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FROM MARKOFF TO OPPENHEIM AND BEYOND.

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OPPENHEIM LECTURE JAN 2026

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A. MARKOFF (1879):

$F(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$ A REAL INDEFINITE BINARY QUADRATIC FORM OF DISC $D(F) > 0$.

$$\mu(F) := \inf_{x \in \mathbb{Z}^2 \setminus \{0\}} \frac{|F(x)|^2}{D(F)} \qquad (*)$$

μ IS SCALE INVARIANT AND $GL_2(\mathbb{Z})$ INVARIANT (LINEAR CHANGE OF VARIABLE IN x)

SO WE CONSIDER THE NUMBERS $\mu(F)$ IN (*) WITH MULTIPLICITIES UP TO THIS EQUIVALENCE.

<u>MARKOFF</u> <u>SPECTRUM</u> ABOVE 1/9:

$\{\mu(F) : \mu(F) > \frac{1}{9}\}$ IS DISCRETE

$$= \frac{1}{5} > \frac{1}{8} > \frac{25}{221} > \cdots \geq \mu_n \geq \cdots \quad > \frac{1}{9}$$

WHERE $\quad \mu_n = \dfrac{M_n^2}{9M_n^2 - 4}$

M_n IS THE n-TH MARKOFF NUMBER CORRESPONDING TO <u>INTEGRAL</u> <u>POINTS</u> ON THE AFFINE CUBIC ("LOG K_3")

$$\left.\begin{array}{l} X: \ x_1^2 + x_2^2 + x_3^2 - 3x_1x_2x_3 = 0 \\ 1 \le x_1 \le x_2 \le x_3 \quad ; \quad \text{"} x_3 = M_n \text{"} \end{array}\right\} (**)$$

THE FIRST FEW M_n'S ARE : 1, 2, 5, 13, 29, ...

$\mu_1 = \frac{1}{5}$ IS ACHIEVED BY $x_1^2 + x_1x_2 - x_2^2$.

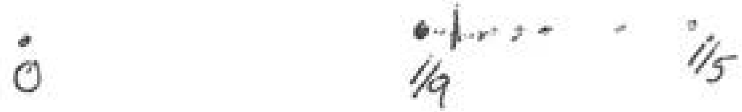
THE DIOPHANTINE ANALYSIS OF LOG K_3'S IS PROBLEMATIC IN GENERAL BUT FOR X AND RELATED CHARACTER VARIETIES WHICH HAVE LARGE AUTOMORPHISMS (WHICH ARE NONLINEAR AND PRESERVE INTEGRALITY) THE RUDEMINTS OF A THEORY HAS BEEN DEVELOPED.

- THE ASYMPTOTICS OF M_n (GURWOOD, ZAGIER, McSHANE-RIVIN, MIRZAKHANI)

$$\#\{M_n \le x\} \sim c(\log x)^2 \quad \text{AS} \quad x \to \infty.$$

IT FOLLOWS THAT FOR $x \to \infty$

$$\#\left\{ \mu_n \geq \frac{1}{9} + \frac{1}{x} \right\} \sim c'(\log x)^2 .$$



THE DIVISIBILITY PROPERTIES OF THE M_n's WERE RAISED BY FROBENIUS

<u>THEOREM</u> (STRONG APPROXIMATION) BOURGAIN-GAMBURD-S W. CHEN (2024)

FOR p LARGE (PROBABLY ALL p) THE ONLY CONGRUENCE OBSTRUCTION THAT M_n SATISFIES IS THAT

$$M \not\equiv 0, \pm\tfrac{2}{3} \bmod p \quad \text{IF} \quad p \equiv 3(4).$$

INDEFINITE TERNARIES:

$$F(x_1,x_2,x_3) = x^t A x \quad ; \quad A = \begin{bmatrix} a_1 & a_4 & a_5 \\ a_4 & a_2 & a_6 \\ a_5 & a_6 & a_3 \end{bmatrix}$$

REAL COEFF AND SIGNATURE $(2,1)$ OVER $\mathbb{R}$.

$D(F) = \det A$;

$$\mu(F) := \inf_{x \in \mathbb{Z}^3 \setminus \{0\}} \frac{|F(x)|^3}{|D(F)|}$$

A. MARKOFF (1902)

$$\mu_1 = \frac{2}{3} \quad , \mu_2 = \frac{2}{5} \quad , \mu_3 = \frac{1}{3}$$

WITH $2x_1^2 - 2x_2^2 - 2x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_1x_3$ ACHIEVING μ_1.

