DISTRIBUTION OF ZEROS AND EIGENVALUES

PETER SARNAK

LECTURE 1: NON-TAME DISTRIBUTION OF ZEROS OF ZETAFUNCTIONS.

LECTURE 2: SPECTRA OF SURFACES AND REGULAR GRAPHS.

PENN-STATE OCT 2025.

- WE REVIEW DEVELOPMENTS AROUND THE DISTRIBUTIONS OF ZEROS OF ZETA FUNCTIONS OVER NUMBER FIELDS, FUNCTION FIELDS AND CLOSELY RELATED PROBLEMS FOR EIGENVALUES OF REGULAR GRAPHS AND RIEMANNIAN SURFACES.

THE BOOK

"ZETA AND L-FUNCTIONS IN NUMBER THEORY AND COMBINATORICS"

WINNIE LI

CBMS 129 (2014)

GIVES AN EXCELLENT INTRODUCTION AND ACCOUNT OF THESE TOPICS.

NOTATION:

RIEMANN EVALUATES FOR $\zeta(s)$ THE SUM OVER THE RECIPROCALS OF THE NON-TRIVIAL ZEROS:

$$\sum_{\rho} \frac{1}{\rho} = \sum_{\rho} \frac{1}{1-\rho} = 1 + \frac{\gamma}{2} - \frac{1}{2}\log\pi - \log 2$$

(HE USED THIS TO COMPUTE A FEW OF THESE ρ's)

$K/\mathbb{Q}$ NUMBER FIELD OF DEGREE n

$\zeta_K(s)$ ITS (DEDEKIND) ZETA FUNCTION

D_K ITS DISCRIMINANT (CONDUCTOR / RAMIFICATION)

h_K CLASS NUMBER; R_K REGULATOR.

"CLASS NUMBER" FORMULA:

$h_K R_K$ IS ESSENTIALLY THE RESIDUE OF $\zeta_K(s)$ AT $s=1$

$$\prod_{\rho} (1-\rho)$$ GEOMETRIC MEAN OF ρ TO 1.

- SEPARATING h_K AND R_K IS NOTORIOUSLY DIFFICULT BUT $h_K R_K$ BEHAVES REGULARLY AT LEAST IN TAME SEQUENCES.

$$0 < I(K) := \sum_{\rho} \frac{1}{1-\rho}$$ ρ NONTRIVIAL ZEROS OF $\zeta_K(s)$

- WE CALL IT THE <u>IHARA</u> CONSTANT AS HE STUDIED THIS AND RELATED INVARIANTS (WHAT HE CALLS EULER-KRONECKER CONSTANTS) AS K VARIES.

<u>DEFN:</u> A SEQUENCE $K_j \to \infty$ IS <u>TAME</u> IF

$$n_K = o(\log D_K^{1/2}).$$

- WE WILL ASSUME GRH — IN FACT ONE GOAL IS TO TEST IT IN EXTREME LIMITS FOR GENERAL L-FUNCTIONS.

<u>SIEGEL-BRAUER:</u> (TAME CASE)

$$\lim_{\substack{K \to \infty \\ K \text{ TAME}}} \frac{\log(h_K R_K)}{\log D_K^{1/2}} = 1$$

$$\lim_{\substack{K \to \infty \\ K\text{-TAME}}} \frac{I(K)}{\log D_K^{1/2}} = 1$$

- IN THIS TAME CASE THE LOW LYING ZEROS (NEAR 1/2) HAVE LOCAL DENSITY $\log D_K^{1/2}$ AND BECOME EQUIDISTRIBUTED WRT dt ON $\mathrm{Re}(s) = \frac{1}{2}$ "DENSITY OF STATES".

• IN THIS <u>TAME</u> CASE WE CAN (AND DO) ALLOW ALL AUTOMORPHIC L-FUNCTIONS OF CUSP FORMS π ON $GL_n/\mathbb{Q}$ AS LONG AS WE USE THE

$\boxed{\text{ANALYTIC CONDUCTOR}}$ IN PLACE OF D_K

WHICH PICKS UP THE COMPLEXITY OF π AT THE ARCHIMEDIAN PLACES.
WITH IT THE THEORY CAN BE FORMULATED AND HAS BEEN WELL STUDIED AND TESTED.

