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On the Existence of a Locally Complete Germ of Deformation of Certain G-Structures

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In [3] we have defined the notion of a *deformation* of a G-structure on a compact manifold X. The purpose of this paper is to construct a *universal germ* of deformation of such a structure with the hypothesis that G be of finite type plus another mild restriction. The pertinent definitions and precise statement of the main Theorem are given in § 1.

The analogous theorem for deformations of complex structure has been proven in [6] by KURANISHI. However, since we are mainly concerned with deformations of *non-integrable* structures, the methods of [6] do not immediately apply to our case. In fact, it is easy to see that in general there exists no differential equation of the *Frobenius type* which, when given a family of G-structures, tells us whether or not the family forms a deformation. In the complex case, given a 1-parameter family of almost-complex structures J_t $(-\varepsilon < t < \varepsilon)$, with J_0 integrable, there is a differential equation Δ such that $\Delta J_t = 0$ implies that the J_t are integrable so that we get a deformation of the complex structure on X. It is this construction which does not generalize to general G-structures.

In fact, the class of G-structures for which such an operator Δ does exist are the so-called *transitive structures*. In §§ 2 and 4 we construct Δ for a special transitive structure, and in the Appendices, Δ is constructed for a general transitive structure of finite type.

In § 3 the main reduction in our Theorem occurs when it is shown the existence of a universal germ of deformation of a general G-structure follows from the existence of a universal germ in a special transitive case. In §§ 4 and 5 the operator Δ is used to construct the universal germ in the special transitive case.

In § 6, the reduction, together with the construction in the transitive situation, are combined to complete the proof of our main result.

In § 7 some examples and applications are given. For instance, a special case of our main theorem gives a generalization of some recent results of H. C. WANG concerning the space of lattices in certain Lie groups. Our transcendental methods give a *local* construction of this space of lattices for any Lie group. The results of WANG saying that this space is, in certain cases, a manifold of a certain dimension then follow from the computation of some cohomology groups. The general calculation of these cohomology groups will presumably give the complete local structure of the space of lattices in any Lie group.

In fact, our results show that the space of lattices in a Lie group, with the equivalence relation that two lattices are identified if they are conjugate, is locally an analytic set. The cohomological result then gives the dimension of this set and, if it exists, the non-singularity.

1. Statement of the Theorem

Throughout our discussion we shall denote by G a connected linear Lie group which is then a subgroup of $GL(n, \mathbf{R})$. Let $f(x) = (f^1(x), \ldots, f^n(x))$ $(x = (x^1, \ldots, x^n))$ be a local diffeomorphism of $\mathbf{R}^n$, and denote by $J_f(x)$ the Jacobian matrix of f at $x \in \mathbf{R}^n$. Recall that G is said to be of *finite type* if there exists an integer $\mu_0 = \mu_0(G)$ such that, for any diffeomorphism f with $J_f(x) \in G$ for all x, $\frac{\partial^{\mu_1+\cdots+\mu_n} f}{(\partial x^1)^{\mu_1}\ldots(\partial x^n)^{\mu_n}} \equiv 0$ whenever $\mu_1 + \cdots + \mu_n > \mu_0$.

Let X be an n-manifold. Then a *G-structure* on X is given by a reduction of the structure group of the tangent bundle of X from $GL(n, \mathbf{R})$ to G. If we let B_G be the manifold of all *G-frames*, then there is a principal fibration $G \to B_G \to X$. The principal bundle $GL(n, \mathbf{R}) \to B \to X$ obtained by extending the structure group of B_G to $GL(n, \mathbf{R})$ is then the principal tangent bundle of all frames on X.

Consider now a local diffeomorphism $f: X \to X$. (By this, we mean that there exist open sets $U, U' \subset X$ and a diffeomorphism $f: U \to U'$.) Such an f has a canonical lifting to a bundle automorphism

$$\begin{array}{rcl} f_*: B & \to & B \\ \downarrow & & \downarrow \\ f: X & \to & X. \end{array}$$

We say that f is a *local G-automorphism* if $f_*(B_G) \subset B_G$ when B_G is considered as a submanifold of B.

Let θ be a local vector field on X. Then θ generates a local one-parameter group $f(t) = \exp(t\theta)$ of local diffeomorphisms of X. We say that θ is an *infinitesimal G-automorphism* if the $f(t)$ are local G-automorphisms. Denote by Θ_G the sheaf on X of germs of infinitesimal G-automorphisms.

Suppose that G is of finite type. Then each stalk $\Theta_G(x)$ $(x \in X)$ is a finite-dimensional real vector space, and we say that the G-structure on X is *normal* if $\dim \Theta_G(x)$ is independent of x. This happens in particular when everything is real analytic.

Let now $U \subset \mathbf{R}^m$ be an open neighborhood of the origin in $\mathbf{R}^m$ with parameter $t = (t^1, \ldots, t^m)$, and let $\mathscr{W} \xrightarrow{\varpi} U$ be a differentiable fibre bundle with fibre X. We may take as structure group of the tangent bundle of $\mathscr{W}$ the group of all matrices $\begin{pmatrix} \gamma & * \\ 0 & \beta \end{pmatrix}$ where $\gamma \in GL(n, \mathbf{R})$, $\beta \in GL(m, \mathbf{R})$ and $* \in \mathrm{Hom}(\mathbf{R}^m, \mathbf{R}^n)$. Let G^* be the linear group of all matrices of the above form where $\gamma \in G$, and let $G' \subset G^*$ consist of the subgroup of all matrices where $* = 0$. Suppose that we have given on $\mathscr{W}$ a G^*-structure. Then on each fibre $X_t = \tilde{\omega}^{-1}(t)$, there is an induced G-structure $G \to B_G(t) \to X_t$.

Before defining what it means for $\mathscr{W}$ to give a deformation of the G-structure of X, we observe that, if $Z \subset X$ is any open set, then there is a natural G'-structure on $Z \times U$ induced from the G-structure on Z. We let $\pi: Z \times \times U \to U$ be the projection mapping.

Now let $\mathscr{W} \xrightarrow{\tilde{\omega}} U$ be as above and suppose that we have on $\mathscr{W}$ a G^*-structure. We say the $\mathscr{W} \xrightarrow{\tilde{\omega}} U$ gives a *deformation of the G-structure on X* if the following two conditions are satisfied: (I) There is a G-diffeomorphism between X and $X_0 = \tilde{\omega}^{-1}(0)$; (II) The fibre bundle $\mathscr{W} \xrightarrow{\tilde{\omega}} U$ is locally G-trivial in the following sense: For each point $x_0 \in \mathscr{W} \cap X_0$, there exists a neighborhood $V(x_0)$ of x_0 in $\mathscr{W}$, a neighborhood $Z(x_0)$ of x_0 in X_0, and a diffeomorphism $f_{x_0}: V(x_0) \to \to Z(x_0) \times U$ such that $\pi \circ f = \tilde{\omega}$ and which transforms the induced G^*-structure on $V(x_0)$ into the natural G'-structure on $Z(x_0) \times U$.

Given deformations $\mathscr{W}' \xrightarrow{\tilde{\omega}} U'$ and $\mathscr{W} \xrightarrow{\tilde{\omega}} U$ of $G \to B_G \to X$, a *mapping of deformations* is given by a pair of mappings $F: \mathscr{W}' \to \mathscr{W}$, $f: U' \to U$ such that

$$\begin{array}{ccc} F: \mathscr{W}' & \to & \mathscr{W} \\ \downarrow \tilde{\omega}' & & \downarrow \tilde{\omega} \\ f: U' & \to & U \end{array}$$

commutes and such that F is a G^*-mapping which is a diffeomorphism on fibres. From this we may define *equivalent deformations* and finally, in a well-known manner, a *germ of deformation*.

For the purposes of this paper, the above definition of deformation is not sufficient, and we must allow that U has singularities (c.f. the examples in § 10 of [3]). Thus let $V \subset U$ be an analytic set through the origin, and suppose that $\mathbf{W} \xrightarrow{\tilde{\omega}} U$ has a G^*-structure such that (I) above is satisfied. Then this data gives a deformation if (II) is satisfied in the following sense: For each $x_0 \in X_0$, there exists a neighborhood $V(x_0)$ of x_0 in $\mathbf{W}$, a neighborhood $Z(x_0)$ of x_0 in X_0, and a diffeomorphism $f_{x_0}: V(x_0) \to Z(x_0) \times U$ with $\pi \circ f = \tilde{\omega}$ and such that $f_{x_0} | \omega^{-1}(t) \cap V(x_0) \to Z(x_0) \times \{t\}$ is a G-isomorphism whenever $t \in V$. We let $\mathscr{W} = \tilde{\omega}^{-1}(V)$ and write such a deformation as $\{\mathscr{W} \xrightarrow{\tilde{\omega}} V\}$, the existence of ambient spaces $\mathbf{W}$ of $\mathscr{W}$ and U of V being understood.

We also say that $\{\mathscr{W} \xrightarrow{\tilde{\omega}'} V\}$ is *effective*, or effectively parametrized, if no restriction of $\mathscr{W}$ to an analytic curve through the origin gives a trivial deformation.

Given $V' \subset U'$, $V \subset U$ and deformations $\{\mathscr{W}' \xrightarrow{\tilde{\omega}'} V'\}$, $\{\mathscr{W} \xrightarrow{\tilde{\omega}} V\}$, a mapping between these deformations is given by a G^*-mapping $F: \mathbf{W}' \to \mathbf{W}$ and a differentiable mapping $f: U' \to U$ such that $f(V) \subset V'$, $f(0) = 0$, and $\tilde{\omega}' \circ F = f \circ \tilde{\omega}$. We write $(F, f) \{\mathscr{W}' \xrightarrow{\tilde{\omega}'} V'\} \to \{\mathscr{W} \xrightarrow{\tilde{\omega}} V\}$. (In case V is non-singular at 0, it is understood that $V = U$.) A germ of deformation $\{\mathscr{W} \xrightarrow{\tilde{\omega}} V\}$ is *locally complete* if, given any non-singular germ $\{\mathscr{W}' \xrightarrow{\tilde{\omega}'} U'\}$, there exists a mapping of germs $(F, f): \{\mathscr{W}' \xrightarrow{\varpi'} U'\} \to \{\mathscr{W} \xrightarrow{\varpi} V\}$. The germ $\{\mathscr{W} \xrightarrow{\varpi} V\}$ is said to be *strongly locally complete* if we may omit the adjective "non-singular" above.

Now let G be of finite type and $G \to B_G \to X$ a G-structure. Recall that a lattice in a Lie group A is a discrete subgroup $\Gamma \subset A$ such that the quotient A/Γ is compact. We say that a normal G-structure $G \to B_G \to X$ is *regular* if the Lie group A of G-automorphisms of the universal covering $\tilde{X}$ of X (with the natural G-structure) has finitely many components and if the fundamental group of X is a lattice in A.

Our main result is the following

Theorem: *Let X be compact and $G \to B_G \to X$ a regular G-structure. Then there exists a locally complete effective germ $\{\mathscr{W} \xrightarrow{\varpi} V\}$ of deformation of this G-structure.*

Remark: The germ which we shall construct is almost certainly strongly locally complete — our methods, however, fail to give this result except in an important special case.

2. Some General Remarks on G-Structures and Deformations

Let $\mathbf{V}$ be a real vector space and $G \subset GL(\mathbf{V})$ a closed linear subgroup.

Proposition: *The set M_G of translation-invariant G-structures on $\mathbf{V}$ forms a manifold which may be identified with $GL(\mathbf{V})/G$.*

This may be seen as follows. By identifying $\mathbf{V}$ with its tangent space at the origin, a translation invariant G-structure on $\mathbf{V}$ is given by a set of frames S such that, if $e, e' \in S$, then $e = e'g$ for some $g \in G$. By choosing a coordinate system, a frame is given by a non-singular matrix, and we let e_0 be the coordinate frame. Define a mapping $\pi: GL(\mathbf{V}) \to M_G$ by setting $\pi(\gamma) = \gamma e_0 G$ for $\gamma \in GL(\mathbf{V})$. Then clearly π establishes an isomorphism $\pi: GL(\mathbf{V})/G \cong M_G$. Q. E. D.