A. OPPENHEIM (1929) GAVE AN EXPOSITION OF $\mu_4 = \frac{8}{25}$ AND MORE IMPORTANTLY GAVE THE FAR REACHING AND DECISIVE INSIGHT THAT THE SET OF μ'S IS DISCRETE IN $(0, \frac{2}{3}]$ (IE THE μ_n'S $\to 0$) AND THAT THE F'S THAT ACHIEVE NON ZERO μ'S ARE EXACTLY THE INTEGRAL ANISOTROPIC F'S.

• F IS <u>ANISOTROPIC</u> IF $F(x) \equiv 0$ FOR $x \in \mathbb{Z}^3 \Rightarrow x = 0$.

IN CELEBRATED WORK MARGULIS (1985) PROVED OPPENHEIM'S CONJECTURE AND SHOWED THAT FOR $X \geq 1$

$$\mathrm{MAR}(X) := \#\left\{ \mu_j \geq \frac{1}{X} \right\} \text{ IS FINITE.}$$

• FOLLOWING SUGGESTIONS BY GRUNEWALD AND MARGULIS, MARTINI (1996) MADE A NUMERICAL STUDY OF $M(X)$ FINDING THAT $\mu_{145} = \frac{16}{245}$ AND SUGGESTED THAT $M(X) \approx X^2$.

FOR THIS PROBLEM THIS DATA IS TOO SMALL AND MISLEADING

<u>THEOREM</u> (A. GAMBURD - A. GHOSH - S - J. WHANG) 2026:

$$\mathrm{MAR}(X) \sim \gamma X \log X \quad \text{AS } X \to \infty.$$

HERE γ IS GIVEN IN TERMS OF AN EXPLICIT BUT COMPLICATED CONVERGENT SUM OF POSITIVE TERMS AND IS APPARENTLY QUITE LARGE.

⑥

- CONCERNING THE ISOTROPIC INTEGRAL F'S;

SERRE (1990) ASKS TO ESTIMATE THE PROBABILITY THAT ~~[illegible]~~ AN F IS ISOTROPIC. GIVEN A LARGE SET OF F'S, SAY THOSE IN $N\Omega$ THE DILATE OF A NICE CONVEX COMPACT DOMAIN $\Omega \subset \mathbb{R}^6$.

USING SIEVE METHODS SERRE SHOWS THAT

$$\mathrm{Iso}(N\Omega) = \#\{F \in N\Omega : F \text{ ISOTROPIC}\} \ll_{\Omega} \frac{N^6}{\sqrt{\log N}}.$$

ALSO USING SIEVE METHODS HOOLEY (2007) GIVES A MATCHING LOWER BOUND:

$$\mathrm{Iso}(N\Omega) \gg_{\Omega} \frac{N^6}{\sqrt{\log N}}.$$

IT TURNS OUT THAT THE ISOTROPICS HAVE A NATURAL DENSITY:

THEOREM (G-G-S-W) 2026:

THE ISOTROPIC F'S HAVE A LOCAL DENSITY

$$\frac{\alpha\, dx_1 dx_2 \cdots dx_6}{\sqrt{\log^+ |D(x)|}}$$

; ON THE CONE OF INDEFINITE x

WHERE $D(x) = \det\begin{bmatrix} x_1 & x_4 & x_5 \\ x_4 & x_2 & x_6 \\ x_5 & x_6 & x_1 \end{bmatrix}$, $\log^+ y = \begin{cases} 1 & \text{IF } y \le e \\ \log y & \text{IF } y \ge e \end{cases}$

AND α IS A CONVERGENT INFINITE PRODUCT OVER ALL PRIMES OF LOCAL MASSES OF AVERAGES OF ISOTROPIC GENERA.

THE SENSE TO WHICH THE ABOVE APPLIES IS

$$I_{JO}(N\Omega) \sim \int_{N\Omega} \frac{\alpha\, dx_1 \cdots dx_6}{\sqrt{\log^+ |D(x)|}} \sim c(\Omega) \frac{N^6}{\sqrt{\log N}}$$

FOR SUITABLE $c(\Omega)$.

