Periods of Integrals on Algebraic Manifolds, I. (Construction and Properties of the Modular Varieties)
Author(s): Phillip A. Griffiths
Source: *American Journal of Mathematics*, Vol. 90, No. 2 (Apr., 1968), pp. 568-626
Published by: The Johns Hopkins University Press
Stable URL: http://www.jstor.org/stable/2373545
Accessed: 25/10/2010 11:46

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at http://www.jstor.org/page/info/about/policies/terms.jsp. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at http://www.jstor.org/action/showPublisher?publisherCode=jhup.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit service that helps scholars, researchers, and students discover, use, and build upon a wide range of content in a trusted digital archive. We use information technology and tools to increase productivity and facilitate new forms of scholarship. For more information about JSTOR, please contact support@jstor.org.

The Johns Hopkins University Press is collaborating with JSTOR to digitize, preserve and extend access to *American Journal of Mathematics.*

http://www.jstor.org

PERIODS OF INTEGRALS ON ALGEBRAIC MANIFOLDS, I.

(Construction and Properties of the Modular Varieties)

By PHILLIP A. GRIFFITHS.*

I. 0. Introduction. (a) The general problem we have in mind is to investigate the periods of integrals on an algebraic variety V defined over a function field $\mathcal{F}$. In practice, this will mean that we are given an algebraic family of algebraic varieties $\{V_t\}_{t\in B}$ where the general member $V=V_t$ of this family is an ordinary polarized, non-singular algebraic manifold, and we wish to study the behavior of the period matrix $\Omega(t)$ of V_t as a function of t. In order to discuss $\Omega(t)$, we should think of the periods as a (not everywhere defined) mapping $\Phi\colon B\rightarrow M$ where M, the *modular variety* associated to V, repersents the totality of inequivalent period matrices satisfying the bilinear relations imposed by the topological manifold underlying V.

In this paper (Part I) we shall study the variety M. Many of he classical results, which arise when V is a curve and M is the Siegel upper-half-space factored by the modular group, will go through. However, there are some striking differences which turn up, and which seem to be best explained by the presence of higher order period relations.

In Part II we shall study the local properties of the period mapping Φ and, in Part III, we shall look into the global behavior of Φ.

Some of these results have been announced in the Proceedings of the National Academy of Sciences (U. S. A.), Vol. 55 (5), 1303-1309; (6), 1392-1395, and Vol. 56 (2), 413-416.

It is my pleasure to express gratitude to several colleagues who, through conversation and correspondence, have been of immense help in studying this question on periods of integrals.

(b) We give now an outline of the results in this paper, which is divided into four sections under the following headings:

1. Period matrices of compact Kähler manifolds;
2. Modular varieties of polarized algebraic manifolds;

Received January 18, 1967.

* Supported in part of Office of Naval Research Contract 3656(14).

3. Some properties of the modular varieties: complex torii associated to the cohomology of an algebraic manifold;
4. Further properties of the modular varieties.

First, in Section 1, we invert the usual presentation of period matrices and view a period matrix Ω as a point in a Grassmannian; this point being the subspace spanned by the rows of Ω. Thus we associate to V the subspaces $H^{q-r,r}(V) \subset H^q(V,\boldsymbol{C})$; the local study of the period mapping is then essentially the study of the variation of the *Hodge decomposition* of the vector space $H^q(V,\boldsymbol{C})$. In order to make the periods vary holomorphically with V_t, we use a filtration on $H^q(V,\boldsymbol{C})$ whose associated graded vector space is $\sum_{r=0}^{q} H^{q-r,r}(V)$. Thus, the *period matrix space* D $(=D_q(V))$ is defined to be a domain on a flag manifold $\boldsymbol{F}$ consisting of points $\Omega = [S_0, S_1, \cdots, S_p]$ where $S_0 \subset S_1 \subset \cdots \subset S_p \subset H^q(V,\boldsymbol{C})$ and where the two *Riemann bilinear relations* $\Omega Q^t\Omega = 0$, $\Omega Q^t\bar{\Omega} > 0$ are satisfied. In case $\Omega = \Omega(V)$ is the period matrix of V, $S_{r+1}/S_r \cong H^{q-r,r}(V)$. After some preliminaries on Kähler varieties, we give in Section 1.(d) the precise definition of the period matrix domains D. In 1.(e) we show that this approach is equivalent, in the case of curves, to using the period matrix directly as a point in the *Siegel upper half space*. In Section 1.(g) it is shown that D is acted on transitively by a real, simple Lie group G and that the isotropy group H is compact; thus $D \cong H\backslash G$. Contrary to the case of curves, D need *not* be an Hermitian symmetric space, but it will be so that D is an open domain on a homogeneous algebraic manifold X, where X is those flags Ω satisfying $\Omega Q^t\Omega = 0$. These period matrix domains are discussed in a general Lie group theoretic manner in [7].

Now the complex structure, with polarization, of V does not give a unique period matrix $\Omega(V) \in D$. This is because there will be polarization-preserving homeomorphisms of V which induce non-trivial action on $H^q(V,\boldsymbol{C})$. Thus there will be defined an arithmetic subgroup $\Gamma \subset G$ such that the *modular variety* $M = D/\Gamma = H\backslash G/\Gamma$ is an analytic space, and V defines a unique point $\Phi(V) \in M$. This modular variety is discussed in Sections 2.(a) and 2.(b).

Suppose now that two polarized algebraic manifolds V, V' are in the same *class* if $h^{p,q}(V) = h^{p,q}(V')$ and if there is a polarization-preserving homeomorphism $f \colon V \rightarrow V'$. Then, in Section 2.(c), we prove (roughly) that $\Phi(V) = \Phi(V')$ if, and only if, the graph $F \in H_{2n}(V \times V', \boldsymbol{Z})$ is of type (n,n). This shows geometrically what it means that two varieties V and V' in the same class should have the same periods.

In Section 3 we discuss complex structures on the real torus

$$H^{2q+1}(V,\boldsymbol{R})/H^{2q+1}(V,\boldsymbol{Z}).$$

One such is given by $A_q(V)$, which is Weil's *intermediate Jacobian* [12]. However, $A_q(V)$ does *not* vary holomorphically with V. The construction of the period matrix space D suggests another complex structure $T_q(V)$ with the following properties: (i) $T_0(V)=A_0(V)$ is the *Albanese variety* and $T_{n-1}(V)=A_{n-1}(V)$ is the *Picard variety*; (ii) $T_q(V)$ varies holomorphically with V (this will be proved in Part II); (iii) the polarization on V induces a *p-convex polarization* (cf. the definition in 3(c)) on $T_q(V)$; (iv) $T_q(V)$ with its polarization is functorial; and (v) the period matrix of the holomorphic 1-forms $\Omega(T_q(V))$ is the point in D_{2q+1} corresponding to $\Omega(V)$. This result is aproved in 3.(c) and 3.(d). The presence of the p-convex polarization on $T_q(V)$ seems to be closely related to failure of the inversion theorem for intermediate cycles ([12], Section 27), which will be discussed in Part II.

Section 4 contains the main properties of the period matrix domains D. We have chosen to use the periods of the 2-forms and 3-forms as being exemplary (cf. 4.(a)), and most results are stated in general form but proved for these domains. The general argument will usually be evident.

Of interest are the *homogeneous line bundles* $\boldsymbol{L}\rightarrow D$; e.g., the canonical bundle $\boldsymbol{K}$, which generalizes the canonical factor of automorphy, is one such. These bundles have G-invariant metrics and the curvature is then a G-invariant form on D. Contrary to the classical case, there will generally be no positive bundles, and so, using $\boldsymbol{K}$, the modular variety M must be considered as an analytic space with a p-convex polarization. Also, instead of *automorphic forms* for D/Γ, we must now expect *automorphic cohomolgy.*

The curvatures, which have been computed group-theoretically in [7], are computed explicitly in 4(b). These explicit results will be used thereafter in several instances. For example, we show in 4(d) that D has an exhaustion function ϕ whose *Levi form* $L(\phi)$ has $n-p$ positive eigenvalues, where $\boldsymbol{K}\rightarrow D$ has a p-convex polarization. In fact, $L(\phi)$ is essentially the curvature of $\boldsymbol{K}$.

One of the geometric reasons for this p-convex behavior is the presence in D of compact subvarieties. In 4.(c) we show that the fibres of the mapping $H\backslash G\rightarrow K\backslash G$ ($K=$ maximal compact subgroup of G) give a family $\{Y_\lambda\}_{\lambda\in K\backslash G}$ of compact, complex submanifolds of D such that $\boldsymbol{K}$ on Y_λ is negative. Combining this with the pseudo-convex exhaustion of D and the Borel-Weil theorem for $H\backslash K$, we show in 4.(d) that $\dim H^p(D,\mathcal{O}(\boldsymbol{K}))=\infty$ ($p=\dim H\backslash K$), and that $H^p(D, \mathcal{O}(\boldsymbol{K}))$ can be expanded in a power series around $Y_0 = H\backslash K$, which is a special case of a theorem of W. Schmid.

To get information on $M = D/\Gamma$, we need the result of W. Schmid that the *absolutely integrable cohomology* (L^1-cohomology) $H_1{}^p(\boldsymbol{K})$ is in some cases a non-empty Banach space. Then, in 4.(e) we prove essentially that the *Poincaré series in cohomology* $\theta(\phi) = \sum_{\gamma \in \Gamma} \gamma(\phi)$ $(\phi \in H_1{}^p(\boldsymbol{K}))$ converges to a Γ-automorphic cohomology class.

To close this introduction, let us mention briefly what relevance these p-convex polarizations, automorphic cohmology, etc. have to the original problem. It will be shown in Part II that the image $\Phi(B)$ lies transversely to the family $\{Y_\lambda\}_{\lambda \in K\backslash G}$ and that $\boldsymbol{K} \mid \Phi(B)$ is positive (0-convex polarization). What this means is that there are *higher order period relations*; i. e. relations of the form $Q(d\Omega, \Omega) = 0$ on $d\Omega$ over $\mathfrak{J}$ which hold universally. By a sort of integration over the fibre, the automorphic cohomology should then give rise to automorphic forms, in the usual sense. These automorphic forms should then be related to fields of moduli, etc.

I. 1. Period matrices of compact Kahler manifolds. (a) Let W be a complex vector space and $e_1{}^*, \cdots, e_b{}^*$ a basis for the dual space W^*. We want to coordinatize $G(h, W)$, the *Grassmann variety* of h-planes through the origin in W. Let $S \in G(h, W)$ be such a subspace and $f_1, \cdots, f_h$ a basis for S. We consider the matrix $\Omega = (\pi_{\alpha\rho})$ where $\pi_{\alpha\rho} = \langle f_\alpha, e_\rho{}^* \rangle$. Clearly Ω has rank h and we assert that the $h \times h$ minors

$$\Omega_{\rho_1 \cdots \rho_h} = \begin{vmatrix} \pi_{1\rho_1} \cdots \pi_{1\rho_h} \\ \vdots \\ \pi_{h\rho_1} \cdots \pi_{h\rho_h} \end{vmatrix} \qquad (\rho_1 < \cdots < \rho_h)$$

give the Plücker coordinates of $S \in G(h, W)$. Indeed, let $e_1, \cdots, e_b$ be the basis of W dual to $e_1{}^*, \cdots, e_b{}^*$; then $f_\alpha = \sum_{\rho=1}^{b} \pi_{\alpha\rho} e_\rho$. It follows that

$$f_1 \wedge \cdots \wedge f_h = \sum_{\rho_1 < \cdots < \rho_h} \Omega_{\rho_1 \cdots \rho_h} e_{\rho_1} \wedge \cdots \wedge e_{\rho_h}$$

so that the $\Omega_{\rho_1 \cdots \rho_h}$ give the Plücker coordinates of S.

If we choose a new basis $\tilde{f}_1, \cdots, \tilde{f}_h$ for S, then $\tilde{f}_\alpha = \sum_{\beta=1}^{h} A_{\alpha\beta} f_\beta$ and $\langle \tilde{f}_\alpha, e_\rho{}^* \rangle = \sum_{\beta=1}^{h} A_{\alpha\beta} \pi_{\beta\rho}$. Thus the matrix Ω is changed into $A\Omega$ by this change of basis. In fact, if we let $P(h, W)$ be the set of all $h \times b$ matrices of rank h, then

$$P(h, W) \to G(h, W)$$

is a *principal bundle* with fibre $GL(h)$ acting on the left.

The group $GL(b) \cong GL(W)$ of automorphisms of W acts on $G(h, W)$ by the rule $T \cdot S = T(S)$ where $T \in GL(b)$, $S \in G(h, W)$, and $T(S)$ is the subspace with basis $T(f_1), \cdots, T(f_h)$. Let $Te_\rho = \sum t_{\rho\sigma} e_\sigma$; then $\langle T(f_\alpha), e_\sigma{}^* \rangle = \sum_{\rho=1}^{b} \pi_{\alpha\rho} t_{\rho\sigma}$ so that, under the action of $GL(b)$, Ω goes into ΩT where $T = (t_{\rho\sigma})$ is the matrix of T.

We remark that T acts on W^* by the rule $Te_\rho{}^* = \sum t_{\sigma\rho} e_\sigma{}^*$.

We suppose now that we are given a non-singular *quadratic form* Q: $W \times W \to \boldsymbol{C}$, which may be either symmetric or skew-symmetric. We let $X \subset G(h, W)$ be the subvariety defined by $X = \{S \mid Q(S, S) = 0\}$. This condition means that we should have $Q(f_\alpha, f_\beta) = 0$ for all α, β. Letting $Q(e_\rho, e_\sigma) = q_{\rho\sigma}$ be the matrix for Q, we have that $\Omega(f_\alpha, f_\beta) = \sum_{\rho,\sigma=1}^{b} \pi_{\alpha\rho} \pi_{\beta\sigma} q_{\rho\sigma}$; so that the condition that $S \in X$ can be written:

$$\Omega Q^t \Omega = 0 \qquad (Q = (q_{\rho\sigma}) \text{ is the matrix of } Q). \tag{1.1}$$

If we let $\tilde{G} \subset GL(b)$ be the orthogonal group for Q;

$$\tilde{G} = \{T \in GL(b) \mid Q(Te, Tf) = Q(e, f) \text{ for } e, f \in W\},$$

then $\tilde{G}$ acts on $G(h, W)$ and preserves X. In fact, the condition that $T = (t_{\rho\sigma})$ belongs to $\tilde{G}$ is:

$$TQ^tT = Q. \tag{1.2}$$

If $S \in X$ is represented by the matrix Ω satisfying (1.1) then $T(S)$ has matrix ΩT and $(\Omega T) Q^t (\Omega T) = \Omega Q^t \Omega = 0$ so that $T \cdot S \in X$.

(b) Let V be a *compact Kähler manifold* and $H^r(V) = H^r(V, \boldsymbol{C}) = W$ the r-th *deRham cohomology group* using differential forms with complex coefficients. We let $\gamma_1, \cdots, \gamma_b$ be an integral basis for the free part of $H_r(V, \boldsymbol{Z})$; we may consider $\gamma_1, \cdots, \gamma_b$ as a basis for W^*.

For fixed p, q with $p + q = r$, the space $H^{p,q} = H^{p,q}(V)$ of harmonic (p, q) forms is a subspace of W. Let $\omega^1, \cdots, \omega^h$ $(h = h^{p,q})$ be a basis for $H^{p,q}$, and form the *period matrix*

$$\Omega = (\pi_{\alpha\rho}) = \begin{bmatrix} \int_{\gamma_1} \omega^1 \cdots & \int_{\gamma_b} \omega^1 \\ \vdots & \vdots \\ \int_{\gamma_1} \omega^h \cdots & \int_{\gamma_b} \omega^n \end{bmatrix} \qquad (\pi_{\alpha\rho} = \int_{\gamma_\rho} \omega^\alpha). \tag{1.3}$$

If we choose a new basis $\phi^{\alpha}=\sum_{\beta=1}^{h}A_{\alpha\beta}\omega^{\beta}$, $|A_{\alpha\beta}|\neq 0$, for $H^{p,q}$, then Ω is changed into $A\Omega$. From the considerations in (a) above, we see that the Plücker coordinates of the period matrix Ω are invariantly defined and these coordinates specify the subspace $H^{p,q}\subset H^{r}(V)$.

However, even as a point in the Grassmannian, the period matrix is not invariantly defined, because if we choose a new integral basis $\mu_1,\cdots,\mu_b$, then

$$\mu_\rho=\sum_{\sigma=1}^{b}\lambda_{\sigma\rho}\gamma_\sigma \tag{1.4}$$

where $\Lambda=(\lambda_{\sigma\rho})$ is unimodular and Ω goes into $\Omega\Lambda$ since

$$\int_{\mu_\rho}\omega^{\alpha}=\sum_{\sigma=1}^{b}\Big(\int_{\gamma_\sigma}\omega^{\alpha}\Big)\lambda_{\sigma\rho}.$$

In other words, thhe integral change of homology basis (1.4) has the same effect on $\Omega\in G(h,W)$ as the transformation $\Lambda=(\lambda_{\sigma\rho})$.

In general, we shall call period matrices Ω and Φ *equivalent*, written $\Omega\sim\Phi$, if

$$\Phi=A\Omega\Lambda \qquad (A\in GL(h),\Lambda\in SL(b,\boldsymbol{Z}))\,; \tag{1.5}$$

cf. Hodge [10], page 199.

The conclusion of this discussion and that in section (a) is that the period matrices for periods of integrals on compact Kähler manifolds should be considered as points in a Grassmann manifold $G(h,W)$ taken modulo the action of a suitable discrete subgroup Γ. What we want is that the totality of all possible period matrices forms a complex manifold D on which Γ acts properly discontinuously; in order to do this, we shall consider only the periods of the primitive harmonic differentials. After some preliminary considerations on Kähler manifolds in (c), we shall give in section (d) the construction of the period matrix space D. The group Γ and its action on D will be discussed in section I.2.(a).

(c) Since V is a compact Kähler manifold, there is a *fundamental class* $\omega\in H^{1,1}\subset H^{2}(V,\boldsymbol{C})$, and we may consider the real operator

$$L\colon H^{q}(V)\rightarrow H^{q+2}(V) \text{ defined by } L\eta=\omega\eta \quad (\eta\in H^{q}(V)).$$

We set $H^{q}(V)_0=\text{kernel }(L^{n-q+1})=\{\eta\in H^{q}(V):\omega^{n-q+1}\eta=0\}$ where $\dim V=n$. The space $H^{q}(V)_0$ is called the space of *primitive cohomology classes of degree* q $(q\leqq n)$.

One of Hodge's fundamental theorems gives the *Lefschetz decomposition*: For $0 < q \leqq n$,

$$(1.6) \qquad H^q(V) = \sum_{0 \leqq r \leqq [q/2]} L^r H^{q-2r}(V)_0.$$

Now we may define a real quadratic form $Q: H^q(V)_0 \times H^q(V)_0 \rightarrow \mathbf{C}$ by

$$(1.7) \qquad Q(\xi, \eta) = (-1)^{q(q+1)/2} \int_V \omega^{n-q} \xi \eta \qquad (\xi, \eta \in H^q(V)_0).$$

For q even, Q is symmetric and Q is alternating for q odd; in either case, Q is non singular.

So far we have made no mention of the complex structure on V; the above are results which hold for the topological manifold V together with the class ω. The complex structure induces an operator $J: H^q(V) \rightarrow H^q(V)$ and we have the *Hodge decomposition*:

$$(1.8) \qquad H^q(V) = \sum_{r+s=q} H^{r,s} \qquad (H^{r,s} = \bar{H}^{s,r}),$$

where $H^{r,s} \subset H^q(V)$ is the subspace spanned by the cohomology classes of type (r, s). Here $J \mid H^{r,s}$ is multiplication by $(\sqrt{-1})^{r-s}$.

Now the decompositions (1.6) and (1.8) are compatible so that, if we set $H_r^q(V) = L^r H^{q-2r}(V)_0$, $H_0^{s,t} = \{\text{kernel of } L^{n-q+1} \text{ on } H^{s,t},\ s+t=q\}$, $H_r^{s,t} = L^r H_0^{s-r,t-r}$, we have, for $0 < q \leqq n$,

$$(1.9) \qquad H^q(V) = \sum_{0 \leqq r \leqq [q/2]} H_r^q(V),$$

$$(1.10) \qquad H^q(V)_0 = \sum_{s+t=q} H_0^{s,t},$$

$$(1.11) \qquad H^{s,t} = \sum_{0 \leqq r \leqq [q/2]} H_r^{s,t},$$

$$(1.12) \qquad H_r^q(V) = \sum_{s+t=q} H_r^{s,t}.$$

The connexion between the decompositions (1.9)-(1.12) and the quadratic form (1.7) is the following:

$$(1.13) \qquad Q(H_0^{q-r,r}, H_0^{s,q-s}) = 0 \qquad (r \neq s)$$

$$(1.14) \qquad Q(H_0^{q-r,r}, \bar{H}_0^{q-r,r}) > 0.$$

Here (1.13) means that $Q(\xi, \eta) = 0$ for $\xi \in H_0^{q-r,r}$, $\eta \in H_0^{s,q-s}$; and (1.14) means that, if we choose a basis $\omega^1, \cdots, \omega^k$ for $H_0^{q-r,r}$, then the matrix

$$(1.15) \qquad \lambda_{\alpha\beta} = \sqrt{-1})^q (-1)^{r+q} Q(\omega^\alpha \overline{\omega}^\beta)$$

is Hermitian positive definite.

The relations (1.13), (1.14) are generalizations of the *Riemann first and second bilinear relations.*

The classical reference for the above is Hodge [10], pages 178-201. We remark that Hodge's "effective" is our "primitive" since we shall use the adjective effective in another context.

(d) What we want to do is to make the set of all possible period matrices for the primitive cohomology $H_0{}^{q-r,r}$ into a complex manifold D. Because of (a) and (b) we can expect that D should be a subset of a Grassmannian; in effect, D will be defined by the bilinear relations (1.13), (1.14).

We now make this construction. Let $W = H^q(V)_0$ be the complex vector space with the bilinear form (1.7). By (1.10), $H^q(V)_0 = \sum_{s+t=q} H_0{}^{s,t}$ and we set: $S^r = H_0{}^{q,0} + H_0{}^{q-1,1} + \cdots + H_0{}^{q-r,r}$, $t = [q-1/2]$. Then $S^0, S^1, \cdots, S^t$ forms an increasing sequence of subspaces of W; $S^0 \subset S^1 \subset \cdots \subset S^t \subset W$. Let $h_0 = h_0{}^{q,0}$, $h_1 = h_0{}^{q,0} + h_0{}^{q-1,1}, \cdots, h_t = h_0{}^{q,0} + \cdots + h_0{}^{q-t,t}$ so that $h_r = \dim H_0{}^{q,0} + \cdots + H_0{}^{q-r,r}$. Let $\boldsymbol{F} = F(h_0, \cdots, h_t, W)$ be the *flag manifold* of all nested sequences of subspaces $S^0 \subset \cdots \subset S^t \subset W$ with $\dim S^r = h_r$. The point in $\boldsymbol{F}$ will be written as $[S^0, \cdots, S^t]$. The Hodge decomposition (1.10) defines then a point $\Omega = \Omega(V)$ in $\boldsymbol{F}$.

Let $G(h, W)$ be the Grassmann variety of h-planes in W. There is a natural embedding $\boldsymbol{F} \subset \boldsymbol{G} = G(h_0, W) \times \cdots \times G(h_t, W)$ defined by sending $[S^0, \cdots, S^t]$ into $(S^0, \cdots, S^t)$ where $S^r \in G(h_r, W)$. We may think of S^r as the *period matrix for the primitive harmonic* $(q-s, s)$ *forms for* $s \leqq r$. Thus $\Omega(V) = [S^0, \cdots, S^t] \in \boldsymbol{F}$ may be thought of as a sequence of period matrices which determine and are determined by the Hodge decomposition (1.10).

The flag $\Omega(V) = [S^0, \cdots, S^t]$ satisfies relations implied by (1.13), (1.14). The first of the relations is $Q(S^0, S^0) = 0$, $Q(S^0, \bar{S}^0) > 0$. Given these relations, we may define S^1/S^0 as a subspace of S^1 by:

$$S^1/S^0 = \{\phi \in S^1 \mid Q(\phi, \bar{S}^0) = 0\}.$$

Then we will have $Q(S^1/S^0, S^1/S^0) = 0$, $Q(S^1/S^0, \bar{S}^1/\bar{S}^0) > 0$. Continuing, we may summarize these relations by:

$$Q(S^r/S^{r-1}, S^r/S^{r-1}) = 0 \qquad \text{for } r \neq q/2, \tag{1.16}$$

$$Q(S^r/S^{r-1}, \bar{S}^r/\bar{S}^{r-1}) > 0. \tag{1.17}$$

Definition. D is the set of all flages $[S^0, \cdots, S^t]$ in $\boldsymbol{F}$ which satisfy the relations (1.16), (1.17).

Definition. X is the set of all flags $[S^0, \cdots, S^t]$ in $\boldsymbol{F}$ which satisfy (1.16) written as:

$$(1.18) \qquad Q(S^r, S^r) = 0 \qquad \text{for } r \neq q/2.$$

Clearly X is an algebraic subvariety of $\boldsymbol{F}$ and $D \subset X$ is an open subset. We shall call D the *period matrix space*; it has been exhibited as an open subset of an algebraic manifold.

The underlying vector space W has the bilinear form (1.7), and we let:

(1.19) $\tilde{G} =$ orthogonal group of the quadratic form Q given by (1.7);

(1.20) $G =$ real linear transformations in $\tilde{G}$.

Clearly $\tilde{G}$ acts on X by $T[S^0, \cdots, S^t] = [T(S^0) \cdots, T(S^t)]$. The embedding $\boldsymbol{F} \subset \boldsymbol{G}$ is equivariant with respect to this action. Furthermore, G acts on X and takes D into D by (1.16), (1.17). We shall see below that $\tilde{G}$ acts transitively on X and G acts transitively on D.

(e) As a first example, we consider a compact *Riemann surface* V of genus q. Then $H^1(V, \boldsymbol{C}) = W$ is a vector space of dimension $2q$ and the space of *Abelian differentials* $H^{1,0} \subset W$ is a subspace of dimension q. The point in $G(q, 2q)$ defined by $H^{1,0} \subset W$ has as Plücker coordinates those of the usual period matrix of V.