Now we may choose coordinates around $e_0 G$ in M_G to be linear coordinates in a neighborhood N of 0 in $gl(n, \mathbf{R})/\mathfrak{g}$[1]). Thus the set of G-structures near a given G-structure form part of a linear space. We now carry this over to manifolds.

Let $G \to B_G \to X$ be a G-structure, and let Σ be the sheaf of germs of smooth sections of the bundle $B_G \times_G gl(\mathbf{V})/\mathfrak{g}$ where G acts on $gl(\mathbf{V})/\mathfrak{g}$ by the *adjoint action* (for $v \in gl(\mathbf{V})/\mathfrak{g}$, $g \in G$, $g \cdot v = gvg^{-1}$). Then, by the above remarks, the G-structures on X near to $G \to B_G \to X$ are given by the "small" sections of the sheaf $\Sigma \to X$. We give some examples of this construction.

(1) $G = O(n)$. Then $GL(n, \mathbf{R})/O(n)$ is the cone $S(n)$ of symmetric positive definite matrices. Furthermore, $gl(n, \mathbf{R})/\mathfrak{O}(n)$ may be identified with the linear space of symmetric matrices.

Let $g(x)$ be a Riemannian metric on X. Then the above says that the Riemannian metrics near to $g(x)$ are of the form $g(x) + \xi(x)$ where $\xi: X \to \Sigma$ is a small symmetric tensor.

(2) $G = GL(n, \mathbf{C}) \subset GL(2n, \mathbf{R})$. Then we may identify $M_{GL(n,\mathbf{C})}$ with the set $\{J \in \operatorname{Hom}(\mathbf{R}^{2n}, \mathbf{R}^{2n}) \mid J^2 = -I\}$. Let $J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$ and define

[1]) Here $\mathfrak{g}$ denotes the linear Lie algebra of G; $gl(n, \mathbf{R}) = \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$.

$\pi : GL(2n, \mathbf{R}) \to M_{GL(n, \mathbf{C})}$ by $\pi(\gamma) = \gamma J_0 \gamma^{-1}$ $(\gamma \in GL(2n, \mathbf{R}))$. Then π establishes an isomorphism $\pi : GL(2n, \mathbf{R})/GL(n, \mathbf{C}) \to M_{GL(n, \mathbf{C})}$.

Let $J \in M_{GL(n, \mathbf{C})}$. Then the tangent space to $M_{GL(n, \mathbf{C})}$ at J is equal to $\{A \in \mathrm{Hom}(\mathbf{R}^{2n}, \mathbf{R}^{2n}) \mid AJ + JA = 0\}$. Note that $M_{GL(n, \mathbf{C})}$ has an almost-complex structure K given by $K(A) = JA$ for $A \in \mathrm{T}(J)$. This structure is integrable, and indeed $M_{GL(n, \mathbf{C})}$ may be identified with an open submanifold of the complex Grassmann variety of complex n-planes in $\mathbf{C}^{2n}$. Let $\mathbf{W}$ be the complexification of $\mathbf{R}^{2n}$ and write $\mathbf{W} = \mathbf{W}_1(J) \otimes \mathbf{W}_2(J)$ where $\mathbf{W}_1(J)$ is the $+\sqrt{-1}$ eigenspace of J and $\mathbf{W}_2(J) = \overline{\mathbf{W}_1(J)}$. The condition $AJ + JA = 0$ means that, when A is extended to $\mathrm{Hom}(\mathbf{W}, \mathbf{W})$, $A = A_1 + \bar{A}_1$ where $A_1 \in \mathrm{Hom}(\mathbf{W}_2(J), \mathbf{W}_1(J))$. Thus the sections $\xi : X \to \Sigma$ are essentially the so-called vector-valued $(0, 1)$ forms $\xi = (\xi^\alpha_{\bar\beta})$ and the nearby almost-complex structures are given by the small vector-valued $(0, 1)$ forms.

This may be done directly as follows. Let $e^\# = (e, \bar{e})$ where $e = (e_1, \dots, e_n)$ be an admissible frame for J. Then a complex frame corresponding to the vector-valued $(0, 1)$ form $\xi(x)$ is given by $(e + \xi\bar{e}, \bar{e} + \bar{\xi}e)$.

(3) $G = I$ is the identity matrix. Then $M_I = GL(n, \mathbf{R})$. An I-structure on X is given by a field of frames $e_1(x), \dots, e_n(x)$ which give a parallelism on X. A section $\sigma : X \to \Sigma$ is a matrix valued function $\sigma(x) = (\sigma(x)^i_j)$, and the corresponding I-structure is given by the field of frames $e'_1(x), \dots, e'_n(x)$ where $e'_j(x) = \sum_{i=1}^{n} \sigma(x)^i_j e_i(x) + e_j(x)$.

Let now $\Xi = H^0(X, \Sigma)$ and suppose that we have on Ξ a norm $\| \ \|$ such that its completion is a Banach space. (Correspondingly, the set of G-structures on X may be made into a Banach manifold whose tangent space at $G \to B_G \to X$ is just Ξ.) Let Ω be a neighborhood of 0 in Ξ. Then there exists a *variation of the G-structure* $\{\mathscr{W} \xrightarrow{\tilde\omega} \Omega\}$ where $\tilde\omega^{-1}(t)$ $(t \in \Omega)$ has the G-structure given by $t \in \Omega$. This space is a sort of *universal model* for the local deformations of $G \to B_G \to X$: Given any deformation $\{\mathscr{W}' \xrightarrow{\tilde\omega} U'\}$, we have a mapping (F, f) : $\{\mathscr{W}' \xrightarrow{\tilde\omega} U'\} \to \{\mathscr{W} \xrightarrow{\tilde\omega} \Omega\}$ as follows: For $t' \in U'$, $f(t') \in \Omega$ is the section of Σ such that $\tilde\omega'^{-1}(t')$ has the G-structure corresponding to $f(t')$, and then F is the identity.

This universal model is unsatisfactory in two respects: (i) the structures $\tilde\omega^{-1}(t)$ $(t \in \Omega)$ are not all locally equivalent so that $\{\mathscr{W} \xrightarrow{\tilde\omega} \Omega\}$ is not a deformation in our sense; (ii) the space $\{\mathscr{W} \xrightarrow{\tilde\omega} \Omega\}$ is not effectively parametrized — we must, in some sense, take the orbit space of Ω under the group of diffeomorphisms of X.

We now give a sort of general procedure for dealing with (i) — this general procedure will be specialized in § 4, and also in the Appendix. Also, in § 5, we shall give a general method for treating (ii). Taking these together will give the Theorem of § 1.

The idea in (i) is to find a (non-linear, of course) differential operator $\Delta = \Delta_G$ on Ξ such that, for $\sigma \in \Omega$, $\Delta\sigma = 0$ is a necessary and sufficient condition that the G-structure $G \to B_G(\sigma) \to X$ determined by σ is locally equivalent to $G \to B_G \to X$. If this has been done, then we set $U = \{\ker \Delta\} \cap \Omega$ and get a bona fide *universal germ of deformation* $\{\mathscr{W} \xrightarrow{\tilde{\omega}} U\}$ of $G \to B_G \to X$.

As an example, recall that $G \to B_G \to X$ is *integrable* if there exists a coordinate covering $\{U_a\}$ of X with coordinates $(x_a^1, \ldots, x_a^n)$ in U_a such that the coordinate frame $\left(\frac{\partial}{\partial x_a^1}, \ldots, \frac{\partial}{\partial x_a^n}\right) \in B_G$. If $G \to B_G \to X$ is integrable, then $\Delta\sigma = 0$ $(\sigma \in \Omega)$ should be the equation of integrability of the "almost G-structure" defined by σ. Such a Δ exists in the complex analytic case ([5]) and also for many other integrable structures ([9]).

The operator Δ has, in the past, been determined from the differential operators in an injective resolution of the sheaf Θ_G. We outline now a procedure for getting the first few terms in such a resolution.

Let now Σ be the sheaf of germs of sections of $B_G \times_G gl(n, \mathbf{R})/\mathfrak{g}$, and let $\mathscr{T}$ be the sheaf of germs of tangent vectors. Define $j : \Theta_G \to \mathscr{T}$ to be injection of the subsheaf Θ_G of $\mathscr{T}$ into $\mathscr{T}$. We now define $D_1 : \mathscr{T} \to \Sigma$ such that $0 \to \Theta_G \xrightarrow{j} \mathscr{T} \xrightarrow{D_1} \Sigma$ is exact. Let $\omega(x) = (\omega^1(x), \ldots, \omega^n(x))$ be a local co-frame for the G-structure $G \to B_G \to X$. Then any other admissible co-frame $\omega'(x)$ may be written as $\omega'(x) = \omega(x)\, g(x)$ where $g(x)$ is a local mapping of X into G. Let θ be a germ in $\mathscr{T}$, and consider $\mathscr{L}_\theta \omega = (\mathscr{L}_\theta \omega^1, \ldots, \mathscr{L}_\theta \omega^n)$[2]). Then we may write $\mathscr{L}_\theta \omega = +\omega \cdot \gamma(x)$ where $\gamma(x) \in gl(n, \mathbf{R})$. We claim that, considering $\gamma(x)$ as an element of $gl(n, \mathbf{R})/\mathfrak{g}$, γ is a section in Σ. Indeed, if ω' is any other co-frame, then $\omega' = \omega \cdot g$ and $\mathscr{L}_\theta(\omega') = \mathscr{L}_\theta \omega \cdot g + \omega \cdot \mathscr{L}_\theta g = +\omega' g^{-1}\gamma g + \omega \mathscr{L}_\theta g = \omega'(+g^{-1}\gamma g + g^{-1}\mathscr{L}_\theta g)$. Since $g^{-1}\mathscr{L}_\theta g$ is a mapping of X into $\mathfrak{g}$[2a]), we see that $\gamma(x)$ is in fact a germ in Σ, and we set $D_1\theta = \gamma \in \Sigma$. Clearly $\Theta_G \xrightarrow{i} \mathscr{T} \xrightarrow{D_1} \Sigma$ is exact.

This may be done more intrinsically as follows. Let B be the bundle of all frames on X, so that $B_G \subset B$ is a sub-manifold, and let $\mathbf{V}$ be a fibre for the tangent bundle of X $(\mathbf{V} \cong \mathbf{R}^n)$. There is on B a $\mathbf{V}$-valued form ω such that, for $h \in GL(\mathbf{V})$ acting on B on the right, $\omega(bh) = h^{-1} \cdot \omega(b)$ $(b \in B)$. Let $\omega_G = \omega \mid B_G$, and let $\theta \in \mathscr{T}$. Then θ defines a right-invariant vector field $\hat{\theta}$ on B, and thus $\mathscr{L}_\theta \omega = +\gamma \cdot \omega$ for some function $\gamma : B \to gl(\mathbf{V})$. From the right invariance of $\hat{\theta}$ and the transformation law on ω, we get that $\gamma(bh) = h^{-1}\gamma(b)h$ $(b \in B,\ h \in GL(\mathbf{V}))$. Letting $\pi : gl(\mathbf{V}) \to gl(\mathbf{V})/\mathfrak{g}$ be the projection, then, on B_G, $\mathscr{L}_\theta \omega_G = \gamma \cdot \omega_G$ and $\pi(\gamma)$ is a function on B_G with values in $gl(\mathbf{V})/\mathfrak{g}$ which transforms by Ad when G acts on the right; i.e. $\pi(\gamma)$ is a section of Σ and $\pi(\gamma) = D_1(\theta)$.