<u>NON-TAME :</u>

IN GENERAL THE DISTRIBUTION OF THE ZEROS NEED NOT BE UNIFORM ON $Re(s)=\frac{1}{2}$ BUT THE POSSIBILITIES ARE RESTRICED BY <u>POSITIVITY</u> IN THE EXPLCIT FORMULA (POSITIVITY OF THE COEFF OF $\zeta_K(s)$ AND POSITIVITY ASSOCIATED WITH GRH) AND THESE ~~HAVE~~ GIVE BOUNDS FOR DISCRIMINANTS THAT ARE STRONGER THAN THOSE FROM THE GEOMETRY OF NUMBERS AND THEY GET CLOSE TO WHAT CONSTRUCTIONS OF UNRAMIFIED TOWERS (HILBERT CLASS FIELDS, GOLOD-SHAFERAVICH) AS WAS OBSERVED BY STARK, ODLYZKO, ...

EXPLICIT FORMULA (POSITIVITY)

$L(s,\pi)$ AUTOMORPHIC L-FUNCTION OF DEGREE n (SUCH AS $\zeta_K(s)$, $\deg(K)=n$) WITH POSITIVE COEFF;

$$-\frac{L'}{L}(s,\pi)=\sum_{m=1}^{\infty}\frac{\Lambda(m)a_\pi(m)}{m^s}, \quad a_\pi(m)\geq 0$$

$\Lambda(m)$ VON-MANGOLT.

Q_π CONDUCTOR OF π;

$\rho_\pi=\frac{1}{2}+i\gamma_\pi$ ZEROS $\gamma_\pi\in\mathbb{R}$ (ASSUME GRH).

FOR $g\in C_0^\infty(\mathbb{R})$ AND $h(r)=\int_{-\infty}^{\infty}g(u)e^{iru}\,du$

$$\sum_{\gamma_\pi}h(\gamma_\pi)+\sum_{m=1}^{\infty}\frac{\Lambda(m)a_\pi(m)}{\sqrt{m}}g(\log m)+\frac{\Lambda(m)\overline{a_\pi(m)}}{\sqrt{m}}g(-\log m)$$

$$=\delta(\pi)\left\{h\left(\tfrac{i}{2}\right)+h\left(-\tfrac{i}{2}\right)\right\}$$

$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}h(r)\left(\log Q_\pi+\sum_{j=1}^{n}\frac{\Gamma_{\mathbb{R}}'}{\Gamma}\left(\frac{1}{2}+\mu_\pi(j)+ir\right)+\sum_{j=1}^{n}\frac{\Gamma_{\mathbb{R}}'}{\Gamma}\left(\frac{1}{2}+\overline{\mu_\pi(j)}-ir\right)\right)dr$$

$\boxed{\delta(\pi)=\text{ORDER OF POLE AT } s=1}$

CHOOSING $h, g\geq 0$ CAREFULLY YIELDS CONSTRAINTS ON THE DISTRIBUTION OF γ_π'S AS Q_π (AND "ANALYTIC CONDUCTOR") GOES TO INFINITY.

- THIS POSITIVITY GIVES A STRINGENT PAUCITY OF (GENERAL) L-FUNCTIONS WHEN ORDERED BY CONDUCTOR. GRH ALSO GIVES A VERY STRONG SEPARATION PROPERTY BETWEEN π AND π' $(\pi \neq \pi')$ IN TERMS OF THE CONDUCTOR.

IN PARTICULAR AS FAR AS I KNOW WE HAVE NOT PROBED THE CONSEQUENCES OF GRH IN THIS EXTREME NONTAME LIMIT FOR THE ILLUSIVE BUT ABUNDANT TRANSCENDENTAL "MAASS CUSP FORMS" WHOSE EXISTENCE IS TIED TO THE PROBLEMATIC ARCHIMEDIAN PLACES.