Let $\gamma_1, \cdots, \gamma_{2q}$ be an integral basis for $H_1(V, \boldsymbol{Z})$ which forms a canonical system of retrosections on the surface V; denote by $\hat{\gamma}^1, \cdots, \hat{\gamma}^{2q} \in H^1(V, \boldsymbol{Z})$ the dual basis. The matrix $Q = (q_{\rho\sigma})$, $q_{\rho\sigma} = \int_V \hat{\gamma}^\rho \wedge \hat{\gamma}^\sigma$, will then be:

$$Q = \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix}.$$

Let $\omega^1, \cdots, \omega^q$ be a basis for $H^{1,0}$ and set $\pi_{\alpha\rho} = \int_{\gamma_\rho} \omega^\alpha$, $\Omega = (\pi_{\alpha\rho})$. The bilinear relations (1.13), (1.14) become (cf. (1.1)):

$$(1.21) \qquad \Omega Q {}^t\Omega = 0,$$

$$(1.22) \qquad \sqrt{-1}(\Omega Q {}^t\bar{\Omega}) > 0.$$

Write $\Omega = (E, F)$ where E, F are $q \times q$ matrices. The relations (1.21), (1.22) become:

$$(1.23) \qquad \begin{gathered} -F^tE + E^tF = 0, \\ \sqrt{-1}(-F^t\bar{E} + E^t\bar{F}) > 0. \end{gathered}$$

The matrix E is non-singular, since if ${}^t\xi E = 0$ for some vector ξ, then $0 < {}^t\xi[\sqrt{-1}(-F{}^t\bar{E} + E{}^t\bar{F})]\bar{\xi} = 0$. Thus $\Omega = (E,F) \sim (I, E^{-1}F) = (I, Z)$. Now the set of $q \times q$ symmetric matrices $Z = X + \sqrt{-1}\,Y$ with $Y > 0$ is *Siegel's upper half space* $\boldsymbol{H}_q$ in genus q, and we have shown:

(1.24) PROPOSITION. *The period matrix space D for a compact Riemann surface of genus q is analytically isomorphic to the Siegel upper half space $\boldsymbol{H}_q$.*

The group $\tilde{G}$ is the $2q \times 2q$ complex matrices T which satisfy $TQ{}^tT = Q$ (cf. (1.2)) where $Q = \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix}$; thus $\tilde{G} = Sp(q, \boldsymbol{C})$ is the complex *symplectic group* (cf. Chevalley [4], page 22). The automorphism group G of $D \cong \boldsymbol{H}_q$ is the real symplectic group $Sp(q) = Sp(q, \boldsymbol{R})$.

In order to identify the action of G on D with the usual action of $Sp(q)$ on $\boldsymbol{H}_q$ given by

$$(1.25) \qquad T(Z) = (AZ + B)(CZ + D)^{-1}, \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix};$$

we define a mapping $\hat{}$: $Sp(q) \to Sp(q)$ by

$$(1.26) \qquad T \to \hat{T} = \begin{pmatrix} D & B \\ C & A \end{pmatrix}, \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Since $T^{-1} = \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$, $\hat{}$ is the composition of the operations $T \to T^{-1}$, $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \to \begin{pmatrix} A & -B \\ -C & D \end{pmatrix}$. We check that $(\widehat{T_1T_2}) = \hat{T}_2\hat{T}_1$.

Now we calculate:

$$\Omega\hat{T} = (I, Z)\begin{pmatrix} D & B \\ C & A \end{pmatrix} = (({}^tD + Z{}^tC), ({}^tB + Z{}^tA))$$

$$\sim (I, ({}^tD + Z{}^tC)^{-1}({}^tB + Z{}^tA)) = (I, (AZ + B)(CZ + D)^{-1}).$$

This shows how the action of G on D corresponds, under the anti-automorphism $\hat{}$, to the usual action (1.25) of $Sp(q)$ on $\boldsymbol{H}_q$.

It is well known that G acts transitively on D, considered as q-dimensional subspaces $S \subset W$ satisfying $Q(S,S) = 0$, $Q(S,\bar{S}) > 0$, and the isotropy group H of a fixed $S \in D$ is the unitary group $U(q)$, so that D is the *coset space* $H \backslash G = U(q) \backslash Sp(q)$.

(f) As a second example, we consider a *compact Kähler* surface V. Letting $W = H^2(V, \boldsymbol{C})_0$ be the primitive cohomology, the construction of the domain D in this case consists in looking at subspaces $S \subset W$, $\dim S = h = h^{2,0}$, which satisfy $Q(S,S)=0$, $Q(S,\bar{S})>0$. A typical such subspace is $H^{2,0}\subset W$. If we write $W=(H^{2,0}\oplus H^{0,2})\oplus H_0{}^{1,1}$ where $H_0{}^{1,1}$ is the primitive cohomology of type; $(1,1)$, then $H^{2,0}\oplus H^{0,2}$ and $H_0{}^{1,1}$ are both defined as real vector spaces; Q on $H^{2,0}\oplus H^{0,2}$ is positive definite and Q on $H_0{}^{1,1}$ is negative definite (*index theorem*). In general, if $S\in D$, then $S+\bar{S}$ and $R=(S+\bar{S})^{\perp}$ are real subspaces on which Q is positive and negative respectively.

Let $k=\dim H_0{}^{1,1}$ and $b=2h+k=\dim W$. Then the real quadratic form Q has signature $(2h,k)$ ($=2h$ positive and k negative signs) so that $G\cong O(2h,k)$ is the orthogonal group of a real, indefinite quadratic form. The group $\tilde{G}$ is just the complex orthogonal group $O(b,\boldsymbol{C})$.

We claim that G acts transitively on D. To see this, let $S\in D$ and $\omega^1,\cdots,\omega^h$ be a basis for S. Then $Q(\omega^\alpha,\omega^\beta)=0$ and the Hermitian matrix $\Lambda=(\lambda_{\alpha\bar{\beta}})$ given by $\lambda_{\alpha\bar{\beta}}=Q(\omega^\alpha,\bar{\omega}^\beta)$ is positive definite. We may choose a new basis $\phi^1,\cdots,\phi^h$ for S such that $Q(\phi^\alpha,\bar{\phi}^\beta)=\delta_\beta{}^\alpha$. Such a basis will be called orthonormal.

Let now $R=(S\oplus\bar{S})^{\perp}=\{\psi\in W\mid Q(\psi,S\oplus\bar{S})=0\}$. Then R is the complexification of a real vector space and we may choose a real basis $\zeta^1,\cdots,\zeta^k$ for R such that $Q(\zeta^i,\zeta^j)=-\delta_j{}^i$. Let $\eta^\alpha=\dfrac{\phi^\alpha-\bar{\phi}^\alpha}{2}$, $\xi^\alpha=\dfrac{\phi^\alpha-\bar{\phi}^\alpha}{2\sqrt{-1}}$. Then $\eta^1,\cdots,\eta^h$; $\xi^1,\cdots,\xi^h$; $\zeta^1,\cdots,\zeta^k$ gives a real basis for W relative to which Q has the matrix $\begin{pmatrix} I_{2h} & 0\\ 0 & -I_h\end{pmatrix}$.

Given $\hat{S}\in D$, we may choose $\{\hat{\omega}^\alpha\}$ $\{\hat{\phi}^\alpha\}$, $\{\hat{\zeta}^j\}$, $\{\hat{\eta}^\alpha\}$, $\{\hat{\xi}^\alpha\}$ as for S and define $T\colon W\rightarrow W$ by $T\eta^\alpha=\eta^\alpha$, $T\xi^\alpha=\hat{\xi}^\alpha$, $T\zeta^j=\hat{\zeta}^j$. Then $T\in G$ and $T(S)=\hat{S}$, so that G acts transitively on D.

Observe that, by letting $T\zeta^1=\pm\hat{\zeta}^1$, we can assure that $\det T=+1$ so that the identity component $SO(2h,k)$ of G acts transitively on D.

Suppose that $S_0\in D$ is fixed and $H=\{T\in G\mid TS_0=S_0\}$ is the *stability group* of S_0. Since T is real, $T\bar{S}_0=\bar{S}_0$ and so $T(S_0\oplus\bar{S}_0)=S_0\oplus\bar{S}_0$. Because T preserves Q, T takes an orthonormal basis for S_0 into an orthonormal basis, and so T on $S_0\oplus\bar{S}_0$ is of the form $A\oplus\bar{A}$ where A is unitary. Also T acts as an isometry in $R=(S_0\oplus\bar{S}_0)^{\perp}$ and so $H\cong U(h)\times O(k)$; in particular, H is compact.

(g) The above examples obviously generalize to give the following:

(1.26) THEOREM. *Let V be a compact Kähler manifold and $W=H^q(V,\boldsymbol{C})_0$ the primitive cohomology in dimension q. We construct the period matrix space D of all possible Hodge decompositions of W satisfying the bilinear relations* (1.13), (1.14) *as in Section* (d). *Then the group G*

of all real linear automorphisms of W preserving the quadratic form Q given by (1.17) is a real simple Lie group which acts transitively on D. The isotropy subgroup of a fixed point $\Omega_0 \in D$ is compact so that D is represented as the open homogeneous complex manifold $H\backslash G$.

Remark. As in the case of the periods of the 2-forms above, we see that the identity component of G acts transitively on D (in particular, D is connected).

I. 2. Modular varieties of polarized algebraic manifolds. (a) The construction of I.1.(d) of the period matrix space D was based on the Hodge decomposition of $H^q(V,\boldsymbol{C})$. If we take a diffeomorphism $f\colon V \rightarrow V$, and if f preserves the Kähler form ω, then we should get equivalent points in D. Another way of saying this is that the period matrix (1.3) is defined up to the equivalence (1.5) which involves right multiplication by a matrix arising from a change of homology basis.

To take this into account, we assume that V is *polarized*; i.e. there exists an *analytic line bundle* $\boldsymbol{L} \rightarrow V$ whose characteristic class is the Kähler form ω. In this case, $\omega \in H^{1,1}(V) \cap H^2(V,\boldsymbol{Z})$ is an integral class, the primitive cohomology $H^q(V,\boldsymbol{C})_0$ is defined over the rational numbers, and the quadratic forms (1.7) are rational.

Consider now the graded ring $H^*(V,\boldsymbol{Z}) = \sum_{q=0}^{2n} H^q(V,\boldsymbol{Z})$. There is an algebra $\boldsymbol{A}$ of operators on $H^*(V,\boldsymbol{Z})$; viz. the cohomology operations (cup product, primary operations, secondary operations when defined, etc.). We now consider the graded isomorphisms $T_*\colon H^*(V,\boldsymbol{Z}) \rightarrow H^*(V,\boldsymbol{Z})$ which satisfy the following: (i) $T_* = \sum_{q=0}^{2n} T_q$ is an automorphism of $H^*(V,\boldsymbol{Z}) = \sum_{q=0}^{2n} H^q(V,\boldsymbol{Z})$ commuting with $\boldsymbol{A}$; and (ii) $T_2(\omega) = \omega$. The set of all such T_* forms a graded group $\boldsymbol{\Lambda}_* = \sum_{q=0}^{2n} \boldsymbol{\Lambda}_q$, which we call the *algebraic automorphism group* of $H^*(V,\boldsymbol{Z})$.

Now a homeomorphism $f\colon V \rightarrow V$ with $f^*\omega = \omega$ induces $T(f)_* \in \boldsymbol{\Lambda}_*$. In this way we get a graded subgroup $\Lambda_* \subset \boldsymbol{\Lambda}_*$; Λ_* is the *geometric automorphism group* of $H^*(V,\boldsymbol{Z})$. We are unable to find much information in the literature on the position of Λ_* in $\boldsymbol{\Lambda}_*$. If V is a curve or abelian variety, then $\Lambda = \boldsymbol{\Lambda}_*$. Results of C. T. C. Wall (Journal London Math. Soc., 39 (1964), 131-140) indicate that, for simply-connected algebraic surfaces, Λ_*

is essentially $\boldsymbol{\Lambda}_*$. The reason for considering both Λ_* and $\boldsymbol{\Lambda}_*$ is that Λ_* contains the geometric information, whereas $\boldsymbol{\Lambda}_*$ is defined algebraically.

Let now $T_* = \sum_{q=0}^{2n} T_q \in \boldsymbol{\Lambda}_*$. Then $T_q \colon H^q(V)_0 \rightarrow H^q(V)_0$ and T_q preserves the Lefschetz decomposition (1.9) and the inner products (1.7); obviously T_q need not preserve the Hodge decomposition (1.10). It follows that, if D $(= D_q)$ is the period matrix space for $H^q(V)_0$ (cf. 1.(d)), then Λ_* and $\boldsymbol{\Lambda}_*$ induce subgroups Λ and $\boldsymbol{\Lambda}$ of G, where G is the transitive group acting on D (cf. Theorem (1.26)). Since the isotropy group $H \subset G$ is compact, Λ and $\boldsymbol{\Lambda}$ are both discrete subgroups of G which act properly discontinuously on D. Thus D/Λ and $D/\boldsymbol{\Lambda}$ are both analytic spaces ([8]) and there is a mapping $D/\Lambda \rightarrow D/\boldsymbol{\Lambda}$. We have:

(2.1) PROPOSITION. *Given the topological manifold V and the line bundle $\boldsymbol{L} \rightarrow V$, we may construct:* (i) *the class $\omega \in H^2(V, \boldsymbol{Z})$*; (ii) *the primitive cohomology spaces and the quadratic forms* (1.7); (iii) *the period matrix space D*; *and* (iv) *the discrete groups Λ and $\boldsymbol{\Lambda}$.*

Given a polarized algebraic structure on $(V, \boldsymbol{L})$, there are defined points $\Phi_\Lambda(V) \in D/\Lambda$ and $\Phi_{\boldsymbol{\Lambda}}(V) \in D/\boldsymbol{\Lambda}$.

(b) It is traditional in moduli questions to have an *arithmetic subgroup* $\Gamma \subset G$ and to let $M = D/\Gamma$ be the *modular variety.* This is primarily for the purpose of constructing automorphic forms, compactifying M, etc. (cf. the Borel-Baily article in Proc. Sym. on Pure Math., Vol. IX, Amer. Math Soc. (1966), 281-296). It seems as though $\boldsymbol{\Lambda}$ need *not* be an arithmetic subgroup of G; the reason is essentially that G does not take into account the cohomology operations on $H^*(V, \boldsymbol{Z})$. Let us give some examples:

(i) If V is a compact Riemann surface of genus $q \geqq 1$, then V is automatically polarized. The period matrix space D is analytically equivalent to the Siegel upper half-space $\boldsymbol{H}_q = \{q \times q$ matrices Z such that ${}^tZ = Z$, $\operatorname{Im} Z > 0\}$. The group G is $Sp(q, \boldsymbol{R})$ acting by $T \cdot Z = (AZ + B)(CZ + D)^{-1}$ $T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. For our purposes, we need only consider the *Siegel modular group* $\Gamma \subset G$ of integral symplectic matrices $(\Lambda = \boldsymbol{\Lambda} = \Gamma)$.

The same is true if V is a normally polarized abelian variety and D is the period matrix space for the 1-forms.

(ii) As as second example, let $(V, \boldsymbol{L})$ be a polarized, simply-connected algebraic surface and D the domain of period matrices for the holomorphic 2-forms on V. Then, if $h = h^{2,0}(V)$, $k = h^{1,1}(V_0)$ (so that $b = 2h + k$ where $b = \dim H^2(V)_0$),

$$(2.2) \qquad D \cong U(h) \times J(k) \backslash O(2h, k), \quad \dim D = \frac{h^2 - h}{2} + kh.$$

The group $\mathbf{\Lambda}$ is described as follows: Let $\phi^1, \cdots, \phi^{b+1}$ be an integral basis for $H^2(V, \boldsymbol{Z})$ (there is no torsion) and write $\omega = \sum_{j=1}^{b+1} \xi_j \phi^j$ $(\xi_j \in \boldsymbol{Z})$. Let $q_{ij} = Q(\phi^i, \phi^j) = \langle \phi^i \cup \phi^j, V \rangle$ and denote by $\mathbf{\Lambda}_*$ the matrices $\tilde{T} = (t_{ij})$ such that $\tilde{T}\omega = \omega$, ${}^t\tilde{T}Q\tilde{T} = Q$. If $\tilde{T} \in \mathbf{\Lambda}_*$, then $\tilde{T}$ induces a linear transformation T on $H^2(V)_0 = \{\phi \in H^2(V, \boldsymbol{C}) \mid Q(\phi, \omega) = 0\}$. The set of all such T gives $\mathbf{\Lambda}$, and $\mathbf{\Lambda}$ is an arithmetic subgroup of G.

(iii) Let $(V, \mathbf{L})$ be a normally polarized abelian variety and D the period matrix space for the holomorphic 2-forms. Then

$$H^2(V)_0 \subset H^1(V, \boldsymbol{C}) \wedge H^1(V, \boldsymbol{C})$$

and

$$H^{2,0}(V) = H^{1,0}(V) \wedge H^{1,0}(V).$$

If $T_2 \in \mathbf{\Lambda}_2$, then $T_2 = T_1 \wedge T_1$ where $T_1 \in \mathbf{\Lambda}_1$. Letting $\tilde{\Gamma}$ be all $\tilde{S}\colon H^2(V, \boldsymbol{Z}) \to H^2(V, \boldsymbol{Z})$ with $\tilde{S}(\omega) = \omega$, then $\tilde{\Gamma}$ induces an arithmetic subgroup $\Gamma \subset G$ (cf. example (ii) above) and $\mathbf{\Lambda}_2 \subset \Gamma$. However, $\mathbf{\Lambda}_2$ is much smaller than Γ and is not an arithmetic subgroup.

The above examples illustrate two points: First, $\mathbf{\Lambda}$ is always contained in an arithmetic subgroup Γ. Secondly, in case $\mathbf{\Lambda}$ is not an arithmetic subgroup, both G and D are too large; i.e., the operations in $H^*(V, \boldsymbol{Z})$ should be built into the definition of D. We shall not do this here but shall assume that, on some grounds, an arithmetic subgroup $\Gamma \subset G$ with $\mathbf{\Lambda} \subset \Gamma$ has been selected and we let $M = D/\Gamma$ be the *modular variety.*

(2.1 bis) PROPOSITION. *M is an analytic space with finite invariant volume, and a polarized algebraic structure on $(V, \boldsymbol{L})$ defines a unique point $\Phi(V) \in M$.*

(c) Suppose now that we have two polarzed algebraic manifolds $(V, \boldsymbol{L})$ and $(V', \boldsymbol{L}')$, of the same type, together with a *polarization-preserving homeomorphism:*

$$(2.3) \qquad f\colon V \to V', \quad f^*(\boldsymbol{L}') = \boldsymbol{L}.$$

Letting $\omega \in H^2(V, \boldsymbol{Z})$ and $\omega' \in H^2(V', \boldsymbol{Z})$ be the characteristic classes of $\boldsymbol{L}$, $\boldsymbol{L}'$, it follows that $f^*(\omega') = \omega$. We note that the *graph* of f defines a homology class

$$F \in H_{2n}(V \times V', \boldsymbol{Z}) \qquad (\dim V = n).$$

We let $M_\Lambda = D_1/\Lambda_1 \times \cdot\cdot\cdot \times D_n/\Lambda_n$ where D_q is the period matrix space for the primitive q-forms on V and $\Lambda = \sum_{q=1}^{n} \Lambda_q$ is the geometric automorphism group; M'_Λ for $(V', \boldsymbol{L}')$ is constructed similarly. From Proposition (2.1), it is clear that we have an isomorphism $f^*: M_\Lambda \rightarrow M_\Lambda$.

(2.4) THEOREM. $\Phi_\Lambda(V) = \Phi_\Lambda(V')$ *if, and only if, the graph* $F \in H_{2n}(V \times V', \boldsymbol{Z})$ *of* f *is of type* (n, n).

Proof. It is clear that $\Phi_\Lambda(V) = \Phi_\Lambda(V')$ if, and only if, with appropriate choices of cohomology bases the induced mapping

$$f^*: H^q(V', \boldsymbol{C})_0 \rightarrow H^q(V, \boldsymbol{C})_0 \tag{2.5}$$

preserves the Hodge decompositions (1.10). Thus we have to show that f^* in (2.5) preserves the Hodge decompositions if, and only if, $F \in H_{2n}(V \times V', \boldsymbol{Z})$ is of type (n, n).

By the general Künneth formula,

$$H^*(V, \boldsymbol{C}) \otimes H^*(V', \boldsymbol{C}) \cong H^*(V \times V', \boldsymbol{C}).$$

Furthermore, if $\phi \in H^{n-q}(V, \boldsymbol{C})$, $\psi \in H^q(V', \boldsymbol{C})$, then

$$\langle \phi \otimes \psi, F \rangle = \int_V \phi \wedge f^*(\psi). \tag{2.6}$$

Let us prove (2.6). Consider the mapping $V \times V \xrightarrow{g} V \times V'$ where $g(x, y) = (x, f(y))$; i.e. $g = i \times f$. Then $F = g_* \Delta$ where $\Delta \in H_{2n}(V \times V, \boldsymbol{Z})$ is the diagonal; and so

$$\langle \phi \otimes \psi, F \rangle = \langle \phi \otimes \psi, g_* \Delta \rangle = \langle g^*(\phi \otimes \psi), \Delta \rangle = \langle \phi \otimes f^* \psi, \Delta \rangle = \int_V \phi \wedge f^* \psi,$$

where the last step uses well-known properties of the cup product.

For Kähler manifolds, we have the Künneth relation:

$$H^{p,q}(V \times V') = \sum_{r,s} H^{p-r,q-s}(V) \otimes H^{r,s}(V'), \tag{2.7}$$

where we notationally agree that $H^{\rho,\sigma} = 0$ for $\rho < 0$ or $\rho > n$. Now $F \in H_{2n}(V \times V', \boldsymbol{Z})$ is of type (n, n) if, and only if,

$$\langle H^{2n-q,q}(V \times V'), F \rangle = 0 \qquad \text{for } q \neq n. \tag{2.8}$$

By (2.7), (2.8) is the same as

$$\langle H^{2n-q-r,q-s}(V) \otimes H^{r,s}(V'), F \rangle = 0 \qquad \text{for } q \neq n. \tag{2.9}$$

From (2.6) and (2.9), we see that F is of type (n, n) if, and only if,

$$(2.10)\qquad \int_V [H^{2n-q-r,q-s}(V)]\wedge f^*[H^{r,s}(V')]=0 \qquad \text{for } q\neq n \text{ and all } r,s.$$

In (2.10) there are three cases:

(i) $r+s>n$,

(ii) $r+s=n$,

(iii) $r+s<n$.

Clearly (i) and (iii) are symmetric. In case (i), $H^{r,s}(V')=\omega'^{r+s-n}H^{n-s,n-r}(V')$ so that (2.10) becomes:

$$(2.11)\qquad \int_V \omega^{r+s-n}[H^{2n-q-r,q-s}(V)]\wedge f^*[H^{n-s,n-r}(V')]=0 \qquad \text{for } q\neq n.$$

Using the decomposition (1.12), (2.11) becomes:

$$(2.12)\qquad \sum_{\mu,\nu}\int_V \omega^{r+s-n}H^{2n-q-r,}{}_{\mu}{}^{q-s}(V)\wedge f^*H^{n-s,}{}_{\nu}{}^{n-r}(V')=0 \qquad \text{for } q\neq n.$$

Now $f^*H_\nu{}^p(V')=H_\nu{}^p(V)$ while we have

$$\int_V \omega^{n-p}H_\mu{}^p(V)\wedge H_\nu{}^p(V)=0 \qquad \text{for } \mu\neq\nu$$

(cf. Hodge [10], page 183). Thus (2.12) becomes:

$$(2.12)\qquad \int_V \omega^{r+s-n}H^{2n-q-r,}{}_{\mu}{}^{q-s}(V)\wedge f^*H^{n-s,}{}_{\mu}{}^{n-r}(V')=0 \qquad \text{for } q\neq n.$$

Since

$$H^{2n-q-r,}{}_{\mu}{}^{q-s}(V)=\omega^{\mu}H^{2n-q-r-\mu,q-s-\mu}(V)_0$$

and

$$f^*H^{n-s,}{}_{\mu}{}^{n-r}(V')=\omega^{\mu}f^*H^{n-s-\mu,n-r-\mu}(V')_0$$

(2.13) may be written as:

$$(2.14)\qquad \int_V \omega^{r+s+2\mu}H^{2n-q-r-\mu,q-s-\mu}(V)_0\wedge f^*H^{n-s-\mu,n-r-\mu}(V')_0=0$$

for $q\neq n$. Finally, (2.14) may be written:

$$(2.15)\qquad Q(H^{2n-q-r-\mu,q-s-\mu}(V)_0, f^*H^{n-s-\mu,n-r-\mu}(V')_0)=0 \qquad \text{for } q\neq n,$$

where Q is the inner product (1.7). By changing indices, (2.15) gives

$$(2.16)\qquad Q(H^{n-r-t,n+t-s}(V)_0, f^*H^{n-s,n-r}(V')_0)=0 \qquad \text{for } t\neq 0.$$

Now (2.16) makes sense in case (ii) and a similar argument to (2.11)-

(2.16) shows that:

$$(2.17)\qquad \begin{cases}\text{The homology class } F \text{ is of type } (n,n) \text{ if, and}\\ \text{only if, (2.16) holds for all } r,s \text{ with } r+s\geqq n.\end{cases}$$

The subspace $S\subset H^{2n-(r+s)}(V)_0$ defined by $Q(H^{n-r-t,n+t-s}(V)_0,S)=0$ for all $t\neq 0$ is precisely $H^{n-s,n-r}(V)_0$, so that we conclude:

$$(2.18)\qquad \begin{cases}\text{The equation (2.16) holds if, and only if,}\\ f^*H^{n-s,n-r}(V')_0=H^{n-s,n-r}(V)_0 \text{ for } r+s\geqq n.\end{cases}$$

Combining (2.17) and (2.18) gives our theorem.

(d) As in section (c) above we let $f\colon(V,\boldsymbol{L})\to(V',\boldsymbol{L}')$ be a homeomorphism of polarized pairs. For simplicity, we assume that $\Lambda=\boldsymbol{\Lambda}=\Gamma$ and $\Lambda'=\boldsymbol{\Lambda}'=\Gamma'$. If $\Phi(V)=\Phi(V')$, and if a suitable version of the Hodge conjecture holds, then from Theorem (2.4) we find that there will exist an algebraic cycle $T\subset V\times V'$ such that $T\sim F$, F being the graph of f. If T is *effective*, i.e., if $T=\sum_{j=1}^{l}n_jW_n$ where $n_j>0$ and W_j are algebraic varieties, then T will induce a *birational correspondence* $T\colon V\to V'$.

The one case in which the Hodge conjecture holds is when V is a curve so that $V\times V'$ is a surface Z. We choose a system $\gamma_1,\cdots,\gamma_q;\gamma_{q+1},\cdots,\gamma_{2q}$ of retrosections for V and set $\gamma'_\alpha=f_*(\gamma_\alpha)$. Then the homology class F of the graph of F is:

$$(2.19)\qquad F=V+V'+\sum_{\alpha=1}^{q}\gamma_\alpha\times\gamma'_{q+\alpha}-\sum_{\alpha=1}^{q}\gamma_{q+\alpha}\times\gamma'_\alpha.$$

We note that the intersection number:

$$(2.20)\qquad (F\cdot F)=2-2q.$$

In this case, it is immediate that F is of type (1,1) if the period matrices $Z=(z_{\alpha\beta})$ and $Z'=(z'_{\alpha\beta})$ of V and V' are equal: Letting ω^α, ω'^β be holomorphic 1-forms on V, V' such that $\int_{\gamma_{q+\beta}}\omega^\alpha=z_{\alpha\beta}=z'_{\alpha\beta}=\int_{\gamma'_{q+\beta}}\omega'^\beta$, the holomorphic 2-forms $\omega^\alpha\wedge\omega'^\beta$ give a basis for $H^{2,0}(V\times V')$. But

$$\int_F\omega^\alpha\wedge\omega'^\beta=z'_{\beta\alpha}-z_{\alpha\beta}=0.$$

Now there will be a line bundle $\boldsymbol{E}\to Z=V\times V'$ with characteristic class F; i.e. the *Poincaré dual* of $c_1(\boldsymbol{E})$ is F. From (2.20), we get $c_1(\boldsymbol{E})\cdot c_1(\boldsymbol{E})=2-2q$. The line bundle $\boldsymbol{E}\to Z$ is not unique but may be written as

$$(2.21)\qquad \boldsymbol{E}(\boldsymbol{\lambda},\boldsymbol{\lambda}')=\boldsymbol{E}\otimes\boldsymbol{E}_{\boldsymbol{\lambda}}\otimes\boldsymbol{E}_{\boldsymbol{\lambda}'}$$

where $\boldsymbol{E}_{\boldsymbol{\lambda}}$ is a line bundle in $H^1(V, \mathcal{O})/H^1(V, \boldsymbol{Z})$ = *Picard variety of* V, and similarly for $\boldsymbol{E}_{\boldsymbol{\lambda}'}$. Obviously we have:

(2.22) PROPOSITION. *The existence of a holmorphic cross section of* $\boldsymbol{E}(\boldsymbol{\lambda}, \boldsymbol{\lambda}')$ *for some* $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}'$ *is equivalent to finding a birational correspondence* $T\colon V \to V'$.