Now the next logical step is to find a sheaf Λ of germs of sections of a bundle and a differential operator $D_2 : \Sigma \to \Lambda$ such that $\mathscr{T} \xrightarrow{D_1} \Sigma \xrightarrow{D_2} \Lambda$ is exact. In practice, once D_2 is found, it is easy to find Δ. We shall carry out this procedure for a special structure in § 4 and in the Appendix for an arbitrary transitive G-structure of finite type.

[2]) Here $\mathscr{L}_\theta(\#)$ is the *Lie derivative* of $\#$ along the vector field θ.

[2a]) Recall that the (left-invariant) Maurer-Cartan form Ω on G is given by $\Omega(g) = g^{-1}dg$.

3. Reduction of the Problem

Let A be a Lie group and $\Gamma \subset A$ a lattice. Denote by $\omega^1, \ldots, \omega^n$ a basis for the right invariant forms on A. Then, on the manifold $Y = A/\Gamma$, there is defined a parallelism, which is a G-structure where $G = I$, defined by the Pfaffians $\omega^1, \ldots, \omega^n$. For convenience, we call a parallelism an *I-structure.*

Let now $G \to B_G \to X$ be a regular G-structure. Denote by $\tilde{X}$ the universal covering manifold of X and let $\Gamma \to \tilde{X} \xrightarrow{\pi} X$ be the covering fibration where $\Gamma = \pi_1(X)$. Over $\tilde{X}$ there is uniquely induced a G-structure by the requirement that π should be a G-mapping. Then, as was shown in [3], the group of G-automorphisms of $G \to \tilde{B}_G \to \tilde{X}$ is a Lie group[3]) A with Lie algebra $\mathfrak{a} = H^0(\tilde{X}, \tilde{\Theta}_G)$. Clearly Γ is a discrete subgroup of A which, by assumption, is a lattice. (The fact that $\Gamma \subset A$ is a lattice is automatic if G is compact. This follows from the easily proved fact that A acting on $\tilde{X}$ has a closed orbit.)

Let $\tilde{A}$ be the universal covering group of A with projection $\sigma : \tilde{A} \to A$. Then $\sigma^{-1}(\Gamma)$ is a lattice $\tilde{\Gamma} \subset \tilde{A}$, and $\tilde{\Gamma}$ is an extension of Γ by the discrete finitely generated kernel of σ. In fact, it will cause no real loss in generality if we assume that A is simply connected — the general situation may be easily derived from this by the methods in [3].

Suppose that P is an open neighborhood of the origin in $\mathbf{R}^m$, that $R \subset P$ is an analytic set, and let $\{\mathscr{S} \xrightarrow{p} R\}$ be a germ of deformation of the I-structure on Y. Then, for each $\tau \in R$, $p^{-1}(\tau) = Y_\tau$ has A as its universal covering manifold. Thus we may associate to $\tau \in R$ the discrete subgroup $\Gamma_\tau \subset A$ such that $Y_\tau = A/\Gamma_\tau$. Clearly then Γ_τ is a lattice in A and we may define $X_\tau = \tilde{X}/\Gamma_\tau$. On X_τ there is a G-structure $G \to B_G(\tau) \to X_\tau$. We *assume* that there exists an open set $U \subset \mathbf{R}^M$ such that $R \subset U$ and a deformation $\{\mathscr{W} \xrightarrow{\tilde{\omega}} R\}$ of the G-structure on X such that $\tilde{\omega}^{-1}(\tau) = X_\tau$. This assumption will be met in the cases we shall consider below.

Theorem A: *If $\{\mathscr{S} \xrightarrow{p} R\}$ is a locally complete germ of deformation of the I-structure on Y, then $\{\mathscr{W} \xrightarrow{\tilde{\omega}} R\}$ is a locally complete germ of deformation of the G-structure on X.*

Proof: Let $\{\mathscr{W}' \xrightarrow{\tilde{\omega}'} U'\}$ be a non-singular germ of deformation of the G-structure $G \to B_G \to X$. By the results of [3], we see that, for each $t' \in U'$, the universal covering of $X_{t'} = \tilde{\omega}'^{-1}(t')$ is $\tilde{X}$ with the G-structure $G \to \tilde{B}_G \to \tilde{X}$ (the assumption of normality is used here). Thus, for each $t' \in U'$, we have a lattice $\Gamma_{t'}$ in A.

Let now $R_0(\Gamma, A)$ be the component of Γ in the space of lattices in A which are, as abstract groups, isomorphic to Γ. Then clearly $R_0(\Gamma, A)$ is an analytic set and we let P' be a neighborhood of Γ in $R_0(\Gamma, A)$ so that $\sigma : \tau \to \sigma(\tau) = \Gamma_\tau \in P'$ is a regular mapping of U' into P'. We now construct a germ of deformation $\{\mathscr{S}' \xrightarrow{p'} P'\}$ of the I-structure on Y by letting $p'^{-1}(\Gamma_\xi) = Y_\xi = A/\Gamma_\xi \quad (\Gamma_\xi \in P')$.

[3]) This statement, is, I believe, due first to Mme. P. Liebermann.

Suppose that $\{\mathscr{S} \xrightarrow{p} R\}$ is a strongly locally complete germ of deformation of the I-structure on Y. Then there exists a mapping $(H, h): \{\mathscr{S}' \xrightarrow{p'} P'\} \to \{\mathscr{S} \xrightarrow{p} R\}$ of the deformations. For each $\Gamma_\xi \in P'$, $H: Y_\xi \to p^{-1}(h(\Gamma_\xi))$ is a mapping of I-structures and lifts to a mapping $H_\xi: A \to A$. For $\gamma \in \Gamma$, we let γ_ξ denote the corresponding element in Γ_ξ acting on $\tilde{X}$ on the right, and we write $h(\Gamma_\xi) = h(\xi)$ $(\Gamma_\xi \in P')$. Then H_ξ satisfies $(H_\xi)^* \omega^\alpha = \omega^\alpha$ $(\alpha = 1, \ldots, n)$ and $H_\xi(a\,\gamma_\xi) = H_\xi(a)\,\gamma_{h(\xi)}$ for $a \in A$. The first condition implies that H_ξ is left translation by some element $a_\xi \in A$, and the second gives that $a_\xi \gamma_\xi a_\xi^{-1} = \gamma_{h(\xi)}$ or $a_\xi \Gamma_\xi a_\xi^{-1} = \Gamma_{h(\xi)}$. Clearly $\xi \to a_\xi$ is a smooth mapping of P' into A.

We now define $(F, f): \{\mathscr{W}' \xrightarrow{\tilde{\omega}} U'\} \to \{\mathscr{W} \xrightarrow{\tilde{\omega}} R\}$. For $\tau \in U'$, we set $f(\tau) = h(\sigma(\tau)) \in R$. In order to define F, it will suffice to define $F_\tau: \tilde{X} \to \tilde{X}$ such that $F_\tau(\tilde{x} \cdot \gamma_\tau) = F_\tau(\tilde{x})\,\gamma_{f(\tau)}$, and such that F depends smoothly on τ. We do this by setting $F_\tau(\tilde{x}) = \tilde{x} \cdot a_{\sigma(\tau)}^{-1}$ (recall that A acts on $\tilde{X}$ on the right). This completes the proof of Theorem A.

In order to complete the proof of our main result, it will suffice to show:

Theorem B: *Let $\Gamma \subset A$ be a lattice where A is simply connected and has finitely many components. Then there exists a strongly locally complete effective germ $\{\mathscr{W} \xrightarrow{\tilde{\omega}} V\}$ of deformation of the I-structure on A/Γ.*

Theorem C: *If this germ is carried over to the deformation of any normal G-structure $G \to B_G \to X$ as in Theorem A, the result is an effective germ of deformation of this G-structure.*

The proof of Theorem B will be given in §§ 4 and 5, and Theorem C together with some other miscellaneous results will be discussed in § 6. We remark that, since the local deformation theory of Γ in A is not changed when we replace Γ by a subgroup of finite index in Γ, we may assume that A is connected.

The proof of Theorem B will be an application to the I-structure on A/Γ of the general program in deformation theory which was outlined in § 2. Indeed, § 4 will mainly be concerned with rectifying the objection (i) in the universal family $\{\mathscr{W} \xrightarrow{\tilde{\omega}} \Omega\}$, and § 5 will deal with (ii). Finally, in the Appendices, we shall show that the techniques of §§ 4 and 5 can be adapted to show that there exists a strongly locally complete germ of deformation of any transitive G-structure of finite type.

4. Construction of a Locally Complete Germ of Deformation

Let now A be a connected, simply connected Lie group, $\Gamma \subset A$ a lattice, and $X = A/\Gamma$. Denote by $\omega^1, \ldots, \omega^n$ a basis for the right-invariant Maurer-Cartan forms on A considered as giving an I-structure on X. We let $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$ be the frame on X dual to the coframe $\omega^1, \ldots, \omega^n$; if $d\omega^\alpha = \frac{1}{2} \sum_{\beta,\gamma} c^\alpha_{\beta\gamma}\, \omega^\beta \wedge \omega^\gamma$, then $[\mathfrak{e}_\alpha, \mathfrak{e}_\beta] = \sum_\gamma c^\gamma_{\alpha\beta} \mathfrak{e}_\gamma$. We designate by $\mathscr{S}^q$ the sheaf of germs of vector-valued q-forms on X and set $\mathrm{T}^q = H^0(X, \mathscr{S}^q)$ = global vector-valued q-forms on X. There is a canonical element $\Omega \in \mathrm{T}^1$ defined by $\Omega = \sum_\alpha \mathfrak{e}_\alpha \otimes \omega^\alpha$. In fact, any germ η in $\mathscr{S}^q$ may be written as $\eta = \sum_\alpha \mathfrak{e}_\alpha \otimes \eta^\alpha$ where the η^α are germs of

q-forms on X. We define a pairing $[\ ,\ ]: \mathscr{S}^p \otimes \mathscr{S}^q \to \mathscr{S}^{p+q}$ as follows: If $\eta = \sum_\alpha \mathrm{e}_\alpha \otimes \eta^\alpha$ and $\sigma = \sum_\beta \mathrm{e}_\beta \otimes \sigma^\beta$, then $[\eta, \sigma] = \sum_{\alpha,\beta} [\mathrm{e}_\alpha, \mathrm{e}_\beta] \otimes \eta^\alpha \wedge \sigma^\beta = \sum_{\gamma,\alpha,\beta} \mathrm{e}_\gamma \otimes \otimes\, c^\gamma_{\alpha\beta} \eta^\alpha \wedge \sigma^\beta$. This pairing makes $\mathscr{S} = \sum_{q \geq 0} \mathscr{S}^q$ a sheaf of *graded Lie algebras* and, in particular, $\mathrm{T} = \sum_{q \geq 0} \mathrm{T}^q$ is itself a graded Lie algebra. If $d : \mathscr{S}^q \to \mathscr{S}^{q+1}$ is defined by $d(\sum \mathrm{e}_\alpha \otimes \eta^\alpha) = \sum_\alpha \mathrm{e}_\alpha \otimes d\eta^\alpha$, then the *Maurer-Cartan equation* is written $d\Omega = \frac{1}{2}[\Omega, \Omega]$.

