety V. If ω is a rational q-form on V which is of the first kind relative to this family, then the abelian differential equation

$$\sum_i \omega(P_i(t))\equiv 0$$

gives an abelian q-equation on the corresponding web in T. Consequently, *the study of webs and their abelian equations includes as a special case the study of the local integral varieties*

$$T\subset V^{(d)}$$

of the differential system

$$\omega(\Gamma)=\omega(P_1)+\cdots+\omega(P_d)=0$$

defined by the regular differentials $\omega\in\Omega^q(V)$ *on an algebraic variety.*

One of the goals of the theory is to approach these questions in algebraic geometry by means of the general theory of webs.

Now it seems at least plausible that a good place to begin with the study of the abelian equations associated to a codimension-k web should be with those of highest degree—i.e., with the abelian k-equations.

If we denote by

$$\Omega_i=\omega_{i,1}\wedge\cdots\wedge\omega_{i,k}$$

the ith web normal, then such an abelian equation is of the form

$$\sum_i F_i(x)\Omega_i(x)\equiv 0 \tag{4.7}$$

where

$$dF_i(x)\wedge\Omega_i(x)=0.$$

When $k=1$ we set $\Omega_i=\omega_i$ and write (4.7) as

$$\sum_i F_i(x)\omega_i(x)\equiv 0. \tag{4.8}$$

Recall that the k-rank is the number of linearly independent equations (4.7). A first question is whether or not this rank is finite, and if so then a natural problem is to study the webs of maximal k-rank. At this juncture we begin to get into some quite interesting mathematics.

For simplicity, let's consider the case $k=1$ of webs whose leaves are hypersurfaces in an open set U in n-space. The 1-rank will be called simply the rank and denoted by r. It is the number of independent equations (4.8) where $dF_i(x)\wedge\omega_i(x)\equiv 0$. It was proved by Blaschke-Bol for $n=2,3$ and by Chern in general that the rank satisfies the upper bound

$$r\leqslant\pi(d,n) \tag{4.9}$$

where

$$\pi(d,n)=m\left\{d-\frac{(m+1)(n-1)}{2}-1\right\}, \qquad m=\left[\frac{d-1}{n-1}\right].$$

This number $\pi(d,n)$ was known from algebraic geometry as being *the maximum genus of a non-degenerate algebraic curve C of degree d in* $\mathbf{P}^n$. For plane curves this is classical (probably due to Cayley), for $n=3$ it was proved by Clifford and Max Noether, and for general n the bound is due to Castelnuovo. We note that the estimate (4.9) implies Castelnuovo's bound by taking the web associated to the algebraic curve and using Abel's theorem. Since (4.9) is proved a local calculation involving successive differentiation of (4.8), we have here again a rather deep theorem in algebraic geometry proved by calculus. In fact, the proof contains the Riemann-Roch theorem for curves as well as Clifford's theorem.

This is just the beginning. Suppose we agree to say that a non-degenerate curve of maximum genus is *extremal*. For $n \leqslant d \leqslant 2n-1$ the extremal curves turn out not to be especially interesting—e.g., every algebraic curve appears here. For $d=2n$ the extremal curves are just the *canonical curves*, and for $d>2n$ they constitute a beautiful class of curves which lie in rather special way on ruled surfaces or scrolls in projective space. For example when $n=3$ they lie on quadric surfaces. Our main result is

THEOREM 2. *Suppose that* $\{\omega_i(x)=0\}$ *defines a d-web of maximum rank* $r=\pi(d,n)$. *Assume moreover that*

$$\begin{cases} d=n+1, \quad d=2n \quad \text{or} \\ d>2n \quad \text{and} \quad n\geqslant 3. \end{cases}$$

Then the web is linearizable, and is equivalent to the web defined by an extremal algebraic curve in $\mathbf{P}^n$.

I should like to comment very briefly on the proof of this theorem, which involves a rather intricate blending of algebraic and differential geometry.

