PERIODS OF INTEGRALS ON ALGEBRAIC MANIFOLDS, II.
(Local Study of the Period Mapping)

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II. 0. Introduction. (a) The purpose of this paper is to study the *local* behavior of the periods of integrals, as functions of the parameters, in a family of polarized algebraic manifolds. The present work is a continuation of "Periods of integrals on algebraic manifolds, I (Construction and properties of the modular varieties)," referred to as I.

Let us call two polarized algebraic manifolds V, V' of the same *type* if there is a polarization-preserving homeomorphism $f\colon V \to V'$. The totality of all possible *period matrices* Ω for the periods of *primitive q-forms* $(0 < q \leqq n = \dim V)$ on polarized algebraic manifolds of the same type forms an open complex manifold $D_q = D$. These *period matrix spaces* D have been studied in I; they are all *homogeneous complex manifolds* of the form $D = G/H$ where G is a real, simple Lie group and $H \subset G$ is a compact subgroup. In many ways, these D are analogous to the *Siegel upper half-spaces* (= period matrix domain for 1-forms), but there are important differences. For example, D is generally *not* an Hermitian symmetric domain, and the classical theory of *automorphic forms* is replaced by *automorphic cohomology.*

If $\{V\}_{t \in \Delta}$ is a complex analytic family of polarized algebraic manifolds parametrized by a polycylinder Δ, then there is defined the *period matrix mapping* $\Phi\colon \Delta \to D$ by $\Phi(t)$ = period matrix of the primitive q-forms on V_t. What we will do below is give the properties of Φ.

(b) We give now an outline of the results in this paper, which is divided into three sections under the following headings:

1. Local study of the period mapping;
2. Complex torii associated with algebraic varieties;
3. Examples of the local period mapping.

The first main theorem, given in Section 1.(a), is that Φ is holomorphic. The idea is the following: Letting $V = V_0$ and $v = [\frac{q-1}{2}]$, there is an embedding $D \subset \boldsymbol{F}$, with $\boldsymbol{F}$ a *flag manifold,* and $\Phi(t)$ is the flag $[S^0(t), \cdots, S^v(t)]$

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where $S^r(t) = H_0^{q,0}(V_t) + \cdots + H_0^{q-r,r}(V_t)$ is a subspace of $W = H^q(V, \boldsymbol{C})_0$. Using the Kodaria-Spencer-Kuranishi theory of deformations of complex structures ([18], [20], [24], and [25]), we prove that $\dfrac{\partial S^r(t)}{\partial \bar{t}} \subset S^r(t)$. This implies that Φ is holomorphic.

The proof of this result also gives a formula for the differential $\Phi_*: T_t(\Delta) \to T_{\Phi(t)}(D)$. To give this formula, at $t=0$, we remark that there is a factorization:

$$\begin{tikzcd} T_0(\Delta) \arrow[rr, "\Phi_*"] \arrow[dr, "\rho"'] & & T_{\Phi(0)}(D) \\ & H^1(V,\Theta) \arrow[ur, "\mu"'] & \end{tikzcd}$$

where ρ is the Kodaira-Spencer *infinitesimal deformation mapping* [18]. To give μ, we use the natural isomorphism:

$$T_{\Phi(0)}(D) \cong \sum_{r=0}^{v} \operatorname{Hom}(H_0^{q-r,r}(V), H_0^{q-r-1,r+1}(V) + \cdots + H_0^{0,q}(V)).$$

Then, for $\theta \in H^1(V,\Theta)$, $\phi \in H_0^{q-r,r}(V)$, $\mu(\theta)(\phi) = \theta \cdot \phi \in H_0^{q-r-1,r+1}(V)$ where $\phi \cdot \theta$ is the *cup product* in cohomology (cf. Section 1.(b)).

This computation of Φ_* gives a practical method of determining when the *periods give local moduli*, and Section 3 is devoted to studying special cases. For example, if $\{V_t\}_{t\in\Delta}$ is the Kuranishi family ([25]; in this case, ρ is the identity), then we find easily that Φ_* is injective if V is a *non-hyperelliptic Riemann surface* (cf. Rauch [27]) or if the *canonical bundle* of V is trivial (cf. Kodaria [17] for *K*3 *surfaces*). Less trivially we find in 3.(c)-3.(f) that Φ_* is injective if V is: (i) a non-singular surface $V \subset P_3$ of degree at least 5; (ii) a *general* non-singular surface on an abelian variety; (iii) a cubic threefold. In 3.(g) we discuss examples when the period mapping degenerates.

There is computational and geometric evidence that we might have: Φ_* is injective if V is a surface with *ample* canonical bundle.

The study of Φ_* also points up a new phenomenon for periods of q-forms ($q>1$). Recall that a point $\Omega = [S^0, \cdots, S^v] \in D$ satisfies the *Hodge bilinear relations* [10]:

$$\begin{cases} \Omega Q^t\Omega = 0. \\ \Omega Q^t\bar{\Omega} > 0. \end{cases}$$

For $q=1$, there are no more relations, but, for $q>1$, there are additional infinitesimal period relations which hold universally. If $q=2$, we have $d\Omega Q^t\Omega = 0$, $d\Omega Q^t d\Omega = 0$. These relations are discussed in 1.(c) and have the following geometric meaning: Letting $K \subset G$ be the maximal compact subgroup, the fibering $D = G/H \xrightarrow{\pi} G/K$ has complex analytic fibres; π is holomorphic only when $H = K$ (this is generally true only when $q = 1$). At each point $\Omega \in D$, there is a G-invariant splitting: $T_\Omega(D) = V_\Omega \oplus H_\Omega$, where V_Ω is the tangent space to the π-fibre through Ω. Then $\Phi_*(T_t(\Delta)) \subset H_{\Phi(t)}$ is the additional period relation. Geometric applications of this transversality theorem are given in Sections 1.(c) and 1.(d).

Section 2 is devoted to complex torri, especially those torii arising from periods of $2p+1$-forms on an algebraic manifold V. Let E_0 be a real $2m$-dimensional vector space, $\Gamma \subset E_0$ a fixed lattice, and $E = E \otimes_{\boldsymbol{R}} \boldsymbol{C}$ the complexification of E_0. We let $\Gamma^* \subset E^*$ be dual to Γ and, for S a subspace of E with $S + S = E$, $S \cap \bar{S} = 0$, we let $E_S = E/S$ and set:

$$\begin{cases} T(S) = E_S/\Gamma \\ T(S)^* = S^*/\Gamma^*. \end{cases}$$

This gives a pair of complex torii depending holomorphically on S. The Kodaira-Spencer mapping ρ and the period mapping Φ are discussed for this family (2.(a)). Also we discuss how a skew-symmetric quadratic form on E_0 induces a polarization on $T(S)^*$ (cf. 2.(c)); this is generally a *q-convex* polarization.

For a family of polarized abelian varieties of dimension at least three, the periods of the 2-forms give the moduli (as well as the customary way of using periods of 1-forms). This gives an equivariant embedding of the Siegel space into a non-symmetric domain, and allows us to determine the additional period relations in a simple case (cf. 2.(d)).

Let now $E_0 = H^{2p+1}(V, \boldsymbol{R})$, $\Gamma = H^{2p+1}(V, \boldsymbol{Z})$. There seem to be two interesting choices of S:

$$\begin{cases} S_1 = \sum_{k \geqq 0} H^{p+k+1, p-k}(V) \\ \\ S_2 = \sum_{k} H^{p+2k+1, p-2k}(V). \end{cases}$$

We set $T_p(V) = T(S_1)^*$, $A_p(V) = T(S_2)^*$. Observe that, if D is the period matrix space for $2p+1$-forms, $\Phi(V) = [S^0, \cdots, S^p]$ with $S^p = S_1$ above. Thus there is a holomorphic family of torii $\mathfrak{J} \xrightarrow{\tilde{\omega}} D$ with $\tilde{\omega}^{-1}(\Phi(V)) = T_p(V)$; in particular, $T_p(V)$ varies holomorphically with V, whereas $A_p(V)$ does not. There are natural polarizations $\boldsymbol{L}_T \to T_p(V)$, $\boldsymbol{L}_A \to A_p(V)$. The torus $A_p(V)$ is *Weil's Jacobian*, and $\boldsymbol{L}_A$ is *positive*, whereas $\boldsymbol{L}_T$ is generally *q-convex*.

In some sense the torii $T_p(V)$ and $A_p(V)$ are *not* fundamentally different. To explain this, we remark that, for $x \in T_p(V)$, the tangent space splits: $T_x = P_x \oplus N_x$. where the curvature ω_T of $\boldsymbol{L}_T$ is positive on P_x and negative on N_x. (This is analogous to the q-convex polarizations on the period matrix spaces D.) Then in 2.(e) we show that there is a *real* linear isomorphism $\xi\colon T_p(V) \to A_p(V)$ such that: (i) $\xi^*(\boldsymbol{L}_A) = \boldsymbol{L}_T$; (ii) if $\vartheta \in H^0(\mathcal{O}(\boldsymbol{L}_A))$ is a holomorphic section, then the C^∞ section $\xi^*(\vartheta)$ of $\boldsymbol{L}_T$ satisfies $\bar{\partial}\xi^*(\vartheta) \mid P_x = 0$; and (iii) if $\omega^1, \cdots, \omega^q$ is a basis for N_0, then the mapping $\vartheta \to \xi^*(\vartheta)\bar{\omega}^1 \wedge \cdots \wedge \bar{\omega}^q$ gives an isomorphism $H^0(\mathcal{O}(\boldsymbol{L}_A)) \cong H^q(\mathcal{O}(\boldsymbol{L}_T))$.