- THE CASE OF INDEFINITE QUADRATIC FORMS IN FOUR VARIABLES IS SIMPLER. OPPENHEIM FOUND THE FIRST FEW VALUES OF THE SPECTRUM IN THE SIGNATURE (2,2) AND (1,3) CASES AND HIS INSIGHT THAT THE SPECTRUM CORRESPONDS EXACTLY TO THE ANISOTROPIC INTEGRAL FORMS FOLLOWS FROM MARGULIS.

- FOR 5 OR MORE VARIABLES (INDEFINITE)

$$\mu(F) = 0 \quad \text{FOR ALL } F$$

BY MARGULIS FOR "IRRATIONAL F" AND MEYER FOR INTEGRAL F.

- TO GIVE THE CONCEPTUAL PICTURE OF HOW THESE RESULTS ARE PROVED IT IS CONVENIENT TO POSE THE PROBLEM IN GENERAL AS WAS DONE BY K. MAHLER.

LET $F(x_1,\dots,x_n)$ BE A HOMOGENEOUS (NON DEGENERATE) REAL POLYNOMIAL OF DEGREE t IN n-VARIABLES.

WE NORMALIZE THE VOLUMES BY THE CONSIDERING THE FORMS GOTTEN BY THE $SL_n(\mathbb{R})$ ACTION:

$$F_g(x) = F(gx)\,, \quad \text{FOR } g \in SL_n(\mathbb{R}).$$

DEFINE

$$\mu(F_g) = \inf_{x \in \mathbb{Z}^n/\{0\}} |F_g(x)| \qquad \text{———}(*)$$

CLEARLY $\mu(F_g)$ IS DEFINED ON THE HOMOGENEOUS SPACE OF LATTICES

$$Y_n = SL_n(\mathbb{R})/SL_n(\mathbb{Z})$$

- MAHLER SHOWS THAT $\mu(F_g)$ IS UPPER SEMI CONTINUOUS FOR $g \in \mathcal{Y}_n$ AND USING HIS COMPACTNESS THEOREM THAT

$$\mu(F_g) \to 0 \quad \text{AS} \quad g \to \partial(\mathcal{Y}_n)$$

THE BOUNDARY OF $\mathcal{Y}_n$.

<u>DEFINITION</u>: SPEC(F) IS THE SET OF VALUES ASSUMED BY $\mu(F_g)$ FOR $g \in \mathcal{Y}_n$ (WITH MULTIPLICITY FOR ISOLATED POINTS).

<u>NB</u>: SPEC(F) DEPENDS ONLY ON THE (REAL) INVARIANTS OF F.

- IF F IS DEFINITE (SAY TAKES POSITIVE VALUES EXCEPT FOR $x = 0$) THEN THE STUCTURE OF SPEC(F) IS CLEAR.

$\mu(F_g)$ IS CONTINUOUS ON $\mathcal{Y}_n$ AND VANISHES AT $\partial\mathcal{Y}_n$ SO

$$\text{SPEC}(F) = [0, M_F]$$

WHERE M_F IS ITS MAX VALUE.

FOR $F_n(x) = x_1^2 + \cdots + x_n^2$ DETERMINING M_{F_n} IS THE WELL STUDIED PROBLEM OF THE OPTIMAL LATTICE SPHERE PACKING IN n-DIMENSIONS WHICH IS KNOWN FOR

$$n = 2, 3, 4, 5, 6, 7, 8, 24.$$

- WHEN F IS INDEFINITE THE PROBLEM OF ITS SPECTRUM IS MUCH MORE SUBTLE AND LITTLE IS KNOWN IN GENERAL.

