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ADDED NOTES TO MY JAN 30 2022 LETTER TO SARAH AND VALERIYA: (1)

The formulation of the several variable Bunyakovsky conjecture in terms of Zariski density is very natural and it places it as an extension of Hilbert's irreducibility theorem. The latter asserts that for any $n \geq 2, 1 \leq r \leq n$, if $f \in \mathbb{Q}[x_1, \dots, x_n]$ is irreducible the specializations of f in $\mathbb{Q}[x_1, \dots, x_r, \alpha_{r+1}, \dots, \alpha_n]$ are irreducible for a Zariski dense set of integers $(\alpha_{r+1}, \dots, \alpha_n)$ in A^{n-r}. The several variable formulation of Bunyakovsky asserts that the same is true over $\mathbb{Z}$ when specializing <u>all</u> of the variables. In what follows unless otherwise stated we assume that the homogeneous part of f is non-degenerate ($n \geq 2$). Also to avoid archimedian conditions we treat $-p$ with p a prime, as prime. So $(\pm p)$ is a prime ideal in $\mathbb{Z}$, and an $x \in \mathbb{Z}^n$ for which $f(x) = \pm p$ is an f-prime.

<u>BUN CONJECTURE:</u>

$f \in \mathbb{Z}[x_1, \dots, x_n]$ irreducible and such that $f(x) \bmod p$ is not identically 0 for every prime p, then $f(\alpha_1, \dots, \alpha_n)$ is prime for a Zariski dense set of $\alpha \in \mathbb{Z}^n$.

BUN in several variables is equivalent to the original Bunyakowsky conjecture which follows by specialization and Hilbert. However unlike the original which is not known for any f of degree bigger than one, there are plenty cases known in several variables; and I summarize below what I found can be deduced from known results.

The BUN conjecture for a set of polynomials $\mathcal{F}$ will be denoted by $BUN(\mathcal{F})$. The quantitative Destagnol-Sofos Bateman-Horne Conjecture (3') of is denoted by $BH(\mathcal{F})$. As we noted $BH(\mathcal{F}) \Rightarrow BUN(\mathcal{F})$.

There is a fundamental difference between $n=1$ and $n \geq 2$ in that for $n=1$ f, while irreducible over $\mathbb{Q}$, factors into linear factors over $\overline{\mathbb{Q}}$. The typical f for $n \geq 2$ is irreducible over $\overline{\mathbb{Q}}$ (absolutely irreducible). If f factors into linear factors in $\overline{\mathbb{Q}}[X_1, \dots, X_n]$ we say that it is factorizable and we denote these polynomials by $\mathcal{P}^{FACT}$. The homogeneous polynomials will be denoted by $\mathcal{P}_{HOM}$. Note that for $n=2$ any $f \in \mathcal{P}_{HOM}$ is factorizable, a feature which is exploited below. The notation $\mathcal{P}(d, n)$ denotes polynomials of degree d in n variables.

Table of known (to me) cases of BUN and BH:

(i) $BH(\mathcal{P}(1,n))$, $n \geqslant 1$. [DIR]

(ii) $BUN(\mathcal{P}(2,n))$, $n \geqslant 2$ [IW]

(iii) $BH(\mathcal{P}_{HOM}(2,n))$, $n \geqslant 2$ [DU].

(iv) $BH(\mathcal{P}_{HOM}(3,2))$ [H-B/M]

(v) $BUN(\mathcal{P}_{HOM}(3,n))$, $n \geqslant 2$.

(vi) $BH(\mathcal{P}^{FAC}_{HOM}(d,n))$, $\frac{3d}{4} \leq n \leq d$, [HERE], [MAY].

(vii) $BH(\mathcal{P}(d,n))$, $n \geqslant \max(4, (d-1)2^{d-1}+1)$, [D-S].

(viii) INVARIANT POLYNOMIALS FOR AFFINE SYMMETRIC SPACES

EXAMPLES:

$BH(\det(X), X$ $n \times n$ matrix$)$

$BH(\det(X), X$ $n \times n$ symmetric matrix$)$

$BH(\mathrm{pfaf}(X): X$ $2n \times 2n$ skew symmetric$)$, [Oh].

(ix) $BH(x_1^2 + x_2^4)$ [F-I].