Both torii are relevant to the study of algebraic p-cycles on V. Let $Z_0 \subset V$ be one such and let $Z \subset V$ be an algebraic p-cycle homologous to Z_0. If $\phi^1, \cdots, \phi^m$ is a basis for S_1, we may define $\phi(Z) \in T_p(V)$ by $\phi(Z) = (\cdots, \int_\sigma \phi^\alpha, \cdots)$ modulo periods, where σ is a $2p+1$ chain with $\partial\sigma = Z - Z_0$. Similarly, we may define $\psi(Z) \in A_p(V)$, and we show (2.(b)): (iv) ϕ and ψ are holomorphic, and the diagram

$$\begin{tikzcd} & T_p(V) \arrow[dd, "\xi"] \\ \{Z\} \arrow[ur, "\phi"] \arrow[dr, "\psi"'] & \\ & A_p(V) \end{tikzcd}$$

commutes; (v) if B is an algebraic parameter space for p-cycles Z, then $\phi_*(T_Z(B)) \subset P_{\phi(Z)}$ (this is the analogue of the infinitesimal period relations for Φ given above).

From (ii), (iv), (v) it follows that $\phi(B)$ is an algebraic manifold (via the sections $\xi\vartheta$, $\vartheta \in H^0(\mathcal{O}(\boldsymbol{L}_A))$), and that $\psi(B)$ varies holomorphically with V.

The differential ϕ_* $(= \psi_*)$ is computed cohomologically in a similar way to Φ_* (cf. Theorem (2.25)). To write the infinitesimal equivalence relation determined by ϕ requires the duality theorem for general coherent sheaves.

To conclude this introduction, let us give the main open problem. We consider both the period matrix mapping $\Phi\colon B \to D$ and the p-cycle mapping $\phi\colon B \to T_p(V)$; in each case, B is a suitable parameter space. Both mappings have a basic similarity. There are natural line bundles $\boldsymbol{L} \to D$, $\boldsymbol{L}_T \to T_p(V)$, each with q-convex polarizations (different q). Now both Φ and ϕ are holomorphic, and they each satisfy additional period relations which imply that $\boldsymbol{L} \mid \Phi(B)$ and $\boldsymbol{L}_T \mid \phi(B)$ are positive. The problem is to construct holomorphic sections of $\boldsymbol{L} \mid \Phi(B)$, $\boldsymbol{L}_T \mid \phi(B)$; these sections would then be generalized *automorphic forms,* resp. *theta functions.*

In the case of $\boldsymbol{L}_T \mid \phi(B)$, by using (i)-(v) above the holomorphic sections are constructed from $H^q(\boldsymbol{\mathcal{O}}(\boldsymbol{L}_T))$, so the problem is done in this case. Now, by Section 4.(e) of I, there is *automorphic cohomology* in $H^q(\boldsymbol{\mathcal{O}}(\boldsymbol{L}))$; as in the classical case where D is a Cartan domain, this cohomology is closely related to $L^2(G)$ $(D = G/H)$. Our problem is to turn $H^q(\boldsymbol{\mathcal{O}}(\boldsymbol{L}))$ into sections of $\boldsymbol{L} \mid \Phi(B)$.

II. 1. Local study of the period mapping. (a) Let $\{V_t\}_{t\in\Delta}$ be an *analytic family of compact, Kähler manifolds* parametrized by a polycylinder $\Delta \subset \boldsymbol{C}^m$. To be precise, we assume given complex manifolds $\boldsymbol{V}$, Δ together with a proper, constant maximal rank holomorphic mapping $\pi: \boldsymbol{V} \to \Delta$ and such that on each $V_t = \pi^{-1}(t)$, we have given a Kähler metric $\omega(t)$ which varies smoothly with t (cf. [3], [18]). We let $V = V_0$ and remark that, if V_0 has a Kähler metric, we can always construct the smooth family $\omega(t)$ postulated above [19]. The data $\boldsymbol{V} \xrightarrow{\pi} \Delta$ will generally be called an *analytic fibre space.*

By passing to a smaller polycylinder if necessary, $\boldsymbol{V} \to \Delta$ will be trivial as a C^∞ family; i.e. we can find a fibre-preserving C^∞ isomorphism

$$\begin{array}{ccc} V_0 \times \Delta & \xrightarrow{\phi} & \boldsymbol{V} \\ \downarrow & & \downarrow \pi \\ \Delta & = & \Delta. \end{array}$$

Letting $W = H^q(V_0, \boldsymbol{C})$, by using the Hodge decomposition

$$H^q(V_t, \boldsymbol{C}) = \sum_r H^{q-r,r}(V_t)$$

and the isomorphism

$$\phi^*: H^q(V_t, \boldsymbol{C}) \longrightarrow H^q(V_0, \boldsymbol{C}),$$

we can define a point $\Omega(t) = [S_0(t), \cdots, S_v(t)]$ $(v = [\frac{q-1}{2}])$ in a *flag manifold* associated to W by letting

$$S_r(t) = \phi^*\{H^{q,0}(V_t) + \cdots + H^{q-r,r}(V_t)\}.$$

We remark that, whereas ϕ is far from unique, ϕ^* is essentially unique. As was discussed in I. 1, $\Omega(t)$ is an invariant way of giving the total *period matrix* of the *harmonic q-forms* on V_t.

(1.1) THEOREM. *$\Omega(t)$ is a holomorphic function of t.*

Remark. We recall here the example in I. 2. (b) where the *Plücker coordinates* of an explicit $\Omega(t)$ were shown to be holomorphic, whereas $\Omega(t)$ was not in terms of the period matrices one most easily writes down.

Proof. What we have to show is that each subspace $S_r(t) \subset W$ varies holomorphically with t. Here we view $S_r(t)$ as a point $\Omega_r(t)$ in a *Grassmann manifold* $G(h_r, W)$ where $h_r = h^{q,0} + \cdots + h^{q-r,r}$ (cf. I. 1. (d)). By Lemma (4. 22) in I. 4. (b), $T_{S_r(t)}(G(h_r, W)) \cong \operatorname{Hom}(S_r(t), W/S_r(t))$ and, by Proposition (4. 27) there, we must show:

$$(1.2) \qquad \bar{\partial} S_r(t) \subset S_r(t),$$

in the following sense: there is a smooth basis $v_1(t), \cdots, v_{h_r}(t)$ for $S_r(t)$ such that $\dfrac{\partial v_j(t)}{\partial \bar{t}^\alpha} \in S_r(t)$ for $j = 1, \cdots, h_r$ and where $t = (t^1, \cdots, t^m) = (\cdots, t^\alpha, \cdots)$. To simplify the notation we assume that $m = 1$ so that Δ is a disc with coordinate t; it will suffice to prove that, for any smooth vector $v(t) \in S_r(t)$,

$$(1.3) \qquad \frac{\partial v(t)}{\partial \bar{t}}\Big]_{t=0} \in S_r = S\ (0).$$

We need to go now into the structure equations for *deformations* of *complex structure* (cf. [20], [24]). First we coordinatize $\boldsymbol{V}$ by a covering U_α of open polycylinders such that, in U_α, we have holomorphic coordinates $(z_\alpha^1, \cdots, z_\alpha^n; t)$ with $\pi(z_\alpha^1, \cdots, z_\alpha^n; t) = t$. This is possible by the implicit function theorem since π_* has constant maximal rank. In $U_\alpha \cap U_\beta$, we have $t = t$ and $z_\alpha^j = f_{\alpha\beta}^j(z_\beta^1, \cdots, z_\beta^n; t)$. Writing $z_\alpha = f_{\alpha\beta}(z_\beta, t)$, we have:

$$(1.4) \qquad f_{\alpha\gamma}(z_\gamma, t) = f_{\alpha\beta}(f_{\beta\gamma}(z_\gamma, t), t) \text{ in } U_\alpha \cap U_\beta \cap U_\gamma.$$

By differentiation, we get from (1. 4) that, if $\theta_{\alpha\beta} = \sum_{j=1}^{n} \dfrac{\partial f_{\alpha\beta}^j}{\partial t}(z_\beta, t) \dfrac{\partial}{\partial z_\alpha^j}$, we have $\theta_{\alpha\beta} + \theta_{\beta\gamma} = \theta_{\alpha\gamma}$ in $U_\alpha \cap U_\beta \cap U_\gamma$. Thus $\theta(t) = \{\theta_{\alpha\beta}(t)\}$ defines an element in $H^1(V_t, \Theta_t)$ which is the *Kodaira-Spenced mapping* [18]:

$$(1.5) \qquad \rho_t\colon T_t(\Delta) \to H^1(V_t, \Theta_t).$$

This mapping, which represents the *infinitesimal variation of the complex structure,* is of fundamental importance in local deformation theory ([3], [21]). For example, if ρ_t is onto, the family $\{V_t\}_{t \in \Delta}$ will be *locally universal,* and V_t will have $\dim H^1(V_t, \Theta_t)$ moduli locally.

We now want to choose the C^∞ trivialization $\phi\colon V_0 \times \Delta \to \boldsymbol{V}$ carefully. To begin with, we can assume that $\phi^{-1}(U_\alpha) = W_\alpha \times \Delta$ where $\{W_\alpha\}$ is a covering of V_0. Thus we can have $W_\alpha = U_\alpha \cap V_0$ and then $z_\alpha = (z_\alpha{}^1, \cdots, z_\alpha{}^n)$ gives a holomorphic coordinate on $W_\alpha \subset V_0$.

(1.6) LEMMA. *We can assume that $\phi^*(z_\alpha) = z_\alpha \circ \phi$ is of the form $\zeta_\alpha(z_\alpha, \bar{z}_\alpha; t)$ where $\zeta_\alpha(z_\alpha, \bar{z}_\alpha; 0) = z_\alpha$ and $\zeta_\alpha(z_\alpha, \bar{z}_\alpha, t)$ is holomorphic in t.*

Proof. This Lemma is implicit in [24], and we know of no elementary proof not involving the sort of estimates given here. In other words, Lemma (1.6) is true for the Kuranishi family, and, by his universality theorem, will be true in general (cf. the remark following Proposition (1.11)).