By the Torelli theorem (cf. [13]), a birational correspondence exists. Thus, for some $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}'$, $H^0(Z, \mathcal{O}(\boldsymbol{E}(\boldsymbol{\lambda}, \boldsymbol{\lambda}'))) \neq 0$. It is not hard to see that $\dim H^0(Z, \mathcal{O}(\boldsymbol{E}(\boldsymbol{\lambda}, \boldsymbol{\lambda}'))) = 1$ and $H^0(\mathcal{O}(\boldsymbol{E}(\boldsymbol{\lambda}, \boldsymbol{\lambda}'))) = 0$ for general points $\boldsymbol{\lambda}, \boldsymbol{\lambda}'$.

Conversely, a direct proof that $H^0(\mathcal{O}(\boldsymbol{E}(\boldsymbol{\lambda}, \boldsymbol{\lambda}'))) \neq 0$ for some $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}'$ would prove the Torelli theorem.

I. 3. Some properties of the modular varieties; complex torii associated to the cohomology of an algebraic manifold. (a) Let V be a polarized algebraic manifold, $q = 2m + 1$ an odd integer, and D_q the period matrix space for the primitive q-forms on V (cf. I. 1(d)). If

$$2b = \dim H^{2m+1}(V, \boldsymbol{C})_0,$$

and if $h_0 = h^{2m+1,0}, h_1 = h_0^{2m,1}, \cdots, h_m = h_0^{2m+1,m}$, then $b = h_0 + \cdots + h_m$ and

$$D_q = U(h_0) \times \cdots \times U(h_m) \backslash Sp(b) = H \backslash G. \tag{3.1}$$

For $m = 0$, $D_1 \cong U(b) \backslash Sp(b) = \boldsymbol{H}_b$ is a *Siegel upper-half-space* of genus b; we may assume that $Q = \begin{pmatrix} -\Delta & 0 \\ 0 & \Delta \end{pmatrix}$ where $\Delta = \begin{pmatrix} \delta_1 & & 0 \\ & \ddots & \\ 0 & & \delta_b \end{pmatrix}$, $1 = \delta_1 \mid \delta_2 \mid \cdots \mid \delta_b$ and so $\Gamma = \Gamma_\Delta$ is a *paramodular group*. In this case the point $\Phi(V) \in M_1 = D_1/\Gamma$ has the following intepretation: We consider the period matrix Ω for the holomorphic 1-forms on V. Thus we have a basis $\omega^1, \cdots, \omega^b$ for $H^{1,0}(V)$ and free generators $\gamma_1, \cdots, \gamma_{2b}$ for $H_1(V, \boldsymbol{Z})$, and we let $\pi_{\alpha\rho} = \int_{\gamma^\rho} \omega^\alpha$ so that $\Omega = (\pi_{\alpha\rho})$. Then the columns of Ω generate a lattice Λ in $\boldsymbol{C}^b$, and $A_1(V)^* = \boldsymbol{C}^b/\Lambda$ is a complex torus, called the *Albanese variety* of V. The relations (1.13), (1.14) become:

$$\begin{cases} \Omega Q {}^t\Omega = 0 \\ i\Omega Q {}^t\bar{\Omega} > 0, \end{cases} \tag{3.2}$$

where $Q = \begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$ is a skew symmetric integral matrix. We may write $\Omega \sim (\Delta, Z)$ where, by (3.2), $Z \in \boldsymbol{H}_b$. The mapping: $A_1(V)^* \to Z \in \boldsymbol{H}_b/\Gamma_\Delta$ is well-defined and we have:

(3.3) PROPOSITION. *The point* $\Phi(V) \in \boldsymbol{H}_b/\Gamma_\Delta$ *is the period matrix of the Albanese variety* $A_1(V)^*$. (*The* $*$ *is used because* $A_1(V)^*$ *is dual to the torus in which we shall be mainly interested.*)

(i) $A_1(V)^*$ *is a complex torus which depends holomorphically on* V;

(ii) *a polarization on* V *induces one on* $A_1(V)^*$;

(iii) $A_1(V)^*$ *is functorial; given a holomorphic mapping* $f\colon V \to V'$, *there is induced a holomorphic homomorphism* $f_*\colon A_1(V)^* \to A_1(V')^*$;

(iv) $\Phi(V) = \Phi(A_1(V)^*) \in M_1 = \boldsymbol{H}_b/\Gamma_\Delta$; *and*

(v) *the mapping* $\psi\colon V \to A_1(V)^*$ *given by*

$$\psi(z) = \begin{pmatrix} \int_{z_0}^{z} \omega^1 \\ \vdots \\ \int_{z_0}^{z} \omega^b \end{pmatrix}$$

is holomorphic and is universal, up to translations.

Now property (v) is special for 1-forms, but we may ask, for general m, if the real torus $T_{2m+1}(V) = H^{2m+1}(V, \boldsymbol{R})/H^{2m+1}(V, \boldsymbol{Z})$ can be given a complex structure such that (i)-(iv) are satisfied? The *intermediate Jacobians* $A_{2m+1}(V)$ of Weil [14] are not entirely satisfactory; $A_{2m+1}(V)$ does not depend holomorphically on V and (iv) is not satisfied. By suitably interpreting (ii), we shall give an affirmative answer to this question; in Part II, we shall also generalize (v) and give the precise relation of our torii to the $A_{2m+1}(V)$.

Remark. Actually, rather than generalizing the Albanese variety above, we shall generalize the construction of the dual torus, which is the Picard variety $A_1(V)$. The conditions (i)-(iv) above will be the same, with the arrow being reversed in (iii).

To construct $A_1(V)$ we let $W = H^1(V, \boldsymbol{C})$ and $S \subset W$ be the subspace $H^{1,0}(V)$. Then $S \cap \bar{S} = 0$, and so the lattice $H^1(V, \boldsymbol{Z}) \subset H^1(V, \boldsymbol{R})$ projects onto a lattice $\Lambda \subset W_S = W/S$. Thus $A_1(V) = W_S/\Lambda$ is a complex torus, and it is this construction we shall generalize.

We remark that, in the usual notation [9], $W_S \cong H^1(V, \mathcal{O})$ and the mapping $H^1(V, \boldsymbol{Z}) \to H^1(V, \mathcal{O})$ is the cohomology mapping arising from the sheaf sequence $0 \to \boldsymbol{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0$.

(b) As an example, suppose that $\dim V = 3$, that $H^3(V, \boldsymbol{C}) = H^3(V, \boldsymbol{C})_0$, and let $W = H^3(V, \boldsymbol{C})$. Define a bilinear form Q on W by:

$$Q(\phi, \psi) = -\left(\frac{\sqrt{-1}}{2}\right)^3 \int_V \phi \wedge \psi; \text{ for } \phi, \psi \in H^3(V, \boldsymbol{C}). \tag{3.4}$$

If $S_1 = H^{3,0}(V)$ and $S_2 = H^{3,0}(V) + H^{2,1}(V)$, then:

$$\begin{cases} Q(S_2, S_2) = 0 \\ Q(S_1, \bar{S}_1) > 0 \\ Q(S_2/S_1, \bar{S}_2/\bar{S}_1) < 0. \end{cases} \tag{3.5}$$

(The change in sign in (3.5) arises from $\omega^1\omega^2\omega^3\overline{\omega}^1\overline{\omega}^2\overline{\omega}^3 = -\omega^1\overline{\omega}^1\omega^2\overline{\omega}^2\omega^3\overline{\omega}^3$ whereas $\omega^1\omega^2\overline{\omega}^3\overline{\omega}^1\overline{\omega}^2\omega^3 = +\omega^1\overline{\omega}^1\omega^2\overline{\omega}^2\omega^3\overline{\omega}^3$.)

We now let $\boldsymbol{D}$ be the flags $[S_1, S_2]$ for which (3.5) is satisfied. Let $\gamma^1, \cdots, \gamma^{2n}$ be an integral basis for $H^3(V, \boldsymbol{Z})$. For each $\Omega \in \boldsymbol{D}$ we may define a complex torus $T(\Omega)$ by:

$$T(\Omega) = \{W/S_2\} \text{ modulo } (\gamma^1, \cdots, \gamma^{2n})_Z, \tag{3.6}$$

where $(\hat{\gamma}^1, \cdots, \hat{\gamma}^{2n})_Z$ is the lattice generated by $\gamma^1, \cdots, \gamma^{2n}$.

In case $\Omega = \Omega(V)$ is the point corresponding to V, we let $T(\Omega) = T_3(V)$. There is an obvious isomorphism $D_3 \cong \boldsymbol{D}$, and the point $\Omega(V)$ corresponding to V is $\Phi_3(V)$.

Corresponding to V there is another torus $A_3(V)$ where we set $R = H^{2,1} + H^{0,3}$ and:

$$A_3(V) = \{W/R\} \text{ modulo } (\gamma^1, \cdots, \gamma^{2n})_Z. \tag{3.7}$$

This $A_3(V)$ is *Weil's intermediate Jacobian*; observe that $Q(R, \bar{R}) > 0$, which is reason for the polarization on $A_3(V)$. It is clear that $T_3(V)$ and $A_3(V)$ are different complex torii unless $H^{3,0} = 0$. If, e.g., V is a *cubic threefold* ($h^{3,0} = 0$, $h^{2,1} = 5$), then they coincide.

(3.8) PROPOSITION. *$A_3(V)$ does not, in general, depend holomorphically on V, whereas $T_3(V)$ does depend holomorphically on V.*

Remark. Clearly $T_3(V) = T_3(\Omega(V))$ depends holomorphically on $\Omega \in \boldsymbol{D}$, and we shall prove in Part II, Section I(a) that $\Omega(V) = \Phi_3(V)$ depends holomorphically on V.

We now give an example where $T(\Omega)$ depends holomorphically on Ω but $A(\Omega)$ does not; this example was suggested by Mattuck.

Let C_1, C_2, C_3 be elliptic curves with holomorphic differentials ω^1, ω^2, ω^3. We suppose that, on C_j, we have cycles γ_{j1}, γ_{j2} with $(\gamma_{j1}, \gamma_{j2}) = +1$, $\int_{\gamma_{j1}} \omega^j = 1$, $\int_{\gamma_{j2}} \omega^j = \tau_j$. Thus $\operatorname{Im} \tau_j > 0$ and C_j has period matrix $(1, \tau_j)$. We let $V = C_1 \times C_2 \times C_3$ and $\omega_{123} = \omega^1\omega^2\omega^3$, $\omega_{12\bar{3}} = \omega^1\omega^2\bar{\omega}^3$, etc., $\phi_j = \omega^j\bar{\omega}^j$. Then $\omega = \phi_1 + \phi_2 + \phi_3$ is a Kähler metric on V, and a basis for the primitive forms in $H^{3,0} + H_0{}^{2,1}$ is:

$$\omega_{123}, \omega_{12\bar{3}}, \omega_{1\bar{2}3}, \omega_{\bar{1}23}, \omega_1(\phi_2 - \phi_3), \omega_2(\phi_1 - \phi_3), \omega_3(\phi_1 - \phi_2).$$

The first four of these differentials form a *reducible set of integrals* (cf. Hodge [10], page 201). We shall compute the period matrix $\hat{\Omega}$ of $\omega_{123}, \omega_{12\bar{3}}, \omega_{1\bar{2}3}, \omega_{\bar{1}23}$ on the cycles

$$\sigma_1 = \gamma_{11} \times \gamma_{21} \times \gamma_{31}, \quad \sigma_2 = \gamma_{11} \times \gamma_{21} \times \gamma_{32}, \quad \sigma_3 = \gamma_{11} \times \gamma_{22} \times \gamma_{31},$$
$$\sigma_4 = \gamma_{11} \times \gamma_{22} \times \gamma_{32}, \quad \sigma_5 = \gamma_{12} \times \gamma_{21} \times \gamma_{31}, \quad \sigma_6 = \gamma_{12} \times \gamma_{21} \times \gamma_{32},$$
$$\sigma_7 = \gamma_{12} \times \gamma_{22} \times \gamma_{31}, \quad \sigma_8 = \gamma_{12} \times \gamma_{22} \times \gamma_{32}.$$

The period matrix of Ω will be a sum of $\hat{\Omega}$ plus a less interesting matrix. We have:

$$\hat{\Omega} = \begin{bmatrix} 1 & \tau_1 & \tau_2 & \tau_3 & \tau_1\tau_2 & \tau_1\tau_3 & \tau_2\tau_3 & \tau_1\tau_2\tau_3 \\ 1 & \bar{\tau}_1 & \tau_2 & \tau_3 & \bar{\tau}_1\tau_2 & \bar{\tau}_1\tau_3 & \tau_2\tau_3 & \bar{\tau}_1\tau_2\tau_3 \\ 1 & \tau_1 & \bar{\tau}_2 & \tau_3 & \tau_1\bar{\tau}_2 & \tau_1\tau_3 & \bar{\tau}_2\tau_3 & \tau_1\bar{\tau}_2\tau_3 \\ 1 & \tau_1 & \tau_2 & \bar{\tau}_3 & \tau_1\tau_2 & \tau_1\bar{\tau}_3 & \tau_2\tau_3 & \tau_1\tau_2\bar{\tau}_3 \end{bmatrix}.$$

Subtracting the second row of $\hat{\Omega}$ from the first and dividing by $(\tau_1 - \bar{\tau}_2)$ gives:

$$\hat{\Omega} \sim \begin{bmatrix} 0 & 1 & 0 & 0 & \tau_2 & \tau_3 & 0 & \tau_2\tau_3 \\ 1 & \bar{\tau}_1 & \tau_2 & \tau_3 & \bar{\tau}_1\tau_2 & \bar{\tau}_1\tau_3 & \tau_2\tau_3 & \bar{\tau}_1\tau_2\tau_3 \\ 1 & \tau_1 & \bar{\tau}_2 & \tau_3 & \tau_1\bar{\tau}_2 & \tau_1\tau_3 & \bar{\tau}_2\tau_3 & \tau_1\bar{\tau}_2\tau_3 \\ 1 & \tau_1 & \tau_2 & \bar{\tau}_3 & \tau_1\tau_2 & \tau_1\bar{\tau}_3 & \tau_2\bar{\tau}_3 & \tau_1\tau_2\bar{\tau}_3 \end{bmatrix}.$$

Subtracting $\bar{\tau}_1$ times the first row from the second gives:

$$\hat{\Omega} \sim \begin{bmatrix} 0 & 1 & 0 & 0 & \tau_2 & \tau_3 & 0 & \tau_2\tau_3 \\ 0 & 1 & \tau_2 & \tau_3 & 0 & 0 & \tau_2\tau_3 & 0 \\ 1 & \tau_1 & \bar{\tau}_2 & \tau_3 & \tau_1\bar{\tau}_2 & \tau_1\tau_3 & \bar{\tau}_2\tau_3 & \tau_1\bar{\tau}_2\tau_3 \\ 1 & \tau_1 & \tau_2 & \bar{\tau}_3 & \tau_1\tau_1 & \tau_1\bar{\tau}_3 & \tau_2\bar{\tau}_3 & \tau_1\tau_2\bar{\tau}_3 \end{bmatrix}.$$

Similar reductions lead to the equivalent matrix:

$$\hat{\Omega} \sim \begin{bmatrix} 0 & 1 & 0 & 0 & \tau_2 & \tau_3 & 0 & \tau_2\tau_3 \\ 1 & 0 & \tau_2 & \tau_3 & 0 & 0 & \tau_2\tau_3 & 0 \\ 0 & 0 & 1 & 0 & \tau_1 & 0 & \tau_3 & \tau_1\tau_3 \\ 0 & 0 & 0 & 1 & 0 & \tau_1 & \tau_2 & \tau_1\tau_2 \end{bmatrix}.$$

Letting $T(\hat{\Omega})$ be the torus whose period matrix is $\hat{\Omega}$, it is clear that $T(\hat{\Omega})$ (and, by an easier computation, $T(\Omega)$) depends holomorphically on τ_1, τ_2, τ_3.

A similar computation shows that $A(\Omega)$ does *not* depend holomorphically on τ_1, τ_2, τ_3: The period matrix of ω_{123}, $\omega_{1\bar{2}3}$, $\omega_{\bar{1}2\bar{3}}$, $\omega_{\bar{1}\bar{2}3}$ over the above eight cycles is:

$$\hat{\Omega}_1 = \begin{bmatrix} 1 & \tau_1 & \tau_2 & \tau_3 & \tau_1\tau_2 & \tau_1\tau_3 & \tau_2\tau_3 & \tau_1\tau_2\tau_3 \\ 1 & \tau_1 & \bar{\tau}_2 & \bar{\tau}_3 & \tau_1\bar{\tau}_2 & \tau_1\bar{\tau}_3 & \bar{\tau}_2\bar{\tau}_3 & \tau_1\bar{\tau}_2\bar{\tau}_3 \\ 1 & \bar{\tau}_1 & \tau_2 & \bar{\tau}_3 & \bar{\tau}_1\tau_2 & \bar{\tau}_1\bar{\tau}_3 & \tau_2\bar{\tau}_3 & \bar{\tau}_1\tau_2\bar{\tau}_3 \\ 1 & \bar{\tau}_1 & \bar{\tau}_2 & \tau_3 & \bar{\tau}_1\bar{\tau}_2 & \bar{\tau}_1\tau_3 & \bar{\tau}_2\tau_3 & \bar{\tau}_1\bar{\tau}_2\tau_3 \end{bmatrix}.$$

Now:

$$\pi_1 = \begin{vmatrix} 1 & \tau_1 & \tau_2 & \tau_3 \\ 1 & \tau_1 & \bar{\tau}_2 & \bar{\tau}_3 \\ 1 & \bar{\tau}_1 & \tau_2 & \bar{\tau}_3 \\ 1 & \bar{\tau}_1 & \bar{\tau}_2 & \tau_3 \end{vmatrix} = (\tau_1 - \bar{\tau}_1)(\tau_2 - \bar{\tau}_2)(\tau_3 - \bar{\tau}_3);$$

$$\pi_2 = \begin{vmatrix} 1 & \tau_2 & \tau_3 & \tau_2\tau_3 \\ 1 & \bar{\tau}_2 & \bar{\tau}_3 & \bar{\tau}_2\bar{\tau}_3 \\ 1 & \tau_2 & \bar{\tau}_3 & \tau_2\bar{\tau}_3 \\ 1 & \bar{\tau}_2 & \tau_3 & \bar{\tau}_2\tau_3 \end{vmatrix} = (\tau_2 - \bar{\tau}_3)(\tau_2 - \bar{\tau}_2)^2.$$

Thus the two Plücker coordinates π_1, π_2 of $\hat{\Omega}_1$ have the ratio $\dfrac{(\tau_2 - \bar{\tau}_2)(\tau_3 - \bar{\tau}_3)}{(\tau_1 - \bar{\tau}_1)}$ so that $\hat{\Omega}_1$ does *not* depend holomorphically on τ_1, τ_2, τ_3.

(c) We consider again the torus $T_3(V)$, constructed above, as regards the properties (i)-(iv) listed below Proposition 3.3. By Proposition 3.8, (i) is satisfied and (iv) is also satisfied. Also, $T_3(V)$ is functorial: given $f\colon V \to V'$, $f^*\colon H^3(V',\boldsymbol{C}) \to H^3(V,\boldsymbol{C})$,

$$f^*\{H^{3,0}(V') + H^{2,1}(V')\} \subset H^{3,0}(V) + H^{2,1}(V),$$

and $f^*\{H^3(V',\boldsymbol{Z})\} \subset H^3(V,\boldsymbol{Z})$ so that we have induced $f^*\colon T_3(V') \to T_3(V)$. So what remains is the question of polarizing $T_3(V)$.

Definition. Let X be a complex manifold of dimension n. A *q-convex polarization* on X is given by a *holomorphic line bundle* $\boldsymbol{L} \to X$ which has a metric whose *characteristic class*, computed from the curvature, is $\omega = \sqrt{-1}\{\sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}}\, dz^\alpha \wedge d\bar{z}^\beta\}$ where $H = (h_{\alpha\bar{\beta}})$ is a non-singular Hermitian matrix with signature $(n-q, q)$.

Remark. If H has signature $(n-q,q)$, then we can locally find 1-forms $\omega^\alpha = \sum_{\beta=1}^{n} A_{\alpha\beta} dz^\beta$ such that:

$$\omega = \sqrt{-1}\{\sum_{\alpha=1}^{n-q} \omega^\alpha \wedge \bar{\omega}^\alpha - \sum_{\beta=n-q+1}^{n} \omega^\beta \wedge \bar{\omega}^\beta\}. \tag{3.9}$$

A *o-convex polarization* is just an ordinary polarization. By passing from $\boldsymbol{L}$ to the dual bundle $\boldsymbol{L}^*$, a q-convex polarization is equivalent to an $(n-q)$-convex polarization

(3.10) PROPOSITION. *$T_3(V)$ has a natural q-convex polarization where $q = h^{2,1}$.*

Proof. We choose an integral basis $\hat{\gamma}^1, \cdots, \hat{\gamma}^{2n}$ for $H^3(V, \boldsymbol{Z})$ and a basis $\omega^1, \cdots, \omega^n$ for $S_2 = H^{3,0}(V) + H^{2,1}(V)$ such that $\omega^1, \cdots, \omega^{n-q}$ is a basis for $H^{3,0}(V)$. Write $\omega^\alpha = \sum_{\rho=1}^{2n} \pi_{\alpha\rho} \hat{\gamma}^\rho$ so that $\Omega = (\pi_{\alpha\rho})$ is the period matrix for $\omega^1, \cdots, \omega^n$. By (3.5), we can find a rational skew-symmetric matrix Q with Q^{-1} integral and such that:

$$\Omega Q^t \Omega = 0; \tag{3.11}$$

$$\Omega Q^t \bar{\Omega} = \sqrt{-1} H_1; \tag{3.12}$$

where H_1 is an Hermitian matrix of signature $(n-q,q)$ whose first $(n-q)$ by $(n-q)$ block is positive definite.

Every vector $\xi \in \boldsymbol{C}^n$ can be written as a *real* linear combination of the column vectors $\xi_1, \cdots, \xi_{2n}$ $\left(\xi_\rho = \begin{pmatrix} \pi_{1\rho} \\ \vdots \\ \pi_{n\rho} \end{pmatrix}\right)$ of Ω, and this gives an isomormorphism $\boldsymbol{C}^n \cong \boldsymbol{R}^{2n}$ with ξ_ρ corresponding to the ρ-th coordinate vector of $\boldsymbol{R}^{2n}$. Letting $x^1, \cdots, x^{2n}$ be the real coordinates on $\boldsymbol{R}^{2n}$ and $z^1, \cdots, z^n$ the complex coordinates on $\boldsymbol{C}^n$, we have $dz^\alpha = \sum_{\rho=1}^{2n} \pi_{\alpha\rho} dx^\rho$. We remark that $dx^1, \cdots, dx^{2n}$ give a basis for $H^1(T_3(V), \boldsymbol{Z})$, where we are using the fact that $T = T_3(V)$ is $\boldsymbol{C}^n$ modulo the lattice $(\xi_1, \cdots, \xi_{2n})_Z$ generated by $\xi_1, \cdots, \xi_{2n}$. In fact we have:

$$0 \to S_2 \to W \to W/S_2 \to 0$$
$$0 \leftarrow S_2{}^* \leftarrow W^* \leftarrow (W/S_2)^* \leftarrow 0.$$

Now $T_3(V)$ is W/S_2 modulo the lattice in W, and the above torus $\boldsymbol{C}^n$ modulo $(\xi_1, \cdots, \xi_{2n})_{\boldsymbol{Z}}$ is $S_2{}^*$ modulo the dual lattice in W^*. But Poincaré duality gives a lattice-preserving isomorphism $W/S_2 \cong S_2{}^*$.

We may now write $dx^\rho = \sum_{\alpha=1}^{n} \psi_\alpha{}^\rho \, dz^\alpha + \bar{\psi}_\alpha{}^\rho \, d\bar{z}^\alpha$ where

$$\sum_{\alpha=1}^{n} (\psi_\alpha{}^\rho \pi_{\alpha\sigma} + \bar{\psi}^{\alpha\rho} \bar{\pi}_{\alpha\sigma}) = \delta_\sigma{}^\rho ;$$

i.e., if $\Psi = (\psi_\alpha{}^\rho)$,

$$(\Psi\bar{\Psi}) \binom{\Omega}{\bar{\Omega}} = I_{2n}. \tag{3.13}$$

Now let $Q^{-1} = (q_{\rho\sigma})$ with $q_{\rho\sigma}$ integral and set: $\omega = \sum_{\rho,\sigma} q_{\rho\sigma} dx^\rho \wedge dx^\sigma$.

(3.14) Lemma. *On $T = T_3(V)$, ω given is $\omega = \sqrt{-1}\,\{\sum_{\alpha,\beta} h_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta\}$, where $H = 2^t H_1{}^{-1}$ and H_1 is given by* (3.12).

Proof.

$$\begin{aligned} \omega &= \sum_{\rho,\sigma} q_{\rho\sigma} dx^\rho \wedge dx^\sigma \\ &= \sum (\psi_\alpha{}^\rho q_{\rho\sigma} \psi_\beta{}^\sigma) \, dz^\alpha \wedge dz^\beta + \sum (\bar{\psi}_\alpha{}^\rho q_{\rho\sigma} \psi_\beta{}^\sigma) \, d\bar{z}^\alpha \wedge d\bar{z}^\beta \\ &\quad + \sum (\psi_\alpha{}^\rho q_{\rho\sigma} \bar{\psi}_\beta{}^\sigma - \bar{\psi}_\beta{}^\rho q_{\rho\sigma} \psi_\alpha{}^\sigma) \, dz^\alpha \wedge d\bar{z}^\beta. \end{aligned}$$

From (3.11) and (3.12), we get:

$$\binom{\Omega}{\bar{\Omega}} Q ({}^t\Omega {}^t\bar{\Omega}) = \begin{pmatrix} 0 & \sqrt{-1}\, H_1 \\ -\sqrt{-1}\, \bar{H}_1 & 0 \end{pmatrix}. \tag{3.15}$$

Taking inverses of (3.15) and using (3.13) gives:

$$\begin{pmatrix} {}^t\Psi \\ {}^t\Psi \end{pmatrix} Q^{-1} (\Psi\bar{\Psi}) = \begin{pmatrix} 0 & \dfrac{\sqrt{-1}}{2}\, \mathrm{H} \\ \dfrac{-\sqrt{-1}}{2}\, \bar{H} & 0 \end{pmatrix}. \tag{3.16}$$

From (3.16) it follows that $\omega = \sqrt{-1} \sum_{\alpha,\beta} h_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ as desired.