We remark that the *Jacobi identity* in a graded Lie algebra reads: For $\eta \in \mathscr{S}^p$, $\sigma \in \mathscr{S}^q$, $\tau \in \mathscr{S}^r$

$$(4.1) \qquad (-1)^{pr}[[\eta, \sigma], \tau] + (-1)^{qr}[[\tau, \eta], \sigma] + (-1)^{pq}[[\sigma, \tau], \eta] = 0\,.$$

If $\eta \in \mathscr{S}^p$, then (4.1) gives

$$(-1)^p[[\Omega, \Omega], \eta] + (-1)^p[[\eta, \Omega], \Omega] - [[\Omega, \eta], \Omega] = 0\,.$$

Applying the *anti-commutation rule* $[\eta, \sigma] = (-1)^{pq+1}[\sigma, \eta]$ to this equation then gives

$$(4.2) \qquad [[\Omega, \Omega], \eta] = 2[\Omega, [\Omega, \eta]]\,.$$

We now set $\Theta = \Theta_I$ and carry out the program of obtaining a resolution

$$(4.3) \qquad 0 \to \Theta \xrightarrow{i} \mathscr{S}^0 \xrightarrow{D_0} \mathscr{S}^1 \xrightarrow{D_1} \mathscr{S}^2 \to \cdots$$

outlined in § 2. Accordingly, for a germ θ in $\mathscr{S}^0$, we define $D_0\theta \in \mathscr{S}^1$ by $D_0\theta = \sum_{\alpha=1}^{n} \mathrm{e}_\alpha \otimes \mathscr{L}_\theta \omega^\alpha$. Then (4.3) is clearly exact at $\mathscr{S}^0$. Since $d\mathscr{L}_\theta \omega^\alpha = \mathscr{L}_\theta d\omega^\alpha = \sum_{\beta,\alpha} c^\alpha_{\beta\gamma} \mathscr{L}_\theta \omega^\beta \wedge \omega^\gamma$, $d(D_0\theta) = [\Omega, D_0\theta]$, and this suggests that we define D_1 by $D_1\eta = d\eta - [\Omega, \eta]$. In fact, if for $q > 1$, we define

$$(4.4) \qquad D_q\eta = d\eta - [\Omega, \eta]\,,$$

then we claim that (4.3) is an exact sequence of sheaves over X. Furthermore, dropping the subscripts and writing D for any D_q, we claim that, for $\eta \in \mathscr{S}^p$, $\sigma \in \mathscr{S}$

$$(4.5) \qquad D[\eta, \sigma] = [D\eta, \sigma] + (-1)^p[\eta, D\sigma]\,.$$

Although this has been proven in [3], we shall for completeness now verify these facts. First, by an explicit calculation, we have

$$(4.6) \qquad d[\Omega, \eta] = \frac{1}{2}[[\Omega, \Omega], \eta] - [\Omega, d\eta]\,.$$

We now check that $D^2 = 0$. Since $D_1D_0 = 0$, it will suffice to show that $D_qD_{q-1} = 0$ $(q > 1)$. We have then that $D(D\eta) = D(d\eta - [\Omega, \eta]) = d^2\eta - d[\Omega, \eta] - [\Omega, d\eta] + [\Omega, [\Omega, \eta]] =$ (by (4.2) and (4.6)) $- \frac{1}{2}[[\Omega, \Omega], \eta] + [\Omega, d\eta] - [\Omega, d\eta] + \frac{1}{2}[[\Omega, \Omega], \eta] = 0$. Also, (4.5) may be checked in a similar fashion.

The Poincaré lemma which then implies the exactness may be verified by the following device. Since we are working locally, we may assume that we are on a germ of a Lie group and that we have left-invariant vector fields $\mathfrak{f}_1, \ldots, \mathfrak{f}_n$ as well as the right-invariant vector fields $\mathfrak{e}_1, \ldots, \mathfrak{e}_n$. Then we may write $\eta \in \mathcal{S}^q$ as $\eta = \sum_\alpha \mathfrak{f}_\alpha \otimes \hat{\eta}^\alpha$ and, when we do this, it can be directly checked that

$$D\eta = \sum_\alpha \mathfrak{f}_\alpha \otimes d\hat{\eta}^\alpha.$$

We now come to the crucial point in our discussion. Let $\eta \in \mathbf{T}^1$ be small. Then $\Omega + \eta$ defines a new I-structure $\omega_\eta^1, \ldots, \omega_\eta^n$ where $\omega_\eta^\alpha = \omega^\alpha + \eta^\alpha$ ($\eta = \sum_\alpha \mathfrak{e}_\alpha \otimes \eta^\alpha$). We wish to verify that this new I-structure is *locally equivalent* to the old one if, and only if,

$$D\eta - \frac{1}{2}[\eta, \eta] = 0 \text{ [4]}. \tag{4.7}$$

For this, we let Δ be the operator $D - \frac{1}{2}[\ ,\ ]$ and verify that $d(\Omega + \eta) - \frac{1}{2}[\Omega + \eta, \Omega + \eta] = 0$ (Maurer-Cartan equation) if, and only if, $\Delta\eta = 0$. But $d(\Omega + \eta) - \frac{1}{2}[\Omega + \eta, \Omega + \eta] = d\Omega - \frac{1}{2}[\Omega, \Omega] + d\eta - [\Omega, \eta] - \frac{1}{2}[\eta, \eta]$. Q. E. D.

Now let $ds^2 = \sum_{\alpha=1}^{n} (\omega^\alpha)^2$ be a Riemannian metric on X. Having introduced this metric, we may form an inner product on $\mathbf{T}^q$ and we let $\mathfrak{T}^q$ be the completion of $\mathbf{T}^q$ in this inner product. Then $\mathfrak{T}^q$ consists of the square-integrable vector-valued q-forms on X. Using this inner product, we define the adjoint $D^* : \mathfrak{T}^q \to \mathfrak{T}^{q-1}$ of D and then, since $\square = DD^* + D^*D$ is strongly elliptic, we have the *Hodge theorem*: There exists a unique completely continuous self-adjoint operator $\mathfrak{G} : \mathfrak{T}^q \to \mathfrak{T}^q$ such that $\square\mathfrak{G} = \mathfrak{G}\square = 0$, $D\mathfrak{G} = \mathfrak{G}D$, $D^*\mathfrak{G} = D^*\mathfrak{G}$, and we have the formula of orthogonal decomposition

$$\mathfrak{T}^q = \mathbf{H}^q \oplus D^*D\mathfrak{G}\mathfrak{T}^q \oplus DD^*\mathfrak{G}\mathfrak{T}^q \tag{4.8}$$

where $\mathbf{H}^q = \ker\square$ is the harmonic space.

Now suppose that we have on $\mathbf{T}^q$ a norm $\|\ \|_q$ and let $\mathbb{T}^q$ be the Banach space obtained by completing $\mathbf{T}^q$ in $\|\ \|_q$. We suppose that this norm $\|\ \|_q$ has the property that D and D^*G are bounded transformations and that $[\ ,\ ]$ is a bounded bi-linear transformation, all in the appropriate norms. Finally we suppose that $\mathbb{T}^q$ is the Banach space product of the sub-Banach spaces obtained by completing $\mathbf{H}^q$, $D^*D\mathfrak{G}\mathbf{T}^q$, and $DD^*\mathfrak{G}\mathbf{T}^q$ in $\|\ \|_q$. We let $\pi_{\mathbf{H}}$, π_{D^*}, and π_D be the respective projection operators so that $\mathbb{T}^q = \pi_{\mathbf{H}}(\mathbb{T}^q) \times \pi_{D^*}(\mathbb{T}^q) \times \pi_D(\mathbb{T}^q)$, and we let $\mathbf{Z}^q$ be the Banach subspace $\pi_{\mathbf{H}}(\mathbb{T}^q) \times \pi_D(\mathbb{T}^q)$. We shall use the technique of *successive approximations* written as an implicit function theorem in Banach spaces ([2]); this approach is motivated by the work of NIJENHUIS and RICHARDSON. We denote by $N(*)$ a generic neighborhood of zero in a Banach space $*$.

We say that $\eta \in \mathbb{T}^1$ is *semi-integrable* if $D\eta - \frac{1}{2}\pi_D[\eta, \eta] = 0$.

[4]) Equation (4.7) should be compared with eqn. (4.10) in [3].

Lemma 1: *There exists $N(\pi_{\mathrm{Z}}(\mathbb{T}^1))$ and $N(\pi_{D^*}(\mathbb{T}^1))$ and a differentiable mapping of Banach spaces $r: N(\pi_{\mathrm{Z}}(\mathbb{T}^1)) \to N(\pi_{D^*}(\mathbb{T}^1))$ such that, if $\varphi \in \pi_{\mathrm{Z}}(\mathbb{T}^1)$ and $\psi \in \pi_{D^*}(\mathbb{T}^1)$, then $\varphi + \psi$ is semi-integrable if, and only if, $r(\varphi) = \psi$.*

Proof: Define $F: \pi_{\mathrm{Z}}(\mathbb{T}^1) \times \pi_{D^*}(\mathbb{T}^1) \to \pi_D(\mathbb{T}^2)$ by $F(\varphi, \psi) = D(\varphi + \psi) - \frac{1}{2}\pi_D[\varphi + \psi, \varphi + \psi]$. Then F is differentiable and $D_2F(0, 0) = D: \pi_{D^*}(\mathbb{T}^1) \to \pi_D(\mathbb{T}^2)$. Thus $D_2F(0, 0)$ is an isomorphism of Banach spaces, and thus there exists $r: N(\pi_{\mathrm{Z}}(\mathbb{T}^1)) \to N(\pi_{D^*}(\mathbb{T}^1))$ such that $F(\varphi, \psi) = 0$ if, and only if, $r(\varphi) = \psi$. Q. E. D.

Define $R: \pi_{\mathrm{H}}(\mathbb{T}^1) \to \mathbb{T}^1$ by $R(\varphi) = \varphi + r(\varphi)$ $(\varphi \in \pi_{\mathrm{H}}(\mathbb{T}^1))$.

Lemma 2: *There exists $N(\pi_{\mathrm{H}}(\mathbb{T}^1))$ such that, if $\varphi \in N(\pi_{\mathrm{H}}(\mathbb{T}^1))$, $\Delta R(\varphi) = 0$ if, and only if, $\pi_{\mathrm{H}}[R(\varphi), R(\varphi)] = 0$.*

Proof: $\Delta R(\varphi) = -\frac{1}{2}\{\pi_{D^*}[R(\varphi), R(\varphi)] + \pi_{\mathrm{H}}[R(\varphi), R(\varphi)]\}$ and if $\Delta R(\varphi) = 0$, then $\pi_{\mathrm{H}}[R(\varphi), R(\varphi)] = 0$ since $\pi_{\mathrm{H}}(\mathbb{T}^2) \cap \pi_{D^*}(\mathbb{T}^2) = 0$.

Assume now that $\pi_{\mathrm{H}}[R(\varphi), R(\varphi)] = 0$. Then, if $\Lambda = \frac{1}{2}\pi_{D^*}[R(\varphi), R(\varphi)]$, $\Lambda = D^*\mathbb{G}[DR(\varphi), R(\varphi)]$ (by (4.5)) $= D^*\mathbb{G}[\Lambda, R(\varphi)]$ (by (4.1)). Thus we get $\|\Lambda\|_2 \leqq c\,\|R(\varphi)\|_1\,\|\Lambda\|_2$, and, since $R(0) = 0$, it follows that $\Lambda = 0$ if $\varphi \in N(\pi_{\mathrm{H}}(\mathbb{T}^1))$. But then $\Delta R(\varphi) = 0$. Q. E. D.

Now let U be a small neighborhood of 0 in the finite dimensional Banach space $\pi_{\mathrm{H}}(\mathbb{T}^1)$. We define a family $\{\hat{\mathrm{W}} \xrightarrow{\tilde{\omega}} U\}$ of I-structures on X by letting $\tilde{\omega}^{-1}(\eta)$ $(\eta \in U)$ have the I-structure given by $\Omega + R(\eta)$. If we then let $V = \{\eta \in U \mid \Delta R(\eta) = 0\}$, V is an analytic set through the origin (Lemma 2) and the structure on $X_\eta = \tilde{\omega}^{-1}(\eta)$ $(\eta \in V)$ is locally isomorphic to the I-structure on X[4a]. Thus, if $\mathscr{W} = \tilde{\omega}^{-1}(V)$, $\{\mathscr{W} \xrightarrow{\tilde{\omega}} V\}$ defines a germ of deformation of the structure on X. In §§ 5 and 6, we shall prove:

Theorem D: *The germ of deformation $\{\mathscr{W} \xrightarrow{\tilde{\omega}} V\}$ is strongly locally complete and effective.*

5. Proof of Theorem D

An I-structure on X, near to the given structure, is uniquely written as $\Omega + \eta$ for some $\eta \in \mathbb{T}^1$. By abuse of language, we simply say that η gives an I-structure, and we define this structure to be *extremal* if $D^*\eta = 0$. By Lemma 1 in § 4, if η is extremal and semi-integrable, then $\eta = R(\varphi) = \varphi + r(\varphi)$ for some $\varphi \in \pi_{\mathrm{H}}(\mathbb{T}^1)$. If $U' \subset \mathbf{R}^m$ is an open set and if for each $t' \in U'$ we have $\eta(t') \in \mathbb{T}^1$, we say that we have a *differentiable family of I-structures* if the local expressions $\eta(x, t')$ are smooth in x and t'.