It follows that, in $W_\alpha \cap W_\beta$, $\zeta_\alpha(z,t) = h_{\alpha\beta}(\zeta_\beta(z,t),t)$ where $h_{\alpha\beta}(\zeta_\beta, t)$ is holomorphic in both variables. Furthermore, $h_{\alpha\beta}(\zeta_\beta(z,0),0) = h_{\alpha\beta}(z_\beta, 0) = f_{\alpha\beta}(z_\beta)$ will be the transition functions on V_0. Write now:

$$d\zeta_\alpha{}^j = \sum_{k=1}^{n} \frac{\partial \zeta_\alpha{}^j}{\partial z_\alpha{}^k} \{ dz_\alpha{}^k + \sum_{l=1}^{n} \Phi_{\alpha,\bar{l}} d\bar{z}_\alpha{}^l \} \tag{1.7}$$

(1.8) LEMMA. *In (1.7), the vector-valued form*

$$\Phi_\alpha(t) = \sum_{k,l} \Phi_{\alpha\bar{l}}{}^k(t) \frac{\partial}{\partial z_\alpha{}^k} \otimes d\bar{z}_\alpha{}^l$$

is a global tensor, which depends holomorphically on t, and which satisfies:

$$\bar{\partial}\Phi(t) - [\Phi(t), \Phi(t)] = 0. \tag{1.9}$$

Proof. By definition, we have $\bar{\partial}\zeta_\alpha{}^j(t) = \sum_{k,l} \frac{\partial \zeta_\alpha{}^j}{\partial z^k} \Phi_{\alpha\bar{l}}{}^k(t) d\bar{z}_\alpha{}^l$. From $0 = \bar{\partial}^2 \zeta_\alpha{}^j(t)$, we get (1.9) by using the definition of [,]. The fact that $\Phi_\alpha(t) = \Phi_\beta(t)$ in $U_\alpha \cap U_\beta$ is seen as follows:

$$\begin{aligned}
\sum_{k,l} \frac{\partial \zeta_\alpha{}^j}{\partial z_\alpha{}^k} \{\Phi_{\alpha\bar{l}}{}^k d\bar{z}_\alpha{}^l\} &= \bar{\partial}\zeta_\alpha{}^j = \sum_k \frac{\partial h_{\alpha\beta}{}^j}{\partial \zeta_\beta{}^k} \bar{\partial}\zeta_\beta{}^k \\
&= \sum_{k,l,m} \frac{\partial h_{\alpha\beta}{}^j}{\partial \zeta_\beta{}^k} \frac{\partial \zeta_\beta{}^k}{\partial z_\beta{}^m} \Phi_{\beta\bar{l}}{}^m d\bar{z}_\beta{}^l = \sum_{m,l} \frac{\partial h_{\alpha\beta}{}^j}{\partial z_\beta{}^m} \Phi_{\beta\bar{l}}{}^m d\bar{z}_\beta{}^l \\
&= \sum_{m,l} \frac{\partial \zeta_\alpha{}^j}{\partial z_\beta{}^m} \Phi_{\beta\bar{l}}{}^m d\bar{z}^l = \sum_{k,l,m} \frac{\partial \zeta_\alpha{}^j}{\partial z_\alpha{}^k} \{ \frac{\partial z_\alpha{}^k}{\partial z_\beta{}^m} \Phi_{\beta\bar{l}}{}^m d\bar{z}_\beta{}^l \}.
\end{aligned}$$

The equality of the two terms in brackets $\{\cdot\ \cdot\ \cdot\}$ is just the equation $\Phi_\alpha = \Phi_\beta$. This proves the Lemma.

Note that $\Phi(0) = 0$; we set $\theta = \frac{\partial \Phi(t)}{\partial t}]_{t=0}$.

(1.10) LEMMA. *The vector-valued form θ satisfies $\bar{\partial}\theta = 0$, and θ is the Doubeault class representing $\theta(0) = \rho_0(\frac{\partial}{\partial t})$ in $H^1(V, \Theta)$.*

Proof. The fact that $\bar\partial\theta = 0$ follows from (1.9) since $\Phi(t) = t\theta + t^2(\cdots)$.

Now we let $\theta_\alpha = \sum_j \frac{\partial \zeta_\alpha{}^j}{\partial t}]_{t=0} \frac{\partial}{\partial z_\alpha{}^j}$; θ_α is a C^∞ vector field in U_α and

$$\bar\partial\theta_\alpha = \sum_{j,k} \frac{\partial^2 \zeta_\alpha{}^j}{\partial t \partial \bar z_\alpha{}^k}]_{t=0} \frac{\partial}{\partial z_\alpha{}^j} \otimes d\bar z_\alpha{}^k$$

$$= \sum_{j,k} \frac{\partial \Phi(t)_{\alpha \bar k}{}^j}{\partial t}]_{t=0} \frac{\partial}{\partial z_\alpha{}^j} \otimes d\bar z_\alpha{}^k = \sum_{j,k} \theta_{\alpha \bar k}{}^j \frac{\partial}{\partial z_\alpha{}^j} \otimes d\bar z_\alpha{}^k = \theta.$$

On the other hand, in $U_\alpha \cap U_\beta$, $\zeta_\alpha(z,t) = h_{\alpha\beta}(\zeta_\beta(z,t),t)$ so that

$$\frac{\partial \zeta_\alpha{}^j}{\partial t}]_{t=0} = \sum_k \frac{\partial h_{\alpha\beta}{}^j}{\partial \zeta_\beta{}^k} \frac{\partial \zeta_\beta{}^k}{\partial t}]_{t=0} + \frac{\partial h_{\alpha\beta}{}^j}{\partial t}]_{t=0}.$$

Using the fact that $\zeta_\alpha(z,0) = z_\alpha$ and $h_{\alpha\beta}(\zeta_\beta, 0) = f_{\alpha\beta}(z_\beta)$, this gives

$$\frac{\partial \zeta_\alpha}{\partial t}]_{t=0} - \sum_k \frac{\partial f_{\alpha\beta}{}^j}{\partial z_\beta{}^k} \frac{\partial \zeta_\beta{}^k}{\partial t}]_{t=0} = \frac{\partial h_{\alpha\beta}{}^j}{\partial t}]_{t=0},$$

or $\theta_\alpha - \theta_\beta = \sum_j \frac{\partial h_{\alpha\beta}{}^j}{\partial t}(z_\beta, 0) \frac{\partial}{\partial z_\beta{}^j} = \theta_{\alpha\beta}(0)$. By the definition of the Dolbeault isomorphism, $\theta = \rho_0(\frac{\partial}{\partial t}) \in H^1(V, \Theta)$.

To collect these results in a systematic statement, we let $\omega_\alpha{}^j = dz_\alpha{}^j + \sum \Phi_{\alpha \bar k}{}^j(t) d\bar z^k$ and have:

(1.11) Proposition. *Let $\{V_t\}_{t\in\Delta}$ be an analytic family of compact, complex manifolds given as an analytic fibre space $\boldsymbol{V} \to \Delta$. Relative to a suitable covering $\{W_\alpha\}$ of $V = V_0$, there exist a family of linearly independent 1-forms $\omega_\alpha{}^1(t), \cdots, \omega_\alpha{}^n(t)$ defined in W_α and satisfying:*

(i) *the $\omega_\alpha{}^j(t)$ give the almost-complex structure on V_t;*

(ii) *$\omega_\alpha{}^j(t)$ depends holomorphically on t;*

(iii) *$\omega_\alpha{}^j(t) = dz_\alpha{}^j + \sum_k \Phi_{\alpha \bar k}{}^j(t) d\bar z_\alpha{}^k$ where $\Phi_\alpha(t) = \sum \Phi_{\alpha \bar k}{}^j(t) \frac{\partial}{\partial z_\alpha{}^j} \otimes d\bar z_\alpha{}^k$ is a vector-valued form depending holomorphically on t with $\Phi(0) = 0$;*

(iv) *if $\lambda = \sum_{a=1}^m \lambda^a \frac{\partial}{\partial t^a} \in T_0(\Delta)$, then $\sum \lambda^a \frac{\partial \Phi(t)}{\partial t^a}]_{t=0} = \rho(\lambda) \in H^1(V, \Theta)$ is the Kodaira-Spencer class (infinitesimal deformation class).*

Remarks. The *Frobenius integrability condition,*

$$d\omega_\alpha{}^j \equiv 0(\omega_\alpha{}^1, \cdots, \omega_\alpha{}^n),$$

is equivalent to the equation (1.9), as is easily verified.

It is also perhaps worth remarking that Proposition (1.11) can be proved fairly easily in a special case, which will cover most of our examples. Namely, *assume* that there are no obstructions to finding deformations with a given class $\theta \in H^1(V,\Theta)$ as tangent. Then the construction of [20] gives a family $V' \to \Delta'$ such that $\rho\colon T_0(\Delta') \to H^1(V,\Theta)$ is onto. It then follows from [21] that $V' \to \Delta'$ is holomorphically universal, the argument here being much simpler than Kuranishi's. Now the same argument as in the proof of Lemma (1.6) will apply.