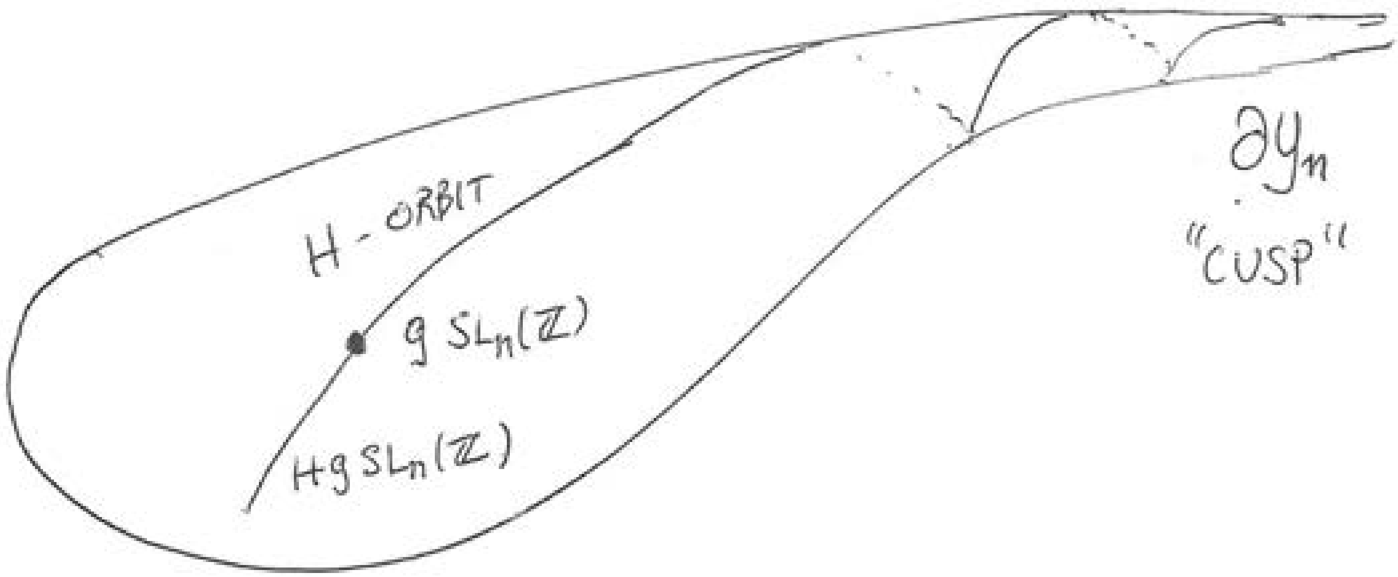
HOMOGENEOUS DYNAMICS:

$$H(\mathbb{R}) = \mathrm{STAB}_F(\mathbb{R}) = \{ g \in SL_n(\mathbb{R}) : F(gx) = F(x) \text{ FOR ALL } x \} \leq SL_n(\mathbb{R})$$

CLEARLY
$$\mu(g) = \mu(F_g) = \mu(hg\gamma)$$
FOR $h \in H(\mathbb{R})$ AND $\gamma \in SL_n(\mathbb{Z})$.

IN PARTICULAR IF H IS NON-COMPACT THE H-ORBITS: $Hg\Gamma$'s IN $\mathcal{Y}$ CARRY POTENTIALLY DECISIVE INFORMATION.

$$\mathcal{Y}_n = SL_n(\mathbb{R}) / SL_n(\mathbb{Z})$$



- IF $HgSL_n(\mathbb{Z})$ RUNS INTO THE CUSP THEN $\mu_F(g) = 0$.

- THE HOWE-MOORE THEOREM ASSERTS THAT IF H IS NON-COMPACT ITS ACTION ON $\mathcal{Y}_n$ IS ERGODIC (WRT HAAR) IN PARTICULAR $\mu_F(g) = 0$ FOR ALMOST ALL g.

- THE ISSUE IS WHAT HAPPENS FOR <u>EVERY</u> ORBIT $H.g.SL_n(\mathbb{Z})$.

- IN PARTICULAR THE "PERIODIC H-ORBITS": $H/H\cap g^{-1}\Gamma g$ IS COMPACT OR FINITE VOLUME PLAY AN IMPORTANT ROLE.

MARKOFF'S BINARY QUADRATICS REVISITED:

$$F(x_1, x_2) = x_1 x_2$$

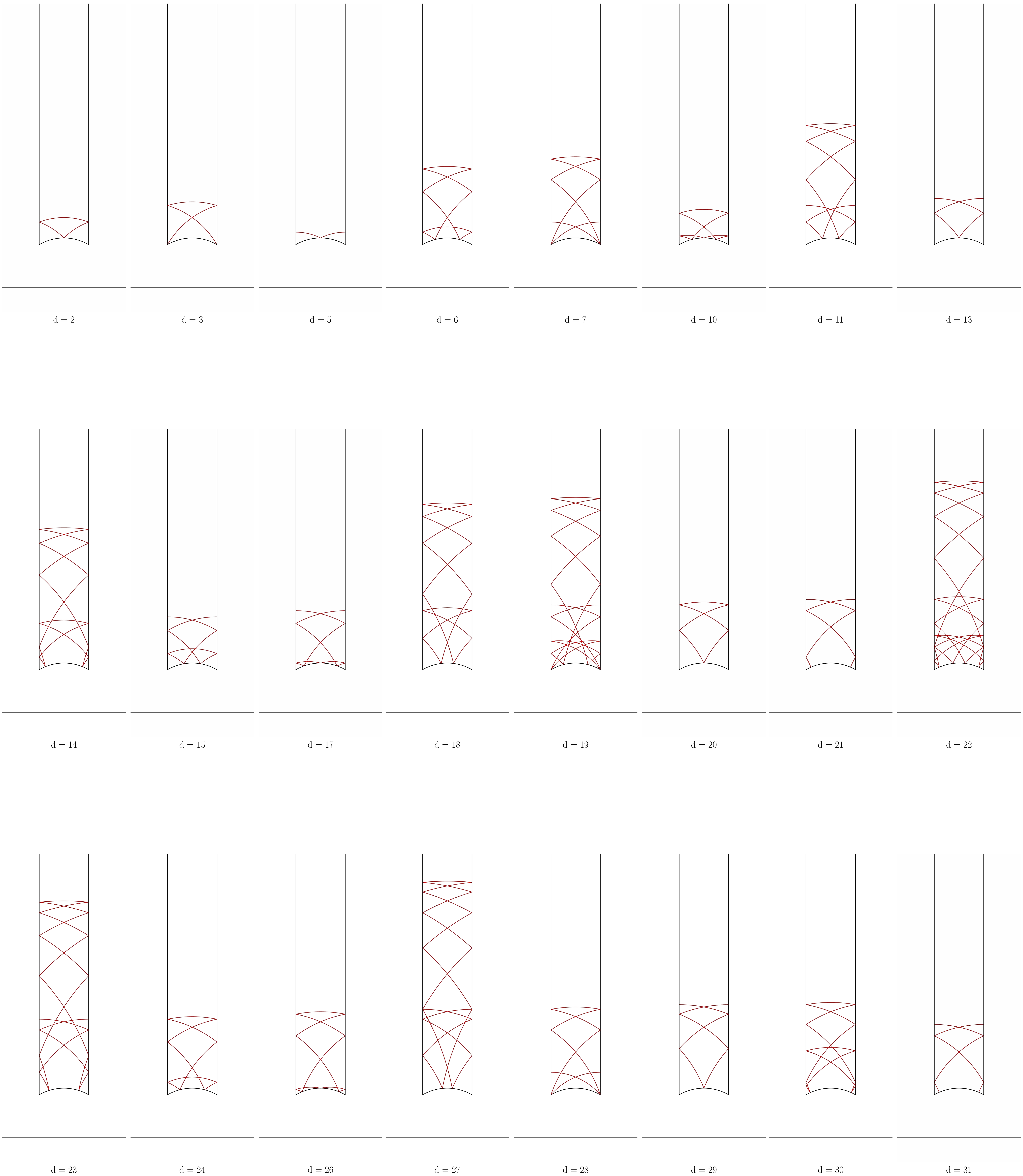
$$H = \left\{ \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{R}^* \right\} \qquad \text{DIAGONAL.}$$

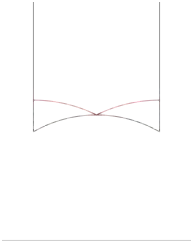
- THE H-PERIODIC ORBITS ARE THE CLOSED GEODESICS CORRESPONDING TO AUTOMORPHS OF <u>INTEGRAL</u> INDEFINITE BINARY FORMS. THE MARKOFF FORMS GIVING EXACTLY THE CLOSED GEODESICS WHOSE MAXIMUM HEIGHT IN THE STANDARD FUNDAMENTAL DOMAIN LIES BELOW $3/2$.

Geodesics on the Modular Surface

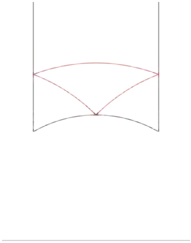
We show closed geodesics on the modular surface, which correspond for varying d to the matrix:

$$x_d = \frac{1}{d^{\frac{1}{4}}} \begin{pmatrix} \frac{d+\sqrt{d}}{2} & \frac{d-\sqrt{d}}{2} \\ 1 & 1 \end{pmatrix}$$