- THE FIRST STEP IS TO GIVE A SUITABLE DEFINITION OF THE ANALYTIC CONDUCTOR WHICH TAKES INTO ACCOUNT THE POSITIVITY AND IS DIMENSION FREE.

NON-TAME K'S (TSFASMAN–VLADUT–IHARA)

USING POSITIVITY TO RESTRICT DISTRIBUTION $\Rightarrow$

$$0.5165 \leq \varliminf_{K\to\infty} \frac{\log(h_K R_K)}{\log D_K^{1/2}} < 0.5839 < 1.0602 < \varlimsup_{K\to\infty} \frac{\log(h_K R_K)}{\log D_K^{1/2}} \leq 1.0938$$

$$0.3647 \leq \varliminf_{K\to\infty} \frac{I(K)}{\log D_K^{1/2}} < \varlimsup_{K\to\infty} \frac{I(K)}{\log D_K^{1/2}} = 1$$

THE NON-TAME EXAMPLES COME FROM EXPLICIT TOWERS OF NUMBER FIELDS WITH LIMITED RAMIFICATION.

- A CONSEQUENCE OF THE POSITIVITY IS THAT AS $K \to \infty$ THE ZEROS OF $\zeta_K(s)$ BECOME DENSE ON $Re(s) = \frac{1}{2}$, ALBEIT THAT THEIR DENSITIES CAN BE EXOTIC (BUT HAVE FULL SUPPORT).

12

CURVES OVER $\mathbb{F}_q$:

X A CURVE OF GENUS g OVER $\mathbb{F}_q$

$Z(X,T)$ IS ZETA FUNCTION HAS $2g$ ZEROS ALL LIE ON $|T| = 1/\sqrt{q}$ (WEIL)

AND THEY COME IN PAIRS $\omega_1, \overline{\omega_1}, \ldots, \omega_g, \overline{\omega_g}$

CLASS NUMBER FORMULA (NO REGULATOR!)

$$h(X) = |\mathrm{JAC}(X)(\mathbb{F}_q)| = \prod_{j=1}^{g} (1-\omega_j \cdot)(1-\overline{\omega_j} \cdot)$$

IHARA'S CONSTANT:

$$I(X) = (q-1)\sum_{j=1}^{g} \frac{1}{(1-\omega_j \cdot)(1-\overline{\omega_j} \cdot)}$$

- GONALITY OF X IS THE LEAST DEGREE COVERING OF $\mathbb{P}^1$ THAT REALIZES X.

$g(X)$ IS THE LOG CONDUCTOR
\# OF ZEROS

- A SEQUENCE X IS TAME IF $\frac{\mathrm{GON}(X)}{g(X)} \to 0$
ALONG THE SEQUENCE.

- THE g ZEROS $\omega = q^{-1/2} e^{i\theta}$, $\theta \in [0, \pi]$

EQUIDISTRIBUTE W.R.T $d\theta$ FOR TAME SEQUENCES OF X's!

$$\Longrightarrow \quad \lim_{\substack{X \to \infty \\ X \text{ TAME}}} \frac{\log_q h(X)}{g(X)} = 1$$

$$\lim_{\substack{X \to \infty \\ X \text{ TAME}}} \frac{I(X)}{g(X) \log q} = 1$$

IN THE GENERAL (NONTAME) CASE THE POSITIVITY OF THE COEFF OF $Z(X,T)$ YIELDS LIMITS ON THE DISTRIBUTIONS OF THE ZEROS.

THESE WERE REALIZED AND EXPLOITED IN WORK OF IHARA - TSFASMAN - VLADUT - DRINFELD TO STUDY THE EXTREME VALUES

(IN THIS CASE THERE ~~ARE~~ NO ARCHIMEDIAN PLACE TO COMPLICATE THE ANALYSIS AND OF COURSE GRH IS NOT AN ASSUMPTION.)

$$1 \leq \underline{\lim}_{X\to\infty} \frac{\log_q h(X)}{g(X)} < \overline{\lim}_{X\to\infty} \frac{\log_q h(X)}{g(X)} \leq 1 + (\sqrt{q}-1)\log_q\left(\frac{q}{q-1}\right)$$

AND

$$1 = \overline{\lim}_{X\to\infty} \frac{I(X)}{g(X)\log q} > \underline{\lim}_{X\to\infty} \frac{I(X)}{g(X)\log q} \geq \frac{\sqrt{q}}{\sqrt{q}+1}$$

MOREOVER WE HAVE EQUALITY ON THE RHS OF EACH IN THE CASE THAT

$X = X_0(N)/\mathbb{F}_q$ AND $q = p^{2a}$ IS A SQUARE

AND $X_0(N)$ IS THE MODULAR CURVE!