With these preliminaries, we can prove Theorem (1.1) by showing that (1.3) holds. This amounts to the following: Given a harmonic (relative to $\omega(0) = \sum_{i,j} g_{ij} dz^i d\bar{z}^j$) $(q-r,r)$ form on V_0, written locally as:

$$\phi = \sum_{i,j} \phi_{i_1\cdots i_{q-r}\bar{j}_1\cdots\bar{j}_r} dz^{i_1} \wedge \cdots \wedge dz^{i_{q-r}} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_r},$$

we have to find harmonic $(q-r,r)$ forms $\phi(t)$ on V_t, relative to $\omega(t) = \sum_{i,j} g_{ij}(t)\omega^i(t)\bar{\omega}^j(t)$, which satisfy:

$$\frac{\partial\phi(t)}{\partial\bar{t}}\Big]_{t=0} \in H^{q,0}(V_0) + \cdots + H^{q-r,r}(V_0). \tag{1.13}$$

By the fundamental *continuity theorems* of Kodarira-Spencer [19], we may assume that $g_{ij}(t)$ is smooth as a function of $z, \bar{z}, t$; and that

$$\phi(t) = \sum \phi_{i_1\cdots i_{q-r}\bar{j}_1\cdots\bar{j}_r}(t)\,\omega^{i_1}(t) \wedge \cdots \wedge \omega^{i_{q-r}}(t) \wedge \bar{\omega}^{j_1}(t) \wedge \cdots \wedge \bar{\omega}^{j_r}(t), \tag{1.14}$$

where $\phi_{i_1\cdots i_{q-r}j_1\cdots j_r}(t)$ is also smooth in $z, \bar{z}, t$.

We now use (iii) in Proposition (1.11) to write: $\phi(t) = \phi_1 + t\phi_2 + \bar{t}\phi_3 + [2]$, where [2] are terms of order 2, ϕ_1 is of type $(q-r,r)$,

$$\phi_3 = \sum_s \theta_{\bar{k}}^{\,j_s} \phi_{i_1\cdots i_{q+r}\bar{j}_1\cdots\bar{j}_r}(0)\, dz^{i_1} \wedge \cdots \wedge dz^{i_{q-r}} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge \underset{(s)}{dz^k} \wedge \cdots \wedge d\bar{z}^{j_r}$$

is of type $(q-r+1, r-1)$, and:

$$\phi_2 = \sum_s \theta_k^{\,i_s} \phi_{i_1\cdots i_{q+r}\bar{j}_1\cdots\bar{j}_r}(0)\, dz^{i_1} \wedge \cdots \wedge \underset{(s)}{d\bar{z}^k} \wedge \cdots \wedge dz^{i_{q-r}} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_r}; \tag{1.15}$$

is of type $(q-r-1, r+1)$. From this it follows that $\dfrac{\partial\phi(t)}{\partial\bar{t}}\Big]_{t=0} \equiv 0$ modulo $H^{q,0}(V_0) + \cdots + H^{q-r,r}(V_0)$ and

$$\frac{\partial\phi(t)}{\partial t}\Big]_{t=0} \equiv \phi_2 (H^{q,0} + \cdots + H^{q-r,r}) \tag{1.16}$$

This proves (1.13) and hence Theorem (1.1).

Remark. We let $\Omega_r(t) \in G(h_r, W)$ be the subspace

$$S_r(t) = H^{q,0}(V_t) + \cdots + H^{q-r,r}(V_t).$$

Then, using Lemma (4.2)) in I.4.(b),

(1.17) $T_{\Omega_r}(G(h_r, W)) \cong \operatorname{Hom}(H^{q,0} + \cdots + H^{q-r,r}, H^{q-r-1,r+1} + \cdots + H^{0,q})$

where $\Omega_r = \Omega\ (0)$ and $H^{q-s,s} = H^{q-s,s}(V_0)$. We want to use (1.15) and (1.16) to compute the differential

(1.18) $(\Omega_r)_* \colon T_0(\Delta) \to \operatorname{Hom}(H^{q,0} + \cdots + H^{q-r,r}, H^{q-r-1,r+1} + \cdots + H^{0,q})$.

To begin with, we have the isomorphism $H^{q-s,s} \cong H^s(V, \Omega^{q-s})$, while the pairing $\Theta \otimes \Omega^{q-s} \to \Omega^{q-s-1}$ gives

(1.19) $$H^1(V, \Theta) \otimes H^s(V, \Omega^{q-s}) \to H^{s+1}(V, \Omega^{q-s-1}).$$

In other words, by using *cup product*, a class $\theta \in H^1(V, \Theta)$ defines an element $\hat{\theta} \in \operatorname{Hom}(H^{q-s,s}, H^{q-s-1,s+1})$.

(1.20) PROPOSITION. *The differential* $(\Omega\ )_*$ *in* (1.18) *is given by* $(\Omega\ )_*(\lambda) = \rho(\hat{\lambda})$, *where* $\lambda \in T_0(\Delta)$ *and* $\rho(\lambda) \in H^1(V, \Theta)$ *is the Kodaira-Spencer class.*

Proof. We have that $(\Omega\ )_*(\frac{\partial}{\partial t})\phi = \frac{\partial \phi(t)}{\partial t}]_{t=0}$, projected into $W/S_r(0) = H^{q-r-1,r+1} + \cdots + H^{0,q}$. Thus, by (1.16), $(\Omega\ )_*(\frac{\partial}{\partial t})\phi = \phi_2$. On the other hand, since $\theta = \frac{\partial \Phi(t)}{\partial t}]_{t=0} = \rho(\frac{\partial}{\partial t})$ by Lemma (1.10), we have that $\phi_2 = \theta \cdot \phi$; i.e., ϕ_2 is the cup product (using differential forms) of θ and ϕ. This says precisely that $(\Omega\ )_*(\frac{\partial}{\partial t}) = \hat{\theta} = \rho(\hat{\frac{\partial}{\partial t}})$.

(1.21) COROLLARY. $(\Omega\ )_*\{T_0(\Delta)\}$ *lies in the subspace* $\operatorname{Hom}(H^{q-r,r}, H^{q-r-1,r+1}) \subset \operatorname{Hom}(H^{q,0} + \cdots + H^{q-r,r}, H^{q-r-1,r+1} + \cdots + H^{0,q})$.

The mapping we want to consider is $\Omega(t) = [S_0(t), \cdots, S_v(t)]$ considered as belonging to a flag manifold $\boldsymbol{F}$. From the embedding:

$$\boldsymbol{F} \subset G(h_0, W) \times \cdots \times G(h_v, W),$$

we have that $T_\Omega(\boldsymbol{F}) \subset \sum_{r=0}^{v} \operatorname{Hom}(S_r, W/S_r)$, where $\Omega = [S_0, \cdots, S_v]$. The condition that $(\phi_0, \cdots, \phi_v)$ with $\phi_r \in \operatorname{Hom}(S_r, W/S_r)$ be a tangent vector in $T_\Omega(\boldsymbol{F})$ is that, for $s < r$, we have a commuting diagram:

$$\begin{array}{ccc} S_s & \longrightarrow & S_r \\ \Big\downarrow{\scriptstyle \phi_s} & & \Big\downarrow{\scriptstyle \phi_r} \\ W/S_s & \longrightarrow & W/S_r \end{array}$$

It follows then that

$$(1.22) \qquad T_\Omega(\boldsymbol{F}) \cong \sum_{r=0}^{v} \operatorname{Hom}(H^{q-r,r}, H^{q-r-1,r+1} + \cdots + H^{0,q}).$$

Combining this with (1.20) and (1.21) we have:

(1.23) THEOREM. *$\Omega_*(T_0(\Delta))$ lies in the subspace*

$$\sum_{r=0}^{v} \operatorname{Hom}(H^{q-r,r}, H^{q-r-1,r+1})$$

of $T_\Omega(\boldsymbol{F})$ given by (1.22). *For $\lambda \in T_0(\Delta)$, $\phi \in H^{q-r,r}$, we have $\Omega_*(\lambda)\phi = \rho(\lambda)\cdot\phi$, where $\rho(\lambda) \in H^1(V,\Theta)$ is the Kodaira-Spencer class and $\rho(\lambda)\cdot\phi$ is the cup product* (1.19).

Examples. (i) When $q=1$, $T_\Omega(\boldsymbol{F}) \cong \operatorname{Hom}(H^{1,0}, H^{0,1})$; (ii) when $q=2$, $T_\Omega(\boldsymbol{F}) \cong \operatorname{Hom}(H^{2,0}, H^{1,1}+H^{0,2})$ and $\Omega_*(T_0(\Delta))$ lies in $\operatorname{Hom}(H^{2,0}, H^{1,1})$; (iii) when $q=3$,

$$T_\Omega(\boldsymbol{F}) \cong \operatorname{Hom}(H^{3,0}, H^{2,1}+H^{1,2}+H^{0,3}) \oplus \operatorname{Hom}(H^{2,1}, H^{1,2}+H^{0,3})$$

and $\Omega_*(T_0(\Delta))$ lies in $\operatorname{Hom}(H^{3,0}, H^{2,1}) \oplus \operatorname{Hom}(H^{2,1}, H^{1,2})$; and (iv) when $q=4$,

$$T_\Omega(\boldsymbol{F}) \cong \operatorname{Hom}(H^{4,0}, H^{3,1}+H^{2,2}+H^{1,3}+H^{0,4}) \oplus \operatorname{Hom}(H^{3,1}, H^{2,2}+H^{1,3}+H^{0,4}),$$

and $\Omega_*(T_0(\Delta))$ lies in the subspace $\operatorname{Hom}(H^{4,0}, H^{3,1}) \oplus \operatorname{Hom}(H^{3,1}, H^{2,2})$.

(b) We assume now that we have a *polarized family of algebraic manifolds*; i.e., over $\boldsymbol{V}$ we have given a line bundle $\mathcal{L} \to \boldsymbol{V}$ such that the restriction $\mathcal{L} \mid V_t = \boldsymbol{L}_t$ is *positive* in the sense of Kodaira [12]. In this cas we choose $\omega(t)$ to be a curvature form representing the *Chern class* of $\boldsymbol{L}_t \to V_t$ ([11]). It follows that, in cohomology, $\phi^*(\omega(t)) = \omega$, where ω on $V = V_0$ is the Kähler form.