Since T is a compact Kähler manifold and $\omega \in H^{1,1} \cap H^2(T, \boldsymbol{Z})$, there will exist a holomorphic line bundle $\boldsymbol{L} \to T$ whose characteristic class is ω (cf. [9]). What we have to do is find $\boldsymbol{L}$ and a metric h in $\boldsymbol{L}$ whose curvature form $\dfrac{\sqrt{-1}}{\pi} (\bar{\partial}\partial \log h) = \omega$.

We write $\boldsymbol{R}^{2n}$ as E_0 with standard basis $e_1,\cdots,e_{2n}$ and define a bilinear form $H(u,v)$ on E_0 by:

$$H(u,v)=\sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}}z^{\alpha}(u)\bar{z}^{\beta}(v),$$

where $z^{\alpha}(u)$ is real linear in u and $z^{\alpha}(e_\rho)=\pi_{\alpha\rho}$. Since $\bar{h}_{\alpha\bar{\beta}}=h_{\beta\bar{\alpha}}$, we have $H(u,v)=\overline{H(v,u)}$. We let $H(z,v)=H_v(z)=\sum h_{\alpha\bar{\beta}}z^{\alpha}\bar{z}^{\beta}(v)$ be a linear holomorphic function on $E_0\cong\boldsymbol{C}^n$ and set

$$\xi_v(z)=\boldsymbol{e}(\frac{\sqrt{-1}}{2}H_v(z)+\frac{\sqrt{-1}}{4}H(v,v))$$

where $\boldsymbol{e}(\alpha)=e^{2\pi i\alpha}$. Denote by Λ the lattice $(e_1,\cdots,e_{2n})_{\boldsymbol{Z}}$.

(3.17) LEMMA. *For $u,v\in\Lambda$, we have:*

$$\xi_{u+v}(z)=\xi_u(z+v)\xi_v(z).$$

Proof. If $i=\sqrt{-1}$, then

$$\begin{aligned}&\xi_{u+v}(z)\{\xi_u(z+v)\xi_v(z)\}^{-1}\\ &=e\{\frac{i}{2}H(z,u+v)+\frac{i}{4}H(u+v,u+v)-\frac{i}{2}H(z+v,u)\\ &\quad-\frac{i}{4}H(u,u)-\frac{i}{2}H(z,v)-\frac{i}{4}H(v,v)\}\\ &=e\{\frac{i}{4}H(v,u)+\frac{i}{4}H(u,v)-\frac{i}{2}H(v,u)\}\\ &=e\{\frac{i}{4}[H(u,v)-H(v,u)]\}.\end{aligned}$$

Thus we must show:

$$\text{(3.18)}\qquad \frac{i}{4}(H(u,v)-H(v,u))\equiv 0\quad(1).$$

Write $u=\sum_{\rho=1}^{2n}\lambda^{\rho}e_{\rho}$, $v=\sum_{\sigma=1}^{2n}\zeta^{\sigma}e_{\sigma}$ where λ^{ρ}, ζ^{σ} are integers. Then

$$H(u,v)=\sum h_{\alpha\bar{\beta}}z^{\alpha}(e_\rho)\bar{z}^{\beta}(e_\sigma)\lambda^{\rho}\zeta^{\sigma}=\sum h_{\alpha\bar{\beta}}\pi_{\alpha\rho}\bar{\pi}_{\beta\sigma}\zeta^{\rho}\lambda^{\sigma}$$

and

$$H(u,v)-H(v,u)=\sum_{\alpha,\beta}h_{\alpha\bar{\beta}}(\pi_{\alpha\rho}\bar{\pi}_{\beta\sigma}-\pi_{\alpha\sigma}\bar{\pi}_{\beta\rho})\lambda^{\rho}\zeta^{\sigma}.$$

Now

$$\begin{aligned}&\sum_{\rho,\sigma}q_{\rho\sigma}dx^{\rho}\wedge dx^{\sigma}=\omega=\omega_{1,1}\\ &=\sqrt{-1}\{\sum h_{\alpha\bar{\beta}}dz^{\alpha}\wedge d\bar{z}^{\beta}\}=\sqrt{-1}\{\sum h_{\alpha\bar{\beta}}(\pi_{\alpha\rho}dx^{\rho}\wedge\bar{\pi}_{\beta\sigma}dx^{\sigma})\}\end{aligned}$$

$$= \frac{\sqrt{-1}}{2}\{ \sum h_{\alpha\bar{\beta}}(\pi_{\alpha\rho}\bar{\pi}_{\beta\sigma} - \pi_{\alpha\sigma}\bar{\pi}_{\beta\rho})\, dx^{\rho} \wedge dx^{\sigma}\}.$$

Thus

$$\frac{i}{4}\{H(u,v) - H(v,u)\} = \tfrac{1}{2}\sum_{\rho,\sigma} q_{\rho\sigma}(\lambda^{\rho}\zeta^{\sigma} - \lambda^{\sigma}\zeta^{\rho}) = \sum_{\rho,\sigma} q_{\rho\sigma}\lambda^{\rho}\zeta^{\sigma} \equiv 0 \quad (1),$$

which proves (3.18).

Now we form the line bundle $\boldsymbol{L} = E_0 \times_{\Lambda} \boldsymbol{C}$ by the equivalence relation:

$$(z, \lambda) \sim (z + u, \xi_u(z)\lambda),$$

where $u \in \Lambda$. Because of Lemma (3.17), this is an equivalence relation, and $\boldsymbol{L} \rightarrow T_3(V)$ is a holomorphic line bundle.

Let now $h(z) = \boldsymbol{e}(-\frac{i}{2}H(z,z)) = e^{\pi\{H(z,z)\}}$.

(3.19) Lemma. *$h(z) = |\xi_v(z)|^2 h(z+v)$ for $v \in \Lambda$.*

Proof.

$$h(z)h(z+v)^{-1} = \boldsymbol{e}\{\frac{i}{2}H(z,z) + \frac{i}{2}H(z+v, z+v)\}$$

$$= \boldsymbol{e}\{\frac{i}{2}[H(z,v) + H(v,z)] + \frac{i}{2}H(v,v)\}.$$

On the other hand,

$$|\xi_v(z)|^2 = \boldsymbol{e}\{\frac{i}{2}H(z,v) + \frac{i}{4}H(v,v) + \frac{i}{2}H(v,z) + \frac{i}{4}H(v,v)\},$$

which proves the lemma.

If now $\phi = (z, \lambda)$ is a point in $\boldsymbol{L}$, we set $\| \phi \|^2 = h(z)|\lambda|^2$. Then $\|(z+v, \xi_v(z)\lambda)\|^2 = h(z+v)|\xi_v(z)|^2|\lambda|^2 = h(z)|\lambda|^2 = \|(z,\lambda)\|^2$ so that we have a metric in $\boldsymbol{L} \rightarrow T_3(V)$. Now

$$\frac{i}{\pi}\partial\bar{\partial}\log h(z) = {}^{i}\,\partial\bar{\partial}\, H(z,z) = {}^{i}\{\sum_{\alpha,\beta} h_{\alpha\bar{\beta}}\, dz^{\alpha} \wedge d\bar{z}^{\beta}\} = \omega,$$

which proves that $\boldsymbol{L} \rightarrow T_3(V)$ gives a q-convex polarization with $q = h^{2,1}$.

(d) Let now E_0 be a real space with basis $e_1, \cdots, e_{2n}$ and Q a skew-symmetric integral bilinear form on E_0. We let $E = E_0 \otimes_{\boldsymbol{R}} \boldsymbol{C}$ be the complexification of E_0. Consider flags $\Omega = [S_1, S_2]$ where $\dim S_1 = n - q$, $\dim S_2 = n$ and let $\boldsymbol{D}$ be the flags Ω for which (3.5) is satisfied. The proof of Proposition (3.10) shows that, for each $\Omega \in \boldsymbol{D}$, there is naturally associated a complex torus $T(\Omega)$ with q-convex polarization. Thus $\{T(\Omega)\}_{\Omega \in \boldsymbol{D}}$ is an analytic family of q-convex polarized torii.

We let $G = Sp(Q)$ be the real symplectic group of Q. Then G acts transitively on $\boldsymbol{D}$ with isotropy group $H = U(n-q) \times U(q)$. The paramodular group Γ_Δ acts properly discontinuously on $\boldsymbol{D}$ from the right, and $M = \boldsymbol{D}/\Gamma_\Delta$ is an analytic space.

(3.20) THEOREM. *Let V be a polarized algebraic manifold of dimension $n \geqq 3$, with $H^3(V,\boldsymbol{C}) = H^3(V,\boldsymbol{C})_0$, and $q = h^{2,1}$. Then:*

(i) *The torus $T_3(V) = H^3(V,\boldsymbol{R})/H^3(V,\boldsymbol{Z})$ carries a complex structure such that the polarization on V induces a q-convex polarization on $T_3(V)$;*

(ii) *$T_3(V)$, together with its polarization is functorial;*

(iii) *The period matrix space D_3 for the 3-forms on V is isomorphic to $\boldsymbol{D}$ and, under this equivalence, $T_3(V)$ is the torus corresponding to the point $\Phi(V)$ in D_3;*

(iv) *In particular, $T_3(V)$ varies holomorphically with V.*

Remark. $\boldsymbol{D} = U(n-q) \times U(q) \backslash Sp(n)$ where $U(n-q) \times U(q)$ is embedded in $Sp(n)$ as follows: On $\boldsymbol{R}^{2n}$ with real coordinates $x^1, \cdots, x^{2n}$, we let $\omega = \sum_{\alpha=1}^{n} dx^\alpha \wedge dx^{n+\alpha}$. Then $Sp(n)$ is the group of real linear transformations which preserve ω. Let now $z^\alpha = x^\alpha + ix^{n+\alpha}$ for $1 \leqq \alpha \leqq n-q$ and $z^\beta = x^\beta - ix^{n+\beta}$ for $n-q+1 \leqq \beta \leqq n$. Then $U(n-q) \times U(q)$ preserves the Herimtian forms $\sum_{\alpha=1}^{n-q} dz^\alpha d\bar{z}^\alpha$ and $\sum_{\beta=n-q+1}^{n} dz^\beta d\bar{z}^\beta$. It follows that $U(n-q) \times U(q)$ preserves ω and this gives the embedding $U(n-q) \times U(q) \subset Sp(n)$.

We note that the Siegel upper half space $\boldsymbol{H}_n \cong U(n) \backslash Sp(n)$.

(3.21) PROPOSITION. *The differentiable fibering $\boldsymbol{D} \xrightarrow{\pi} \boldsymbol{H}_n$, induced from the inclusion $U(n-q) \times U(q) \subset U(n)$, is not holomorphic if $q > 0$ and is not anti-holomorphic if $n-q > 0$.*

Remark. This is the analogue of Proposition (3.8), since the point $\pi(\Phi(V)) \in \boldsymbol{H}_n$ is the point corresponding to $A_3(V)$.

A group-theoretic proof, based upon the root structures, may be given; and this will be discussed in I.4 below. However, by giving π geometrically, the Proposition will become clear.

We can identify $\boldsymbol{H}_n$ with the n-dimensional subspaces $S \subset E$ satisfying $Q(S,S) = 0$, $\sqrt{-1}\, Q(S,\bar{S}) > 0$ (cf. (1.21) and (1.22)). Then $\pi[S_1,S_2] = (S_1 + \bar{S}_2/\bar{S}_1)$, so that π is holomorphic if $S_1 = S_2$ and anti-holomorphic if $S_1 = 0$.

To state the general theorem, which will have essentially the same proof as Theorem 3.20, we let V be a polarized algebraic manifold of dimension n. For each m with $m=[\frac{n-1}{2}]$ we set:

$$\begin{cases} p_m=\frac{1}{2}\dim H^{2m+1}(V,\boldsymbol{C})_0; \\ q_m=\sum\limits_{k\leqq[\frac{m-1}{2}]} h_0{}^{2m-2k,2k+1}; \\ r_m=p_m-q_m. \end{cases}$$

Furthermore, we set $\boldsymbol{D}_m=D_1\times D_3\times\cdot\cdot\cdot\times D_{2m+1}$, where D_q is the period matrix space for the primitive q-forms.

(3.22) THEOREM. (i) *To each* $\Omega\in\boldsymbol{D}_{(m)}$ *there is naturally associated a complex torus* $T(\Omega)$ *with q-convex polarization where*

$$q=p_m-r_{m-2}+p_{m-4}\cdot\cdot\cdot;$$

(ii) *If* $\Omega_m(V)=(\Phi_1(V),\cdot\cdot\cdot,\Phi_{2m+1}(V))\in\boldsymbol{D}_m$, *then there is a natural complex structure on* $T_{2m+1}(V)=H^{2m+1}(V,\boldsymbol{R})/H^{2m+1}(V,\boldsymbol{Z})$ such that $T_{2m+1}(V)=T(\Omega_m(V))$;

(iii) *The complex torus* $T_{2m+1}(V)$ *varies holomorphically with* V.

Remark. Part (iii) will be proved in Part II below.

We record here one property of the q-convex polarized torii $T(\Omega)$.

(3.23) PROPOSITION. *Let* $T=T(\Omega)$ *and* $\boldsymbol{L}\rightarrow T$ *the line bundle with characteristic class* ω *given by Lemma* (3.14). *Then the sheaf cohomology groups* $H^r(T,\mathcal{O}(\boldsymbol{L}))=0$ *for* $r\neq q$ *and* $\dim H^q(T,\mathcal{O}(\boldsymbol{L}))=P(Q)$, *where* $P(Q)$ *is the Pfaffian of* Q.

Proof. Let ω have the form (3.9) where $dz^1,\cdot\cdot\cdot,dz^n$ are the holomorphic 1-forms on T. Then, for any constants c_1, c_2,

$$ds^2=c_1\sum_{\alpha=1}^{n-q}\omega^\alpha\bar{\omega}^\alpha-c_2\sum_{\beta=n-q+1}^{n}\omega^\beta\bar{\omega}^\beta$$

will give a Kähler metric on T. We now use the argument of Theorem 7.1, Section VII, of [6] to show that $H^r(T,\mathcal{O}(\boldsymbol{L}))=0$ for $r\neq q$. By the Hirzebruch-Riemann-Roch theorem [9], we have then that

$$(-1)^q\dim H^q(T,\mathcal{O}(\boldsymbol{L}))=\chi(T,\mathcal{O}(\boldsymbol{L}))=\boldsymbol{T}(T,\boldsymbol{L})$$

where $\boldsymbol{T}(T,\boldsymbol{L})$ is the *Todd genus* of $\boldsymbol{L}\rightarrow T$. Since all the Chern classes of T

are zero, $\boldsymbol{T}(T,\boldsymbol{L}) = \frac{1}{n!}\int_{\boldsymbol{T}} \omega^n$ while $\omega^n = \pm\, n!\, P(Q)\, dx^1 \wedge \cdots \wedge dx^{2n}$. Thus $\boldsymbol{T}(T,\boldsymbol{L}) = \pm\, P(Q)$ and so $\dim H^q(T, \mathcal{O}(\boldsymbol{L})) = P(Q)$ as required.

Remark. This generalizes a well-known result in *Abelian functions* (cf. Conforto [5], also [14]). For $q = 0$, $H^0(T, \mathcal{O}(\boldsymbol{L}))$ is the vector space of entire functions $\vartheta(z)$ defined on E_0 and satisfying $\vartheta(z+\gamma) = \xi_\gamma(z)\vartheta(z)$ for $\gamma \in \Lambda$. These are the *theta-functions*. For general q, the classes in $H^q(T, \mathcal{O}(\boldsymbol{L}))$ are given by C^∞ differential forms:

$$= \frac{1}{q!} \sum_I \vartheta(z,\bar{z})_I \, d\bar{z}^I, \tag{3.24}$$

where $I = (\alpha_1, \cdots, \alpha_q)$ and $d\bar{z}^I = d\bar{z}^{\alpha_1} \wedge \cdots \wedge d\bar{z}^{\alpha_q}$, $\vartheta(z,\bar{z})_I$ is skew-symmetric in $\alpha_1, \cdots, \alpha_q$, and such that:

$$\begin{cases} \vartheta(z+\gamma)_I = \xi_\gamma(z)\vartheta(z)_I \\ (\Box\vartheta)_I = 0, \end{cases} \tag{3.25}$$

where

$$(\Box\vartheta)_I = \sum_\alpha \frac{\partial^2 \vartheta_I}{\partial z^\alpha \partial \bar{z}^\alpha} + \pi \sum_\beta h_{\alpha\bar{\beta}} \bar{z}^\beta \frac{\partial \vartheta_I}{\partial \bar{z}^\alpha} - \pi \sum_{\alpha, j} h_{\alpha\bar{\alpha}_j} \vartheta_{\bar{\alpha}_1 \cdots \underset{j}{(\bar{\alpha})} \cdots \bar{\alpha}_q}$$

is the *Laplacian* of ϑ.

These cohomology groups will be given a geometric interpretation in Part II below, where they will be shown to serve as sort of theta-functions for the intermediate cycles on V.

I. 4. Some further properties of modular varieties. (a) We shall discuss some further properties of the period matix domains as constructed in I. 1. (d). We shall work with two special cases, but the theorems will be true for all period matrix domains. To give these examples, we begin with a real n-dimensional vector space W_0, defined over $\boldsymbol{Q}$; W_0 will correspond to $H^2(V,\boldsymbol{R})_0$ and $H^3(V,\boldsymbol{R})_0$ in the two special cases. We let W be the complexification of W_0 and $Q\colon W \otimes W \to \boldsymbol{C}$ a non-singular, rational quadratic form.

Case 1. Q is symmetric and D consists of all h-dimensional subspaces $S \subset W$ which satisfy:

$$\begin{cases} Q(S,S) = 0, \\ Q(S,\bar{S}) > 0. \end{cases} \tag{4.1}$$

We set $R_S = (S \oplus \bar{S})^\perp = \{w \in W \mid Q(S \oplus \bar{S}, w) = 0\}$, and we assume that Q is negative definite on the real vector space underlying R_S.

As a coset space (cf. Theorem (1.26)), $D \cong SO(2h,k)/U(h)\times SO(k)$ where $2h+k=n$; D is a period matrix space for the periods of holomorphic 2-forms.

If we let X be all $S \subset W$ satisfying $Q(S,S)=0$, then X is an algebraic subvariety of $G(h,W)$ and $D \subset X$ is an open set.

Case 2. Q is skew-symmetric and D consists of all flags $[S_1, S_2]$ where $S_1 \subset S_2$, $\dim S_2 = n$, and:

$$(4.2) \qquad \begin{cases} Q(S_2,S_2)=0, \\ \sqrt{-1}\,Q(S_1,\bar{S}_1) > 0, \\ \sqrt{-1}\,Q(S_2/S_1,\bar{S}_2/\bar{S}_1) < 0; \end{cases}$$

(cf. (3.2)).

As a coset space, $D \cong U(n-q)\times U(q)\backslash Sp(n)$ where $\dim S_1 = n-q$, although we recall (Proposition (3.21)) that D does *not* fibre holomorphically over $\boldsymbol{H}_n = U(n)\backslash Sp(n)$. We let X be all flags $[S_1,S_2]$ with $Q(S_2,S_2)=0$; then X is an algebraic subvariety of $G(n-q,W)\times G(n,W)$ and $D\subset X$ is an open set. The domain D is a period matrix space for the periods of the 3-forms.

(4.3) THEOREM. *Let $\boldsymbol{G} = SO(Q,\boldsymbol{C})$ be the complex simple Lie group of all linear automorphisms of W which preserve Q. Then $\boldsymbol{G}$ acts transitively on X so that X is a rational, homogeneous algebraic manifold. The Lie group G of real transformations preserving Q is a real form of $\boldsymbol{G}$ and $D\subset X$ is an open G-orbit with compact isotropy group.*

Proof. We shall prove this theorem in case D is the first type of domain listed above. The main step in the proof is:

(4.4) LEMMA. *Let $S\subset W$ be an h-plane with $Q(S,S)=0$. Then we can choose a basis $v_1,\cdots,v_h$; $u_1,\cdots,u_h$; $w_1,\cdots,w_k$ for W such that:* (i) *$v_1,\cdots,v_h$ is abasis for S, and* (ii) *the matrix of Q in this basis is*

$$\begin{bmatrix} 0 & I_h & 0 \\ I_h & 0 & 0 \\ 0 & 0 & I_k \end{bmatrix}.$$

Proof. Choose a basis $v_1,\cdots,v_h$ for S and complete this set of vectors to a basis $v_1,\cdots,v_h$; $z_1,\cdots,z_{h+k}$ for W. Then $Q(v_\alpha,v_\beta)=0$ since $Q(S,S)=0$. Because Q is non-singular, the $h\times(h+k)$ matrix $q_{\alpha\rho}=Q(v_\alpha,z_\rho)$ has rank h; we may assume that $(q_{\alpha\beta})_{1\leqq\alpha,\beta\leqq h}$ has rank h. Suppose that

$\sum_{\beta=1}^{h} q_{\alpha\beta}A_{\beta\gamma}=\delta_{\gamma}{}^{\alpha}$ and let $y_{\gamma}=\sum_{\beta=1}^{h} A_{\beta\gamma}z_{\beta}$. Then $v_1,\cdots,v_h;\ y_1,\cdots,y_h;\ z_{h+1},\cdots,z_k$ gives a basis for W relative to which Q has the matrix $\begin{bmatrix}0 & I_h & * \\ I_h & * & * \\ * & * & *\end{bmatrix}$. Let $x_{\rho}=z_{h+\rho}-\sum_{\beta} q_{\beta,h+\rho}y_{\beta}$ for $1\leqq\rho\leqq k$. Then $Q(v_{\alpha},x_{\rho})=q_{\alpha,h+\rho}-q_{\alpha,h+\rho}=0$ so that, relative to the basis $v_1,\cdots,v_h;\ y_1,\cdots,y_h;\ x_1,\cdots,x_k$, Q has matrix $\begin{bmatrix}0 & I_h & 0 \\ I_h & * & * \\ 0 & * & *\end{bmatrix}$. The matrix $Q(y_{\alpha},y_{\beta})$ is symmetric, so that we can let $Q(y_{\alpha},y_{\beta})=-B_{\alpha\beta}-B_{\beta\alpha}$ for some matrix $B=(B_{\alpha\beta})$. We let $u_{\alpha}=y_{\alpha}+\sum_{\gamma=1}^{h}B_{\alpha\gamma}v_{\gamma}$. Then $Q(u_{\alpha},u_{\beta})=Q(y_{\alpha},y_{\beta})+B_{\alpha\beta}+B_{\alpha\beta}=0$ so that, relative to the basis $v_1,\cdots,v_h;\ u_1,\cdots,u_h;\ x_1,\cdots,x_k$, Q has the matrix $\begin{bmatrix}0 & I_h & 0 \\ I_h & 0 & * \\ 0 & * & *\end{bmatrix}$. By replacing x_{ρ} by $x_{\rho}-\sum_{\beta=1}^{h}Q(u_{\beta},x_{\rho})v_{\beta}$, the matrix of Q becomes $\begin{bmatrix}0 & I_h & 0 \\ I_h & 0 & 0 \\ 0 & 0 & Q_1\end{bmatrix}$. Since Q_1 is non-singular, we may choose then a basis $v_1,\cdots,v_h;\ u_1,\cdots,u_h;\ w_1,\cdots,w_k$ such that the matrix of Q is $\begin{bmatrix}0 & I_h & 0 \\ I_h & 0 & 0 \\ 0 & 0 & I_k\end{bmatrix}$ as required.

Proof of Theorem 4.3. If $S,\hat{S}\in X$, then we can choose bases $v_1,\cdots,v_h;\ u_1,\cdots,u_h;\ w_1,\cdots,w_k$ and $\hat{v}_1,\cdots,\hat{v}_h;\ \hat{u}_1,\cdots,\hat{u}_h;\ \hat{w}_1,\cdots,\hat{w}_k$ corresponding to S and $\hat{S}$ respectively. Then $T\colon W\rightarrow W$ defined by $Tv_{\alpha}=\hat{v}_{\alpha}$, $Tu_{\alpha}=\hat{u}_{\alpha}$, $Tw_{\rho}=\hat{w}_{\rho}$ will be an element of $\boldsymbol{G}$ such that $TS=\hat{S}$. This shows that $\boldsymbol{G}$ acts transitively on X, and the theorem follows.

(b) Over the Grassmann variety $G(l,W)$ there is a canonical *holomorphic vector bundle* $\boldsymbol{F}\rightarrow G(l,W)$ given as follows: The fibre $\boldsymbol{F}_S$ at an l-plane $S\in G(l,W)$ is the vector space $S\subset W$. There is a natural bundle mapping $\boldsymbol{F}\rightarrow\boldsymbol{W}$ where $\boldsymbol{W}=G(l,W)\times W$ is the trivial bundle, and we have the exact sequence:

$$0\rightarrow\boldsymbol{F}\rightarrow\boldsymbol{W}\rightarrow\boldsymbol{E}\rightarrow 0, \tag{4.5}$$

where $\boldsymbol{E}_S=W/S$. We observe that the line bundle $\boldsymbol{L}=\det\boldsymbol{E}$ is a positive bundle with dual bundle $\boldsymbol{L}^*=\det\boldsymbol{F}$. For example, let us prove:

(4.6) PROPOSITION. *The holomorphic sections of* $\boldsymbol{L}$ *give the Plücker coordinates on* $G(l,W)$.

Proof. $\boldsymbol{L}=(\det\boldsymbol{F})^*=(\Lambda^l\boldsymbol{F})^*=\operatorname{Hom}(\Lambda^l\boldsymbol{F},\boldsymbol{C})$. Let $e_1^*,\cdots,e_n^*$ give a basis for W^* (cf. I.1.(a)). Then, for $\rho_1,<\cdots<\rho_l$ we define a section $\Omega_{\rho_1\cdots\rho_l}$ of $\boldsymbol{L}=\operatorname{Hom}(\Lambda^l\boldsymbol{F},\boldsymbol{C})$ by:

$$\Omega_{\rho_1\cdots\rho_l}(f)=\langle f,e^*_{\rho_l}\wedge\cdots\wedge e^*_{\rho_l}\rangle, \tag{4.7}$$

where $f\in\Lambda^l\boldsymbol{F}_S\subset\Lambda^lW$. In other words, if $f=f_1\wedge\cdots\wedge f_l\in\Lambda^l\boldsymbol{F}_S$ $(f_1,\cdots,f_l\in S)$, then $f=\sum_{\rho_1<\cdots<\rho_l}\Omega_{\rho_1\cdots\rho_l}(f)e_{\rho_1}\wedge\cdots\wedge e_{\rho_l}$, so that the $\Omega_{\rho_1\cdots\rho_l}$ are exactly the Plücker coordinates on $G(l,W)$.