(x) $BUN(x_1 x_2 \cdots x_n + x_1^2 + \cdots + x_n^2)$, $n \geqslant 2$ [IW]

COMMENTS

I have not checked the full details in the table and there may be small conditions omitted in places (see below).

(i) can be proved in many ways. If $n=1$ it is just the prime number theorem for primes in arithmetic progressions. For $n \geq 2$ it can be proved as in the letter using the solution of the Linnik problem in this linear case which is elementary. In fact (iii) and (viii) are proved this way. For (iii) when $n=2$, that is a homogeneous binary quadratic form BH follows from Hecke's equidistribution of primes in sectors theorem. For $n=3$ it follows from Duke's solution of Linnik's problem for general quadrics in 3-variables. For $n \geq 4$ one can use the circle method as in (vii) and even deal with non-homogeneous quadratic f's. For (viii) in the case of det of symmetric matrices Oh [Oh] shows that the Linnik problem is "Hardy-Littlewood" and I expect, but did not check, that the sum over product of densities $w(p)$ can be executed over primes p.

For (ii) and $n=2$, Iwaniec's half dimensional sieve gives the asymptotic count with upper and lower bounds of the same order (see [F-I 2] Section 21.2) which yilds $BUN(\mathcal{P}(2,2))$. Hilbert together Specialization arguments than give $BUN(\mathcal{P}(2,n))$, $n \geq 2$. (iv) is a striking result of Heath-Brown and Moroz. It exploits that elements of $\mathcal{P}_{HOM}(3,2)$ factorize over $\overline{\mathbb{Q}}$ into three linear forms and treat the problem as a linear hyperplane section of a norm form in three variables. One deals with the type I and type II bilinear sieve sums which are highly structured, by following the breakthrough of Friedlander and Iwaniec [F-I 1] who deal with (ix). This establishes $BUN(P_{HOM}(3,2))$ and then using Hilbert and specialization argements with 2-dimensional linear subspaces one deduces $BUN(\mathcal{P}_{HOM}(3,n))$, $n \geq 2$.

The case (vi) when $n=d$ is again Hecke's prime in sectors theorem and The extension of Heath-Brown/Moroz to these incomplete norm forms is due to Maynard [MA] (He has some added conditions on the resulting number fields that he imposes and I didn't check if these can be relaxed).

(X) is the Markoff-Hurwitz type polynomial and it can be proved by specializations to the BUN(P(2,2)) case [ii].

It appears that the several variable versions of both BUN and BH are ripe for further exploration by new methods. Some immediate case that come to mind are

(A) From (V) BUN($x_1^3+x_2^3+x_3^3$) is true. It would be interesting to see if the conditional results of Wang [Wa] can be extended to BH($x_1^3+x_2^3+x_3^3$).

(B) Similarly for the Markoff form, BUN($x_1x_2x_3+x_1^2+x_2^2+x_3^2$) holds according to (X), can one extend the analysis of Ghosh-Sarnak [G-S] to establish BH($x_1x_2x_3+x_1^2+x_2^2+x_3^2$)? Note that essentially

$$\mathrm{MARKOFF}(x_1,x_2,x_3) = \det\begin{bmatrix} 2 & x_1 & x_2 \\ x_1 & 2 & x_3 \\ x_2 & x_3 & 2 \end{bmatrix}.$$

(c) Let

$$f(x_1, x_2, x_3, x_4) = 18 x_1 x_2 x_3 x_4 + x_2^2 x_3^2 - 4 x_1 x_3^3 - 4 x_2^3 x_4 - 27 x_1^2 x_4^2$$

be the discriminant of binary cubic form whose coefficients are x_1, x_2, x_3, x_4. f is generates the ring of invariants for the SL_2 action on binary cubics. Note that setting $x_2 = x_4 = 0$, $x_3 = 1$ shows that $f(\mathbb{Z}^4) \supset 4\mathbb{Z}$. This illustrates why we need to at least impose the Zariski density condition when seeking f-primes. From the point of view of Davenport [Da] who counts cubic forms with $|f| \leq F$, the above forms are singular and he also removes the reducible ones as well. The problem is to prove BUN(f) or better still BH(f).
A well known related open problem is BUN$(x_1^2 + x_2^3)$ and ~~its~~ quantification BH$(x_1^2 + x_2^3)$.

Best regards

Peter.

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