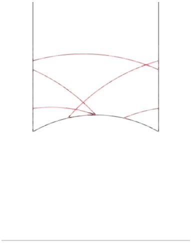




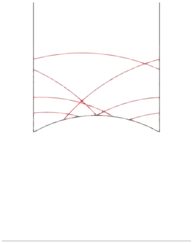
$M_1 = 1$
$y_{\max} = 1.1180$



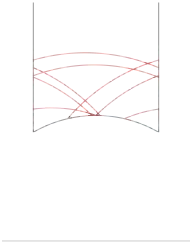
$M_2 = 2$
$y_{\max} = 1.4142$



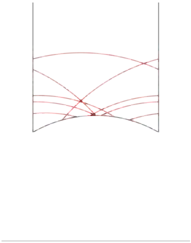
$M_3 = 5$
$y_{\max} = 1.4866$



$M_4 = 13$
$y_{\max} = 1.4980$



$M_5 = 29$
$y_{\max} = 1.4996$



$M_6 = 34$
$y_{\max} = 1.4997$

INDEFINITE TERNARIES REVISITED:

$$F(x_1, x_2, x_3) = x_1^2 - x_2 x_3$$

$H(\mathbb{R}) \simeq SO(2,1)$ LOCALLY $\simeq SL_2(\mathbb{R})$,

IS NOT ONLY NON-COMPACT BUT IS GENERATED BY UNIPOTENTS.

• RAGHUNATHAN CONJECTURED THAT IN THE CONTEXT OF HOMOGENEOUS DYNAMICS THE CLOSURES OF UNIPOTENT ORBITS ARE RIGID BOTH TOPOLOGICALLY AND MEASURE THEORETICALLY. (IE. CORRESPOND TO INTERMEDIATE PERIODIC ORBITS).

• FOR THE CASE AT HAND MARGULIS PROVED THAT THE CLOSURE OF ANY H ORBIT WHICH IS CONTAINED IN A COMACT SUBSET OF $\mathcal{Y}_3$ MUST CORRESPOND TO A COMPACT H-PERIODIC ORBIT, IE TO AN <u>INTEGRAL</u> ANISOTROPIC FORM, THUS PROVING OPPENHEIM.

- IN HER CELEBRATED WORK M. RATNER (1991) ESTABLISHED THE FULL RAGHUNATHAN UNIPOTENT TOPOLOGICAL AND MEASURE RIGIDITY CONJECTURES.

- THE LOCAL DENSITY FOR ISOTROPIC TERNARIES RELIES ON THEOREMS OF ESKIN AND OH (2002) AND OH (2004) (WHICH IN TURN RELY ON RATNER) WHICH ASSERT THAT THE <u>INDIVIDUAL</u> PERIODIC H - ORBITS IN THIS INDEFINITE TERNARY CASE BECOME EQUIDISTRIBUTED IN Y_3 AS THEIR VOLUMES GO TO INFINITY.

- KEY TO OUR RESULTS FOR INDEFINITE TERNARIES ARE SOME NEW WEIGHTED AVERAGES OVER CLASSES OF INTEGRAL FORMS.

CLASS INVARIANTS AND AVERAGES

THE INTEGRAL F's COME IN CLASSES UNDER THE ACTION OF $SL_3(\mathbb{Z})$ WHICH WE DENOTE BY c, AND THEIR TOTALITY BY $\mathcal{C}$, AND THE ISOTROPIC c' BY $\mathcal{C}^{iso}$.

FOR EACH c ; $D(c)$ = DET(F) IS A CLASS INVARIANT.

AS IS

$$K(c) = \text{MIN} \{ |F(x)| : x \in \mathbb{Z}^3/\{0\} \}.$$

IN HIS WORK ON HIS "MASS FORMULA" SIEGEL INTRODUCES AN ARCHIMEDEAN GLOBAL CLASS INVARIANT

$$\rho(c)$$

DEFINED AS FOLLOWS: FOR $F \in c$

$$\psi_F(Z) = Z^t F Z \quad ; \psi : \text{MAT}(3\times 3, \mathbb{R}) \to \text{Sym}^2(\mathbb{R}^3)$$

FOR T A SMALL NBH OF F IN $\text{Sym}^2(\mathbb{R}^3)$

$$Y = \{ Z : \psi(Z) \in T \}, \quad Y_0 := Y/\text{AUT}_F(\mathbb{Z})$$

$$\rho(c) = \rho(F) = \lim_{T \to F} \frac{\text{VOL}(Y_0)}{\text{VOL}(T)}.$$

• IT IS WELL KNOWN THAT THE NUMBER $h(d)$ OF CLASSES $e \in \mathcal{C}$ WITH $D(e) = d$ IS FINITE.