Remark. The bundles in (4.5) are all *homogeneous vector bundles* [3]; i.e. the action of the linear group $GL(W)$ lifts to bundle automorphisms.

Let now D be a period matrix domain as constructed in I.1.(d); then D is an open set on a homogeneous algebraic manifold X (cf. Theorem 4.3). There is an *equivariant embedding* $X\subset G(h_0,W)\times\cdots\times G(h_t,W)$ which, in the two examples above, reduces to:

$$\begin{cases}\text{(i)} & X\subset G(h,W);\\ \text{(ii)} & X\subset G(n-q,W)\times G(n,W).\end{cases}$$

From each of the factors $G(h_r,W)$, there is induced a *homogeneous line bundle* $\boldsymbol{L}_r\to X$ and we set $\boldsymbol{L}(\alpha_0,\cdots,\alpha_t)=\boldsymbol{L}_0^{\alpha_0}\otimes\cdots\otimes\boldsymbol{L}_t^{\alpha_t}$.

(4.8) THEOREM. *The homogeneous line bundle* $\boldsymbol{L}(\alpha_0,\cdots,\alpha_t)\to D$ *has a unique G-invariant p-convex polarization for a suitable integer p. The canonical bundle* $\boldsymbol{K}\to D$ *is a homogeneous bundle* $\boldsymbol{L}(\alpha_0,\cdots,\alpha_t)$ *where each* $\alpha_j<0$.

Proof. Set $\boldsymbol{L}=\boldsymbol{L}(\alpha_0,\cdots,\alpha_t)$. Since $D=H\backslash G$ where H is compact, the line bundle $\boldsymbol{L}\to D$ carries a G-invariant metric which is unique up to a constant factor. The curvature of this metric will then give the G-invariant p-convex polarization.

The rule for finding p in terms of the roots of G and H has been given in [7]. We shall give, for the period matrix domains of types 1 and 2 above, the explicit computation of the p-convex polarization and of the canonical bundle $\boldsymbol{K}$.

Remark. We observe that, if $Z(H)$ is the center of H, then $\dim_{\boldsymbol{R}}Z(H)=t+1$ and the group $\mathscr{L}$ of line bundles of the form $\boldsymbol{L}(\alpha_0,\cdots,\alpha_t)$ is $\underbrace{\boldsymbol{Z}\oplus\cdots\oplus\boldsymbol{Z}}_{t+1}$. (Clearly we have

$$\boldsymbol{L}(\alpha_0,\cdots,\alpha_t)\otimes\boldsymbol{L}(\beta_0,\cdots,\beta_t)=\boldsymbol{L}(\alpha_0+\beta_0,\cdots,\alpha_t+\beta_t).)$$

The homogeneous algebraic manifold is of the form $X=\boldsymbol{B}\backslash\boldsymbol{G}$, where $\boldsymbol{B}\subset\boldsymbol{G}$ contains a *Borel subgroup* $\boldsymbol{P}$. The *unipotent radical* $\boldsymbol{N}$ of $\boldsymbol{B}$ will be an analytic subgroup of $\boldsymbol{B}$ and $\boldsymbol{B}/\boldsymbol{N}\cong\boldsymbol{H}$, the complexification of H ([3]). Given a *character* λ of H, λ extends to a *holomorphic homomorphism* $\lambda\in\operatorname{Hom}(\boldsymbol{H},\boldsymbol{C}^*)$, which then extends to $\lambda\in\operatorname{Hom}(\boldsymbol{B},\boldsymbol{C}^*)$. This λ then gives the homogeneous line bundle $\boldsymbol{L}=\boldsymbol{G}\times_{\boldsymbol{B}}\boldsymbol{C}$ over $X=\boldsymbol{B}\backslash\boldsymbol{G}$ (cf. [3]). In this way the *character group* $\chi(H)$ parametrizes all homogeneous line bundles over X. But $\chi(H)\cong\underbrace{\boldsymbol{Z}\oplus\cdots\oplus\boldsymbol{Z}}_{t+1}$ and so $\chi(H)/\mathcal{L}$ is a finite group; in this sense, $\mathcal{L}$ gives almost all line bundles over X.

Example 1. In case 1, $t=0$, $H\cong U(h)\times SO(k)$, and $\chi(H)\cong\boldsymbol{Z}$. Thus the homogeneous line bundles in $\mathcal{L}$ over D are of the form $\boldsymbol{L}^\alpha\to D$ where α is an integer.

(4.9) PROPOSITION. *The line bundle* $\boldsymbol{L}^\alpha\to D$ $(\alpha<0)$ *has a p convex polarization where* $p=\dfrac{h^2-h}{2}$. *In particular, the canonical bundle* $\boldsymbol{K}=\boldsymbol{L}^{-(h+k-1)}$ *has such a p-convex polarization.*

Proof. Choose a basis $e_1,\cdots,e_n$ for W and let $\boldsymbol{P}$ be the space of $h\times n$ matrices $\Omega=(\pi_{\alpha\rho})$ of rank h and satisfying $\Omega Q^t\Omega=0$. As explained in I.1.(a), $\boldsymbol{P}\xrightarrow{\pi}X$ is a holomorphic principal bundle with group $GL(h)$. Clearly $\boldsymbol{P}$ is the principal bundle of $\boldsymbol{F}\to X$ (cf. (4.5)), and so $\boldsymbol{L}^{-1}=\boldsymbol{P}\times_{GL(h)}\boldsymbol{C}$, where $GL(h)$ operates on $\boldsymbol{C}$ by sending A into $\det(A)^{-1}$. A metric in $\boldsymbol{L}^{-1}$ is given by a positive real function $\psi(\Omega)$ which satisfies $\psi(A\Omega)=|\det A|^2\psi(\Omega)$. One such $\psi(\Omega)$ is given by:

$$\psi(\Omega)=\sum_{\rho_1<\cdots<\rho_n}|\Omega_{\rho_1\cdots\rho_n}|^2. \tag{4.10}$$

This metric is invariant under a unitary charge of $e_1,\cdots,e_n$, and so is invariant under the *maximal compact subgroup* $M\subset\boldsymbol{G}$, but is *not* invariant under the full complex group $\boldsymbol{G}$.

For example, if $h=1$, then X is a quadric in $\boldsymbol{P}_{n-1}$. The metric $\psi(\Omega)=\psi(\pi_1,\cdots,\pi_n)=\sum_{\alpha=1}^{n}|\pi_\alpha|^2$ exists on $\boldsymbol{P}_{n-1}$, and the *curvature form* [6] ω of $\psi(\Omega)$ is given by:

$$\omega=-\partial\bar{\partial}\log\psi(\Omega).$$

On the open set $\pi_n \neq 0$, we let $z^\alpha = \pi_\alpha/\pi_n$ for $1 \leqq \alpha \leqq n$. Then $\psi(\Omega) = |\pi_n|^2\{(z,z)+1\}$ where $(z,z) = \sum_{\alpha=1}^{n-1} z^\alpha \bar{z}^\alpha$. Thus

$$\partial\bar{\partial}\log\psi(\Omega) = \partial\bar{\partial}\log\{(z,z)+1\} = \sum_{\alpha,\beta} h_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$$

where

$$h_{\alpha\bar{\beta}} = \frac{1}{\{(z,z)+1\}^2}\{(z,z)+1-\bar{z}^\alpha z^\beta\}.$$

It follows that

$$\sum_{\alpha,\beta} h_{\alpha\bar{\beta}}\xi^\alpha\bar{\xi}^\beta = \frac{1}{\{(z,z)+1\}^2}\{(z,z)(\xi,\xi) - (z,\xi)\overline{(\xi,z)} + (\xi,\xi)\} \geqq \frac{1}{\{(z,z)+1\}}(\xi,\xi).$$

The conclusion is that ω is negative on P_{n-1}, hence is negative on $X \subset P_{n-1}$. Obviously then the curvature form of the metric (4.10) in $\boldsymbol{L}^{-1} \to X$ is, for general h, negative. This is consistent with the fact that $\boldsymbol{L} \to X$ is positive and serves to check the signs and notation.

We now let $P \subset \boldsymbol{P}$ be those Ω satisfying $\Omega Q^t\bar{\Omega} = H(\Omega) > 0$. Then $P = \pi^{-1}(D)$ and G acts as fibre-preserving automorphisms in $P \to D$. However, G does not leave the metric (4.10) invariant; to get a G-invariant metric we take:

$$\psi(\Omega) = \det H(\Omega). \tag{4.11}$$

Since, for $T \in G$, $H(\Omega) = \Omega Q^t\bar{\Omega} = \Omega T Q^t(\Omega T) = H(\Omega T)$, $\psi(\Omega)$ is a G-invariant metric in $\boldsymbol{L}^{-1}$ and

$$\omega = -\partial\bar{\partial}\log(\det H) \tag{4.12}$$

will be the G-invariant curvature in $\boldsymbol{L}^{-1}$.

Now $\dim_{\mathbf{C}} D = \frac{h^2-h}{2} + hk$ $(2h+k=n)$ and we want to show that, if $z^1, \cdots, z^n$ are local coordinates on D, then $\omega = \sum_{\alpha,\beta=1}^{m} h_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ where $(h_{\alpha\bar{\beta}})$ has signature $(hk, \frac{h^2-h}{2})$.

We assume that $e_1, \cdots, e_n$ have been chosen as a real basis for $W_0 \subset W$ and such that the matrix of Q is $Q = \begin{pmatrix} I_{2h} & 0 \\ 0 & -I_k \end{pmatrix}$, $2h+k=n$. Then $\boldsymbol{P}$

is those matrices $\Omega = (A, B, C)$ with A and B each $h \times h$, C an $h \times k$ matrix, and with

$$(4.13) \qquad \Omega Q^t\Omega = A^tA + B^tB - C^tC = 0.$$

On P we have that:

$$(4.14) \qquad H(\Omega) = \Omega Q^t\bar{\Omega} = A^t\bar{A} + B^t\bar{B} - C^t\bar{C} > 0.$$

Since $\psi(\Omega)$ and ω are G-invariant, it will suffice to compute ω at $\Omega_0 = (I, \sqrt{-1}\, I, 0)$ so that $H(\Omega_0) = 2I_h$. The reason we choose Ω_0 is that the stability group $H = \{T \in G = SO(2h, k) : \Omega_0 T \sim \Omega_0\}$ of $\pi(\Omega_0) \in D$ is just $U(h) \times SO(k) \subset SO(2h) \times SO(k) = K$, the imbedding $U(h) \subset SO(2h)$ being

$$\alpha + i\beta \cdot \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

Now $\pi(\Omega_0)$ belongs to the *Zariski open set* $\Omega_{1\cdots h} \neq 0$ on X, and we may choose the local holomorphic section $\Omega(B, C) = (I, B, C)$ where B is close to $\sqrt{-1}\, I$, C is close to 0, and, by (4.13),

$$(4.15) \qquad I + B^tB - C^tC = 0.$$

By (4.14) we have that $H(\Omega) = H(B, C)$ is given by:

$$(4.16) \qquad H(B, C) = I + B^t\bar{B} - C^t\bar{C} > 0.$$

Writing $B = (b_{\alpha\beta})$ and $C = (c_{\alpha\rho})$, we shall evaluate ω as a differential form in $db_{\alpha\beta}$, $dc_{\alpha\rho}$, subject to the relations (4.15).

Write now $\Delta = \det H$ and $H = (h_{\alpha\beta})$. Then

$$\partial\bar{\partial} \log \Delta = \partial\left(\frac{\bar{\partial}\Delta}{\Delta}\right) = \frac{\partial\bar{\partial}\Delta}{\Delta} - \frac{\partial\Delta \wedge \bar{\partial}\Delta}{\Delta^2}.$$

We have:

$$\bar{\partial}\Delta = \sum_{\alpha,\beta} \frac{\partial\Delta}{\partial h_{\alpha\beta}} \bar{\partial} h_{\alpha\beta}$$

and

$$\partial\bar{\partial}\Delta = \sum_{\left\{\substack{\alpha,\beta \\ \lambda,\mu}\right.} \frac{\partial^2\Delta}{\partial h_{\alpha\beta}\partial h_{\lambda\mu}} \partial h_{\alpha\beta} \wedge \partial h_{\lambda\mu} + \sum_{\alpha,\beta} \frac{\partial\Delta}{\partial h_{\alpha\beta}} \bar{\partial} h_{\alpha\beta}.$$

Since $H(\Omega_0) = H(I, \sqrt{-1}\, I, 0) = 2I$, we want to evaluate $\dfrac{\partial\Delta}{\partial h_{\alpha\beta}}$ and $\dfrac{\partial^2\Delta}{\partial h_{\alpha\beta}\partial h_{\lambda\mu}}$ at $H_0 = \begin{pmatrix} \xi_1 & & 0 \\ & \ddots & \\ 0 & & \xi_h \end{pmatrix}$ and then take $\xi_\alpha = 2$. The formulae are:

$$(4.17)\qquad \begin{aligned} \frac{\partial\Delta}{\partial h_{\alpha\beta}}\Big]_{H_0} &= \begin{cases} 0 \text{ if } \alpha \neq \beta, \\ \xi_1 \cdots \hat{\xi}_\alpha \cdots \xi_n \text{ if } \alpha = \beta; \end{cases} \\ \frac{\partial^2\Delta}{\partial h_{\alpha\beta}\partial h_{\lambda\mu}}\Big]_{H_0} &= \begin{cases} \xi_1 \cdots \hat{\xi}_\alpha \cdots \hat{\xi}_\lambda \cdots \boldsymbol{\xi}_n \text{ if } \alpha = \beta, \lambda = \mu, \alpha \neq \lambda, \\ -\xi_1 \cdots \hat{\xi}_\alpha \cdots \hat{\xi}_\lambda \cdots \xi_n \text{ if } \alpha = \mu, \beta = \lambda, \alpha \neq \beta, \\ 0 \quad \text{otherwise.} \end{cases} \end{aligned}$$

Thus we get from (4.17) that:

$$\begin{aligned} \omega_{\Omega_0} &= -\partial\bar{\partial}\log\Delta]_{\Omega_0} = -\frac{\partial\bar{\partial}\Delta}{\Delta}]_{\Omega_0} + \frac{\partial\Delta\wedge\bar{\partial}\Delta}{\Delta^2}]_{\Omega_0} \\ &= -\tfrac{1}{4}\sum_{\alpha\neq\beta}\partial h_{\alpha\alpha}\wedge\bar{\partial}h_{\beta\beta} + \tfrac{1}{4}\sum_{\alpha\neq\beta}\partial h_{\alpha\beta}\wedge\bar{\partial}h_{\beta\alpha} - \tfrac{1}{2}\sum_{\alpha}\partial\partial h_{\alpha\alpha} \\ &\qquad + \tfrac{1}{4}\sum_{\alpha,\beta}\partial h_{\alpha\alpha}\wedge\bar{\partial}h_{\beta\beta} \text{ or:} \end{aligned}$$

$$(4.18)\qquad \omega_{\Omega_0} = -\tfrac{1}{2}\sum_{\alpha}\partial\bar{\partial}h_{\alpha\alpha} + \tfrac{1}{4}\sum_{\alpha\neq\beta}\partial h_{\alpha\beta}\wedge\bar{\partial}h_{\beta\alpha} + \tfrac{1}{4}\sum_{\alpha}\partial h_{\alpha\alpha}\wedge\bar{\partial}h_{\alpha\alpha}.$$

From (4.16), $h_{\alpha\beta} = \delta_\beta{}^\alpha + \sum_\gamma b_{\alpha\gamma}\bar{b}_{\beta\gamma} - \sum_\rho c_{\alpha\rho}\bar{c}_{\beta\rho}$ and so

$$\partial\bar{\partial}h_{\alpha\alpha} = \sum_\gamma db_{\alpha\gamma}\wedge\overline{db}_{\alpha\gamma} - \sum_\rho dc_{\alpha\rho}\wedge\overline{dc}_{\alpha\rho},\ \partial h_{\alpha\beta}\wedge\overline{\partial h}_{\beta\alpha} = db_{\alpha\beta}\wedge\overline{db}_{\alpha\beta} \ \text{(at } \Omega_0).$$

Thus, by this and (4.18),

$$(4.19)\qquad \begin{aligned} \omega_{\Omega_0} &= -\tfrac{1}{2}\{\sum_\gamma db_{\alpha\gamma}\wedge d\bar{b}_{\alpha\gamma} - \sum_\rho dc_{\alpha\rho}\wedge d\bar{c}_{\alpha\rho}\} \\ &\qquad + \tfrac{1}{4}\{\sum_{\alpha\neq\beta} db_{\alpha\beta}\wedge db_{\alpha\beta}\} + \tfrac{1}{4}\sum_\alpha \partial h_{\alpha\alpha}\wedge\partial h_{\alpha\alpha}. \end{aligned}$$

By (4.15) we have

$$\sum_\gamma (db_{\alpha\gamma}b_{\beta\gamma} + b_{\alpha\gamma}db_{\beta\gamma}) - \sum_\rho (dc_{\alpha\rho}c_{\rho\alpha} + c_{\alpha\rho}dc_{\rho\alpha}) = 0$$

which, at $\Omega_0 = (I, \sqrt{-1}\,I, 0)$, gives $db_{\alpha\beta} + db_{\beta\alpha} = 0$. Combining this with (4.19) gives:

$$(4.20)\qquad \omega_{\Omega_0} = \tfrac{1}{2}\{\sum_\rho dc_{\alpha\rho}\wedge d\bar{c}_{\alpha\rho} - \sum_{\alpha<\beta} db_{\alpha\beta}\wedge\overline{db}_{\alpha\beta}\}.$$

Since the $dc_{\alpha\rho}$ give hk linearly independent 1-forms and the $db_{\alpha\beta}$ $(\alpha<\beta)$ gives $\frac{h^2-h}{2}$ independent 1-forms, it follows from (4.20) that ω_{Ω_0} has signature $(hk, \frac{h^2-h}{2})$ as required.

To complete the proof of Proposition 4.9, we need to show that $\boldsymbol{K} = \boldsymbol{L}^{-(h+k-1)}$ where $\boldsymbol{K}\to X$ is the canonical bundle. We let $Y = G(h, W)$ and $\boldsymbol{Q}$ be the space of $h\times n$ matrices $\Omega = (\pi_{\alpha\rho})$ of rank h. Then $\boldsymbol{Q}\to Y$ is a principal bundle with group $GL(h)$ and we have:

$$\begin{tikzcd} \boldsymbol{P} \arrow[r, phantom, "\subset"] \arrow[d, "\pi"] & \boldsymbol{Q} \arrow[d] \\ X \arrow[r, phantom, "\subset"] & Y, \end{tikzcd}$$

where $\boldsymbol{P}$ is $\{\Omega \mid \Omega Q {}^t\Omega = 0\}$. We let $\boldsymbol{N} \rightarrow X$ be the normal bundle of X in Y.

(4.21) LEMMA. $\boldsymbol{N}^* = \boldsymbol{P} \times_{GL(h)} \boldsymbol{C}^{\frac{h(h+1)}{2}}$ *where* $\boldsymbol{C}^{\frac{h(h+1)}{2}}$ *is the space of* $h \times h$ *symmetric matrices* B *and* $GL(h)$ *acts by* $A(B) = AB{}^tA$.

Proof. We set $B(\Omega) = \Omega Q {}^t\Omega$. Then ${}^tB(\Omega) = B(\Omega)$ and $B(\Omega)$ is a matrix-valued holomorphic function. Also $B(A\Omega) = AB(\Omega){}^tA$ and $B(\Omega) = 0$ defines $\boldsymbol{P} \subset \boldsymbol{Q}$. The normal bundle of $\boldsymbol{P} \subset \boldsymbol{Q}$ *is* $\pi^{-1}(\boldsymbol{N})$, and the differentials in $dB(\Omega)$ along $B(\Omega) = 0$ give a holomorphic frame for the dual normal bundle $\pi^{-1}(\boldsymbol{N}^*)$ of $\boldsymbol{P} \subset \boldsymbol{Q}$. Now

$$dB(A\Omega) = dAB(\Omega){}^tA + AdB(\Omega){}^tA + AB(\Omega){}^tdA$$

so that, along $\boldsymbol{P}$, $dB(A\Omega) = AdB(\Omega){}^tA$. It follows that $\boldsymbol{N}^* = \boldsymbol{P} \times_{GL(h)} \boldsymbol{C}^{\frac{h(h+1)}{2}}$ as claimed.

(4.22) LEMMA. *Let* $\boldsymbol{T}(Y) \rightarrow Y$ *be the holomorphic tangent bundle of* $G(h, W)$. *Then* $\boldsymbol{T}(Y) \cong \mathrm{Hom}(\boldsymbol{F}, \boldsymbol{E})$ *where* $\boldsymbol{E}$, $\boldsymbol{F}$ *are given by* (4.5).

Proof. Let $S \in G(h, W;)$ we define a linear mapping

$$\boldsymbol{T}_S(Y) \rightarrow \mathrm{Hom}(\boldsymbol{F}_S, \boldsymbol{E}_S)$$

as follows: Given $\theta \in \boldsymbol{T}_S(Y)$, choose a holomorphic curve $S_t \subset G(h, W)$ with $S_0 = S$ and with tangent θ. If $\xi \in \boldsymbol{F}_S \subset W$, choose $\xi_t \in S_t$ with $\xi_0 = \xi$ and we let:

$$\theta(\xi) = \text{projection of } \frac{\partial \xi_t}{\partial t}\Big]_{t=0} \text{ on } \boldsymbol{E}_S = W/S. \tag{4.23}$$

We claim that this gives the bundle isomorphism $\boldsymbol{T}(Y) \cong \mathrm{Hom}(\boldsymbol{F}, \boldsymbol{E})$.

To begin with, choose $v_1(t), \cdots, v_h(t)$ which give a basis for S_t and set $v_\alpha = v_\alpha(0)$. If $u_1(t), \cdots, u_h(t)$ is another basis, then $u_\alpha(t) = \sum_{\beta=1}^{h} a_{\alpha\beta}(t) v_\beta(t)$ and

$$\frac{\partial u_\alpha(t)}{\partial t}\Big]_{t=0} \equiv \sum_{\beta=1}^{h} a_{\alpha\beta} \frac{\partial v_\beta(t)}{\partial t}\Big]_{t=0} \text{ modulo } S, \tag{4.24}$$

where $a_{\alpha\beta} = a_{\alpha\beta}(0)$.

If $\xi = \sum_{\alpha=1}^{h} \xi_\alpha v_\alpha \in S$, then $\xi_t = \sum_{\alpha=1}^{h} \xi_\alpha(t) v_\alpha(t)$ and $\frac{\partial \xi_t}{\partial t}\Big]_{t=0} \equiv \sum_{\alpha=1}^{h} \xi_\alpha \frac{\partial v_\alpha}{\partial t}\Big]_{t=0}$

modulo S. From this and (4.24), it follows that $\frac{\partial\xi_t}{\partial t}]_{t=0}$ in W/S depends only on $\xi=\xi_0\in S$. Since $(\xi+\zeta)_t=\xi_t+\zeta_t$ it follows that θ is linear, $\theta\in\mathrm{Hom}(\boldsymbol{F}_S,\boldsymbol{E}_S)$.

The remaining steps necessary to check (4.22) are routine and will be omitted.

(4.25) LEMMA. *Let* $\boldsymbol{T}=\boldsymbol{T}(Y)\,|\,X$. *Then*

$$\boldsymbol{K}=(\det\boldsymbol{T})\otimes(\det\boldsymbol{N}).$$

Proof. This follows from the exact sequence:

$$0\to\boldsymbol{T}(X)\to\boldsymbol{T}\to\boldsymbol{N}\to 0. \tag{4.26}$$

In fact, (4.26) gives $\det\boldsymbol{T}=\boldsymbol{K}^*\otimes(\det\boldsymbol{N})$ since $\boldsymbol{K}=\det\boldsymbol{T}(X)$.

We may now prove Proposition 4.9. From (4.22),

$$\det\boldsymbol{T}=(\det\boldsymbol{F}^*)^{h+k}\cdot(\det\boldsymbol{E})^h=\boldsymbol{L}^{(2h+k)}$$

so that $\boldsymbol{K}=(\det\boldsymbol{N})\otimes\boldsymbol{L}^{-(2h+k)}$. We claim that $\det\boldsymbol{N}=\boldsymbol{L}^{(h+1)}$or, equivalently, $\det\boldsymbol{N}^*=\mathbf{L}^{-(h+1)}$. Let $e_{\alpha\beta}$ $(\alpha\leqq\beta)$ be the $h\times h$ matrix with 1 in the α, β and β, α slots, zeroes elsewhere. Then, if $A=\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_h\end{bmatrix}\in GL(h)$, $Ae_{\alpha\beta}{}^tA$ $=\lambda_\alpha\lambda_\beta e_{\alpha\beta}$ and so $\bigwedge\limits_{\alpha\leqq\beta}(Ae_{\alpha\beta}{}^tA)=\{\prod\limits_{\alpha=\beta}^{h}\lambda_\alpha{}^{h+1}\}\bigwedge\limits_{\alpha\leqq\beta}e_{\alpha\beta}$. From this and (4.21), we find $\det\boldsymbol{N}^*=\boldsymbol{L}^{-(h+1)}$ as desired.

Remark. For later use, we record here the following offshoot of Lemma (4.22). Let $\Delta\subset\boldsymbol{C}^m$ be a neighborhood of the origin with coordinates $t^1,\cdots,t^m$ and let $\Phi\colon\Delta\to G(h,W)$ be a C^∞ mapping. Let $S_t=\Phi(t)$ and let $v_1(t),\cdots,v_h(t)$ be vectors, depending differentiably on t, and such that the $v_\alpha(t)$ span S_t.

(4.27) PROPOSITION. *If* $\frac{\partial v_\alpha(t)}{\partial\bar{t}^\rho}\equiv 0$ *modulo* S_t *for all* t, *then* Φ *is holomorphic.*

Proof. Let $\xi=\sum\limits_{\alpha=1}^{h}\xi_\alpha v_\alpha(t)\in S_t=\boldsymbol{F}_{S_t}$. Then, by (4.23),

$$\Phi_*(\frac{\partial}{\partial\bar{t}^\rho})(\xi)=\sum_{\alpha=1}^{h}\xi_\alpha\frac{\partial v_\alpha(t)}{\partial\bar{t}^\rho}$$

projected in W/S_t so that $\Phi_*(\frac{\partial}{\partial\bar{t}^\rho})=0$ throughout Δ. Thus Φ is holomorphic.