Recall that the web associated to an algebraic curve C in $\mathbf{P}^n$ is given in suitable open sets U of the *dual* projective space $\mathbf{P}^{n^*}$ by writing

$$A\cdot C=\sum_i P_i(A) \qquad (A\in\mathbf{P}^{n^*}),$$

and then the ith web hypersurface passing through A_0 is defined by

$$P_i(A)=P_i(A_0).$$

Now, as is true in any projective space, the tangent directions to $\mathbf{P}^{n^*}$ at A may be naturally identified with the lines passing through A. These lines are just the linear pencils of hyperplanes in $\mathbf{P}^n$ which contain A, and as such are uniquely given by their axes which are hyperplanes in $A\cong\mathbf{P}^{n-1}$.

Dualizing, we find that the *projectivized cotangent space to* $\mathbf{P}^{n^*}$ *at A is naturally identified with A itself*. When this is done the web normals correspond to the points $P_i(A)\in A$.

Now suppose that C is an extremal algebraic curve of degree $d>2n$ where $n\geqslant 3$, and let S be the ruled surface on which the curve lies. Then $S\cap A$ is a *rational normal curve* in $A\cong P^{n-1}$ which contains the web normals $P_i(A)$. This last statement that the web normals $\omega_i(x)$ lie on a rational normal curve $E(x)$ in the projectivized cotangent spaces $\mathbf{P}(T_x^*)$ to U at x makes sense for any web, and may be proved to hold for webs of maximal rank when $d>2n$ and $n\geqslant 3$. The proof is obtained by differentiating the abelian equation (4.8) as in the argument for Chern's theorem and concluding that the web normals lie on $\infty^{(n-1)(n-2)/2}$ independent quadrics on $\mathbf{P}(T_x^*)\cong\mathbf{P}^{n-1}$. Any set of $d>2n$ points lying on such a linear system of quadrics then lies on a unique rational normal curve $E(x)$. In a sense what we have done here is to infinitesimally reconstruct a hyperplane section of Castelnuovo's ruled surface S.

Now any rational normal curve E in $\mathbf{P}^{n-1}$ is projectively equivalent to the standard one given parametrically by $[1,t,\ldots,t^{n-1}]$. Thus the field of rational normal curves $E(x)\subset\mathbf{P}(T_x^*)$ gives a special type of G-structure, one which has the distinguished property of having a large number of completely integrable cross-sections $\omega_i(x)\in E(x)$. Using this one may deduce the existence of a unique *projective connection* with certain local properties and for which the web hypersurfaces are totally geodesic. Further argument then implies that this connection is projectively flat, which then yields the linearization theorem. The global algebraicity is a consequence of the Sophus Lie theorem.

Actually, it is quite reasonable that projective differential geometry should enter into the picture since this provides the basic infinitesimal calculus dealing with a general linear structure. More precisely, our web of maximal rank leads to a system of ordinary differential equations

$$\left(\frac{dx_\beta}{dt}\right)\cdot\left[\frac{d^2x_\alpha}{dt^2}+\sum_{\lambda,\gamma}\Gamma_\alpha^{\lambda\gamma}\frac{dx_\lambda}{dt}\frac{dx_\gamma}{dt}\right]=\left(\frac{dx_\alpha}{dt}\right)\left[\frac{d^2x_\beta}{dt^2}+\sum_{\lambda,\gamma}\Gamma_\beta^{\lambda\gamma}\frac{dx_\lambda}{dt}\frac{dx_\gamma}{dt}\right] \tag{4.10}$$

whose solution curves give a system of paths in U. Were our linearization theorem to be true, this system of paths would be diffeomorphic to the system found by the straight lines in $\mathbf{P}^n$. Equivalently, there would be a change of coordinates $y_\alpha = y_\alpha(x)$ and change of parameter $s = s(t)$ converting (4.10) into the system

$$\frac{d^2 y_\alpha}{ds^2} = 0. \tag{4.11}$$

Now the basic theorem of local projective differential geometry is that one may attach an intrinsic projective connection to the system (4.10) in much the same way as the Levi-Civita connection is associated to a Riemannian metric, and moreover the system (4.10) is equivalent to (4.11) exactly when the projective curvature tensor is zero ($n \geqslant 3$).