Recalling now the notions of Kähler varieties as reviewed in I.1.(c), it follows that ϕ^* preserves all the cohomology structure of $H^q(V,\boldsymbol{C})$, except those notions dealing with type. In particular, $\phi^*\{H^q(V_t,\boldsymbol{C})_0\} = H^q(V,\boldsymbol{C})_0$ ($=$ *primitive cohomology* in dimension q) and we may take $W = H^q(V,\boldsymbol{C})_0$, $S_r(t) = \phi^*\{H_0{}^{q,0}(V_t) + \cdots + H^{q-r,r}(V_t)\}$ and pursue the same development as in Section II.1.(a) above. The point $\Omega(t) = [S_0(t),\cdots,S_v(t)]$

will not only lie on a flag manifold $\boldsymbol{F}$, but, in fact, $\Omega(t)$ will lie on the domain $D \subset \boldsymbol{F}$ given by the bilinear relations (1.16) and (1.17) of I.1.(d). In that section, D was called the *period matrix space* and its properties are given in part I. We recall that $T_\Omega(D) \subset T_\Omega(\boldsymbol{F})$ is the subspace of $T_\Omega(\boldsymbol{F})$ in (1.22) given by:

$$(1.24) \qquad T_\Omega(D) \cong \sum_{r=0}^{v} \mathrm{Hom}_Q(H_0{}^{q-r,r}, H_0{}^{q-r-1,r+1} + \cdots + H_0{}^{0,q}),$$

where, by definition, $\phi \in \mathrm{Hom}_Q(H_0{}^{q-r,r}, H_0{}^{r,q-r})$ if, and only if,

$$(1.25) \qquad Q(\phi(\xi),\zeta) + Q(\xi,\phi(\zeta)) = 0 \quad (\xi \in H_0{}^{q-r,r}, \zeta \in H_0{}^{q-r,r}).$$

Here, Q is the quadratic form on $H^q(V,\boldsymbol{C})_0$ given by (1.7) of I.1.(c).

We let now $H^1(V,\Theta)_\omega \subset H^1(V,\Theta)$ be those classes θ satisfying $\theta \cdot \omega = 0$ in $H^2(V,\boldsymbol{\mathcal{O}})$, where $\omega \in H^1(V,\Omega^1)$ is the Kähler class. If $\theta = \rho(\lambda)$ for some $\lambda \in T_0(\Delta)$, then $\theta \in H^1(V,\Theta)_\omega$. In fact, $H^1(V,\Theta)_\omega$ is just the subspace of $H^1(V,\Theta)$ *which infinitesimally preserve the polarization.* If $\theta \in H^1(V,\Theta)_\omega$, then $\theta \cdot H^q(V,\boldsymbol{C})_0 \subset H^q(V,\boldsymbol{C})_0$ and we have, in place of (1.19), that:

$$(1.26) \qquad H^1(V,\Theta)_\omega \otimes H_0{}^{q-r,r} \to H_0{}^{q-r-1,r+1}.$$

Having noted now the additional relations which appear when we consider a polarized family, we may give the main theorem, whose proof already follows from Theorems (1.1) and (1.23).

(1.27) THEOREM. (i) *The period matrix mapping* $\Omega\colon \Delta \to D$ *given by* $\Omega(t) = [S_0(t),\cdots,S_v(t)]$ *where* $S_r(t) = H_0{}^{q,0}(V_t) + \cdots + H_0{}^{q-r,r}(V_t)$ *is holomorphic*; (ii) $\Omega_*(T_0(\Delta)) \subset \sum_{r=0}^{v} \mathrm{Hom}_Q(H_0{}^{q-r,r}, H_0{}^{q-r-1,r+1}) \subset T_\Omega(D)$ *given by* (1.24) ; *and* (iii) *if* $\phi \in H_0{}^{q-r,r}$ *and* $\lambda \in T_0(\Delta)$, $\Omega_*(\lambda)\phi = \rho(\lambda)\cdot\phi \in H_0{}^{q-r-1,r+1}$ *where* $\rho(\lambda) \in H^1(V,\Theta)_\omega$ *is the Kodaira-Spencer class and* $\rho(\lambda)\cdot\phi$ *is the cup product* (1.26).

Examples of (ii). We give the analogues of the examples following Theorem (1.23). When $q=1$, $T_\Omega(D) \cong \mathrm{Hom}_Q(H^{1,0}, H^{0,1})$ and there are no restrictions on $\Omega_*\{T_0(\Delta)\}$. When $q=2$, $T_\Omega(D) \cong \mathrm{Hom}_Q(H^{2,0}, H_0{}^{1,1} + H^{0,2})$ and $\Omega_*\{T_0(\Delta)\} \subset \mathrm{Hom}(H^{2,0}, H_0{}^{1,1})$. When $q=3$,

$$T_\Omega(D) \cong \mathrm{Hom}_Q(H^{3,0}, H_0{}^{2,1} + H_0{}^{1,2} + H^{0,3}) \quad \oplus \mathrm{Hom}_Q(H_0{}^{2,1}, H_0{}^{1,2} + H^{0,3})$$

and $\Omega_*\{T_0(\Delta)\} \subset \mathrm{Hom}(H^{3,0}, H_0{}^{2,1}) \oplus \mathrm{Hom}_Q(H_0{}^{2,1}, H_0{}^{1,2})$.

We want to give now a cohomological condition in order that the *periods should give local moduli.* To do this, we observe that $H^{n-1}(V,\Omega^1 \otimes \Omega^n)$ is the dual space to $H^1(V,\Theta)$ and, if we let $H^1(V,\Theta)_\omega{}^\natural \subset H^{n-1}(V,\Omega^1\otimes\Omega^n)$ be the annihilator of $H^1(V,\Theta)_\omega$, then:

$$(1.28)\qquad H^{n-1}(V,\Omega^1\otimes\Omega^n)_\omega = H^{n-1}(V,\Omega^1\otimes\Omega^n)/H^1(V,\Theta)_\omega{}^\natural,$$

is the dual space to $H^1(V,\Theta)_\omega$. We shall consider families $\{V_t\}_{t\in\Delta}$ which are subfamilies of the *Kuranishi univeral family* ([25]); in practice, this will mean that $\Delta\subset H^1(V,\Theta)_\omega$ and $\rho\colon T_0(\Delta)\to H^1(V,\Theta)$ is the identity mapping. The *periods will be said to give local moduli* if, for any such family, the differential Ω_* of the period matrix mapping is of maximal rank.

(1.29) THEOREM. *The periods of the primitive q-forms give local moduli if the cup product:*

$$(1.30)\qquad \sum_{r=0}^{v} H_0{}^{q-r,r}\otimes H_0{}^{n-q+r+1,n-r-1} \xrightarrow{\mu} H^{n-1}(V,\Omega^1\otimes\Omega^n)_\omega,$$

is surjective.

Proof. This follows simply by dualizing the condition that

$$T_0(\Delta)\xrightarrow{\Omega_*}\sum_{r=0}^{v}\mathrm{Hom}_Q(H_0{}^{q-r,r},H_0{}^{q-r-1,r+1})$$

should be injective.

Remarks. An important special case occurs when the *canonical bundle* $\boldsymbol{K}\to V$ is positive. Then any family preserves this polarization and so $H^1(V,\Theta)_\omega = H^1(V,\Theta)$, $H^{n-1}(V,\Omega^1\otimes\Omega^n)_\omega = H^{n-1}(V,\Omega^1\otimes\Omega^n)$, and (1.30) becomes:

$$(1.31)\qquad \sum_{r=0}^{v} H_0{}^{q-r,r}\otimes H_0{}^{n-q+r+1,n-r-1} \xrightarrow{\mu} H^{n-1}(V,\Omega^1\otimes\Omega^n).$$

If we ignore polarizations, the condition that the periods of the q-forms give local moduli is that the cup product:

$$(1.32)\qquad \sum_{r=0}^{v} H^{q-r,r}\otimes H^{n-q+r+1,n-r-1}\to H^{n-1}(V,\Omega^1\otimes\Omega^n)$$

should be onto.

(c) Let now $\boldsymbol{V}\xrightarrow{\pi} B$ be a family of polarized algebraic manifolds where B is assumed *simply connected.* Then we can globally define the *period mapping*:

$$\Omega\colon B\to D,$$

where D is the period matrix space for the primitive harmonic q-forms (cf. I.1.(d)). Conerning the domain D, we recall the following facts:

(i) D is a *homogeneous complex manifold*; $D = H\backslash G$ where G is real, simple Lie group and $H \subset G$ is a compact subgroup (I. 1, Theorem (1.26));

(ii) The *canonical bundle* $\boldsymbol{K} \to D$ has a unique G-invariant p-convex polarization (I. 3.(c) and I. 4.(b), Theorem (4.8));

(iii) If $K \subset G$ is the *maximal compact subgroup* and $P = K\backslash G$, then the fibres in the fibering $D \xrightarrow{\tilde{\omega}} P$ (given by $H\backslash G \to K\backslash G$) are compact, complex subvarieties of D. If we set $Y_\lambda = \tilde{\omega}^{-1}(\lambda)$, then $p = \dim Y_\lambda$ and the canonical bundle $\boldsymbol{K}$ is negative on Y_λ (I. 4.(c) Theorem (4.41)); and

(iv) For each $\Omega \in D$, there is a unique G-invariant splitting:

$$T_\Omega(D) = V_\Omega \oplus H_\Omega, \tag{1.33}$$

where V_Ω is the tangent space to the fibre of $\tilde{\omega}$ passing through Ω. The curvature form ω of $\boldsymbol{K}$ is negative on V_Ω and positive on H_Ω, and the *Levi form* $L(\phi)$ of an $(m-p)$-pseudo-convex exhaustion function ϕ of D is positive on H_Ω ($m = \dim D$) (cf. I. 4.(f), Proposition (4.66) and I. 4.(d), Lemma (4.46)).