A direct argument in case $m = 1$, $t \in \Delta \subset \boldsymbol{C}$ is as follows: We want to choose $u_\alpha(t) = \sum_{\beta=1}^{h} a_{\alpha\beta}(t) v_\beta(t)$ with $\det(a_{\alpha\beta}(t)) \neq 0$ and $\frac{\partial u_\alpha(t)}{\partial \bar{t}} = 0$. We are given that $\frac{\partial v_\alpha(t)}{\partial \bar{t}} = \sum_{h}^{\beta=1} c_{\alpha\beta}(t) v_\beta(t)$. If we let $a_{\alpha\beta}(t)$ be a solution of the *differential equation*:

$$(4.28) \qquad \frac{\partial a_{\alpha\beta}(t)}{\partial \bar{t}} + \sum_{\gamma=1}^{h} a_{\alpha\gamma}(t) c_{\gamma\beta}(t) = 0, \; a_{\alpha\beta}(0) = \delta_\beta{}^\alpha;$$

then $\frac{\partial u_\alpha(t)}{\partial \bar{t}} = \sum_{\beta} \{ \frac{\partial a_{\alpha\beta}(t)}{\partial \bar{t}} v_\beta(t) + \sum_{\gamma} a_{\alpha\gamma}(t) c_{\gamma\beta}(t) v_\beta(t) \} = 0.$

We shall abbreviate (4.27) by saying:

$$(4.29) \qquad \bar{\partial} S_t \subset S_t \text{ implies that } \Phi(t) \text{ is holomorphic.}$$

Example 2. In this example, $D \cong U(n-q) \times U(q) \backslash Sp(n)$ and $X \subset G(n-q, W) \times G(n, W)$.

(4.30) PROPOSITION. *The line bundles* $\boldsymbol{L}(-\alpha_1, -\alpha_2) \to D$ *have signature* $(n\frac{(n+1)}{2}, q(n-q))$ *if* $\alpha_1 > 0$, $\alpha_2 > 0$. *In particular the canonical bundle* $\boldsymbol{K} = \boldsymbol{L}(-n, \; -n-1)$ *has a p-convex polarization with* $p = q(n-q)$.

Proof. We choose a real basis $e_1, \cdots, e_{2n}$ for $W_0 \subset W$ such that Q has matrix $Q = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. We let $\boldsymbol{P}$ be the $(n \times 2n)$ matrices Ω of rank n which satisfy $\Omega Q {}^t\Omega = 0$. Included in $\boldsymbol{P}$ is the open set P of thost Ω such that $\sqrt{-1}\, \Omega Q {}^t\bar{\Omega} = H$, where the Hermitian matrix H has signature $(n-q, q)$ and the first $(n-q) \times (n-q)$ block of H is positive definite. The group $GL(n-q, q)$ of non-singular matrices

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} (A_{11} \text{ is } (n-q) \times (n-q))$$

operates on $\boldsymbol{P}$ by $A(\Omega) = A\Omega$. Since

$$\sqrt{-1}\, (A\Omega Q {}^t\Omega {}^t\bar{A}) = AH {}^t\bar{A} = \begin{pmatrix} A_{11} H_{11} {}^t\bar{A}_{11} & * \\ * & * \end{pmatrix}$$

where $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ we see that P is invariant under $GL(n-q, q)$. Writing $\Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}$ where Ω_1 is $(n-q) \times 2n$, $A\Omega = \begin{pmatrix} A_{11}\Omega_1 \\ A_{21}\Omega_1 + A_{22}\Omega_2 \end{pmatrix}$. It follows that $GL(n-q, q) \backslash \boldsymbol{P} = X$ and $GL(n-q, q) \backslash P = D$. The representation $A \to (\det A_{11})^{\alpha_1} (\det A)^{\alpha_2}$ gives the line bundle $\boldsymbol{L}(\alpha_1, \alpha_2) \to X$ where $\boldsymbol{L}(\alpha_1, \alpha_2) = \boldsymbol{P} \times_{GL(n-q,q)} \boldsymbol{C}$. The positive real functions:

$$(4.31) \qquad \psi_1(\Omega) = \det(H_{11}),$$

$$(4.32) \qquad \psi_2(\Omega) = (-1)^q \det(H),$$

satisfy $\psi_1(A\Omega) = |\det A_{11}|^2\psi_1(\Omega)$, $\psi_2(A\Omega) = |\det A|^2\psi_2(\Omega)$ and so give metrics in $\boldsymbol{L}(1,0)$ and $\boldsymbol{L}(0,1)$ respectively. We shall compute the curvatures ω_1 and ω_2 of these metrics. Because of invariance under the group $G = Sp(n)$, it will suffice to compute ω_1 and ω_2 at

$$\Omega_0 = \left(I_n \left| \begin{matrix} \sqrt{-1}\, I_{n-q} & 0 \\ 0 & -\sqrt{-1}\, I_q \end{matrix} \right. \right).$$

(4.33) LEMMA. *For B near zero and C near* $\begin{pmatrix} \sqrt{-1}\, I_{n-q} & 0 \\ 0 & -\sqrt{-1}\, I_q \end{pmatrix}$, *the holomorphic mapping* $\Omega(B,C) = \left(\begin{matrix} I_{n-q} & B \\ 0 & I_q \end{matrix} \middle| \, C \right)$ *gives a holomorphic section of $P \to D$, provided that $\Omega(B,C)Q^t\Omega(B,C) = 0$.*

Proof. Writing $\Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}$, the conditions $(\Omega_1)_{1\cdots n-q} \neq 0$, $\Omega_{1\cdots n} \neq 0$ define a Zariski open set U containing Ω_0. If $\Omega \in U$, we can find a square matrix A_1 such that $A_1\Omega_1 = (I_{n-q}C_1)$ where C_1 is of size $(n-q) \times (n+q)$. Then

$$\begin{pmatrix} A_1 & 0 \\ 0 & I_q \end{pmatrix}\begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix} = \begin{pmatrix} I_{n-q} & C_1 \\ D_1 & E_1 \end{pmatrix}.$$

Now

$$\begin{pmatrix} I & 0 \\ -D_1 & I \end{pmatrix}\begin{pmatrix} I & C_1 \\ D_1 & E_1 \end{pmatrix} = \begin{pmatrix} I & C_1 \\ 0 & E_2 \end{pmatrix}$$

and, since $\Omega_{1\cdots n} \neq 0$, $(E_2)_{1\cdots q} \neq 0$. Thus we will have, for some $A_2 \in GL(q)$,

$$\begin{pmatrix} I & 0 \\ 0 & A_2 \end{pmatrix}\begin{pmatrix} I & C_1 \\ 0 & E_2 \end{pmatrix} = \left(\begin{matrix} I & B \\ 0 & I \end{matrix} \middle| \, C \right).$$

In summary, if $\Omega \in U$, we can find $A \in GL(n-q, q)$ such that

$$A\Omega = \left(\begin{matrix} I & B \\ 0 & I \end{matrix} \middle| \, C \right)$$

and so $A\Omega = \Omega(B,C)$. This proves the Lemma.

Writing $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$, the relation $\Omega Q^t\Omega = 0$ becomes:

$$(4.34) \qquad \begin{cases} C_{11} + C_{12}{}^tB = {}^t(C_{11} + C_{12}{}^tB) \\ C_{12} = {}^t(C_{21} + C_{22}{}^tB) \\ C_{22} = {}^tC_{22}. \end{cases}$$

The matrix $H(B,C)=\sqrt{-1}\,\{\Omega(B,C)Q^t\bar{\Omega}(B,C)\}$ is given by:

$$(4.35)\quad H=\sqrt{-1}\begin{pmatrix} {}^t(\bar{C}_{11}+\bar{C}_{12}{}^tB)-(C_{11}+C_{12}{}^t\bar{B}) & {}^t\bar{C}_{21}+B{}^t\bar{C}_{22}-C_{12} \\ {}^t\bar{C}_{12}-(C_{21}+C_{22}{}^t\bar{B}). & {}^t\bar{C}_{22}-C_{22} \end{pmatrix}$$

Write $H=\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ and observe that $H(\Omega_0)=2\begin{pmatrix} I_{n-q} & 0 \\ 0 & -I_q \end{pmatrix}=H_0$. For notation, we let $H_0=\begin{bmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{bmatrix}$, $\Delta_1=\det(H_{11})$, $\Delta_2=(-1)^q\det H$, and we agree on the ranges of indices $1\leqq\alpha$, $\beta\leqq n$, $1\leqq i$, $j\leqq n-q$, $n-q+1\leqq\rho$, $\sigma\leqq n$. We have to compute $\omega_1=-\partial\bar{\partial}\log\Delta_1$ and $\omega_2=-\partial\bar{\partial}\log\Delta_2$ at Ω_0. We calculate:

$$\omega_2=-\partial\partial\log\Delta_2=\frac{-\partial\bar{\partial}\Delta}{\Delta}]_{\Omega_0}+\frac{\partial\Delta\wedge\bar{\partial}\Delta}{\Delta^2}]_{\Omega_0}$$

$$=-\frac{1}{\Delta}\sum_{\left\{\substack{\alpha,\beta\\ \lambda,\mu}\right.}\frac{\partial^2\Delta}{\partial h_{\alpha\beta}\partial h_{\lambda\mu}}]_{H_0}\partial h_{\alpha\beta}\wedge\bar{\partial}h_{\lambda\mu}-\frac{1}{\Delta}\sum_{\alpha,\beta}\frac{\partial\Delta}{\partial h_{\alpha\beta}}]_{H_0}\partial\bar{\partial}h_{\alpha\beta}$$

$$+\frac{1}{\Delta^2}\sum_{\left\{\substack{\alpha,\beta\\ \lambda,\mu}\right.}\frac{\partial\Delta}{\partial h_{\alpha\beta}}]_{H_0}\frac{\partial\Delta}{\partial h_{\lambda\mu}}]_{H_0}\partial h_{\alpha\beta}\wedge\bar{\partial}h_{\lambda\mu}=(\text{by }(4.17))$$

$$-\sum_{\alpha\neq\beta}\frac{1}{h_\alpha h_\beta}\partial h_{\alpha\alpha}\wedge\bar{\partial}h_{\beta\beta}+\sum_{\alpha\neq\beta}\frac{1}{h_\alpha h_\beta}\partial h_{\alpha\beta}\wedge\bar{\partial}h_{\beta\alpha}-\sum_{\alpha}\frac{1}{h_\alpha}\partial\bar{\partial}h_{\alpha\alpha}$$

$$+\sum_{\alpha,\beta}\frac{1}{h_\alpha h_\beta}\partial h_{\alpha\alpha}\wedge\bar{\partial}h_{\beta\beta}$$

which gives:

$$(4.36)\qquad \omega_2=\sum_{\alpha}\frac{1}{(h_\alpha)^2}\partial h_{\alpha\alpha}\wedge\bar{\partial}h_{\alpha\alpha}+\sum_{\alpha\neq\beta}\frac{1}{h_\alpha h_\beta}\partial h_{\alpha\beta}\wedge\bar{\partial}h_{\beta\alpha}-\sum_{\alpha}\frac{1}{h_\alpha}\partial\bar{\partial}h_{\alpha\alpha}.$$

Similarly, we have:

$$(4.37)\qquad \omega_1=\sum_{i}\frac{1}{(h_i)^2}\partial h_{ii}\wedge\bar{\partial}h_{ii}+\sum_{i\neq j}\frac{1}{h_i h_j}\partial h_{ij}\wedge\bar{\partial}h_{ji}-\sum_{i}\frac{1}{h_i}\partial\bar{\partial}h_{ii}.$$

From (4.35), we have at Ω_0: $\partial h_{ij}=-\sqrt{-1}\,dc_{ij}$, $\partial h_{\rho\sigma}=-\sqrt{-1}\,dc_{\rho\sigma}$, $\partial h_{i\rho}=\sqrt{-1}(\sqrt{-1}db_{i\rho}-dc_{i\rho})$, $\partial h_{\rho i}=-\sqrt{-1}\,dc_{\rho i}$,

$$\partial\bar{\partial}h_{ii}=\sqrt{-1}\,(\sum_{\rho}db_{i\rho}\wedge d\bar{c}_{i\rho}-dc_{i\rho}\wedge d\bar{b}_{i\rho}),$$

$$\bar{\partial}h_{ij}=\sqrt{-1}\,d\bar{c}_{ji},\ \bar{\partial}h_{\rho\sigma}=\sqrt{-1}\,d\bar{c}_{\sigma\rho},\ \bar{\partial}h_{i\rho}=\sqrt{-1}\,d\bar{c}_{\rho i},$$

$\bar{\partial}h_{\rho i}=\sqrt{-1}\,(d\bar{c}_{i\rho}+\sqrt{-1}\,d\bar{b}_{i\rho})$. From (4.34) we have at Ω_0: $dc_{ij}=dc_{ji}$, $dc_{i\rho}=dc_{\rho i}-\sqrt{-1}\,db_{i\rho}$, $dc_{\rho\sigma}=dc_{\sigma\rho}$. Combining gives:

$$\omega_2 = \tfrac{1}{4}\{\sum_{\alpha} dc_{\alpha\alpha} \wedge d\bar{c}_{\alpha\alpha} + \sum_{i \neq j} dc_{ij} \wedge d\bar{c}_{ij} + \sum_{\rho \neq \sigma} dc_{\rho\sigma} \wedge d\bar{c}_{\rho\sigma}\}$$
$$+ \tfrac{1}{2} \sum_{\rho, i} dc_{\rho i} \wedge d\bar{c}_{\rho i} + 2 \sum_{i, \rho} dc_{i\rho} \wedge d\bar{c}_{i\rho} - \sum_{i, \rho} dc_{\rho i} \wedge d\bar{c}_{i\rho} - \sum_{\rho, i} dc_{i\rho} \wedge d\bar{c}_{\rho i};$$

that is:

$$\omega_2 = \tfrac{1}{4}\{\sum_{\alpha} dc_{\alpha\alpha} \wedge d\bar{c}_{\alpha\alpha} + 2 \sum_{i<j} dc_{ij} \wedge d\bar{c}_{ij} + 2 \sum_{\rho<\sigma} dc_{\rho\sigma} \wedge d\bar{c}_{\rho\sigma}\} \tag{4.38}$$
$$+ \sum_{\rho, i} \{ (\frac{dc_{\rho i}}{\sqrt{2}} - \sqrt{2}\, dc_{i\rho}) \wedge (\frac{d\bar{c}_{\rho i}}{\sqrt{2}} - \sqrt{2}\, d\bar{c}_{i\rho}) \}.$$

Similarly,

$$\begin{aligned} \omega_1 &= \tfrac{1}{4}\{\sum_{i} dc_{ii} \wedge d\bar{c}_{ii} + 2 \sum_{i<j} dc_{ij} \wedge d\bar{c}_{ij}\} + \sum_{i, \rho} dc_{i\rho} \wedge d\bar{c}_{i\rho} \\ &\quad - \tfrac{1}{2} \{ \sum_{\rho, i} dc_{\rho i} \wedge d\bar{c}_{i\rho} + dc_{i\rho} \wedge d\bar{c}_{\rho i} \}. \end{aligned} \tag{4.39}$$

Now we set $\omega(\alpha_1, \alpha_2) = \alpha_1\omega_1 + \alpha_2\omega_2$. Then:

$$\begin{aligned} \omega(\alpha_1, \alpha_2) &= \phi(\alpha_1, \alpha_2) + (\alpha_1 + 2\alpha_2)\{\sum_{i, \rho} dc_{i\rho} \wedge d\bar{c}_{i\rho}\} \\ &\quad + \frac{\alpha_2}{2}\{\sum dc_{\rho i} \wedge d\bar{c}_{\rho i}\} + (-\frac{\alpha_1}{2} - \alpha_2)\{\sum_{\rho, i} dc_{\rho i} \wedge d\bar{c}_{i\rho} + dc_{i\rho} \wedge d\bar{c}_{\rho i}\}, \end{aligned} \tag{4.40}$$

where

$$\phi(\alpha_1, \alpha_2) = \tfrac{1}{4}\{(\alpha_1 + \alpha_2) \sum_{i} dc_{ii} \wedge d\bar{c}_{ii} + \alpha_2 \sum_{\rho} dc_{\rho\rho} \wedge d\bar{c}_{\rho\rho}$$
$$+ 2(\alpha_1 + \alpha_2) \sum_{i<j} dc_{ij} \wedge d\bar{c}_{ij} + 2\alpha_2 \sum_{\rho<\sigma} dc_{\rho\sigma} \wedge d\bar{c}_{\rho\sigma}\}$$

is positive definite. Since

$$\frac{(n-q)(n-q+1)}{2} + \frac{q(q+1)}{2} + q(n-q) = \frac{n(n+1)}{2},$$

it will follow that $\omega(\alpha_1, \alpha_2)$ has signature $(\frac{n(n+1)}{2}, q(n-q))$ for $\alpha_1 > 0$, $\alpha_2 > 0$ if we show that the matrix

$$M = \begin{bmatrix} \alpha_1 + 2\alpha_2 & -\frac{\alpha_1}{2} - \alpha_2 \\ -\frac{\alpha_1}{2} - \alpha_2 & \frac{\alpha_2}{2} \end{bmatrix}$$

has signature $(1,1)$. But $\alpha_1 + 2\alpha_2 > 0$ and

$$\det M = \frac{\alpha_1\alpha_2}{2}\sqrt{-1} + \alpha_2{}^2 - \frac{\alpha_1{}^2}{4} - \alpha_1\alpha_2 - \alpha_2{}^2 = -\frac{\alpha_1}{2}(\frac{\alpha_1}{2} + \alpha_2) < 0.$$

This proves that $\boldsymbol{L}(-\alpha_1,-\alpha_2)$ has signature $(\frac{n(n+1)}{2}, q(n-q))$ for $\alpha_1>0$, $\alpha_2>0$.

The proof that $\boldsymbol{K}=\boldsymbol{L}(-n,-n-1)$ is similar to the argument in Example 1 and will be omitted.

Remark. It can easily be checked that the possible signatures of the curvature forms of the bundles $\boldsymbol{L}(\alpha_1,\alpha_2)$ are:

$$(\frac{n(n+1)}{2}, q(n-q)),\ (\frac{n(n+1)}{2}-q(n-q), 2q(n-q)),$$

$$(\frac{q}{2}(2n-q+1), \frac{n-q}{2}(n+q+1)),$$

$$(2q(n-q)+\frac{q(q+1)}{2}, \frac{(n-q)(n-q+1)}{2}),$$

and the negatives of these. In particular, none of the bundles $\boldsymbol{L}(\alpha_1,\alpha_2)$ is positive.

(c) Let $D\cong H\backslash G$ be a period matrix domain and $K\subset G$ the maximal compact subgroup, $Y_0=H\backslash K$ the K-orbit of the origin in $H\backslash G$.

(4.41) THEOREM. (i) *$Y_0\subset D$ is a compact complex submanifold and the family of analytic subvarieties $\{gY_0\}_{g\in G}$ gives a fibering of D with compact, complex analytic fibres, but with a generally non-holomorphic parameter space.* (ii) *If $p=\dim Y_0=\dim H\backslash K$,* then the *canonical bundle $\boldsymbol{K}\to D$ has a p-convex polarization and $\boldsymbol{K}\mid Y_0$ is negative.*

Proof. A proof along group-theoretic lines has been indicated in [7]. However, we shall discuss the two examples as the explicit form of the compact subvarieties will be needed.

Case 1. We consider $D\subset G(h,W)$ as given by the relations (4.1). Recall that, for $S\in D$, we had set $R_S=(S\oplus\bar{S})^{\perp}$, and we let: $X_S=\{S'\in D, Q(S',R_S)=0\}$. Obviously $X_S\subset D$ is an analytic subvariety passing through S. We observe that $S'\in X_S$ if, and only if,

$$S'\oplus\bar{S}'=S\oplus\bar{S}=R_S{}^{\perp}. \tag{4.42}$$

Thus, if X_S meets $X_{S'}$, then $X_S=X_{S'}$ so that the subvarieties $\{X_S\}$ give a fibering of D by complex analytic subvarieties.

Now let $S_1\in X_S$ so that $S_1\oplus\bar{S}_1=S\oplus\bar{S}$. Then there exists $T\in G$ $(\cong SO(2h,k;\boldsymbol{R}))$ with $T(S)=S_1$ Thus $T\colon W\to W$ and $T(S\oplus\bar{S})=S\oplus\bar{S}$ so that T splits: $T=T_1\oplus T_2$ where $T_1=T\mid S\oplus\bar{S}$, $T_2=T\mid R_S$. Furthermore $T_1(S) = S_1$ so that, if G_S is the group of transformations $T_1\colon S \oplus \bar{S} \to S \oplus \bar{S}$ induced by $T\colon W \to W$ which are in G and which split, then G_S acts transitively on X_S. But $G_S \cong SO(2h, \boldsymbol{R})$ is compact and so $X_S \subset D$ is compact. Since $K = SO(2h) \times SO(k)$, $H = U(h) \times SO(k)$, it is clear that the varieties X_S are just the G-translates of $H\backslash K$. Observe that

$$\dim_{\boldsymbol{C}} H\backslash K = \tfrac{1}{2}\{\frac{2h(2h-1)}{2} - h^2\} = \frac{h^2 - h}{2}.$$

We now give Y_0 in terms of period matrices: we follow the notation in the proof of Proposition 4.9 so that $Q = \begin{pmatrix} I_{2h} & 0 \\ 0 & -I_k \end{pmatrix}$. The origin has period matrix $\Omega_0 = (I, \sqrt{-1}\, I, 0)$ and we claim that Y_0 are just the points with period matrixes $\Omega = (A, B, 0)$, $A^tA + B^tB = 0$ (note that the relation $A^t\bar{A} + B^t\bar{B} > 0$ is automatic). In fact, $S_0 \oplus \bar{S}_0$ is the vector space with basis $e_1, \cdots, e_{2h}$ and so R_{S_0} has basis $e_{2h+1}, \cdots, e_n$. Thus, Y_0 consists of all subspaces S whose period matrix Ω satisfies $\Omega Q \xi_\rho = 0$ where $\xi_\rho = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ has a one in the $2h + \rho$ position. But $\Omega Q \xi_\rho$ is the $2h + \rho$-th column of Ω, which proves our assertion.

Referring to (4.20), since Y_0 is given by $c_{\alpha\rho} = 0$, ω_{Ω_0} restricted to the tangent space to Y_0 is:

$$-\tfrac{1}{2} \sum_{\alpha<\beta} db_{\alpha\beta} \wedge d\bar{b}_{\alpha\beta} \tag{4.43}$$

and so $\omega \mid Y_0$ is negative definite. Thus $\boldsymbol{L}^{-1} \mid Y_0$ is negative, as is $\boldsymbol{K} = \boldsymbol{L}^{-(h+k-1)}$.

Remark. An intrinsic description of Y_0 is the following. Let E_0 be a real $2h$-dimensional vector space on which we have a positive quadratic form Q. If $E = E_0 \otimes_{\boldsymbol{R}} \boldsymbol{C}$ is the complexification of E, we let $Y \subset G(h, E)$ be those h-dimensional subspaces $R \subset E$ which satify $Q(R, R) = 0$. Then $Y_0 \cong Y \cong SO(2h)/U(h)$.

Case 2. We consider $D \subset G(n - q, W) \times G(n, W)$ given by the relations (4.2). For $S = [S_1, S_2] \in D$, we set $R_S = S_1 \oplus \bar{S}_2/\bar{S}_1$ where

$$S_2/S_1 = \{v \in S_2 \mid Q(v, \bar{S}_1) = 0\}.$$

Then $Q(R_S, R_S) = 0$ and $\sqrt{-1}\, Q(R_S, \bar{R}_S) > 0$ (cf. Proposition 3.21). We let: $X_S = \{S' \in D \mid R_S = R_{S'}\}$. We claim that $X_S \subset D$ is an analytic subvariety passing through S.

Let $S' \in X_S$; $S' = [S_1', S_2']$. Then $S_1' \subset R_S$ and $\sqrt{-1}\,Q(S_1', \bar{S}_1') > 0$. We define $R_S/S_1' = \{w \in R_S \mid Q(w, \bar{S}_1') = 0\}$ and claim that $S_2' = S_1' \oplus \bar{R}_S/\bar{S}_1'$. In fact, $R_S = R_{S'} = S_1' \oplus \bar{S}_2'/\bar{S}_1' = S_1' \oplus R_S/S_1'$ so that $\bar{S}_2'/\bar{S}_1' = R_S/S_1'$ or $S_2' = S_1' \oplus \bar{R}_S/\bar{S}_1'$. Thus $S' \in X_S$ is uniquely given by $S_1' \subset R_S$; i.e., $X_S \cong G(n-q, R_S)$. This shows that X_S is a complex subvariety, isomorphic to a Grassmannian $G(n-q, \mathbf{C}^n)$.

The argument used for Case 1 shows that the group G_S of automorphisms of X_S induced by automorphisms in G acts transitively on X_S and is compact; $G_S \cong U(n)$ and $X_S \cong U(n-q) \times U(q) \backslash U(n) = H \backslash K$.

The mapping $H\backslash G \to K \backslash G$ is the (non-holomorphic) mapping $D \to \mathbf{H}_n$ discussed in Proposition 3.21 given by $S \to R_S$.

Let now S be the point with period matrix

$$\Omega_0 = \left(I \;\middle|\; \begin{matrix} \sqrt{-1}\,I_{n-q} & 0 \\ 0 & -\sqrt{-1}\,I_q \end{matrix} \right).$$

We want to parametrize the variety $X_{S_0} \subset D$ passing through S_0. We let $\xi_\alpha = e_\alpha + \sqrt{-1}\,e_{n+\alpha}$; the vectors ξ_α give a basis for R_{S_0} and S_0 corresponds to the $(n-q)$ plane S_1 in R_{S_0} with basis $\xi_1, \cdots, \xi_{n-q}$. Let $S' \in X_{S_0}$ be a point close to S_0; then S is given by an $(n-q)$ plane $S_1' \subset R_{S_0}$ with S_1' close to S_1. We may assume then that S_1' has a basis $\zeta_1, \cdots, \zeta_{n-q}$ where $\zeta_i = \xi_i + \sum_{\rho=n-q+1}^{n} b_{i\rho}\xi_\rho$. Thus S_1' has period matrix

$$\Omega_1' = (I_{n-q} B \sqrt{-1}\,I_{n-q} \sqrt{-1}\,B)$$

where $B = (b_{i\rho})$.