THE TRACE FORMULA GIVES THE DISTRIBUTION OF THE ZEROS

- FOR $q = p$ IT IS THE PLANCHEREL MEASURE μ_{p+1}
- FOR $q = p^2$ IT IS THE MEASURE $\left(1 - \sum_{n=1}^{\infty} \frac{\cos(n\theta)}{q^{n/2}}\right) d\theta$

NOT RANDOM! GOPPA CODES

WHICH REMARKABLY SATURATES THE ABOVE INEQUALITIES.

- ONE CAN THE RATE OF CONVERGENCE OF THESE $N \to \infty$ AND THEIR STATISTICS (ZUBRILINA - S)

MORE PRECISELY (ZUBRILINA – S):

FOR N PRIME AND $(n, N) = 1$, $H(N)$ THE CUSP FORMS OF WEIGHT 2 ON $X_0(N)$.

$$a_f(n) = n^{1/2} \lambda_f(n)$$

COEFF OF $f \in H(N)$

$$\frac{1}{|H(N)|} \sum_{f \in H(N)} \lambda_f(n) = \sum_{k^2 = n} \frac{1}{k} + O_\varepsilon\left(\frac{n^{1/8}}{\sqrt{N}} (nN)^\varepsilon\right)$$

$\varepsilon > 0$

LEADS TO: FOR p FIXED, $N \to \infty$

$$\log_p |J_0(N)(\mathbb{F}_p)| = \left[1 + \frac{p-1}{2} \log_p\left(\frac{p^2}{p^2-1}\right)\right] g_0(N) + O_p\left(N^{2/5}\right)$$

$$\log_{p^2} |J_0(N)(\mathbb{F}_{p^2})| = \left[1 + (p-1) \log_{p^2}\left(\frac{p^2}{p^2-1}\right)\right] g_0(N) + O_p\left(N^{2/5}\right)$$

ABELIAN VARIETIES A OVER $\mathbb{F}_q$:

- FROM THE POSITIVITY CONSTRAINTS ON THE DISTRIBUTION OF THE ZEROS

$$\omega_1^{-1}, \omega_2^{-1}, \ldots, \omega_g^{-1}, \bar{\omega}_1^{-1}, \ldots, \bar{\omega}_g^{-1}$$

OF $Z(T, X) = Z(T, \mathrm{JAC}\,X)$ FOR CURVES X OVER $\mathbb{F}_q$, IT FOLLOWS THAT THE ZEROS BECOME DENSE IN THE CIRCLE

$$C_q = \{ z : |z| = \sqrt{q} \}$$

AS $g(X) \to \infty$ (TSFASMAN – VLADUT).

FOR ABELIAN VARIETIES $A/\mathbb{F}_q$ THIS RIGIDITY DISAPPEARS AND GAPS CAN BE CREATED FOR THE ZEROS AS $\dim(A) \to \infty$.

- THE EXACT CONDITION ON $K \subset C_q$ (CLOSED) TO CONTAIN ALL THE ZEROS OF THE ZETA FUNCTION OF AN INFINITE SEQUENCE OF A's $/\mathbb{F}_q$ WAS DETERMINED BY SERRE (2018) AND DEPENDS ON THE CAPACITY OF K.

- FOR $K \subset \mathbb{C}$ COMPACT, ITS TRANSFINITE DIAMETER OR CAPACITY IS DEFINED BY

$$n \geq 1, \quad d_n(K) = \max_{z_1, \dots, z_n \in K} \left(\prod_{j<k} |z_j - z_k| \right)^{2/n(n-1)}$$

GEOMETRIC MEAN.