If $f: X \to X$ is a diffeomorphism near the identity, then f transforms $\Omega + \eta$ into a new I-structure written as $\Omega + \eta(f)$. Clearly η is integrable if, and only if, $\eta(f)$ is.

If $\xi \in N(\mathbb{T}^0)$, we define a diffeomorphism $e(\xi)$ to be $\exp(t\xi)]_{t=1}$. Theorem D will follow from

[4a]) $\mathbb{T}$ being a linear space has a real analytic structure. Then $\pi_{\mathrm{H}}, \pi_D, \pi_{D^*}, D, \ldots$ are real analytic, as is R; and consequently V is a real analytic set. In fact, the power series defining V may be explicitly written down.

Theorem E: *There exists a differentiable mapping* $s : N(\mathbb{T}^1) \to N(\pi_{D^*}(\mathbb{T}^0))$ *such that, for* $\varphi \in N(\mathbb{T}^1)$, $\xi \in N(\pi_{D^*}(\mathbb{T}^0))$, $\varphi(e(\xi))$ *is extremal if, and only if,* $\xi = s(\varphi)$. *Furthermore, if* $\varphi(t')$ *depends smoothly on* t', *then so does* $\xi(t') = s(\varphi(t'))$.

Proof of Theorem D: Let $\{\mathscr{W}' \xrightarrow{\tilde{\omega}} V'\}$ be a deformation of the I-structure on X and let $U' \supset V'$ be the ambient space so that we have $\{\mathbf{W}' \xrightarrow{\tilde{\omega}'} U'\}$ with $\tilde{\omega}'^{-1}(V') = \mathscr{W}'$. For each $t' \in U'$, $\eta(t') \in \mathbb{T}^1$ defines the I-structure on $X_{t'} = \tilde{\omega}'^{-1}(t')$ and $\eta(t')$ depends smoothly on t'. Then there exists an open subset U of a finite-dimensional subspace $\Lambda \subset \pi_{\mathrm{H}}(\mathbb{T}^1) \times \pi_{D^*}(\mathbb{T}^1)$ such that $\eta(t')\big(e(\xi(t'))\big) \in U$ for all $t' \in U'$, where now $\xi(t') = s(\eta(t'))$. Since $\eta(t')$ is integrable for $t' \in V'$, $\eta(t')\big(e(\xi(t'))\big) \in V$ for $t' \in V'$, and we may define (F, f): $: \{\mathbf{W}' \xrightarrow{\tilde{\omega}'} U'\} \to \{\mathbf{W} \xrightarrow{\tilde{\omega}} U\}$ by $f(t') = \eta(t')\big(e(\xi(t'))\big)$ and $F|\tilde{\omega}'^{-1}(t') = e(\xi(t'))$. Then $f(V') \subset V$ and we in fact have a mapping of deformations. Q. E. D.

Set now $\varphi(e(\xi)) = \varphi + D\xi + R(\varphi, \xi)$ for $\varphi \in N(\mathbb{T}^1)$, $\xi \in N(\mathbb{T}^0)$. Theorem E will follow from

Proposition A: *R is a differentiable mapping of* $N(T^1) \times N(\pi_{D^*}(T^0)) \to$ $\to N(T^1)$ *which satisfies* $D_2\mathscr{R}(0, 0) = 0$.

Proof of Theorem E: Define $S : N(\mathbb{T}^1) \times N(\pi_{D^*}(\mathbb{T}^0)) \to N(\pi_D(\mathbb{T}^1))$ by $S(\varphi, \xi) = D\xi + \pi_D\{\varphi + \mathscr{R}(\varphi, \xi)\}$. Then, by Proposition A, S is a differentiable mapping of Banach spaces and $D_2(0, 0) = D$. By the implicit function theorem, there exists $s : N(\mathbb{T}^1) \to N(\pi_{D^*}(\mathbb{T}^0))$ such that $S(\varphi, \xi) = 0$ if, and only if, $\xi = s(\varphi)$. But $S(\varphi, \xi) = 0$ if, and only if, $D^*S(\varphi, \xi) = 0$ which in turn is equivalent to $D^*D\xi + D^*\varphi + D^*\mathscr{R}(\varphi, \xi) = D^*\big(\varphi(e(\xi))\big) = 0$.

The smooth dependence on parameters is also easily established by using again successive approximations[5]). Q. E. D.

So far we have avoided mention of what the norm $\| \;\|_q$ on $\mathbf{T}^q$ should be. Given a norm $\| \;\|_q^Z$ on vector-valued q-forms on an open subset $Z \Subset \mathbf{R}^n$, we may take a finite coordinate covering $\{U_\alpha\}$ of X and define $\| \;\|_q^{\{U_\alpha\}} = \sum_\alpha \| \;\|_q^{U_\alpha}$. Furthermore, for any two finite coverings $\{U_\alpha\}$, $\{U'_\alpha\}$, the norms $\| \;\|_q^{\{U_\alpha\}}$, $\| \;\|_q^{\{U'_\alpha\}}$ will be equivalent. Thus, having defined $\| \;\|_q^Z$, we may define $\| \;\|_q$ as an *equivalence class* of norms obtained from finite open coverings. With this in mind, we choose $k > n + 3$ and let $\|\eta\|_q^Z$ be the *Sobolev norm* $\|\eta\|_{k-q}^Z$ on a vector-valued q-form η on Z. Thus, if

$$\eta = \sum \eta^i_{j_1 \ldots j_q} \frac{\partial}{\partial x^i} \otimes dx^{j_1} \wedge \cdots \wedge dx^{j_q}, \; (\|\eta\|_{k-q}^Z)^2 = \sum_{\substack{0 \le \mu \le k-q \\ i; j_1, \ldots, j_q}} \int_Z |D^\mu \eta^i_{j_1 \ldots j_q}|^2 \, dx,$$

where D^μ runs over all partial derivatives of order μ.

Clearly then $D : \mathbf{T}^q \to \mathbf{T}^{q+1}$ is a bounded transformation and also $D^*\mathbf{G} : \mathbf{T}^q \to$ $\to \mathbf{T}^{q-1}$ is bounded, due to the property $\|\mathbf{G}\eta\|_{l+2}^Z \le c\|\eta\|_l^{Z'}$ where $Z \subset Z'$ ([7]). Finally, $[\,,\,] : \mathbf{T}^p \otimes \mathbf{T}^q \to \mathbf{T}^{p+q}$ is clearly a bounded bi-linear transformation. Thus it remains to verify Proposition A with our choice of norms.

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[5]) It can be shown, by a direct convergence argument, that if $\varphi(t)$ depends real-analytically on t, then so does $\xi(t) = s(\varphi(t))$.

Suppose therefore that $Z \Subset \mathbf{R}^n$ and that we have on Z an I-structure given by $\omega^1, \ldots, \omega^n$. Fix $\delta < 1$ and suppose that we have a vector-valued form $\varphi = \{\varphi^\alpha\}$ and vector fields $\xi = \{\xi^\alpha\}$, $\xi' = \{\xi'^\alpha\}$ with $\|\varphi\|_k^Z < \delta$, $\|\xi\|_k^Z < \delta$, $\|\xi'\|_k^Z < \delta$. Let $Z' \subset Z$ be an open set with $e(\xi)Z' \subset Z$, $e(\xi')Z' \subset Z$. Then, writing $\varphi(e(\xi)) = \varphi + D\xi + \mathscr{R}(\varphi, \xi)$, it will suffice to prove:

Lemma A: *For* $Z'' \subset Z' \subset Z$,

$$(5.1) \quad \|\mathscr{R}(\varphi, \xi) - \mathscr{R}(\varphi, \xi')\|_{k-1}^{Z''} \leqq c(\|\varphi\|_k^{Z'} + \|\xi\|_k^{Z'} + \|\xi'\|_k^{Z'})(\|\xi - \xi'\|_k^{Z'}).$$

We prove (5.1). First observe that $(\exp\xi)^*(\omega^\alpha + \varphi^\alpha) = \omega^\alpha + \varphi^\alpha + \mathscr{L}_\xi(\omega^\alpha + \varphi^\alpha) + \sum R^\alpha_{\beta\gamma}(\varphi, \xi)\xi^\beta\xi^\gamma$. From this it follows that $\varphi(e(\xi)) = \varphi + D\xi + \mathscr{L}_\xi\varphi + \sum R_{\beta\gamma}(\varphi, \xi)\xi^\beta\xi^\gamma$. Thus we may write

$$(5.2) \qquad \mathscr{R}(\varphi, \xi) - \mathscr{R}(\varphi, \xi') = D(\xi - \xi') + \mathscr{L}_{(\xi-\xi')}\varphi + T(\varphi, \xi, \xi')$$

where $T(\varphi, \xi, \xi')$ can be determined. Clearly $\|D(\xi - \xi')\|_{k-1}^{Z''} \leqq c(\|\xi - \xi'\|_k^{Z'})$. To derive an estimate

$$\|\mathscr{L}_{(\xi-\xi')}\varphi\|_{k+1}^{Z''} \leqq c(\|\varphi\|_k^{Z'} + \|\xi\|_k^{Z'} + \|\xi'\|_k^{Z'})(\|\xi - \xi'\|_k^{Z'}),$$

using the *Cartan identity* $\mathscr{L}_{(\xi-\xi')}\varphi = di(\xi - \xi')\varphi + i(\xi - \xi')\,d\varphi$, it will clearly suffice to have an estimate of the sort

$$(5.3) \qquad \|fg\|_{k-1}^{Z''} \leqq c\|f\|_{k-1}^{Z'}\|g\|_{k-1}^{Z'}$$

for functions f, g on Z' with finite norm $\|\ \|_{k-1}^{Z'}$. However, for $k - 1 > n$, (5.3) is an easy consequence of *Sobolev's lemma* ([7]): For $l > n/2$, $x \in Z''$, $|D^\mu h(x)| \leqq c\|h\|_{\mu+l}^{Z'}$ where h is a function on Z' with finite norm $\|\ \|_{\mu+l}^{Z'}$.