Now S_1' determines uniquely $S_2' \in G(n, W)$ such that $S' = [S_1', S_2']$, and the period matrix Ω' of S_2' will be:

$$\Omega' = \left(\begin{matrix} I_{n-q} & B \\ \alpha & I_q \end{matrix} \;\middle|\; \begin{matrix} \sqrt{-1}\,I_{n-q} & \sqrt{-1}\,B \\ \gamma & \delta \end{matrix} \right) = \begin{pmatrix} \Omega_1' \\ \Omega_2' \end{pmatrix}.$$

Here Ω_2' should be the period matrix of S_2'/S_1'; i.e. $\Omega_2' Q^t \bar{\Omega}_1' = 0$. The condition $S_1' \oplus \bar{S}_2'/\bar{S}_1' = R_{S'} = R_S$ can be written: $Q(R_S, S_1' \oplus \bar{S}_2'/\bar{S}_1') = 0$ or $\begin{pmatrix} \Omega_1' \\ \bar{\Omega}_2' \end{pmatrix} Q^t \begin{pmatrix} \Omega_1 \\ \bar{\Omega}_2 \end{pmatrix} = 0$, where $\Omega_0 = \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}$. Finally we must have $\Omega' Q^t \Omega' = 0$. The condition $\Omega_2' Q^t \bar{\Omega}_1' = 0$ is: $-\gamma - \delta^t\bar{B} - \sqrt{-1}\,\alpha - \sqrt{-1}\,{}^t\bar{B} = 0$. Since $\begin{pmatrix} \Omega_1 \\ \bar{\Omega}_2 \end{pmatrix} = (I_n, \sqrt{-1}\,I_n)$, then condition $\begin{pmatrix} \Omega_1' \\ \bar{\Omega}_2' \end{pmatrix} Q^t \begin{pmatrix} \Omega_1 \\ \bar{\Omega}_2 \end{pmatrix} = 0$ becomes: $\sqrt{-1}\,\bar{\alpha} - \bar{\gamma} = 0$, $\sqrt{-1}\,I - \delta = 0$. This gives $\delta = -\sqrt{-1}\,I$ and $\gamma = -\sqrt{-1}\,\alpha$. Note that these equations give

$$\begin{aligned}&-\gamma-\delta^t\bar{B}-\sqrt{-1}\,\alpha-\sqrt{-1}\,{}^t\bar{B}\\&\qquad=\sqrt{-1}\,\alpha+\sqrt{-1}\,{}^t\bar{B}-\sqrt{-1}\,\alpha-\sqrt{-1}\,{}^t\bar{B}=0.\end{aligned}$$

The condition $\Omega' Q^{t\prime}\Omega=0$ gives:

$$-\sqrt{-1}\,{}^t\alpha-\sqrt{-1}\,B+{}^t\gamma+B^t\delta=0,\quad -\gamma^t\alpha-\delta+\alpha^t\gamma+{}^t\delta=0.$$

Thus $\delta=-\sqrt{-1}\,I$, $\gamma=-\sqrt{-1}\,\alpha$, $-\gamma^t\alpha-\delta+\alpha^t\gamma+{}^t\delta=0$ and the first condition is

$$0=-\sqrt{-1}\,{}^t\alpha-\sqrt{-1}\,B+{}^t\gamma-\sqrt{-1}\,B=-2\sqrt{-1}\,B-2\sqrt{-1}\,{}^t\alpha$$

or $\alpha=-{}^tB$, $\gamma=-\sqrt{-1}\,{}^tB$. Consequently:

$$\Omega'=\left(\begin{array}{cc|cc} I & B & \sqrt{-1}\,I & \sqrt{-1}\,B\\ -{}^tB & I & \sqrt{-1}\,{}^tB & -\sqrt{-1}\,I\end{array}\right).$$

Now

$$\begin{aligned}\Omega'&\sim\begin{pmatrix} I & 0\\ {}^tB & I\end{pmatrix}\begin{pmatrix} I & B & \sqrt{-1}\,I & \sqrt{-1}\,B\\ -{}^tB & I & \sqrt{-1}\,{}^tB & -\sqrt{1}\end{pmatrix}\\&=\begin{pmatrix} I & B & \sqrt{-1}\,I & \sqrt{-1}\,B\\ 0 & {}^tBB+I & 2\sqrt{-1}\,{}^tB & \sqrt{-1}\,({}^tBB-I)\end{pmatrix}\\&\sim\begin{pmatrix} I & B & \sqrt{-1}\,I & \sqrt{-1}\,B\\ O & I & \dfrac{2\sqrt{-1}\,{}^tB}{({}^tBB+I)} & \dfrac{\sqrt{-1}\,({}^tBB-I)}{({}^tBB-I)}\end{pmatrix},\end{aligned}$$

which puts Ω' in the form given by Lemma (4.33). Thus $B=B$,

$$C_{11}=\sqrt{-1}\,I,\ C_{12}=\sqrt{-1}\,B,\ C_{21}=\frac{2\sqrt{-1}\,{}^tB}{({}^tBB+I)},\ C_{22}=\frac{\sqrt{-1}\,({}^tBB-I)}{({}^tBB+1)}.$$

Now, at $B=0$, $dc_{ij}=0$, $dc_{i\rho}=\sqrt{-1}\,db_{i\rho}$, $dc_{\rho i}=2\sqrt{-1}\,db_{i\rho}$, $dc_{\rho\sigma}=0$. By (4.40), $\omega(\alpha_1,\alpha_2)$ on the tangent space to X_{S_0} at S_0 becomes:

$$(-\alpha_1)\sum_{i,\rho}(db_{i\rho}\wedge d\bar{b}_{i\rho}). \tag{4.44}$$

For the canonical bundle, $\alpha_1=n$ so that $\boldsymbol{K}\mid X_{S_0}$ is negative as desired.

(d) Let $D=H\backslash G$ be a period matrix domain and $\boldsymbol{E}=\boldsymbol{K}^\nu\to D$ a fixed positive power of the canonical bundle. Then, by Theorem (4.41), $\boldsymbol{E}$ has a G-invariant p-convex polarization where $p=\dim Y_0$, $Y_0=H\backslash K$.

Given a discrete subgroup $\Gamma\subset G$, Γ acts properly discontinuously on D and the quotient space $D/\Gamma=M$ is an analytic space. Furthermore, $\boldsymbol{E}/\Gamma=\boldsymbol{E}\to M$ is a line bundle with p-convex polarization. Now then one is led to expect that $H^q(M, \mathcal{O}(\boldsymbol{E})) = 0$ for $q \neq p$ and that $H^p(M, \mathcal{O}(\boldsymbol{E}))$ should become large as $\nu \rightarrow \infty$ (cf. Proposition 3.23). In case M is compact, this theorem on *automorphic cohomology* has been proved in [7], and we now prove it in the other extreme where $\Gamma = \{e\}$.

(4.45) THEOREM. *$H^q(D, \mathcal{O}(\boldsymbol{E})) = 0$ for $q > p$ and* $\dim H^p(D, \mathcal{O}(\boldsymbol{E}))$ $= \infty$. *In fact, $H^p(D, \mathcal{O}(\boldsymbol{E}))$ can be "expanded in a power series around Y_0."*

Proof. The following Lemma, due to W. Schmid, is crucial:

(4.46) LEMMA. *There exists an exhaustion function ϕ on D such that the E. E. Levi form $L(\phi)$ has everywhere $n - p$ positive eigenvalues.*

Proof. Let D be a period matrix domain of type 1 or 2 (the general argument is similar). In the first case we consider the line bundle $\boldsymbol{L}^{-1} \rightarrow D$ (cf. Proposition 4.9) and in the second case $\boldsymbol{L}(-1, -1) \rightarrow D$ (cf. Proposition 4.30). In each case we have a principal bundle $P \rightarrow D$ and metrics $\psi(\Omega)$ $(\Omega \in P)$ for the line bundles $\boldsymbol{L}^{-1}, \boldsymbol{L}(-1, -1)$. In case 1, $\psi(\Omega) = \det(\Omega Q^t \bar{\Omega})$ and, in case 2, $\psi(\Omega) = \det(\sqrt{-1}\,\Omega_1 Q^t \bar{\Omega}_1) \cdot (-1)^q \det(\sqrt{-1}\,\Omega Q^t \bar{\Omega})$.

Now D is covered by finitely many Zariski open sets U_α and there exist holomorphic cross-sections $\sigma_\alpha \colon U_\alpha \rightarrow P$ over U_α. We let $\psi_\alpha = \psi \circ \sigma_\alpha$ and $\phi_\alpha = -\log \psi_\alpha$. Then the Levi form $\partial\bar{\partial}\phi_\alpha = -\partial\bar{\partial}\log\psi_\alpha$ is the curvature in $\boldsymbol{L}^{-1}$, respectively $\boldsymbol{L}(-1, 1)$. Thus $\partial\bar{\partial}\phi_\alpha = \partial\bar{\partial}\phi_\beta$ in $U_\alpha \cap U_\beta$and $\omega = \{\partial\bar{\partial}\phi_\alpha\}$ is a G-invariant form with signature $(n - p, p)$.

Suppose now that $z \notin U_\alpha$ and $\{z_n\} \subset U_\alpha$ is a sequence with $z_n \rightarrow z$. We claim that $\phi_\alpha(z_n) \rightarrow -\infty$ as $z_n \rightarrow z$. For example, take D to be of type 1 and U_α to be given by $\Omega_{1 \cdots n} \neq 0$. Then z has period matrix (A, B, C) with $\det(A) = 0$, and $z_n = (A_n, B_n, C_n) = \Psi_n$ with $\det(A_n) \neq 0$. We may suppose that $(A_n, B_n, C_n) \rightarrow (A, B, C)$ so that $\det(A_n) \rightarrow 0$. Then $\sigma_\alpha(z_n) = \Omega_n$ $= (I, A_n^{-1}B_n, A_n^{-1}C_n)$ and $\phi_\alpha(z_n) = -\log\psi_\alpha(z_n)$ where

$$\psi_\alpha(z_n) = \det(\Omega_n Q^t \bar{\Omega}_n) = |\det(A_n)|^{-2} \det(\Psi_n Q^t \Psi_n).$$

Thus $\lim\limits_{z_n \rightarrow z} \psi_\alpha(z_n) = +\infty$ and so $\lim\limits_{z_n \rightarrow z} \phi_\alpha(z_n) = -\infty$ as desired.

It follows that, if we set $\phi_\alpha(z) = -\infty$ for $z \notin U_\alpha$, $\phi(z) = \sum\limits_\alpha e^{\phi_\alpha(z)}$ is a globally defined C^∞ function on D. The Levi form

$$L(\phi) = \partial\bar{\partial}\phi = \sum_\alpha L(\phi_\alpha) e^{\phi_\alpha(z)} + \sum_\alpha \partial\phi_\alpha \wedge \bar{\partial}\phi_\alpha e^{\phi_\alpha(z)}$$

$$= \phi(z)\omega + (\text{positive semi-definite form}) \geqq \phi(z)\omega.$$

Thus $L(\phi)$ has everywhere at least $n - p$ positive eigenvalues.

It remains to show that $\phi(z)$ is an exhaustion function; i.e. the sets $D_c = \{z \in D : \phi(z) < c\}$ are relatively compact and $\bigcup_c D_c = D$. This involves looking at $\phi(z)$ near $\partial D = \bar{D} - D$. Suppose that $z \in \partial D \cap U_\alpha$ and let $\{z_n\} \subset U_\alpha$ be a sequence with $z_n \rightarrow z$. Looking again at the above example, we will have $z_n = (I, B_n', C_n') = \Omega_n$ with $H_n = \Omega_n Q^t \bar{\Omega}_n$ and $H_n > 0$. Since z_n tends toward the boundary, H_n becomes singular so that $\det(H_n) \rightarrow 0$. But $\phi_\alpha(z_n) = -\log\{\det(H_n)\}$ so that $\phi_\alpha(z_n) \rightarrow +\infty$ as $z_n \rightarrow z$. It now follows that $\phi(z)$ is an exhaustion function.

Now the extension by Andreotti-Grauert [1] of Theorem B gives:

(4.47) $H^q(D, \mathcal{S}) = 0$ for $q > p$ and $\mathcal{S}$ any *coherent sheaf* over D.

We now use the method of [6] to expand $H^p(D, \mathcal{O}(\boldsymbol{L}))$ around $Y_0 \subset D$. Let $I^\mu(\boldsymbol{E}) \subset \mathcal{O}(\boldsymbol{E})$ be the sections of $\boldsymbol{E}$ vanishing to order μ along Y_0; $I^\mu(\boldsymbol{E}) = I^\mu \otimes_{\mathcal{O}} \mathcal{O}(\boldsymbol{E})$ where $I \subset \mathcal{O}_D$ is the ideal sheaf of Y_0. Then, if $\boldsymbol{N} \rightarrow Y_0$ is the *normal bundle* and $\boldsymbol{N}^{*(\mu)}$ the μ-th symmetric product of $\boldsymbol{N}^*$, we have:

(4.48) $0 \rightarrow I^{\mu+1}(\boldsymbol{E}) \rightarrow I^\mu(\boldsymbol{E}) \rightarrow \mathcal{O}_{Y_0}(\boldsymbol{N}^{*(\mu)} \otimes \boldsymbol{E}) \rightarrow 0.$

From (4.47) we have:

(4.49) $H^p(I^{\mu+1}(\boldsymbol{E})) \rightarrow H^p(I^\mu(\boldsymbol{E})) \rightarrow H^p(\mathcal{O}_{Y_0}(\boldsymbol{N}^{*(\mu)} \otimes \boldsymbol{E})) \rightarrow 0.$

This gives:

(4.50) LEMMA. *$H^p(D, \mathcal{O}(\boldsymbol{E}))$ has a decreasing filtration*

$$F_\mu \subset H^p(D, \mathcal{O}(\boldsymbol{E}))$$

and the associated graded module is: $\sum_{\mu=0}^{\infty} H^p(Y_0, \mathcal{O}_{Y_0}(\boldsymbol{N}^{*(\mu)} \otimes \boldsymbol{E}))$.

Now $\boldsymbol{E} \mid Y_0 = \boldsymbol{K}^\nu \mid Y_0$ is a *negative line bundle* (cf. the proof of Proposition (4.9)) while $\boldsymbol{N} \rightarrow Y_0$ is spanned by its global sections (cf. [7]). Thus $H^r(\mathcal{O}_{Y_0}(\boldsymbol{N}^{*(\mu)} \otimes \boldsymbol{E})) \cong H^{p-r}(\mathcal{O}_{Y_0}(\boldsymbol{E}^* \otimes \boldsymbol{N}^{(\mu)} \otimes \boldsymbol{K}_Y))$, and it follows [7] that $H^r(\mathcal{O}_{Y_0}(\boldsymbol{N}^{*(\mu)} \otimes \boldsymbol{E})) = 0$ for $0 \leqq r < p$ and $\dim H^p(\mathcal{O}_{Y_0}(\boldsymbol{N}^{*(\mu)} \otimes \boldsymbol{E})) \rightarrow \infty$ as $\mu \rightarrow \infty$. This completes the proof of Theorem 4.45.

Remark. It can be shown that $\dim H^p(\mathcal{O}_{Y_0}(\boldsymbol{N}^{*(\mu)} \otimes \boldsymbol{E})) = c\mu^p +$ (lower order terms) where $c > 0$.

(e) In the above discussions (cf. Theorems (3.20) and (4.8)) it has appeared that certain contructions in transcendental algebraic geometry lead naturally to varieties with a p-convex polarization, where p need not be zero as in the classical case. The cohomology groups in dimension p, instead of being viewed as obstructions, then become primary objects of interest. This is, at least at first glance, an unfortunate state of affairs because it is not easy to see the geometric relevance of these groups.

In this section we shall give further evidence that the cohomology groups are natural invariants and also give a possible geometric interpretation of them. Basically, what we want to prove is:

(*) THEOREM. *If* $\Gamma \subset G$ *is a discrete group acting properly discontinuously on* D, *and if* $\phi \in H^p(D, \boldsymbol{E})$ *is an absolutely integrable cohomology class, then the Poincaré series in cohomology:*

$$\theta(\phi) = \sum_{\gamma \in \Gamma} \gamma^*(\phi), \tag{4.51}$$

converges to an automorphic cohomology class in $H^q(D/\Gamma, \mathcal{O}(\boldsymbol{E}))$ (*as above,* $\boldsymbol{E} = \boldsymbol{K}^\nu$ *is a power of the canonical bundle*).

What we shall actually prove is something slightly weaker which will, however, interpret the cohomology as sections of a bundle. We proceed in several steps.

(i) Let $D = H\backslash G$ be a period matrix space and $Y_0 = H\backslash K$. We let $\boldsymbol{G}$ be the complexification of G and consider the set of all subvarieties gY_0 where $gY_0 \subset D$ and g is in the complex group $\boldsymbol{G}$. This makes sense since $D \subset X$ and $\boldsymbol{G}$ acts on X (cf. Theorem 4.3). In this way we get an analytic family $\{Y_\lambda\}_{\lambda \in \mathfrak{B}}$ of compact, complex submanifolds $Y_\lambda \subset D$ (cf. [6], Section III.2) whose parameter space $\mathfrak{B}$ is an open complex manifold. If $Y_0 = Y_{\lambda_0}$, then $\boldsymbol{T}_{\lambda_0}(\mathfrak{B}) \cong H^0(Y_0, \mathcal{O}(\boldsymbol{N}))$, where $\boldsymbol{N} \to Y_0$ is the holomorphic normal bundle. It is clear that G acts on $\mathfrak{B}$; in fact, it is proved in [7] that $\mathfrak{B} \subset \boldsymbol{K}\backslash\boldsymbol{G}$ where $\boldsymbol{K} \subset \boldsymbol{G}$ is the stabilizer of Y_0. In general, G does *not* act transitively on $\mathfrak{B}$.

Over $\mathfrak{B}$ we may construct an analytic fibre space $\mathfrak{D} \xrightarrow{\pi} \mathfrak{B}$ such that $\pi^{-1}(\lambda) \cong Y_\lambda$ and such that there is a holomorphic mapping $\tilde{\omega}\colon \mathfrak{D} \to D$ where $\tilde{\omega}(\pi^{-1}(\lambda)) = Y_\lambda$; (cf. [6], Theorem 3.1). We thus get a diagram:

$$\begin{tikzcd} \mathfrak{D} \arrow[r, "\tilde{\omega}"] \arrow[d, "\pi"] & D \\ \mathfrak{B}, & \end{tikzcd}$$

and we set $\mathcal{E} = \tilde{\omega}^{-1}(\boldsymbol{E})$.

Over $\mathfrak{B}$ we construct a holomorphic vector bundle $\mathfrak{F} \xrightarrow{\sigma} \mathfrak{B}$ with $\sigma^{-1}(\lambda) = H^p(Y_\lambda, \mathcal{O}_{Y_\lambda}(\boldsymbol{E}))$; clearly we have that $\mathcal{O}_{\mathfrak{B}}(\mathfrak{F}) = R_\pi{}^p(\mathcal{E})$ where $R_\pi{}^p(\mathcal{E})$ is the p-th *direct image sheaf* of the proper, holomorphic mapping π

$\mathfrak{D} \longrightarrow \mathfrak{B}$ relative to the sheaf $\mathcal{O}_{\mathfrak{D}}(\mathcal{E})$. This finally gives then a linear transformation:

$$\xi\colon H^p(D,\mathcal{O}(\boldsymbol{E}))\to H^0(\mathfrak{B},\mathcal{O}(\mathfrak{J})) \tag{4.52}$$

given by $\xi(\phi)_\lambda \in H^p(Y_\lambda, \mathcal{O}_{Y_\lambda}(\boldsymbol{E}))$. More formally: $\xi(\phi)=\pi_*\tilde{\omega}^*(\phi)$ where $\tilde{\omega}^*(\phi)\in H^p(\mathfrak{D},\mathcal{O}_{\mathfrak{D}}(\mathcal{E}))$ and $\pi_*\tilde{\omega}^*(\phi)\in H^0(\mathfrak{B},R_\pi^p(\mathcal{E}))$.

We observe that ξ is G-equivariant and that the range of ϕ is an infinite-dimensional subspace of $H^0(\mathfrak{B},\mathcal{O}(\mathfrak{J}))$ (cf. Lemma (4.50)). It is via the mapping ξ that we interpret cohomology as holomorphic sections of a bundle.

Remark. There is some evidence that ξ is an isomorphism. In fact, $\mathfrak{B}$ is an open set in an affine variety and may well be holomorphically convex. If this were the case, then, since $R_\pi^q(\mathcal{E})=0$ for $q\neq p$, we would have, by the *Leray spectral sequence*:

$$H^p(\mathfrak{D},\mathcal{O}_{\mathfrak{D}}(\mathcal{E}))\cong H^0(\mathfrak{B},\mathcal{O}_{\mathfrak{B}}(\mathfrak{J})). \tag{4.53}$$

On the other hand, $\mathfrak{D}\to D$ is a holomophic fibering whose fibres seem to be Stein manifolds, and then it might follow that $\tilde{\omega}^*\colon H^p(D,\mathcal{O}_D(\boldsymbol{E}))\to H^p(\mathfrak{D},\mathcal{O}_{\mathfrak{D}}(\mathcal{E}))$ is an isomorphism, in which case ξ would be.

In any event, the kernel of ξ consists of the cohomology classes $\phi\in H^p(D,\mathcal{O}(\boldsymbol{E}))$ which *vanish to infinite order along* Y_0, in the sense that $\phi\in H^p(I^\mu(\boldsymbol{E}))$ for all $\mu\geqq 0$ (cf. 4.49)).

What we shall prove is:

(4.53) THEOREM. *If the cohomology class $\phi\in H^p(D,\mathcal{O}(\boldsymbol{E}))$ is absolutely integrable (cf. the definition below), then the Poincaré series*:

$$\theta(\xi(\phi))=\sum_{\gamma\in\Gamma}\gamma^*(\xi(\phi)) \tag{4.54}$$

converges to a Γ-invariant section of $\mathfrak{J}\to\mathfrak{B}$.

Remark. If $\Gamma'\subset\Gamma$ is any finite subset, then

$$\xi\Big(\sum_{\gamma\in\Gamma'}\gamma^*\phi\Big)=\sum_{\gamma\in\Gamma'}\gamma^*(\xi(\phi)),$$

so that Theorem (*) gives Theorem (4.53). The converse is almost true; we need that ξ is injective plus the fact that the range of ξ is closed (in an appropriate topology).

(ii) We want now to speak of what it means that $\phi\in H^q(D,\mathcal{O}(\boldsymbol{E}))$ should be absolutely integrable, or, more generally, should be in the analogue of the L^p space.

In the line bundle $\boldsymbol{E} \rightarrow D$ there is a G-invariant metric and we take a G-invariant Hermitian metric in the *holomorphic tangent bundle* $\boldsymbol{T}$ of D. There is a pointwise inner product $\langle\phi, \psi\rangle$ on the space $C^q(\boldsymbol{E})$ of C^∞ $\boldsymbol{E}$-valued $(0, q)$ forms (cf. [9]) and we let $|\phi| = \langle\phi, \phi\rangle^{\frac{1}{2}}$. It is then clear how to define the space $\boldsymbol{C}_p{}^q(\boldsymbol{E})$ of measurable $\boldsymbol{E}$-valued $(0, q)$ forms ϕ for which $\int |\phi|^p d\mu < \infty$, where $d\mu$ is the G-invariant volume element on D. $\boldsymbol{C}_p{}^q$ is a Banach space on which there is a densely defined unbounded operator $\bar{\partial}$. In fact, if $C_c{}^q(\boldsymbol{E})$ are the forms with compact support, then, for $\phi \in C_c{}^q(\boldsymbol{E})$ we let $\| \phi \|_p = (\int_D |\phi|^p d\mu)^{1/p}$, and $\boldsymbol{C}_p{}^q$ is the completion of $C_c{}^q(\boldsymbol{E})$ in this norm. Clearly $\bar{\partial}$ is defined on $C_c{}^q(\boldsymbol{E})$ and we let $\boldsymbol{Z}_p{}^q(\boldsymbol{E}) \subset \boldsymbol{C}_p{}^q(\boldsymbol{E})$ be the kernel of $\bar{\partial}$.

There is a natural mapping: $\boldsymbol{Z}_p{}^q(\boldsymbol{E}) \cap C^q(\boldsymbol{E}) \rightarrow H^q(D, \mathcal{O}(\boldsymbol{E}))$ (via the *Dolbeault theorem*), and we shall say that a cohomology class is in L^p if it appears in the range of the above cohomology classes. We let $H_p{}^q(\boldsymbol{E})$ be the L^p-cohmology classes.

Let us discuss *square-integrable cohomology* for a moment. On $C_c{}^q(\boldsymbol{E})$ we can define an inner product $(\phi, \psi) = \int_D \langle\phi, \psi\rangle d\mu$, and $\boldsymbol{C}_2{}^q(\boldsymbol{E})$ is the Hilbert space completion of $C_c{}^q(\boldsymbol{E})$ relative to $(\ ,\ )$. The adjoint $\bar{\partial}^*$ of $\bar{\partial}$ is defined on $C_c{}^q(\boldsymbol{E})$ and we define the *Laplacian* $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. The operators $\bar{\partial}$, $\bar{\partial}^*$, $\square$ are densely defined operators on $\boldsymbol{C}_2{}^q(\boldsymbol{E})$ (cf. [2], [11]), and we define the sqare integrable cohomology space:

$$(4.55) \qquad \boldsymbol{H}_2{}^q(\boldsymbol{E}) = \{\phi \in \boldsymbol{C}_2{}^q(\boldsymbol{E}) \text{ with } \square\phi = 0\};$$

i. e., $\boldsymbol{H}_2{}^q(\boldsymbol{E})$ is the space of L^2 harmonic forms. Since a harmonic form is C^∞ (*Weyl lemma*), an L^2 cohomology class in the sense of (4.55) is also an L^2 class in the way previously defined; in particular, there is a mapping $\boldsymbol{H}_2{}^q(\boldsymbol{E}) \rightarrow H^q(D, \mathcal{O}(\boldsymbol{E}))$. Note that $\boldsymbol{H}_2{}^q(\boldsymbol{E})$ is a unitary G-module.

By using techniques in several complex variables and group representations, together with the methods of [2] and curvature calculations, W. Schmid has proved (in his Berkeley thesis) the following important result:

(*) THEOREM (W. Schmid). (a) *The L^2-cohomology space $\boldsymbol{H}_2{}^q(\boldsymbol{E}) = 0$ for $q \neq p$ where $\boldsymbol{E} \rightarrow D$ has a p-convex polarization; and, more important,* (b) *$\boldsymbol{H}_2{}^p(\boldsymbol{E})$ is an irreducible unitary G-module and, as a K-module, $\boldsymbol{H}_2{}^p(\boldsymbol{E})$ is algebraically equivalent to (cf. Lemma (4.50)): $\sum_{\mu=0}^{\infty} H^p(Y_0, \mathcal{O}_{Y_0}(\boldsymbol{N}^{*(\mu)} \otimes \boldsymbol{E}))$.*

Remark. If $g \in G$ and $x \in D$, then $g^*: \boldsymbol{E}_{gx} \otimes \Lambda^q \bar{T}^*{}_{gx} \rightarrow \boldsymbol{E}_x \otimes \Lambda^q \bar{T}^*{}_x$ is given by $g^* = g^{-1} \otimes \tilde{g}^*$ where $g^{-1}\colon \boldsymbol{E}_{gx} \to \boldsymbol{E}_x$ and $\tilde{g}^*$ is the mapping on forms. In this way, $g^*(\phi)$ for $\phi \in \boldsymbol{C}_p{}^q(\boldsymbol{E})$ is defined and $\| g^*\phi \|_p = \| \phi \|_p$, $(g^*\phi, g^*\psi) = (\phi, \psi)$ for $\phi, \psi \in \boldsymbol{C}_2{}^q(\boldsymbol{E})$.

The arguments which Schmid has used will prove:

(4.56) PROPOSITION. *$H_1{}^p(\boldsymbol{E})$ is non-empty. In particular, there exists an infinite dimensional space of absolutely integrable classes in $H^p(D, \mathfrak{O}(\boldsymbol{E}))$ with $\xi(\phi) \neq 0$, provided that $\boldsymbol{E} = \boldsymbol{K}^\nu$ and ν is large.*

Thus Theorem (4.53) is not vacuous, although it is of course possible that $\theta(\xi(\phi)) = 0$ even though $\xi(\phi) \neq 0$. However, at least when D/Γ is compact, we can show that the sections $\theta(\xi(\phi))$ span the fibres of $\mathfrak{F} \to \mathfrak{B}$ except, perhaps, on a proper subvariety of $\mathfrak{B}$ (cf. [7]).