$d_n(K)$ IS DECREASING AS n INCREASES AND $d(K) = \lim_{n \to \infty} d_n(K)$ IS THE TRANSFINITE DIAMETER.

THEOREM (FEKETE 1930)

IF $\mathrm{CAP}(K) < 1$ THEN

$\{ \alpha : \alpha$ IS AN ALGEBRAIC INTEGER ALL OF WHOSE GALOIS CONJUGATES ARE IN $K \}$ IS FINITE.

CONVERSE TO FEKETE:
RAPHAEL ROBINSON PROVES AN ESSENTIAL CONVERSE: FOR $K \subset \mathbb{R}$ IF $\mathrm{CAP}(K) > 1$ THEN K CONTAINS INFINITELY MANY SUCH α's.

USING THIS AND HONDA-TATE THEORY WHICH ASSERTS THAT THE CONDITION TO BE THE ZEROS OF A ZETA FUNCTION OF ~~$\mathbb{F}_q$~~ A OVER $\mathbb{F}_q$ IS THAT ALL THE GALOIS CONJUGATES OF THE ALGEBRAIC INTEGER ω ALL HAVE $|\omega| = \sqrt{q}$;

SERRE PROVES: IF $K \subset C_q$ HAS $\mathrm{CAP}(K) > q^{1/4}$ THEN THERE IS A SEQUENCE OF A's OVER $\mathbb{F}_q$ WITH $\dim(A) \to \infty$ ALL ITS ZEROS ARE IN K.

NOTE $\mathrm{CAP}(C_q) = \sqrt{q}$.

- QUANTITATIVE EQUIDISTRIBUTION MEASURES ASSOCIATED TO THIS CONVERSE TO FEKETE HAVE BEEN DEVELOPED BY

SMYTH, SERRE, SMITH, ORLOSKI-SARDARI.

SPECIFICALLY BOUNDS (HIGHLY NON-TRIVIAL) FOR THE ~~SERRE~~ SCHUR-SIEGEL-SMYTH NUMBER λ_{SSS} WHICH IS THE LIMINF OF NORMALIZED TRACES OF TOTALLY POSITIVE ALGEBRAIC INTEGERS

$$1{,}8023.. \leq \lambda_{SSS} \leq \overset{1.8216\ldots}{\cancel{1.8983..}} < 2$$

ORLOSKI, SARDARI, SMITH   SMITH SARDARI, ORLOSKI

APPLICATION:

B. KADETS (FIX q A SQUARE ~~q~~ AND FOR A SIMPLE)

$$q + 2\sqrt{q} + 1 - \lambda_{sss} \leq \varlimsup_{A \to \infty} \frac{\log |A(\mathbb{F}_q)|}{\dim A} \leq q + 2\sqrt{q}$$

$$q - 2\sqrt{q} + 2 \leq \varliminf_{A \to \infty} \frac{\log |A(\mathbb{F}_q)|}{\dim A} \leq q - 2\sqrt{q} + 1 + \lambda_{sss}$$

(THE BOUNDS INVOLVING λ_{sss} ARE FOR GROWING q)

LOCAL DISTRIBUTION OF ZEROS AND LOCAL STRUCTURE FOR JAC(X) FOR X IN A RANDOM FAMILY?

- THESE WERE STUDIED AND DETERMINED IN KATZ-S IN THE DOUBLE LIMIT

$$\underset{g\to\infty}{\overset{\text{SECOND}}{\lim}}\ \underset{q\to\infty}{\overset{\text{FIRST}}{\lim}}$$

(FOR ANY GEOMETRIC FAMILY).

THE KEY IS TO USE THE LEFSCHETZ TRACE FORMULA TO DO THE COUNTING OF THE STATISTICAL QUANTITIES. TAKING $q\to\infty$ FIRST ALLOWS US TO GET BY WITH MINIMAL INFORMATION ABOUT THE COHOMOLOGY OF THE CORRESPONDING PARAMETER SPACE OF THE FAMILY (DELIGNE'S PURITY IS DECISIVE WHEN $q\to\infty$). IN THIS WAY WE ARRIVE AFTER THE SECOND LIMIT ($g\to\infty$) WITH ONE OF THE UNIVERSAL FOUR TYPES.