To complete the proof, we must derive an estimate

$$(5.4) \qquad \|T(\varphi, \xi, \xi')\|_{k-1}^{Z''} \leqq c(\|\varphi\|_k^{Z'} + \|\xi\|_k^{Z'} + \|\xi'\|_k^{Z'})(\|\xi - \xi'\|_k^{Z'}).$$

Since $\|T(\varphi, \xi, \xi')\| = \{(\exp\xi)^*\varphi - (\exp\xi')^*\varphi\} + \{(\exp\xi)^*\omega - (\exp\xi')^*\omega\} + \mathscr{L}_{(\xi-\xi')}\varphi$, we claim that to prove (5.4), it will suffice to show

Lemma B: *Let $Z'' \Subset Z' \Subset Z \Subset \mathbf{R}^n$ be open sets with Z convex. Suppose that we have mappings $f\colon Z' \to Z$, $g\colon Z' \to Z$ which satisfy $\|f - I\|_{k-1}^{Z'} < \delta$, $\|g - I\|_{k-1}^{Z'} < \delta$ where $\delta < 1$. Assume that h is a function on Z with $\|h\|_k^Z < \delta$. Then there exists $c = c(Z'', Z', Z, \delta)$ such that*

$$(5.5) \quad \|h \circ f - h \circ g\|_{k-1}^{Z''} \leqq c(\|h\|_k^Z + \|f - I\|_{k-1}^{Z'} + \|g - I\|_{k-1}^{Z'})(\|f - g\|_{k-1}^{Z'}).$$

Indeed, suppose that (5.5) is satisfied. Then, for example,

$$(\exp\xi)^*\varphi - (\exp\xi')^*\varphi = \sum \frac{\partial e(\xi)(x)}{\partial x^\alpha}\varphi^\alpha(e(\xi)) - \frac{\partial}{\partial x^\alpha}e(\xi')(x)\,\varphi^\alpha(e(\xi'))$$

$$= \sum \frac{\partial e(\xi)}{\partial x^\alpha}\varphi^\alpha(e(\xi)) - \frac{\partial e(\xi)}{\partial x^\alpha}\varphi^\alpha(e(\xi')) + \frac{\partial e(\xi)}{\partial x^\alpha}\varphi^\alpha(e(\xi')) - \frac{\partial}{\partial x^\alpha}e(\xi')\,\varphi^\alpha(e(\xi'))$$

and thus

$$\|(\exp\xi)^*\varphi - (\exp\xi')^*\varphi\|_{k-1}^{Z''} \leqq c\Big\{\|e(\xi)\|_{k-1}^{Z'} \cdot \|\varphi^\alpha(e(\xi)) - \varphi^\alpha(e(\xi'))\|_{k-1}^{Z'} + \\ + \|\varphi^\alpha(e(\xi'))\|_{k-1}^{Z'} \cdot \left\|\frac{\partial e(\xi)}{\partial x^\alpha} - \frac{\partial e(\xi')}{\partial x^\alpha}\right\|_{k-1}^{Z'}\Big\}$$

(by (5.3)). But, if (5.5) holds, we then get $\|(\exp\xi)^*\varphi - (\exp\xi')^*\varphi\|_{k-1}^{Z''} \leqq$ $\leqq c\{(\|\varphi\|_k^{Z'} + \|\xi\|_k^{Z'} + \|\xi'\|_k^{Z'})\ (\|\xi - \xi'\|_k^{Z'})\}$. Similarly, the other terms in $T(\varphi, \xi, \xi'')$ may be estimated.

The proof of (5.5), which is due to KURANISHI, can be found in the Appendix to [4]. This completes the proof of Lemma A and hence of Theorem D.

6. Completion of the Proof of the main Theorem

We first finish the proof of Theorem B. Suppose on X that we have an I-structure given by $\omega^1, \ldots, \omega^n$ where $d\omega^\alpha = \frac{1}{2}\sum_{\beta,\gamma} c^\alpha_{\beta\gamma}\omega^\beta \wedge \omega^\gamma$ and the $c^\alpha_{\beta\gamma}$ are constant. In the notations of § 4, let U be an open neighborhood of the origin in $\pi_{\mathrm{H}}(\mathbb{T}^1)$ and construct the family of I-structures $\{\mathrm{W} \xrightarrow{\tilde{\omega}} U\}$ where $\tilde{\omega}^{-1}(\varphi)$ $(\varphi \in U)$ has the I-structure corresponding to $R(\varphi) = \varphi + r(\varphi)$ in Lemma 1 of § 4. If then $V = \{\varphi \in U \mid \Delta R(\varphi) = 0\}$, $\mathscr{W} = \tilde{\omega}^{-1}(V)$, then $\{\mathscr{W} \xrightarrow{\tilde{\omega}} V\}$ is a strongly locally complete germ of deformation of the I-structure on X. We want to show that this germ is effective.

Suppose then that $\{\mathscr{W}' \xrightarrow{\tilde{\omega}'} V'\}$ is any germ of deformation of the I-structure on X. Let $\mathscr{Z}_0(V') = \mathscr{Z}_0'$ be the Zariski tangent space to V' at the origin. We define the linear mapping $\varrho : \mathscr{Z}_0' \to H^1(X, \Theta)$ as follows: Let $\gamma(s)$ $(s \in (-\varepsilon, \varepsilon))$ be an analytic curve in V', and let $\varphi(s)$ be the element in $\mathbb{T}^1$ which defines the I-structure on $\tilde{\omega}'^{-1}(\gamma(s))$. Then, if $\varphi = \frac{d\varphi(s)}{ds}\Big]_{s=0}$, since $\Delta\varphi(s) = 0$, $D\varphi = 0$ and we may thus assign to $\gamma(s)$ the element $\varphi \in H^1(X, \Theta)$. Since $\mathscr{Z}_0'$ is spanned by tangents to analytic curves in V', we may thus define $\varrho : \mathscr{Z}_0' \to H^1(X, \Theta)$. It is easy to see that, if $\{\mathscr{W}' \xrightarrow{\tilde{\omega}'} V'\}$ is a trivial germ of deformation, then ϱ is the zero map. (For our purposes, we may assume that V' is non-singular, in which case the statement is obvious.)

Returning to the case of our strongly locally complete germ $\{\mathscr{W} \xrightarrow{\tilde{\omega}} V\}$, the Zariski tangent space $\mathscr{Z}_0$ to V at the origin is a linear subspace of $\pi_{\mathrm{H}}(\mathbb{T}^1)$, and ϱ is the identity map. Then clearly $\{\mathscr{W} \xrightarrow{\tilde{\omega}} V\}$ is effective. This proves Theorem B.

To prove Theorem C, we let $G \to B_G \to X$ be a regular G-structure of finite type. Let A be the Lie group of automorphisms of the induced G-structure on the universal covering of X, and let $\Gamma \subset A$ be the fundamental group. Set $Y = A/\Gamma$. If $\mathscr{E}$ is the set of germs of deformation of the I-structure on Y and if $\mathscr{F}$ is the set of germs of deformation of the G-structure on X, then we have defined $\eta : \mathscr{E} \to \mathscr{F}$. Associated with each germ γ in $\mathscr{E}$ or $\mathscr{F}$, we let $\mathscr{Z}_0(\gamma)$ be the Zariski tangent space to the base space of γ at the origin. Then, for $\gamma \in \mathscr{E}$, we have defined $\varrho : \mathscr{Z}_0(\gamma) \to H^1(Y, \Theta_I)$. Similarly, if $\gamma \in \mathscr{F}$, we may define $\sigma : \mathscr{Z}_0(\gamma) \to H^1(X, \Theta_G)$. Now, as was shown in [3], there are natural isomorphisms $H^1(Y, \Theta_I) \cong H^1(\Gamma, \mathfrak{a})$, $H^1(\Gamma, \mathfrak{a}) \cong H^1(X, \Theta_G)$ where $H^1(\Gamma, \mathfrak{a})$ is group cohomology and Γ acts on $\mathfrak{a}$ by Ad. Thus there is an isomorphism $\tau : H^1(Y, \Theta_I) \xrightarrow{\sim}$ $\xrightarrow{\sim} H^1(X, \Theta_G)$.

Now, if $\gamma \in \mathscr{E}$, it is easy to check that the following diagram commutes:

$$\begin{array}{ccc} \mathscr{X}_0(\gamma) & \xrightarrow{\varrho} & H^1(Y, \Theta_I) \\ \downarrow & & \downarrow{\scriptstyle\tau} \\ \mathscr{X}_0(\eta(\gamma)) & \xrightarrow{\sigma} & H^1(X, \Theta_G)\,. \end{array}$$

Then the same argument as in Theorem B proves Theorem C.

This completes the proof of our main Theorem.

7. Applications and Examples

Let A be a connected, simply connected Lie group and Γ a discrete group. A *lattice* in A is a faithful representation $\varrho : \Gamma \to A$ such that $\varrho(\Gamma)$ is discrete and $A/\varrho(\Gamma)$ is compact. Let $R(\Gamma, A)$ be the space of lattices of Γ in A. Since Γ is finitely generated and finitely related, $R(\Gamma, A)$ has locally the structure of an analytic space. In fact, let $\varrho_0 \in R(\Gamma, A)$ and let $\gamma_1, \ldots, \gamma_N$ be generators of Γ. Then there exists a neighborhood U of the unit $e \in A$ such that $U\gamma_i \cap U\gamma_j = \phi (i \neq j)$. Then a neighborhood of ϱ_0 in $R(\Gamma, A)$ is a subset of $\underbrace{U \times \cdots \times U}_{N}$; in fact, $R(\Gamma, A) \cap \{\underbrace{U \times \cdots \times U}_{N}\}$ is clearly the zero locus of finitely many analytic functions corresponding to the relations of Γ.

Now let $\varrho_0 \in R(\Gamma, A)$ and let $R_0(\Gamma, A)$ be the component of Γ in $R(\Gamma, A)$. Then A acts on $R_0(\Gamma, A)$ by sending a lattice into its conjugate; it is well known that $R_0(\Gamma, A)/A$ need not even be Hausdorff. However, our Theorem B is easily seen to imply the following

Theorem: *There exists a neighborhood U of ϱ_0 in $R_0(\Gamma, A)$ and Z of e in A such that, if $\varrho, \varrho' \in U$ are declared equivalent if $\varrho = \gamma\varrho'\gamma^{-1}$ for some $\gamma \in Z$, then the quotient space has the structure of an analytic subset of $H^1(\Gamma, a)$.*

Thus, although $R_0(\Gamma, A)/A$ is poorly behaved globally, it is in some sense a locally analytic space. We denote by $V(\varrho_0)$ the germ "U factored by the equivalence relation in the above theorem." Clearly, if $V(\varrho_0)$ is an analytic set without singularities, then a neighborhood of ϱ_0 in $R(\Gamma, A)$ is a manifold, but not conversely.

As for examples, it has been shown by H. Garland that, if A is *nilpotent*, then $H^1(\Gamma, A) \cong H^1(\mathfrak{a}, \mathfrak{a})$ independently of the lattice Γ. In this case, for the locally complete germ $\{\mathscr{W} \xrightarrow{\bar{\omega}} V\}$, V is non-singular and is in fact a neighborhood of the identity in the outer automorphism group. Furthermore, due to the stable nature of $H^1(X, \Theta_I)$ under a deformation, an easy continuation argument shows that all lattices in A are obtained by applying an automorphism of A to Γ. This is a theorem of Malcev ([10]).

In fact, by computing an appropriate cohomology group, the results of Wang [10] should follow from our construction — his Theorem that $R_0(\Gamma, A)$ is, in some cases, a manifold should be a consequence of the *cohomological* result that there are *no obstructions* to constructing a deformation through $\theta \in H^1(X, \Theta_I)$[6]).

On the other hand, we have in [3] given two examples of a construction of a strongly locally complete family $\{\mathscr{W} \overset{\varpi}{\to} V\}$ where V had singularities (in one case a quadratic and in the other a cubic singular locus). It would perhaps seem now as though local the theory of deformations of a lattice in a Lie group could be in some sense completed if a uniform method of computing $H^1(\Gamma, \mathfrak{a}) \cong \cong H^1(X, \Theta_I)$ could be found.

Appendix I: Transitive and homogeneous G-structures

We keep the notations of § 2. Let $G \subset GL(n, \mathbf{R})$ be a connected linear Lie group. It is well-known ([1]) that we may associate to G a sequence $G_0 = G, G_1, \ldots$ of linear groups, where G_μ is the μ^{th} prolongation of G. In fact, if $\dim G = r$, then $G_1 \subset GL(n + r, \mathbf{R})$ is the abelian linear group whose linear Lie algebra $\mathfrak{g}_1$ consists of all matrices of the form

$$\begin{array}{c c} & \begin{array}{cc} \overbrace{}^{r} & \overbrace{}^{n} \end{array} \\ \begin{array}{c} r\{ \\ n\{ \end{array} & \begin{pmatrix} 0 & \xi \\ 0 & 0 \end{pmatrix} \end{array}$$

where $\xi \in \operatorname{Hom}(\mathbf{R}^n, \mathfrak{g})$ is written as $\xi^i_{jk} = \xi^i_{kj}$ and, for each fixed k, $\xi^i_{jk} = \xi^i_j(k) \in \mathfrak{g}$. Then G_μ is defined inductively by $G_\mu = (G_{\mu-1})_1$.