<u>THEOREM</u> (G-G-S-W) AS $N \to \infty$

$$\sum_{\substack{e \in \mathcal{C} \\ |D(e)| \leq N}} 1 = \sum_{|d| \leq N} h(d) \sim \frac{18}{5} \frac{\zeta(2)}{\zeta(4)} N \log N$$

$$\sum_{\substack{e \in \mathcal{C} \\ |D(e)| \leq N k^3(e)}} 1 \sim \gamma N \log N$$

$$\sum_{\substack{e \in \mathcal{C}^{iso} \\ |D(e)| \leq N}} |D(e)| \rho(e) \sim \frac{\alpha N}{\sqrt{\log N}}$$

THE LAST SHOULD BE COMPARED WITH SIEGEL (1944)

$$\sum_{\substack{e \in \mathcal{C} \\ |D(e)| \leq N}} |D(e)| \rho(e) \sim \frac{\zeta(2)\zeta(3)}{2} N .$$

• THE MOST CHALLENGING IS THE SUM INVOLVING $k(e)$. WATSON (1959) SHOWED THAT $k(e) = O_\varepsilon(|D(e)|^{1/4+\varepsilon})$ AND HIS METHOD IS OUR STARTING POINT. WE DEVELOP SOME COMBINATORIAL AND SIEVE TOOLS TO TAME $k(e)$.

RETURNING TO THE GENERAL MAHLER PROBLEM FOR F'S OF DEGREE 3 OR HIGHER.

BINARY FORMS OF DEGREE $d \geq 3$

FOR $F(x_1, x_2)$ A NON-SINGULAR HOMOGENEOUS FORM OF DEGREE $d \geq 3$

G. KOTSOVOLIS (P.U. THESIS 2024):

$$\mathrm{SPEC}(F) = [0, M_F] \quad \text{AN INTERVAL}$$

FOR $d = 3$ THERE ARE TWO F'S UP TO $\mathbb{R}$ EQUIV:

$$M_F = \sqrt[4]{\frac{-D_F}{23}} \quad \text{IF} \quad D_F < 0$$

$$M_F = \sqrt[4]{D_F/49} \quad \text{IF} \quad D_F > 0.$$

$D_F = \mathrm{DISC}(F)$.

NOTE: FOR THESE THE STABILIZERS H ARE COMPACT (EVEN FINITE) SO DYNAMICS CANNOT BE APPLIED DIRECTLY.