(1.34) THEOREM. *Let $\Omega\colon B \to D$ be the holomorphic period mapping. Then, for $t \in B$,*

$$\Omega_*(T_t(B)) \subset H_{\Omega(t)}.$$

Thus $\boldsymbol{K}$ is a positive bundle on $\Omega(B)$ and $\phi \mid \Omega(B)$ is a pseudo-convex function. Furthermore, $\Phi(B)$ is transverse to the compact subvarieties Y_λ in (iii) *above.*

Proof. In keeping with the arguments of I. 4., we shall prove this result for 2-forms and 3-forms. For 2-forms, by (4.70) in I. 4.(f), we have that

$$H_{\Omega(t)} = \mathrm{Hom}(H^{2,0}(V_t), H_0^{1,1}(V_t)); \tag{1.35}$$

for 3-forms, by (4.73) in I. 4.(f) we see that:

$$\begin{aligned} H_{\Omega(t)} = {} & \mathrm{Hom}(H^{3,0}(V_t), H_0^{2,1}(V_t)) \oplus \mathrm{Hom}_Q(H_0^{2,1}(V_t), H_0^{1,2}(V_t)) \\ & + \mathrm{Hom}_Q(H^{3,0}(V_t), H^{0,3}(V_t)). \end{aligned} \tag{1.36}$$

From the examples following Theorem (1.27), we see that, for 2-forms,

$$\Omega_*(T_t(B)) \subset \mathrm{Hom}(H^{2,0}(V_t), H_0^{2,1}(V_t)); \tag{1.35$'$}$$

and, for 3-forms,

(1.36)′ $\Omega_*(T_t(B)) \subset \mathrm{Hom}(H^{3,0}(V_t), H_0{}^{2,1}(V_t)) \oplus \mathrm{Hom}_Q(H_0{}^{2,1}(V_t), H_0{}^{1,2}(V_t))$.

Comparing (1.35) with (1.35)′ and (1.36) with (1.36)′ gives the theorem.

Remarks. The condition $\Omega_*(T_t(B)) \subset H_{\Omega(t)}$ can be phrased as a bilinear relation $Q_1(\Omega(t), d\Omega(t)) = 0$. The family of subspaces $H_\Omega \subset T_\Omega(D)$ gives a *non-integrable distribution* in the complex tangent bundle; the tangent spaces to $\Omega(B)$ give an integrable subdistribution, which means that relations of the form $Q_2(d\Omega(t), d\Omega(t)) = 0$ hold (compare (1.36) and (1.36)′). Thus:

(1.37) CONCLUSION. *Let V be a polarized algebraic variety defined over a function field $\mathfrak{F}$. Then, if Ω is the period matrix of V, Ω satisfies the Hodge bilinear relations $Q(\Omega, \Omega) = 0$, $Q(\Omega, \bar{\Omega}) > 0$ plus additional relations $Q_1(\Omega, d\Omega) = 0$ and $Q_2(d\Omega, d\Omega) = 0$, where $d\Omega$ is defined over $\mathfrak{F}$.*

It should be emphasized that these new relations are *universal*, as opposed, e.g., to the *special* $\frac{(g-2)(g-3)}{2}$ relations satisfied by a curve of genus g, but not satisfied by the periods of the 1-forms of a general V having the Siegel space as period matrix domain.

We shall now give these additional relations explicitly for periods of 2-forms and 3-forms. These are the two cases discussed at length in I. 4. (b) and I. 4. (c).

Examples. (i) Let V be a polarized algebraic manifold and $h = h^{2,0}(V)$, $k = h_0{}^{1,1}(V) = h^{1,1}(V) - 1$. Then there will be a $(2h+k) \times (2h+k)$ symmetric matrix Q and the period matrix space D will consist of all $h \times (2h+k)$ matrices Ω which satisfy:

(1.38)
$$\begin{cases} \Omega Q^t\Omega = 0 \\ \Omega Q^t\bar{\Omega} > 0; \end{cases}$$

(1.39) PROPOSITION. *The period matrix of V over $\mathfrak{F}$ satisfies the additional bilinear relations:*

(1.40)
$$\begin{cases} d\Omega Q^t\Omega = 0. \\ d\Omega Q^t d\Omega = 0. \end{cases}$$

Remark. We first observe that (1.40) makes sense; if we replace Ω by $A\Omega$, then $d(A\Omega)Q^t(A\Omega) = dA\Omega Q^t\Omega^t A + Ad\Omega Q^t\Omega A = A(d\Omega Q^t\Omega)^t A$ (by (1.38)). By a similar calculation, we see that both equations in (1.40) make sense on D.

Proof. To prove the first relation in (1.40), we have to show that, if Δ is any disc with parameter t and $\Omega(t)$ the variable period matrix, then

$\Omega'(t)Q\Omega(t)=0$. If $W=H^2(V,\boldsymbol{C})_0$, then $\Omega(t)$ gives a subspace $S_t\subset W$ ($S_t=H^{2,0}(V_t)$; cf. I.1.(a)) and $\Omega'(t)$ gives the tangent to the curve (S_t) as follows: Write $\Omega(t)=\begin{pmatrix}\pi_1(t)\\ \vdots\\ \pi_h(t)\end{pmatrix}$ where the row vectors $\pi_1(t),\cdots,\pi_h(t)$ give a basis for S_t. We then define $\frac{\partial}{\partial t}\in \operatorname{Hom}(S_t,W/S_t)$ by sending $\pi_\alpha(t)$ into $\frac{d\pi_\alpha(t)}{dt}$ (modulo S_t).

By Corollary (1.21), $\frac{\partial}{\partial t}\in\operatorname{Hom}(H^{2,0}(V_t),H_0^{1,1}(V_t))$ and then $\frac{d\pi_\alpha(t)}{dt}\in H^{2,0}(V_t)+H_0^{1,1}(V_t)$ so that $\Omega'(t)Q\Omega(t)=0$ as desired.

The second condition in (1.40) follows by taking the exterior derivative of $d\Omega Q^t\Omega=0$.

(ii) As above, V is a polarized algebraic manifold and we let $2n=\dim W$ where $W=H^3(V,\boldsymbol{C})_0$, $q=h_0^{2,1}(V)$, $n-q=h^{3,0}(V)$. To describe the period matrix space D, we are given a rational skew-symmetric matrix Q and we consider $n\times 2n$ matrices $\Omega=\begin{pmatrix}\Omega_1\\ \Omega_2\end{pmatrix}$ where Ω_1 is $(n-q)\times 2n$. With equivalence relation $\Omega\sim A\Omega$ where $A=\begin{pmatrix}A_{11} & 0\\ A_{12} & A_{22}\end{pmatrix}$ (A_{11} is $(n-q)\times(n-q)$), the bilinear relations giving D are:

$$(1.41)\qquad \begin{cases}\Omega Q^t\Omega=0\\ \sqrt{-1}\,\Omega_1Q^t\bar{\Omega}_1>0\\ \sqrt{-1}\,\Omega Q^t\bar{\Omega}\text{ has signature }(n-q,q).\end{cases}$$

Here Ω_1 represents the space $H^{3,0}(V)\subset W$ and Ω the space $H^{3,0}(V)+H_0^{2,1}(V)$.

(1.42) PROPOSITION. *The period matrix Ω of V over $\mathfrak{J}$ satisfies the additional bilinear relations:*

$$(1.43)\qquad \begin{cases}d\Omega_1Q^t\Omega=0\\ d\Omega Q^t\Omega_1=0\\ d\Omega_1Q^td\Omega=0.\end{cases}$$

The proof of this Proposition is basically the same as that of Proposition (1.39).

(d) As an application of Theorem (1.34), we have:

(1.44) THEOREM. *Let $V\xrightarrow{\pi}B$ be an analytic fibre space of polarized* *algebraic manifolds where B is compact and simply connected. Then the period mapping $\Phi: B \to D$ is constant.*

Proof. Let $\phi: D \to \boldsymbol{R}$ be the exhaustion function for D, and $\psi = \phi \cdot \Phi$. If ξ, η are tangent vectors of type $(1,0)$ to B, we claim that $L(\psi)(\xi, \bar{\eta}) = L(\phi)(\Phi_*(\xi), \overline{\Phi_*(\eta)})$ where $L(\psi)$ and $L(\phi)$ are the E. E. Levi forms. This is a straightforward computation using the fact that Φ is holomorphic. Since $L(\phi) > 0$ on $H_{\Phi(t)} \subset T_{\Phi(t)}(D)$, it follows that $L(\psi) \geqq 0$ and $L(\psi) = 0$ if, and only if, ϕ is constant (i.e., $\Phi_* \equiv 0$). But a pseudo-convex function on a compact manifold is necessarily constant, which proves the theorem.

II. 2. Complex torii associated with algebraic varieties. (a) We first discuss general complex torii. Let E_0 be a real, $2m$-dimensional vector space with basis $e_1, \cdots, e_{2m}$, and let $E = E_0 \otimes_{\boldsymbol{R}} \boldsymbol{C}$ be the complexification of E. Let $B \subset G(m, E)$ be the open subset of all n-dimensional subspaces $S \subset E$ with $S \cap \bar{S} = 0$. We construct a family of complex torii $\boldsymbol{V} \xrightarrow{\pi} B$ as follows: Over $G(m, E)$ we let $\boldsymbol{E} \to G(m, E)$ be the holomorphic universal bundle with fibre $E_S = E/S$ at $S \in G(m, E)$. Over B, the lattice Γ generated (over $\boldsymbol{R}$) by $e_1, \cdots, e_{2m}$ projects onto a lattice Γ_S in E_S, and we let $T_S = E_S/\Gamma_S$. In this *way* we get an analytic fibre space of complex torii $\boldsymbol{V} \xrightarrow{\pi} B$ with $\pi^{-1}(S) = T_S$.