Let now $G \to B_G \to X$ be a G-structure. Then ([1] and also below) there is naturally induced on B_G a G_1-structure $G_1 \to B_{G_1} \to B_G$; furthermore, on B_{G_1} there is a natural G_2-structure, and so forth. Clearly G is of finite type if, and only if, $\mathfrak{g}_{\mu_0} = 0$ for some μ_0, which happens if, and only if, there is induced on $G_{\mu_0-1} \to B_{G_{\mu_0-1}} \to B_{G_{\mu_0-2}}$ a linear connexion.

Thus, given $G \to B_G \to X$, we have a sequence of bundles $\{G_\mu \to B_{G_\mu} \to B_{G_{\mu-1}}\}$ and clearly any G-automorphism f of X "lifts" to a sequence $\{f^\mu\}$ of G_μ-automorphisms of $B_{G_{\mu-1}}$. ($f^0 = f$ and $f^1 = f_*$ is the differential of f, etc.) Conversely, given any G_μ-automorphism h of $B_{G_{\mu-1}}$, there exists an G-automorphism f of X such that $f^\mu = h$.

If $\mathbf{T}$ is a set $\{t\}$ of local diffeomorphisms of a manifold Y, we say that $\mathbf{T}$ acts *locally transitively* on Y if, given $y, y' \in Y$ which are sufficiently close, there exists $t \in \mathbf{T}$ such that $t(y) = y'$.

Let $G \to B_G \to X$ be a G-structure, and let Γ be the set of local G-automorphisms of X. Then Γ induces a set Γ^μ of local G-automorphisms on $B_{G_{\mu-1}}$. We say that $G \to B_G \to X$ is *locally homogeneous* if Γ acts locally transitively on X, and we define the G-structure $G \to B_G \to X$ to be *transitive* if all Γ^μ act locally transitively on B_{G_μ}. There exists a μ_0 such that $G \to B_G \to X$ is transitive if, and only if, Γ^μ acts locally transitively on B_{G_μ} for $0 \leqq \mu \leqq \mu_0$.

Examples: (i) Any integrable structure is transitive, and B_{G_μ} may be thought of as the bundle of jets of order μ of local G-automorphisms. (ii) If

[6]) The necessary cohomology results have now been obtained by H. GARLAND in his thesis at the University of California, Berkeley.

$G = O(n)$, then local homogeneity means that X is covered by a homogeneous Riemannian manifold, provided, of course, that the Riemannian structure on X is complete. Transitivity of the Riemannian structure means nothing more nor less than *constant curvature.*

Generalizing the notion of constant curvature, we shall now describe the most general transitive G-structure of finite type. (Their importance for us is that they are probably the most general structures for which we can find the operator Δ of § 2.)

Let A be a connected Lie group and $B \subset A$ a closed, connected subgroup. Set $V = \mathfrak{a}/\mathfrak{b}$ and let $G(A, B) \subset GL(V)$ be the linear group of all transformations $\{\mathrm{Ad}(b)\}$ $(b \in B)$ acting on $\mathfrak{a}/\mathfrak{b}$. Then $X = A/B$ clearly has an A-invariant $G(A, B)$ structure, and, in fact, this is the smallest A-invariant G-structure on X.

Set $B_1 = \mathrm{Ker}\{\mathrm{Ad}\}$ where Ad is the above linear representation of B. Then obviously $B_{G(A,B)} = A/B_1$, and, if $V^1 = \mathfrak{a}/\mathfrak{b}_1$, then on A/B_1 we have the A-invariant $G(A, B_1)$ structure. Clearly $G(A, B)_1 \subset G(A, B_1)$. In this way, we may construct the bundles $B_{G(A,B)_\mu} = A/B_{\mu+1}$, and this process terminates with some integer μ_0 where $B_{\mu_0+1} = \{1\}$ and then $B_{G(A,B)_{\mu_0}} = A$. Thus we have

Proposition: *The canonical $G(A, B)$ structure on A/B is a transitive G-structure of finite type.*

Remark: The condition $G(A, B)_1 = \{1\}$ is implied by the condition that A/B be a *reductive coset space* ([8]). By this, we mean that the representation Ad of B on $\mathfrak{a}$ admits a complementary subspace to $\mathfrak{b} \subset \mathfrak{a}$.

Now we shall prove conversely the

Theorem: *Any transitive G-structure $G \to B_G \to X$ of finite type is locally equivalent to the $G(A, B)$ structure on A/B for some A and B.*

For simplicity we shall assume that $\mathfrak{g}_1 = 0$ (i.e. $G_1 = \{1\}$); the general argument is the same. Let $\dim G = r$. We agree on the ranges of indices $1 \leqq i, j, k \leqq n$, and $1 \leqq \varrho, \sigma, \tau \leqq r$. Let $\{\mathfrak{e}_\varrho\}$ be a basis for the linear Lie algebra $\mathfrak{g}$. By choosing the usual basis for $gl(n, \mathbf{R})$, we may write $\mathfrak{e}_\varrho$ as the matrix $a^i_{\varrho j}$. Let ω be the canonical $\mathbf{R}^n$-valued form on B_G which satisfies $\omega(bg) = g^{-1}\omega(b)$ $(b \in B, g \in G)$. In § 2, ω was denoted by ω_G. Then $\omega = (\omega^1, \ldots, \omega^n)$ and we may locally write

$$d\omega^i = -\sum_{\varrho,j} a^i_{\varrho j}\omega^j \wedge \pi^\varrho + \frac{1}{2}\sum_{j,k} c^i_{jk}(b)\,\omega^j \wedge \omega^k \tag{A.1}$$

where the π^ϱ are locally defined forms on B_G which restrict to the left-invariant Maurer-Cartan forms on each fibre. The forms $(\pi^1, \ldots, \pi^n;\ \omega^1, \ldots, \omega^n)$ define a local co-frame on B_G, and any other local co-frame is of the form $(\pi'^1, \ldots, \pi'^r;\ \omega^1, \ldots, \omega^n)$ where $\pi'^\varrho = \pi^\varrho + \sum_k b^\varrho_k\omega^k$. The b^ϱ_k define an element $\xi(b) \in \mathrm{Hom}(\mathbf{R}^n, \mathfrak{g})$ by $\xi(b) = (\xi^i_{jk})$ where $\xi^i_{jk} = \sum_\varrho a^i_{\varrho j} b^\varrho_k$. The new functions c'^i_{jk} in (A.1) are given by $c'^i_{jk} = c^i_{jk} + \sum_\varrho (a^i_{\varrho j}b^\varrho_k - a^i_{\varrho k}b^\varrho_j)$, and, since these functions are global, it follows that $\xi^i_{jk} = \xi^i_{kj}$. This proves that on B_G

there is a natural G_1-structure. Clearly if $G \to B_G \to X$ is transitive, then the functions c^i_{jk} are *constant*.

It is clear that for a general G, we may treat the G_1-structure on B_G in the same fashion, and so on. However, suppose that $G_1 = \{1\}$. Then the π^ϱ may be assumed to be globally defined, and $\{\pi^1, \ldots, \pi^r; \omega^1, \ldots, \omega^n\}$ gives a parallelism on B_G. We set $c^i_{\varrho j} = a^i_{\varrho j}$ so that $d\omega^i = - \Sigma c^i_{\varrho j} \omega^j \wedge \pi^\varrho + \frac{1}{2} \Sigma c^i_{jk} \omega^j \wedge \omega^k$. Furthermore, since the π^ϱ clearly give a connexion in $G \to B_G \to X$, $d\pi^\varrho - - \frac{1}{2} \Sigma c^\varrho_{\sigma\tau} \pi^\sigma \wedge \pi^\tau = \frac{1}{2} \Sigma c^\varrho_{ij} \omega^i \wedge \omega^j$ where the c^ϱ_{ij} are functions on B_G which essentially give the curvature of the connexion, and where $c^\varrho_{\sigma\tau}$ are the constants of structure of g. Now if Γ^1 acts transitively on B_G, then clearly the c^i_{jk} and c^ϱ_{ij} are constants, as well as the $c^\varrho_{\sigma\tau}$ and $c^i_{\varrho j}$.

Then the Pfaffians $\{\pi^1, \ldots, \pi^r; \omega^1, \ldots, \omega^n\}$ define locally on B_G the structure of a Lie group, and it is easy to see that $G \to B_G \to X$ is locally equivalent to the $G(A, B)$ structure on A/B for some chosed, connected subgroup $B \subset A$. Furthermore, in case A/B is a reductive coset space, and the connexion given by $\pi^1, \ldots, \pi^r$ on B_G is just the canonical connexion of the first kind in [8].

Example: Let $X = P_n(\mathbf{R})$ be the real projective space written as a coset space $X = A/B$ where $A = SL(n+1, \mathbf{R})$ is the projective group. Then $G(A, B)_1 \neq \{1\}$ but $G(A, B)_2 = \{1\}$, so that $G(A, B)$ may be said to be of *order* 2. The $G(A, B)_1$ structure on A/B_1 is a so-called *projective-connexion* and just as the spheres are models for Riemannian geometry of constant positive curvature, X is the model for the projective connexion of zero projective curvature. The transitivity of the $G(A, B)$-structure on $P_n(\mathbf{R})$ is the statement that the projective group acts transitively on the bundle of tangent directions over $P_n(\mathbf{R})$. One may clearly make similar models for the (Weyl) conformal connexions, etc.

Appendix II: A resolution of the infinitesimal automorphisms of a transitive G-structure of finite type

Since the problem is local, and by the result in Appendix I, it will suffice to treat the case of the canonical $G(A, B)$-structure on a coset space A/B where A is a connected Lie group and $B \subset A$ is a closed connected subgroup. We begin with some preliminary calculations. Again we assume that $G(A, B)_1 = \{1\}$ — the general case will be clear from this.

For any manifold Y, we let $\mathbf{T}(Y)$ be the tangent bundle and $\mathbf{T}(Y)^*$ the dual cotangent bundle. If $A/B = X$, then $\mathbf{T}(A)/B$ is a vector bundle over X, and we let Σ^q be the sheaf of germs of sections of the bundle $\mathbf{T}(A)/B \otimes \Lambda^q \mathbf{T}(X)^*$. We may describe Σ^q as follows: Let $\mathfrak{a}$ be the Lie algebra of *left*-invariant vector fields (= infinitesimal *right* translations) on A, and set $\mathbf{V}^q = \mathfrak{a} \otimes \Lambda^q(\mathfrak{a}/\mathfrak{b})^*$. There is a natural representation ϱ^q of B on $\mathbf{V}^q$, derived from the adjoint representation of B on $\mathfrak{a}$, and then $\mathbf{T}(A)/B \otimes \Lambda^q \mathbf{T}(X)^* = A \times_B \mathbf{V}^q$.

Let now $\mathscr{S}^q$ be the sheaf on A of germs of vector-valued q-forms. Then there is an injection $j: \Sigma^q \to \mathscr{S}^q$ which is induced from the injection $j: \mathfrak{a} \otimes \Lambda^q(\mathfrak{a}/\mathfrak{b})^* \to$

$\to \mathfrak{a} \otimes \Lambda^q(\mathfrak{a})^*$. On $\mathscr{S} = \bigoplus_{q \geq 1} \mathscr{S}^q$, we have defined in § 4 a differential operator $D : \mathscr{S}^q \to \mathscr{S}^{q+1}$ and a bracket $[\, ,\,] : \mathscr{S}^p \otimes \mathscr{S}^q \to \mathscr{S}^{p+q}$.