WE RAISED THE QUESTION ABOUT SWITCHING THE ORDER (EVEN FIX q !).

THERE HAVE BEEN MANY WORKS IN THE HYBRID SETTING — SAWIN HAS PROBED THESE DEEPLY.

THE STRIKING BREAKTHROUGHS COME FROM TOPOLOGY AND IN PARTICULAR COHOMOLOGICAL STABILITY OF CERTAIN MODULI SPACES OF CURVES.

$\mathcal{H}_g$: THE GONALITY 2 CURVES OF GENUS g;
IE HYPERELLIPTIC CURVES

THE RECENT RESULTS OF MILLER-PATZT-PETERSEN-RANDAL-WILLIAMS

ON THE UNIFORM COHOMOLOGICAL STABILITY (ALLOWING LOCAL TWISTING) OF $\mathcal{H}_g$ ALLOWS FOR THE SWITCHING OF THE ORDER $g \to \infty$ FIRST AND THEN q (THE LATTER SLOWLY).

IN PARTICULAR GIVING THE

- MOMENTS OF Z_K AT THE CENTRAL POINT (BERGSTROM, DIACONU, PETERSEN, WESTERLAND)

- LOCAL SPACINGS RATIOS (V. WANG) ARE "SYMPLECTIC"

- THE LOCAL (COHEN-LENSTRA) STATISTICS FOR THE GROUP JAC(X), $X \in \mathcal{H}_g$ IN THE LIMIT $g \to \infty$ AND THEN $q \to \infty$ WAS ACHIEVED BY USING THE LEFSCHETZ FORMULA TO DO THE COUNTING, BY ELLENBERG – VENKATESH – WESTERLAND, AND STRENGTHENED SIGNIFICANTLY RECENTLY BY LANDESMAN AND LEVI.

$\mathcal{M}_g$? (MODULI SPACE OF CURVES OF GENUS g)

PERHAPS THE MOST PRESSING PROBLEM CONCERNING THE LOCAL STATISTICS AND JAC(X) LOCAL GROUP STATISTICS, IS FOR THE NON-TAME CASE OF $\mathcal{M}_g / \mathbb{F}_q$, IN THE $g \to \infty$ LIMIT.

THE TOPOLOGICAL APPROACH HAS A NONTAME COUNTER PART WHICH IS THAT THE EULER CHARACTERISTIC GROWS LIKE g^g (HARER-ZAGIER).

EVEN THE PROBLEM OF THE ASYMPTOTIC SIZE OF $|\mathcal{M}_g(\mathbb{F}_q)|$ AS $g \to \infty$ IS PROBLEMATIC.

SO ITS NOT CLEAR WHAT THE DENSITY OF STATES IS AND THE ASYMPTOTIC OF $|\mathrm{JAC}(X)(\mathbb{F}_q)|$ FOR RANDOM $X \in \mathcal{M}_g(\mathbb{F}_q)$.

ONE CAN MODEL THE PROBLEM USING TOPOLOGICAL METHODS AND INCORPORATE ONLY THE STABLE COHOMOLOGY CLASSES WHICH ARE KNOWN (MADSEN-WEISS) AND MAKE SOME PREDICTIONS FOR DENSITIES (ACHTER, ERMAN, KEDLAYA, WOOD, ZUREICK-BROWN).

- I EXPECT THAT THE LOCAL SPACING STATISTICS FOLLOW THE UNIVERSALITY SYMPLECTIC CLASS AND THAT COHEN-LENSTRA STATISTICS APPLY TO $\mathrm{JAC}(X)(\mathbb{F}_q)$, AS $g \to \infty$.

- ONE CAN THESE STATISTICS FOR THE (VERY NONRANDOM IN $\mathcal{M}_g$) CURVES $X_0(N)/\mathbb{F}_q$ WITH N VARYING TO SOME EXTENT (ZUBRILINA - S) THE SPACINGS (LOCAL) ARE POISSON LIKE!