(2.1) *Example.* Let $\{V_t\}_{t \in \Delta}$ be a family of polarized algebraic manifolds of dimension n. We choose $0 \leqq p \leqq n-1$ and let: $E_0 = H^{2n-2p-1}(V, \boldsymbol{R})$; $e_1, \cdots, e_{2m}$ be free generators of $H^{2n-2p-1}(V, \boldsymbol{Z})$; and $E = H^{2n-2p-1}(V, \boldsymbol{C})$. For $t \in \Delta$, we let $S_t \subset E$ be the subspace given by:

$$S_t = H^{2n-2p-1,0}(V_t) + \cdots + H^{n-p,n-p-1}(V_t). \tag{2.2}$$

Then $S_t \cap \bar{S}_t = 0$ and so $S_t \in B$. The resulting complex torus T_{S_t} will be denoted by $T_p(V_t)$. The mapping $\Phi_p: \Delta \to B$ given by $\Phi_p(t) = S_t$ is holomorphic (Theorem (1.1)), and so the torus $T_p(V_t)$ depends holomorphically on V_t. We shall see in Section II.4 below that $T_p(V)$ is related to the *algebraic p-cycles on V*.

We now look at some special cases of this construction:

(i) $p = n-1$. Then $E = H^1(V, \boldsymbol{C})$ and $S = H^{1,0}(V)$. Thus $E/S \cong H^{0,1}(V)$ and $T_{n-1}(V) = H^{0,1}(V)/H^1(V, \boldsymbol{Z})$ is the *Picard variety* ([28]) of V.

(ii) $p = 0$. Then $E = H^{2n-1}(V, \boldsymbol{C})$ and $S = H^{n,n-1}(V)$. Thus E/S

$\cong H^{n-1,n}(V)$ and $T_0(V)$ will be seen (Proposition (2.16)) to be the *Albanese variety* ([2]) of V.

(iii) $n=3$, $p=1$. Then $E=H^3(V,\boldsymbol{C})$ and $S=H^{3,0}(V)+H^{2,1}(V)$. The torus $T_1(V)\cong H^{1,2}(V)+H^{0,3}(V)/(V,\boldsymbol{Z})$ is *not Weil's intermediate Jacobian* unless $H^{3,0}(V)=0$ (cf. I.3.(b) and II.2.(e) below).

Returning now to the general picture, we want to give local coordinates in $\boldsymbol{V}\xrightarrow{\pi}B$. Fix S and choose a basis $\xi_1,\cdots,\xi_m$ for S. Then $\xi_\alpha=-\sum_{\rho=1}^{2m}\pi_{\rho\alpha}e_\rho$ where the $2m\times m$ matrix $(\pi_{\rho\alpha})$ has rank m. We may assume then that $\det(\pi_{m+\beta,\alpha})\neq 0$ and choose a basis $\xi_1,\cdots,\xi_m$ for S so that

$$\xi_\alpha=-\sum_{\beta=1}^{m}\pi_{\beta\alpha}e_\beta+e_{m+\alpha}. \tag{2.3}$$

Then $e_1,\cdots,e_m$ project onto a basis for W/S and, from (2.3), we get $e_{m+\alpha}\equiv\sum_\beta\pi_{\beta\alpha}e_\beta(S)$. Thus, if we identify W/S with $\boldsymbol{C}^m$ by using $e_1,\cdots,e_m$, then Γ_S is generated by the $2m$ column vectors in the matrix (I,Π) $(\Pi=(\pi_{\beta\alpha}))$.

Now the $(-\pi_{\beta\alpha})$ in (2.3) give local coordinates on B around S, and the complex torus $T_S=\boldsymbol{C}^m/\Gamma_S$ where Γ_S is the lattice generated by the $2m$ column vectors in the matrix

$$\Omega(S)=(I,\Pi). \tag{2.4}$$

This is the analytic space point of view: locally, $\boldsymbol{V}\xrightarrow{\pi}B$ is a family of torii depending on m^2 parameters $\pi_{\alpha\beta}$ whose period matrix is given by (2.4).

The other point of view in deformations is to fix the real manifold and let the complex structure vary; we want to show how this works here. Let then $\boldsymbol{R}^{2m}$ with basis $f_1,\cdots,f_{2m}$ be fixed, let Γ be the lattice generated by $f_1,\cdots,f_{2m}$ over $\boldsymbol{Z}$, and let T be the real torus $\boldsymbol{R}^{2m}/\Gamma$. We let $x^1,\cdots,x^{2m}$ be the real linear coordinates on $\boldsymbol{R}^{2m}$ dual to $f_1,\cdots,f_{2m}$ and define a (linear) complex structure on $\boldsymbol{R}^{2m}$ by letting:

$$dz^\alpha=dx^\alpha+\sum_{\beta=1}^{m}\pi_{\alpha\beta}dx^{m+\beta}. \tag{2.5}$$

This gives then a complex structure on T and the resulting complex torus will have period matrix $\Omega(S)$ given by (2.4); in other words, this is the torus T_S.

Fix now S_0 with period matrix $\Omega(S_0)=(I,\Pi)$ and let $\Psi=(\psi_{\alpha\beta})$ be a matrix close to zero. We define dz^α by (2.5) and define:

$$dw^{\alpha} = dx^{\alpha} + \sum_{\beta} (\pi_{\alpha\beta} + \psi_{\alpha\beta}) dx^{m+\beta}. \tag{2.6}$$

This gives a holomorphic family of torii centered at T_{S_0} and we want to compute the *Kodaira-Spencer mapping* ρ (cf. (1.5)). To do this we write:

$$\sum_{\beta=1}^{m} A_{\beta}{}^{\alpha} dw^{\beta} = dz^{\alpha} + \sum_{\beta=1}^{m} \Phi_{\bar{\beta}}{}^{\alpha}(\psi) d\bar{z}^{\beta}, \tag{2.7}$$

and seek to determine $\Phi_{\bar{\beta}}{}^{\alpha}(\psi)$ (cf. (1.7)). In terms of matrices, (2.5), (2.6), and (2.7) give:

$$A = I + \Phi$$
$$A(\Pi + \Psi) = \Pi + \Phi\bar{\Pi}.$$

This reduces to give $(I+\Phi)\Psi = \Phi(\bar{\Pi} - \Pi)$. Now $\bar{\Pi} - \Pi = \Lambda^{-1}$ for some matrix Λ, so we get that:

$$(I+\Phi)\Psi\Lambda + I = I + \Phi$$

or

$$I = (I+\Phi)(I - \Psi\Lambda)$$

which gives the formula:

$$\Phi(\Psi) = (I - \Psi\Lambda)^{-1} - I = \Psi\Lambda + (\Psi\Lambda)^2 + \cdots. \tag{2.8}$$

It follows then that

$$\rho\left(\frac{\partial}{\partial\psi_{\alpha\beta}}\right) = \sum_{\gamma} \Lambda_{\gamma}{}^{\beta} \frac{\partial}{\partial z^{\alpha}} \otimes d\bar{z}^{\gamma} \qquad (\Lambda = (\bar{\Pi} - \Pi)^{-1}). \tag{2.9}$$

In particular, ρ is an isomorphism.

Remark. Assume that $m=1$ and $\pi = \sqrt{-1}$. Then $dz = dx^1 + \sqrt{-1}\,dx^2$ and we write $dw = dx^1 + \alpha dx^2$. Solving the equation $\lambda dw = dz + \beta d\bar{z}$ gives the reciprocal relations:

$$\alpha = i\left(\frac{1-\beta}{1+\beta}\right), \qquad \beta = \frac{i+\alpha}{i-\alpha}. \tag{2.10}$$

Here, as α varies over the upper half-plane, β varies over the unit disc and vice-versa. The approach to moduli of elliptic curves writing $dw = dx^1 + \alpha dx^2$ ($\mathrm{Im}\,\alpha > 0$) is that of varying the lattice generated by 1, α in $\boldsymbol{C}$; the approach given by writing $\lambda dw = dz + \beta d\bar{z}$ ($|\beta| < 1$) and keeping the lattice fixed is the one via quasi-conformal mapping.

We now ask if the periods give local coordinates in the family $\boldsymbol{V} \to B$:

constructed above. Since, for each $S \in B$, the mapping $\rho\colon T_S(B) \to H^1(T_S, \Theta)$ is an isomorphism (cf. (2.9)), the periods of the q-forms will give local coordinates on B if the cup product (1.32) is onto. Actually, it will be the case that:

$$H^{q,0} \otimes H^{m-q+1,m-1} \xrightarrow{\mu} H^{m-1}(T_S, \Omega^1 \otimes \Omega^m) \to 0. \tag{2.11}$$

To see this, we observe that a basis for $H^{m-1}(T_S, \Omega^1 \otimes \Omega^m)$ consists of forms $dz^\alpha \otimes dz^A \otimes d\bar{z}^J$ where $A = (1, \cdots, m)$, $dz^A = dz^1 \wedge \cdots \wedge dz^m$, $J = (\alpha_1, \cdots, \alpha_{m-1})$, $d\bar{z}^J = d\bar{z}^{\alpha_1} \wedge \cdots \wedge d\bar{z}^{\alpha_q}$. But then, if say $\alpha \leqq q$,

$$\mu(dz^1 \wedge \cdots \wedge dz^q \otimes \{dz^\alpha \wedge dz^{q+1} \wedge \cdots \wedge dz^m \otimes d\bar{z}^J\}) = \pm\, dz^\alpha \otimes dz^A \otimes d\bar{z}^J$$

so that μ is onto in (2.11).