To prove Theorem (4.53), we shall show:

(4.53)′ THEOREM. *Let $\phi \in C^p(D, \boldsymbol{E}) \cap \boldsymbol{Z}_1{}^p(\boldsymbol{E})$ be a $\bar{\partial}$-closed form with $\| \phi \|_1 = \int_D | \phi | \, d\mu < \infty$. Then $\xi(\phi) \in H^0(\mathfrak{B}, \mathfrak{O}(\mathfrak{F}))$ and the series $\sum_{\gamma \in \Gamma} \gamma^*(\xi(\phi))$ converges uniformly on compact sets to a Γ-invariant holomorphic section $\theta(\xi(\phi)) \in H^0(\mathfrak{B}, \mathfrak{O}(\mathfrak{F}))$.*

(iii) We shall prove first that $\sum_{\gamma \in \Gamma} \gamma^*(\xi(\phi))$ converges pointwise to a section $\theta(\xi(\phi))$ of $\mathfrak{F} \to \mathfrak{B}$. To do this we let

$$\mathfrak{F}_\lambda{}^* = H^p(Y_\lambda, \mathfrak{O}_{Y_\lambda}(\boldsymbol{E}))^* \cong H^0(Y_\lambda, \mathfrak{O}_{Y_\lambda}(\boldsymbol{K}_\lambda \otimes \boldsymbol{E}^*))$$

be the dual space to the fibre $\mathfrak{F}_\lambda$ (the $\cong$ follows by the *duality theorem*). Then, for $\psi_\lambda \in \mathfrak{F}_\lambda{}^*$, we shall show that $\sum_{\gamma \in \Gamma} \langle \gamma^*(\xi(\phi))_\lambda, \psi_\lambda \rangle$ converges absolutely as a series of complex numbers. This will follow from:

(4.57) PROPOSITION. *$\sum_{\gamma \in \Gamma'} | \langle \gamma^*(\xi(\phi))_\lambda, \psi_\lambda \rangle | \leqq c \| \phi \|_1$ where $\| \phi \|_1 = \int_D | \phi | d\mu$, c depends only on ψ, and $\Gamma' \subset \Gamma$ is any finite subset.*

We need two Lemmas, the first of which is:

(4.58) LEMMA. *Given a compact set $C \subset D$, there exists a number $\beta = \beta(C)$ such that each point $x \in D$ meets at most β translates γC for $\gamma \in \Gamma$.*

Proof. It will suffice to prove that there are at most a finite number of $\gamma \in \Gamma$ such that γC meets C. If this were false, there is a sequence $\{\gamma_n\} \subset \Gamma$ of distinct elements such that $\gamma_n C \cap C \neq \phi$. Choose $y_n \in \gamma_n C \cap C$ and set $y_n = \gamma_n x_n$ for $x_n \in C$. By passing to a subsequence, we may suppose that

$x_n \to x$ for some $x \in C$, and then, by passing again to a subsequence, we may assume that $y_n \to y$ for some $y \in C$. Then we have $y_n = \gamma_n x_n$, $x_n \to x$, $y_n \to y$.

Suppose $y \neq \gamma x$ for $\gamma \in \Gamma$; then, by proper discontinuity, there are neighborhoods $N(x)$ and $N(y)$ such that $\Gamma\{N(x)\}$ does not meet $N(y)$. So, in this case, we have a contradiction.

If $y = \gamma x$ for some $\gamma \in \Gamma$, we set $z_n = \gamma^{-1}\gamma_n x_n = \xi_n x_n$ where $\xi_n = \gamma^{-1}\gamma_n$ is a sequence of distinct elements in Γ. Then we have $z_n = \xi_n x_n$, $x_n \to x$, $z_n \to x$. This again contradicts the proper discontinuity of Γ on D.

Remark. By (4.58), for any finite subset $\Gamma' \subset \Gamma$, we have

$$(4.59) \qquad \beta \| \phi \|_1 = \beta \int_D | \phi | \, d\mu \geqq \sum_{\gamma \in \Gamma'} \int_{\gamma C} | \phi | \, d\mu.$$

(4.60) LEMMA. *The dual of the restriction mapping*

$$H^p(D, \mathcal{O}(\mathbf{E})) \xrightarrow{\rho_\lambda} H^p(Y_\lambda, \mathcal{O}_{Y_\lambda}(\boldsymbol{E}))$$

is of the form:

$$H^0(Y_\lambda, \mathcal{O}_{Y_\lambda}(\boldsymbol{K}_\lambda \otimes \boldsymbol{E}^*)) \xrightarrow{\sigma_\lambda} H_c^{n-p}(D, \mathcal{O}(\boldsymbol{K} \otimes \boldsymbol{E}^*)),$$

where $H_c^{n-p}(\cdot\ \cdot\ \cdot)$ is cohomology with compact support.

Remark. This will give the formula:

$$\langle \rho_\lambda(\phi), \psi_\lambda \rangle = \langle \phi, \sigma_\lambda(\psi_\lambda) \rangle = \int_D \phi \wedge \sigma_\lambda(\psi_\lambda),$$

where $\phi \in H^p(D, \mathcal{O}(\boldsymbol{E}))$ and $\psi_\lambda \in H^0(Y_\lambda, \mathcal{O}_{Y_\lambda}(\boldsymbol{K}_\lambda \otimes \boldsymbol{E}^*))$.

Proof. This lemma is a special case of the situation

$$(4.61) \qquad H^q(D, \mathcal{O}_D(\boldsymbol{E})) \xrightarrow{\rho} H^q(Y, \mathcal{O}_Y(\boldsymbol{E})),$$

where $Y \subset D$ is a compact submanifold, $\boldsymbol{E} \to D$ is a holomorphic bundle, and q is any integer. If $\dim Y = m$, $\dim D = m + r$, we shall show that the dual to (4.61) is:

$$(4.62) \qquad H^{m-q}(Y, \mathcal{O}(\boldsymbol{K}_Y \otimes \boldsymbol{E}^*)) \xrightarrow{\sigma} H_c^{m+r-q}(D, \mathcal{O}(\boldsymbol{K}_D \otimes \boldsymbol{E}^*)).$$

Suppose first that $r = \operatorname{codim} Y = 1$ and choose an open covering $\{U_\alpha\}$ of D such that $Y \cap U_\alpha$ is given by $f_\alpha = 0$, where f_α is a holomorphic function in U_α. Then we choose C^∞ (1, 0) form ξ_α in U_α such that $d \log f_{\alpha\beta} = \xi_\alpha - \xi_\beta$, and then $\{\bar{\partial}\xi_\alpha\} = \zeta$ will be the Dolbeault representative of a class in

$H^1(D,\Omega_D{}^1)$ which represents the *Chern class* of the line bundle determined by $Y\subset D$. Observe that we may assume that ζ has compact support; $\zeta\in H_c{}^1(D,\Omega_D{}^1)$; this is in line with the fact that ζ is the dual cohomology class to $Y\in H_{2m}(D,\partial D)$.

Let now $\psi\in H^{m-q}(Y,\mathbf{\mathcal{O}}(\boldsymbol{K}_Y\otimes\boldsymbol{E}^*))=H^{m,m-q}(Y,\mathbf{\mathcal{O}}(\boldsymbol{E}^*))$ and represent ψ by a $\bar{\partial}$-closed $(m,m-q)$ form on Y with values in $\boldsymbol{E}^*$. We choose a C^∞ $\boldsymbol{E}^*$-valued $(m,m-q)$ form ψ on D with compact support and with $\psi\mid Y$ the original ψ. We let $\omega=\{\xi_\alpha-\frac{f_\alpha}{df_\alpha}\}$ be the global $(1,0)$ form with a first order pole on Y and set $\sigma(\psi)=\bar{\partial}(\psi\wedge\omega)$. Then $\sigma(\psi)$ is an $\boldsymbol{E}^*$-valued, $\bar{\partial}$-closed $(m+1,m+1-q)$ form and we have to show that $\sigma(\psi)$ is C^∞ and that, for $\phi\in H^q(D,\mathbf{\mathcal{O}}(\boldsymbol{E}))$,

$$\int_D\phi\wedge\sigma(\psi)=\int_Y\rho(\phi)\wedge\psi. \tag{4.63}$$

Now $\sigma(\psi)=\bar{\partial}\psi\wedge\omega+\psi\wedge\zeta$ (since $\bar{\partial}\omega=\zeta$) and so we have to see that $\bar{\partial}\psi\wedge\omega$ is C^∞. Locally, we may choose coordinates $z_1,\cdots,z_{m+1}$ on D such that $\omega=\frac{dz_{m+1}}{z_{m+1}}+$ (C^∞ term), and Y is given by $z_{m+1}=0$. We can write $\bar{\partial}\psi=\psi_1\wedge d\bar{z}_{m+1}+\psi_2\wedge dz_{m+1}+\psi_3$, where ψ_3 involves only $dz_1,\cdots,dz_m$, $d\bar{z}_1,\cdots,d\bar{z}_m$ and $\psi_3$vanishes on Y. Then

$$\bar{\partial}\psi\wedge\omega=\psi_1\wedge\frac{dz_{m+1}\wedge d\bar{z}_{m+1}}{z_{m+1}}+\psi_3\wedge\frac{dz_{m+1}}{z_{m+1}}+(C^\infty\text{ terms}),$$

and so $\bar{\partial}\psi\wedge\omega$ is C^∞ as required.

To prove (4.63), we have $\phi\wedge\sigma(\psi)=\phi\wedge\bar{\partial}(\psi\wedge\zeta)=\pm d(\phi\wedge\psi\wedge\zeta)$ (since $\bar{\partial}\phi=0)=\pm d(\phi\wedge\psi\wedge\zeta)$. Thus, if B_ϵ is the ϵ-tube around Y,

$$\int_D\phi\wedge\sigma(\psi)=\pm\int_{dB\epsilon}\phi\wedge\psi\wedge\zeta$$

for any $\epsilon>0$. Thus

$$\int\phi\wedge\sigma(\psi)=\pm\lim_{\epsilon\to0}\int\phi\wedge\psi\wedge\zeta=\int\rho(\phi)\wedge\psi.$$

Now we observe the mapping σ in (4.62) still exists so that (4.64) holds even if Y is non-compact, provided that we take the compactly supported cohomology $H_c{}^{m-q}(Y,\mathbf{\mathcal{O}}(\boldsymbol{K}_Y\otimes\boldsymbol{E}^*))$ (of course, it need not be true that $H_c{}^{m-q}(Y,\mathbf{\mathcal{O}}(\boldsymbol{K}_Y\otimes\boldsymbol{E}^*))$ is still the dual space of $H^q(Y,\mathbf{\mathcal{O}}(\boldsymbol{K}_Y\otimes\boldsymbol{E}^*)))$.

Suppose then that $Y=Y_1\cap Y_2$ where Y_1, Y_2 are submanifolds of D, each of codimension one, and which meet transversely along Y. We assume that Y is compact, even though Y_1 and Y_2 need not be. Then we have:

$$\begin{tikzcd} H^q(D, \mathcal{O}(\boldsymbol{E})) \arrow[r] \arrow[rr, bend left, "\rho"] & H^q(Y_1, \mathcal{O}(\boldsymbol{E})) \arrow[r] & H^q(Y, \mathcal{O}(\boldsymbol{E})), \end{tikzcd}$$

$$\begin{tikzcd} H^{m-q}(Y, \mathcal{O}(\boldsymbol{K}_Y \otimes \boldsymbol{E}^*)) \arrow[r] \arrow[rr, bend left, "\sigma"] & H_c{}^{m-q+1}(Y_1, \mathcal{O}(\boldsymbol{K}_{Y_1} \otimes \boldsymbol{E}^*)) \arrow[r] & H^{m-q+2}(D, \mathcal{O}(\boldsymbol{K}_D \otimes \boldsymbol{E}^*)), \end{tikzcd}$$

where σ still satisfies (4.63) so that σ is dual to ρ. In other words, Lemma (4.60) holds if Y_λ is a complete intersection of hypersurfaces on D. But we can easily see from Section I, 4.(c) that Y_λ is such a complete intersection; e.g., if $Y_\lambda = X_S$ where D is of type 1 (cf. (4.42)), then X_S is defined by $Q(S', R) = 0$, which is $hk = \operatorname{codimen} X_S$ equations ($\dim R_S = k$, $\dim S' = h$).

Proof of Proposition (4.57). We let $\psi = \sigma_\lambda(\psi_\lambda)$ where, by (4.60), ψ is an $(n - q)$-form with support in a compact set $C \subset D$. Then

$$\begin{aligned} \langle \gamma^*(\xi(\phi))_\lambda, \psi_\lambda \rangle &= \langle \rho_\lambda(\gamma^*\phi), \psi_\lambda \rangle = \langle \gamma^*\phi, \psi \rangle \\ &= \int_C \gamma^*\phi \wedge \psi = \int_C \gamma^*\phi \wedge \gamma(\gamma^{*-1}\psi) \\ &= \int_C \gamma^*(\phi \wedge \gamma^{*-1}\psi) = \int_{\gamma C} \phi \wedge \gamma^{*-1}\psi. \end{aligned}$$

This gives:

$$\begin{aligned} | \langle \gamma^*(\xi(\phi))_\lambda, \psi_\lambda \rangle | &= \left| \int_{\gamma C} \phi \wedge \gamma^{*-1}\psi \right| \\ &\leqq \int_{\gamma C} | \phi | \, | \gamma^{*-1}\psi | \, d\mu \leqq \alpha \int_{\gamma C} | \phi | \, d\mu \end{aligned}$$

where $\alpha = \sup_{x \in C} | \psi |_x = \sup_{y \in \gamma C} | \gamma^{*-1}\psi |_y$. Combining this with (4.59), we obtain: $\sum_{\gamma \in \Gamma'} | \langle \gamma^*(\xi(\phi))_\lambda, \psi_\lambda \rangle | \leqq \alpha\beta \, \| \phi \|_1$, which proves Proposition (4.57).

We now prove the uniform convergence on compact sets of $\theta(\xi(\phi))$. It will suffice to have:

(4.64) PROPOSITION. *Let* $\{\psi_\lambda\}$ *be a* C^∞ *section of the dual bundle* $\mathfrak{F}^* \to \mathfrak{B}$ *and* $\lambda_0 \in \mathfrak{B}$ *be given. Then there exists a compact neighborhood* U *of* λ_0 *such that, given* $\epsilon > 0$, *we can find a finite subset* $\Gamma' \subset \Gamma$ *with* $\sum_{\gamma \in \Gamma - \Gamma'} | \langle \gamma^*(\xi(\phi))_\lambda, \psi_\lambda \rangle | < \epsilon$ for $\lambda \in U$.

Proof. Let U be a sufficiently small compact neighborhood of λ_0; the varieties Y_λ for $\lambda \in U$ will then all lie in a compact neighborhood of $Y_{\lambda_0} \subset D$. Moreover, for each $\lambda \in U$, let $\psi(\lambda) \in H_c{}^{n-p}(D, \mathcal{O}(\boldsymbol{K}^* \otimes \boldsymbol{E}^*))$ be given by $\psi(\lambda) = \sigma_\lambda(\psi_\lambda)$. Then there will be a compact set $C \subset D$ such that all $\psi(\lambda)$

have support in C. We let $\alpha = \sup_{\left\{\substack{x \in C \\ \lambda \in U}\right.} |\psi(\lambda)|_x$; clearly $\alpha < \infty$. Furthermore, we let β be the number corresponding to C in Lemma (4.58).

Choose now a compact set $\hat{C} \subset D$ such that $\int_{D-C} |\phi| d\mu < \frac{\epsilon}{\alpha\beta}$. Let $\Gamma' = \{\gamma \in \Gamma \text{ such that } \gamma C \cap \hat{C} \neq \phi\}$; i.e. Γ' is those γ for which γC meets $\hat{C}$. As in Lemma (4.58), we can prove that Γ' is finite. Now for $\lambda \in U$,

$$\sum_{\gamma \in \Gamma - \Gamma'} |\langle \gamma_0(\xi(\phi))_\lambda, \psi_\lambda \rangle| \leqq \sum_{\gamma \in \Gamma - \Gamma'} \int_{\gamma C} |\phi| \, |\gamma^{*-1}\psi(\lambda)| \, d\mu$$

$$\leqq \alpha \sum_{\gamma \in \Gamma - \Gamma'} \int_{\gamma C} |\phi| \, d\mu \leqq \alpha\beta \int_{D-C} |\phi| \, d\mu < \epsilon$$

This proves Proposition (4.64) and completes the proof of Theorem (4.53).

(f) In addition to Proposition (4.27) (cf. (4.29)), we want to record one other property of the period matrix domains to be used in Part II on the local study of the period mapping. Recall that a period matrix domain D is of the form $D = H \backslash G$ (cf. Theorem (1.26)) where G is a real simple Lie group and $H \subset G$ is a compact subgroup. If we let $B = K \backslash G$ where $K \subset G$ is a maximal compact subgroup, then the fibering:

$$D \xrightarrow{\pi} B, \tag{4.65}$$

given group theoretically by $H \backslash G \to K \backslash G$, is a fibering of the complex manifold D with compact, complex submanifolds of D as fibres (cf. Theorem (4.41)). However, even if $B = K \backslash G$ happens to carry a complex structure, the fibering (4.65) is *not* an analytic fibre space; the holomorphic normal bundle $N \to Y_0$ is *not* trivial (cf. Proposition (3.21)).

(4.66) PROPOSITION. (i) *For each $x \in D$, there is a unique G-invariant splitting of the holomorphic tangent space:*

$$T_x(D) = V_x \oplus H_x,$$

where V_x is the holomorphic tangent space to the compact subvariety Y_λ passing through x. (ii) if ω is the curvature in the canonical bundle, then $\omega \mid H_x$ is positive definite for each $x \in D$.

Proof. As usual, (4.66) has a direct proof using the structure theory of Lie groups, but we shall check it on the two types of period matrix domains as the explicit forms of V_x and H_x will be needed in Part II.

Case 1 (cf. the proof of Theorem (4.41)). In this case, $D \subset G(h, W)$ and, by Proposition (4.22),

$$T_S(D) \subset \mathrm{Hom}(S, W/S).$$

By using the quadratic relations (4.1), $W \cong S \oplus R_S \oplus \bar{S}$, and this splitting is preserved by G. Thus, $W/S \cong R_S \oplus \bar{S}$ and

$$T_S(D) \subset \mathrm{Hom}(S, R_S \oplus \bar{S}) = \mathrm{Hom}(S, R_S) \oplus \mathrm{Hom}(S, \bar{S}). \tag{4.67}$$

We claim that $T_S(D)$ consists of all $\theta \in \mathrm{Hom}(S, W/S)$ which satisfy:

$$Q(\theta(\xi), \zeta) + Q(\xi, \theta(\zeta)) = 0 \qquad (\xi, \zeta \in S). \tag{4.68}$$

In fact, looking at the proof of Lemma (4.22), we choose a curve $\{S_t\}$ with $S_0 = S$ and tangent θ. Letting $\xi_t, \zeta_t \in S_t$ with $\xi_0 = \xi$, $\zeta_0 = \zeta$, we have $Q(\xi_t, \zeta_t) = 0$. By differentiation, we have $Q(\frac{\partial \xi_t}{\partial t}]_{t=0}, \zeta) + Q(\xi, \frac{\partial \zeta_t}{\partial t}]_{t=0}) = 0$. Since $\theta(\xi) =$ projection of $\frac{\partial \xi_t}{\partial t}]_{t=0}$ in W/S and $Q(S, S) = 0$, we get (4.68).

By combining (4.67) and (4.68) we obtain:

$$T_S(D) \cong \mathrm{Hom}(S, R_S) \oplus \mathrm{Hom}_Q(S, \bar{S}), \tag{4.69}$$

where $\mathrm{Hom}_Q(S, \bar{S})$ is all $\theta \in \mathrm{Hom}(S, \bar{S})$ with (4.68) being satisfied. The G-invariant splitting (4.69) gives the required decomposition where $V_S = \mathrm{Hom}_Q(S, \bar{S})$ and $H_S = \mathrm{Hom}(S, R_S)$. The assertion about the curvature ω of $\boldsymbol{K}$ is clear from (4.69), (4.43), and (4.20) (the $dc_{\alpha\rho}$ are dual to H_S and the $db_{\alpha\beta}$ dual to V_S).

Remark. In the actual case of periods when $W = H^2(V, \boldsymbol{C})_0$ for some algebraic manifold V, $S = H^{2,0}(V)$, $\bar{S} = H^{0,2}(V)$, $R_S = H^{1,1}(V)_0$ and (4.69) becomes:

$$T_{\Phi(V)}(D) \cong \mathrm{Hom}(H^{2,0}, H^{1,1}{}_0) \oplus \mathrm{Hom}_Q(H^{2,0}, H^{0,2}). \tag{4.70}$$

Case 2. Now we have $D \subset G(n-q, W) \times G(n, W)$ and so

$$T_S(D) \subset \mathrm{Hom}(S_1, W/S_1) \oplus \mathrm{Hom}(S_2, W/S_2).$$

This is one general relation which defines the tangent space to the flag manifold in a proluct of Grassmannians; in this case, if $\theta_1 \in \mathrm{Hom}(S_1, W/S_1)$ and $\theta_2 \in \mathrm{Hom}(S_2, W/S_2)$, then θ_2 will equal θ_1 as mappings in W/S_2. In other words, the following diagram commutes:

$$\begin{tikzcd} S_1 \arrow[r, "\theta_1"] \arrow[d] & W/S_1 \arrow[d] \\ S_2 \arrow[r, "\theta_2"] & W/S_2. \end{tikzcd} \tag{4.71}$$

By using the quadratic relations (4.2), we have $Q(\theta(\xi),\zeta)+Q(\xi,\theta(\zeta))=0$ where θ is either θ_1 or θ_2. This is by the same reasoning which gave (4.68). Also, we have a G-invariant splitting: $W=S_1\oplus S_2/S_1\oplus \bar{S}_2/\bar{S}_1\oplus \bar{S}_1$. For example, in the period matrix case, $W=H^3(V,\boldsymbol{C})_0$ and this splitting becomes $W=H^{3,0}\oplus H^{2,1}{}_0\oplus H^{1,2}{}_0\oplus H^{0,3}$ since $S_1=H^{3,0}$ and $S_2=H^{3,0}+H^{2,1}$.

Let $R_S=S_1\oplus \bar{S}_2/\bar{S}_1$. Then $\operatorname{Hom}(S_1,R_S/S_1)$ is a summand of $\operatorname{Hom}(S_1,W/S_1)$ and we claim that, using the bilinear relation, $\operatorname{Hom}(S_1,R_S/S_1)$ determines $\operatorname{Hom}(S_2/S_1,\bar{S}_1)$. This is because, given $\theta\in\operatorname{Hom}(S_2/S_1,\bar{S}_1)$ and $\xi\in S_2/S_1$, $\zeta\in S_1$, we will have $Q(\theta(\xi),\zeta)+Q(\xi,\theta(\zeta))$ and, since $Q(S_1,\bar{S}_1)>0$, $\theta(\xi)$ is determined by knowing $\theta(\zeta)$ for all $\zeta\in S_1$.

We may now describe $T_S(D)$. Using the decomposition of W and (4.71), we have:

$$T_S(D)\subset\{\operatorname{Hom}(S_1,S_2/S_1)\oplus\operatorname{Hom}(S_2/S_1,\bar{S}_2/\bar{S}_1)\oplus\operatorname{Hom}(S_2,\bar{S}_1)\}\\ \oplus\operatorname{Hom}(S_1,R_S/S_1).$$

Using then the bilinear relation, we have:

$$T_S(D)\cong\{\operatorname{Hom}(S_1,S_2/S_1)\oplus\operatorname{Hom}_Q(S_2/S_1,\bar{S}_2/\bar{S}_1)\\ \oplus\operatorname{Hom}_Q(S_1,\bar{S}_1)\}\oplus\operatorname{Hom}(S_1,R_S/S_1), \tag{4.72}$$

where the meaning of $\operatorname{Hom}_Q(\cdot,\cdot)$ is clear. Taking $H_S=\{\cdot\ \cdot\ \cdot\}$ and $V_S=\operatorname{Hom}(S_1,R_S/S_1)$ in (4.72), we get the desired G-invariant decomposition. In the case of periods, (4.73) becomes:

$$T_{\Phi(V)}(D)\cong\{\operatorname{Hom}(H^{3,0},H_0{}^{2,1})\oplus\operatorname{Hom}_Q(H_0{}^{2,1},H_0{}^{1,2})\\ \oplus\operatorname{Hom}_Q(H^{3,0},H^{0,3})\}\oplus\operatorname{Hom}(H^{3,0},H_0{}^{1,2}). \tag{4.73}$$

To check the curavture assertion, we follow the notations in the proof of Proposition (4.30). Combining the relations $dc_{i\rho}=dc_{\rho i}-\sqrt{-1}\,db_{i\rho}$ which define $T_S(D)$ (by (4.34)) and the relation $dc_{i\rho}+\sqrt{-1}\,db_{i\rho}=0$ which defines H_S (cf. just above (4.44)), we find by (4.40) that

$$\omega\mid H_S=\phi(\alpha_1,\alpha_2)+(\alpha_1+2\alpha_2)\{\sum_{i,\rho}dc_{i\rho}\wedge d\bar{c}_{i\rho}\}$$

so that $\omega\mid H_S$ is positive definite as required.

UNIVERSITY *of* CALIFORNIA, BERKELEY,
PRINCETON UNIVERSITY.

REFERENCES.

[1] A. Andreotti and H. Grauert, "Théorèmes de finitude pour la cohomologie des espaces complexes, *Bulletin de la la Société Mathématique de France,* vol. 91 (1963), pp. 1-38.

[2] A. Andreotti and E. Vesentini, "Carleman estimates for the Laplace-Beltrami operator on complex manifolds," *Institut des Hautes Études Scientifiques,* vol. 25 (1965), pp. 313-362.

[3] R. Bott, "Homogeneous vector bundles," *Annals of Mathematics,* vol. 66 (1967), pp. 203-248.

[4] C. Chevalley, *Theory of Lie groups,* Princeton University Press (number 8), Princeton, N. J.

[5] F. Conforto, *Abelsche funktionen und algebraische geometrie,* Springer (1956), Berlin.

[6] P. Griffiths, "The extension problem in complex analysis II; embeddings with positive normal bundle," *American Journal of Mathematics,* vol. 88 (1966), pp. 366-445.

[7] ———, "Curvature and the cohomology of locally homogeneous vector bundles," to appear.

[8] R. Gunning and H. Rossi, *Analytic functions of several complex variables,* Prentice-Hall, Inc. (1965).

[9] F. Hirzebruch, "Neue topologische methoden in der algebraischen geometrie," *Ergebnisse der Mathematik,* vol. 9 (1956), Springer.

[10] W. V. D. Hodge, *The theory and applications of harmonic integrals,* Cambridge University Press (1959).

[11] J. J. Kohn, "Harmonic integrals on strongly pseudo-convex manifolds I," *Annals of Mathematics,* vol. 78 (1963), pp. 112-148.

[12] A. Weil, "On Picard varieties," *Amehican Journal of Mathematics,* vol. 74 (1952), pp. 865-894.

[13] ———, "Zum beweis des Torellischen satzes," *Nachrichten der Akademie Wissenschaften in Göttingen* (1957), pp. 33-53.

[14] ———, *Variétés Kähleriennes,* Hermann (6), Paris.