SPECIAL TERNARY CUBIC FORMS

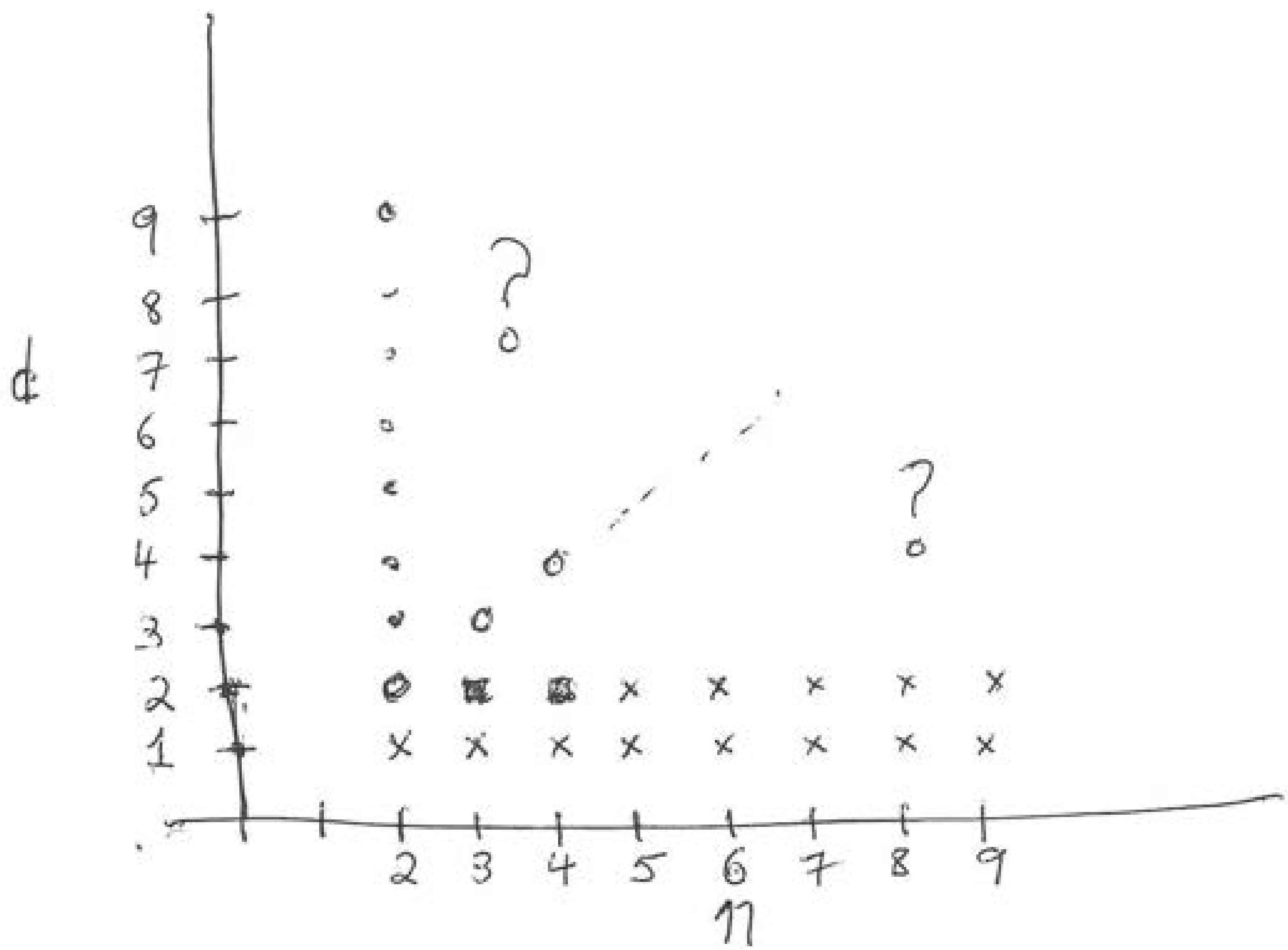
FOR $F(x_1, x_2, x_3) = x_1 x_2 x_3$ THE HOLY GRAIL CONJECTURE IS DUE TO CASSELS AND SWINNERTON-DYER (1955):

$\mathrm{SPEC}(x_1 x_2 x_3)$ IS RIGID, IE CONSISTS OF DISCRETE POINTS ACCUMULATING AT 0 CORRESPONDING TO PERIODIC H ORBITS.

HERE $H = \left\{ \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & 1/\lambda_1\lambda_2 \end{pmatrix} : \lambda_1, \lambda_2 \in \mathbb{R}^* \right\}$ IS NONCOMPACT AND CORRESPONDS TO A HIGHER RANK DIAGONAL ACTION.

- MEASURE RIGIDITY CONJECTURES FOR SUCH ACTIONS IMPLY THE HOLY GRAIL CONJ.
- THE ADVANCES BY EINSIEDLER-KATOK-LINDENSTRAUSS (2006) ON SUCH HIGHER RANK DIAGONAL RIGIDITY IMPLY THAT THE HAUSDORFF DIMENSION OF $\mathrm{SPEC}(x_1 x_2 x_3)$ IS ZERO.

SPEC(F) F-INDEFINITE.



o DIAGONAL MIXED BEHAVIOR

x SPECTRUM IS $\{0\}$

▩ SPECTRUM IS RIGID (MARGULIS)

• SPECTRUM IS AN INTERVAL (KOTSOVOLIS)

THE TABLE BEGS TO BE FILLED.

THE PROBLEM OF SPEC(F) CAN BE INTERPRETED AS THE 'BASS-NOTE' SPECTRUM OF THE LINEAR DIFFERENTIAL OPERATOR CORRESPONDING TO F AS ONE VARIES OVER THE LOCAL GEOMETRIES WHICH IN THIS SETTING ARE FLAT TORI. AS SUCH THEY CAN BE FORMULATED IN THE SETTING OF GENERAL LOCALLY HOMOGENEOUS GEOMETRIES.

THE RIGIDITY PROPERTIES OF SUCH SPECTRA AS FIRST RECOGNIZED BY OPPENHEIM IS A CENTRAL THEME IN GENERAL.