Proposition: $D(\Sigma^q) \subset \Sigma^{q+1}$ *and* $[\Sigma^p, \Sigma^q] \subset \Sigma^{p+q}$.

Proof: Let $\mathscr{R}^q \subset \mathscr{S}^q$ be the subsheaf consisting of germs of functions $\eta : A \to \mathfrak{a} \otimes \Lambda^q(\mathfrak{a})$ such that $(R_b \eta) = \varrho^q(b)\eta$ where R_b = right translation by $b \in B$ and ϱ^q is the adjoint representation of B on $\mathfrak{a} \otimes \Lambda^q(\mathfrak{a})^*$. Then $\Sigma^q \subset \mathscr{R}^q$.

We now assume that $\dim A = m$, $\dim B = r$, $m - r = n$, and we agree on the ranges of indices $1 \leq i, j, k \leq m$; $1 \leq \alpha, \beta, \gamma \leq r$, and $r + 1 \leq \varrho, \sigma, \tau \leq \leq r + n = m$. Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_m$ be a basis for $\mathfrak{a}$ such that $\mathfrak{e}_1, \ldots, \mathfrak{e}_r$ is a basis for $\mathfrak{b}$. Furthermore, let $\omega^1, \ldots, \omega^m$ be the dual basis for the Maurer-Cartan forms on A. Then the canonical form ω on A/B is just $\sum_\varrho \mathfrak{e}_\varrho \otimes \omega^\varrho$; ω is a global section of Σ^1. Also, $\Omega = \sum_i e_i \otimes \omega^i$ is a section of $\mathscr{R}^1$. A germ η in $\mathscr{R}^q$ is written $\eta = \sum_i \mathfrak{e}_i \otimes \eta^i$ where η^i is a germ of a q-form on A. Then the infinitesimal right translation is given by $\mathscr{L}_{\mathfrak{e}_\alpha} \eta^i = \Sigma\, c^i_{\alpha j} \eta^j$ where the c^i_{jk} are the structure constants of $\mathfrak{a}$.

If $\eta = \Sigma\, \mathfrak{e}_i \otimes \eta^i$, $\xi = \Sigma\, \mathfrak{e}_j \otimes \xi^j$, then $[\eta, \xi] = \Sigma\, \mathfrak{e}_i \otimes c^i_{jk} \eta^j \wedge \xi^k$. To show that $[\mathscr{R}^p, \mathscr{R}^q] \subset \mathscr{R}^{p+q}$, we must show that

$$\mathscr{L}_{\mathfrak{e}_\alpha}(\Sigma\, c^i_{jk} \eta^j \wedge \xi^k) = \Sigma\, c^i_{\alpha l} c^l_{jk} \eta^j \wedge \xi^k . \tag{A.2}$$

Using the relations $\mathscr{L}_{\mathfrak{e}_\alpha}(\eta^j) = \Sigma\, c^j_{\alpha l} \eta^l$, $\mathscr{L}_{\mathfrak{e}_\alpha} \xi^k = \Sigma\, c^k_{\alpha l} \xi^l$ (since $\eta, \xi \in \mathscr{R}^p, \mathscr{R}^q$ respectively), (A.2) reduces to showing that $\sum_k c^i_{kl} c^k_{\alpha j} + c^i_{jk} c^k_{\alpha l} = \sum_k c^i_{\alpha k} c^k_{jl}$, which is just the Jacobi-identity. This clearly implies that $[\Sigma^p, \Sigma^q] \subset \Sigma^{p+q}$.

Now let $\eta = \Sigma\, \mathfrak{e}_i \otimes \eta^i \in \Sigma^q$ $(q \geq 1)$. Then $\eta^i \equiv 0 (\operatorname{mod} \omega^\varrho)$. By definition, $D\eta = d\eta - [\Omega, \eta]$, and, by the above paragraph, $D\eta \in \mathscr{R}^{q+1}$. To complete the proof, we show that $D\eta \equiv 0 (\operatorname{mod} \omega^\varrho)$; i.e. $i(\mathfrak{e}_\alpha) D\eta = 0$ $(\alpha = 1, \ldots, r)$. But $i(\mathfrak{e}_\alpha)\, d\eta^i + d\, i(\mathfrak{e}_\alpha)\eta^i = \mathscr{L}_{\mathfrak{e}_\alpha} \eta^i$ and, since $i(\mathfrak{e}_\alpha)\eta^i = 0$, $i(\mathfrak{e}_\alpha)\, d\eta^i = \Sigma\, c^i_{\alpha k} \eta^k$. Since $D\eta = \Sigma\, \mathfrak{e}_i \otimes \{d\eta^i - c^i_{jk} \omega^j \wedge \eta^k\}$, $i(\mathfrak{e}_\alpha)\, D\eta = \Sigma\, \mathfrak{e}_i \otimes \{i(\mathfrak{e}_\alpha)\, d\eta^i - c^i_{\alpha k} \eta^k\} = 0$. Q. E. D.

This Proposition easily implies that, over X, we have the exact sequence of sheaves

$$\Sigma^1 \xrightarrow{D} \Sigma^2 \to \cdots \to \Sigma^q \xrightarrow{D} \Sigma^{q+1} \to \cdots . \tag{A.3}$$

In § 2, we have derived a sequence

$$0 \to \Theta_G \xrightarrow{i} \mathscr{T} \xrightarrow{D_1} \Sigma . \tag{A.4}$$

Our program will be completed by showing

Theorem: *There exists a canonical isomorphism* $j : \Sigma \rightrightarrows \Sigma^1$ *such that:*

(i) $\mathscr{T} \xrightarrow{j \circ D_1} \Sigma^1 \xrightarrow{D} \Sigma^2$ *is exact*

(ii) *For* $\sigma \in H^0(X, \Sigma^1)$, *if* $\Delta\sigma = D\sigma - \frac{1}{2}[\sigma, \sigma]$, *then the equation* $\Delta\sigma = 0$ *defines those G-structures on X which are locally equivalent to the given transitive G-structure of finite type, provided that* σ *is small.*

Proof: Let $\mathrm{W} = \mathfrak{a}/\mathfrak{b}$, and let $\mathfrak{b}(\mathrm{W}) \subset gl(\mathrm{W})$ be the linear space generated by the transformations $\{\mathrm{ad}(b)\}$ $(b \in \mathfrak{b})$. Then the vector space $gl(\mathrm{W})/\mathfrak{b}(\mathrm{W})$ is a B-module, and Σ was the sheaf of germs of sections of $A \times_B gl(\mathrm{W})/\mathfrak{b}(\mathrm{W})$. Now, if $\Xi \subset H^0(X, \Sigma)$ is a neighborhood of zero, we had interpreted the elements in Ξ as giving the $G(A, B)$ structures on $X = A/B$ which were near to the given structure. Let $\sigma \in \Xi$ and let $G(A, B) \to B_G(\sigma) \to X$ be the corresponding $G(A, B)$ structure. Then $B_G(\sigma)$ is *differentiably equivalent* to A, and so we may think of $B_G(\sigma)$ as being A with a different parallelism $\{\mathrm{e}_i(\sigma)\}$ where $\mathrm{e}_i(0) = \mathrm{e}_i$. But this parallelism is not arbitrary, for there should exist an action σ of B on the right on A so that the parallelism $\{\mathrm{e}_i(\sigma)\}$ is that which results from the canonical $G(A, B)$ structure on the quotient $X_\sigma = A/\sigma(B)$. Let $\{\omega^i(\sigma)\}$ be the dual parallelism to $\{\mathrm{e}_i(\sigma)\}$. Then the above condition is met if the following equations are satisfied:

(i) $[\mathrm{e}_\alpha(\sigma), \mathrm{e}_\beta(\sigma)] = \Sigma\, c^\gamma_{\alpha\beta}\mathrm{e}_\gamma(\sigma)$

(ii) $\mathscr{L}_{\mathrm{e}_\alpha(\sigma)}\omega^\varrho(\sigma) = \Sigma\, c^\varrho_{\alpha\tau}\omega^\tau(\sigma)$.

Indeed, (i) means that B acts on A on the right by an action $\sigma(B)$, and (ii) guarantees that the parallelism $\{\mathrm{e}_i(\sigma)\}$ is the parallelism arising from the $G(A, B)$-structure on $A/\sigma(B) = X_\sigma$.

But, and this is the whole point, if σ is sufficiently small, then there exists a function $\xi(\sigma): A \to \mathrm{Hom}(\mathfrak{a}/\mathfrak{b}, \mathfrak{a})$ such that $\omega^\varrho(\sigma) = \xi(\sigma) \cdot \omega^\varrho$ where $\omega(\sigma) = (\omega^1(\sigma), \ldots, \omega^n(\sigma))$ and $\omega = (\omega^1, \ldots, \omega^n)$ are the canonical forms associated with respect to the $G(A, B)$ structures on X_σ and X respectively. Then $\xi(\sigma)$ is a section of $\mathscr{S}^1$, and, from (i) and (ii), we see that $\xi(\sigma)$ is in fact a section of Σ^1. This whole argument is reversible: Given a section ξ of Σ^1, we may define an element σ_ξ of Σ by the rule $\omega(\sigma_\xi) = \xi \cdot \omega$ such that (i) and (ii) will be satisfied for the parallelism $\{\mathrm{e}(\sigma_\xi)_i\}$ determined by $\omega(\sigma_\xi)$. Then $\xi(\sigma_\xi) = \xi$ and $\sigma_{\xi(\sigma)} = \sigma$, so that we in fact have an isomorphism of sheaves $j: \Sigma \overset{\approx}{\to} \Sigma^1$ $(j(\sigma) = \xi(\sigma))$.

Now it is clear from (4.7) that (ii) in the Theorem is satisfied. Indeed, if $d\omega^\alpha(\sigma) - \frac{1}{2}\Sigma\, c^\alpha_{\beta\gamma}\omega^\beta(\sigma) \wedge \omega^\gamma(\sigma) = \frac{1}{2}\Sigma\, c^\alpha_{\varrho\tau}(\sigma)\, \omega^\varrho(\sigma) \wedge \omega^\tau(\sigma)$ and if $d\omega^\varrho(\sigma) - \Sigma\, c^\varrho_{\alpha\tau}\omega^\alpha(\sigma) \wedge \omega^\tau(\sigma) = \frac{1}{2}\Sigma\, c^\varrho_{\tau\varphi}(\sigma)\, \omega^\tau(\sigma) \wedge \omega^\varphi(\sigma)$, the equation $\Delta\sigma = 0$ is just the same as saying that the structure functions $c^\alpha_{\varrho\tau}(\sigma)$ and $c^\varrho_{\tau\varphi}(\sigma)$ are constant.

It remains only to check (i). Now on A the sequence $\mathscr{S}^0 \xrightarrow{D} \mathscr{S}^1 \xrightarrow{D} \mathscr{S}^2$ described in § 4 is exact. Furthermore, there is a mapping of sheaves $k: \mathscr{T} \to \mathscr{S}^0$ obtained by sending a vector field θ on $X = A/B$ into its "lift" on $B_G = A$ with respect to the connexion determined by the $G(A, B)$ structure. It is then easy to see that the diagram

$$\begin{array}{ccccc} \mathscr{S}^0 & \xrightarrow{D} & \mathscr{S}^1 & \xrightarrow{D} & \mathscr{S}^2 \\ {\scriptstyle k}\uparrow & & \uparrow & & \uparrow \\ \mathscr{T} & \xrightarrow{j\circ D_1} & \Sigma^1 & \xrightarrow{D} & \Sigma^2 \end{array}$$

is commutative, and this gives (i). Q. E. D.

Corollary: *Let X be a compact manifold on which there is a transitive G-structure of finite type $G \to B_G \to X$. Then there exists a strongly locally complete germ of deformation of this structure.*

The proof may now be done exactly as for a transitive I-structure.

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