Finally this admittedly very sketchy discussion of the proof would not be complete without mentioning the main algebro-geometric input. This method is due to Poincaré[6] which provides a web-theoretic analogue of the canonical curve associated to an algebraic curve. Namely, suppose that

$$\sum_{i=1}^{d} F_{\lambda,i}(x)\omega_i(x) = 0 \qquad (\lambda = 1,\dots,r) \tag{4.12}$$

give a basis for the abelian equations associated to our maximal rank web. The point

$$Z_i(x) = [F_{1,i}(x),\dots,F_{r,i}(x)] \in \mathbf{P}^{r-1}$$

is intrinsically defined, and as x varies it describes an arc C_i since the $F_{\lambda,i}(x)$ are constant on the web hypersurfaces. The equations (4.12) may be written as

$$\sum_i Z_i(x)\omega_i(x) = 0,$$

and so the points $Z_1(x),\dots,Z_d(x)$ span a $\mathbf{P}^{d-n-1}(x)$ in $\mathbf{P}^{r-1}$. The space $\mathbf{P}^{r-1}$ has an intrinsic linear structure (whereas U does not), and the infinitesimal structure of the *Poincaré map*

$$x \longrightarrow \mathbf{P}^{d-n-1}(x) \subset \mathbf{P}^{r-1}$$

turns out to reflect the field of rational normal curves $E(x) \subset \mathbf{P}(T_x^*)$. In the

[6] "Sur les surfaces de translation et les fonctions abeliennes", *Bull. Soc. Math France*, Vol. 29 (1901), pp. 61–86.

end, it is the counterpoint relation between this algebro-geometric configuration and the projective differential geometry which leads to the proof of our theorem.

The reason for giving this brief sketch is to provide some motivation for previously mentioned hope of using these geometric methods for dealing with at least the extremal integral varieties of the abelian differential equations (1.1). In this context I would like to close by mentioning one particular result in higher dimension. Suppose that S is an irreducible quintic surface in $\mathbf{P}^3$ given by an equation $f(x,y,z)=0$. The differentials

$$\omega = \frac{p(x,y,z)\,dx \wedge dy}{\partial f/\partial z(x,y,z)} \qquad (\deg p \leqslant 1) \tag{4.13}$$

are those which are of the first kind relative to the family $\{L \in \mathbf{G}(1,3)\}$ of lines in $\mathbf{P}^3$. Thus, $\mathbf{G}(1,3)$ gives one 4-dimensional integral variety in $S^{(5)}$ for the differential system $\omega = 0$ induced by the forms ω given by (4.13) on S. Our result in the converse:

PROPOSITION. *Suppose that $T \subset \mathbf{C}^4$ is an open set parametrizing a family*

$$\Gamma(t) = P_1(t) + \cdots + P_5(t) \qquad (t \in T)$$

of zero-cycles on S and which has the properties:

(i) *the points $P_i(t)$ vary in an open set on S*
(ii) *the abelian differential equations*

$$\sum_i \omega(P_i(t)) \equiv 0$$

are satisfied for all ω given by (4.13). *Then T is an open set on $\mathbf{G}(1,3)$ and the zero-cycles $\Gamma(t)$ are residual intersections of S with lines.*

In other words, any *local* 4-dimensional integral variety $T \subset S^{(5)}$ of the differential system $\omega = 0$ which is in general position is necessarily global algebraic and describes the geometric property "5 points lie on a line" in the surface S. With some considerable luck, this result may be part of a general pattern.