Of course, if $q = 1$, (2.11) reduces to

$$H^{1,0} \otimes H^{m,m-1} \xrightarrow{\mu} H^{m-1}(T_S, \Omega^1 \otimes \Omega^m), \tag{2.12}$$

and μ is an isomorphism. This should be so since we are, in effect, using the periods of the holomorphic 1-forms to give the family $\boldsymbol{V} \to B$. Using the notation of (2.9), we have that (cf. Proposition (1.20)):

$$\Omega_*\left(\frac{\partial}{\partial \psi_{\alpha\beta}}\right) \in \operatorname{Hom}(H^{1,0}, H^{0,1})$$

is given by:

$$\begin{cases} \Omega_*\left(\dfrac{\partial}{\partial \psi_{\alpha\beta}}\right)(dz^\alpha) = \sum \Lambda_\gamma{}^\beta d\bar{z}^\gamma \\ \Omega_*\left(\dfrac{\partial}{\partial \psi_{\alpha\beta}}\right)(dz^\lambda) = 0 \text{ for } \lambda \neq \alpha. \end{cases} \tag{2.13}$$

Thus it is clear that Ω_* is an isomorphism.

(b) There is another family of torii $\boldsymbol{V}^* \xrightarrow{\pi^*} B$ which is constructed as follows: If $S \in B$, we have an exact sequence $0 \to S \to E \to E/S \to 0$ and its dual: $0 \leftarrow S^* \leftarrow E^* \leftarrow (E/S)^* \leftarrow 0$. The lattice Γ gives a dual lattice $\Gamma^* \subset E^*$ and Γ^* projects onto a lattice $\Gamma_S{}^*$ in S^*. We set $T_S{}^* = S^*/\Gamma_S{}^* = \pi^{*-1}(S)$.

To coordinate $T_S{}^*$, we choose a basis $\xi_1, \cdots, \xi_m$ for S and write $\xi_\alpha = \sum_{p=1}^{2m} \pi_{\rho\alpha} e_\rho$. We claim that $T_S{}^* \cong \boldsymbol{C}^m/\Gamma_S{}^*$ where $\boldsymbol{C}^m$ are row-vectors and $\Gamma_S{}^*$ is the lattice generated by the rows of the $2m \times m$ matrix $\Pi = (\pi_{\rho\alpha})$. In fact, if $\xi_\alpha{}^* \in S^*$ is dual to ξ_α, then it will suffice to show that the projection e_ρ^* of e_ρ^* into S^* satisfies $e_\rho^* = \sum \pi_{\rho\alpha}\xi_\alpha^*$. This amounts to the equality $\langle e_\rho^*, \xi_\beta\rangle = \langle \sum_{\alpha=1}^{m} \pi_{\rho\alpha}\xi_\alpha^*, \xi_\beta\rangle = \pi_{\rho\beta}$. Thus proves:

(2.14) PROPOSITION. *To find the period matrix for T_S^*, take a basis $\xi_1, \cdots, \xi_m$ for S and form the $2m \times m$ matrix $\langle e_\rho^*, \xi_\alpha\rangle = \pi_{\rho\alpha}$. Then the row vectors in this matrix generate a lattice in $\boldsymbol{C}^m$, and the resulting complex torus is T_S^*.*

(2.15) *Example.* We follow the example (2.1) where $E = H^{2n-2p-1}(V, \boldsymbol{C})$ and $S = H^{2n-2p-1,0}(V) + \cdots + H^{n-p,n-p-1}(V)$. The basis $\xi_1, \cdots, \xi_m$ for S means now that we take cohomology cases $\xi_1, \cdots, \xi_m$ giving a basis for S, and dual basis $e_1^*, \cdots, e_{2m}^*$ means a free system of integral generators for the homology group $H_{2n-2p-1}(V, \boldsymbol{Z})$. The matrix $\langle e_\rho^*, \xi_\alpha\rangle = \int_{e_\rho^*} \xi_\alpha$, so that $T_S^* = T_p(V)^*$ is the complex torus $\boldsymbol{C}^m/\Gamma^*(V)$, where $\Gamma^*(V)$ is the lattice generated by periods of $\xi_1, \cdots, \xi_m$.

For example, when $p = n-1$, $E = H^1(V, \boldsymbol{C})$, $S = H^{1,0}(V)$, $\xi_1, \cdots, \xi_m$ are a basis for the holomorphic 1-forms on V, and $T_{n-1}(V)^*$ is $\boldsymbol{C}^m$ modulo the periods of the holomorphic 1-forms on V. This $T_{n-1}(V)^*$ is what is usually called the *Albanese variety* of V. If we fix $p_0 \in V$, the mapping ϕ: $V \to T_{n-1}(V)^*$ given by $\phi(p) = (\int_{p_0}^{p} \xi_1, \cdots, \int_{p_0}^{p} \xi_m)$ is well-defined, and is the standard mapping of a variety into its Albanese variety.

(2.16) PROPOSITION. *The complex torii $T_p(V)$ and $T_{n-p-1}(V)^*$ are naturally isomorphic.*

Proof. Let $E_p = H^{2n-2p-1}(V, \boldsymbol{C})$ and

$$S_p = H^{2n-2p-1,0}(V) + \cdots + H^{n-p,n-p-1}(V).$$

Then, by Poincaré duality,

$$S_p^* \cong H^{p,p+1}(V) + \cdots + H^{0,2p+1}(V) \cong E_{n-p-1}/S_{n-p-1};$$

i. e. we have a natural isomorphism:

$$E_p/S_p \cong S^*_{n-p-1}. \tag{2.17}$$

Under the isomorphism (2.17), the lattices Γ_{S_p} and $\Gamma^*_{S_{n-p-1}}$ go into one aother, and so the corresponding complex torii are naturally isomorphic. Q. E. D.

By using the isomorphism $T_0(V) \cong T_{n-1}(V)^*$, we get a holomorphic mapping ϕ: $V \to T_0(V)$, which is unique up to translation. This generalizes as follows: Let $Z_0 \subset V$ be a fixed *algebraic p-cycle*; i. e., $Z_0 = \sum_{j=1}^{k} n_j S_j$ where the n_j are integers and $S_j \subset V$ is an irreducible p-dimensional subvariety. We let $B(Z_0)$ be the set of algebraic p-cycles Z which are homologous to Z_0 and define

$$\phi\colon B(Z_0) \to T_p(V), \tag{2.18}$$

by the following method: Choose a basis $\xi_1, \cdots, \xi_m$ for $H^{2p+1,0} + \cdots + H^{p+1,p}$ and write $Z - Z_0 = \partial C$ where C is a $2p+1$ chain. Then set

$$\phi(Z) = (\int_C \xi_1, \cdots, \int_C \xi_m) \tag{2.19}$$

(2.20) THEOREM. *Let $\{Z_\lambda\}_{\lambda \in B}$ be an algebraic family of effective subvarieties with B non-singular. Set $Z_0 = Z_{\lambda_0}$ and define $\phi\colon B \to T_p(V)$ by $\phi(\lambda) = \phi(Z_\lambda)$. Then ϕ is holomorphic.*

Proof. We shall first treat the case where the Z_λ are analytic submanifolds forming a *continuous system* in the sense of Kodaira [16]. Because the problem is local, we may assume that Δ is a polycylinder and $\lambda_0 = 0$ is the origin. We recall that Kodaira [16] has defined the *characteristic map*:

$$\rho_\lambda\colon T_\lambda(\Delta) \to H^0(Z_\lambda, \mathbf{\mathcal{O}}(\mathbf{N}_\lambda)), \tag{2.21}$$

where $\mathbf{N}_\lambda \to Z_\lambda$ is the *normal bundle* of Z_λ in V.

Now we can find an analytic fibre space $\mathbf{Z} \xrightarrow{\pi} \Delta$ with $\pi^{-1}(\lambda) = Z_\lambda$ and and a holomorphic mapping $F\colon \mathbf{Z} \to V$ such that F is the identity on each Z_λ [8]. In fact, we will have $\mathbf{Z} \subset V \times \Delta$ and F, π are induced by the projections; we remark that $\mathbf{Z} \cdot V \times \{\lambda\} = Z_\lambda \times \{\lambda\}$.

On $\mathbf{Z}$ there is an obvious chain (modulo cycles) $\mathbf{C}_\lambda$ with $\partial \mathbf{C}_\lambda = Z_\lambda - Z_0$; we simply take a curve $\gamma_\lambda \subset \Delta$ connecting 0 and λ, and let $\mathbf{C}_\lambda = \bigcup_{\zeta \in \gamma_\lambda} Z_\zeta$. Then, letting C_λ be the corresponding chain on V, we will have

$$\phi(\lambda) = (\int_{C_\lambda} \xi_1, \cdots, \int_{C_\lambda} \xi_m) = (\int_{\mathbf{C}_\lambda} F^*(\xi_1), \cdots, \int_{\mathbf{C}_\lambda} F^*(\xi_m)).$$

There is perhaps the foundational question of in what sense is C_λ a chain. However, such problems are much easier than similar questions which have been treated successfully in [23] and will not be dwelt on here.

Now we let $F^*(\xi_\alpha) = \omega_\alpha$, we assume that Δ is a disc with coordinate λ, and we shall examine how $\pi_\alpha(\lambda) = \int_{\mathbf{C}_\lambda} \omega_\alpha$ varies with λ. Recall that, on $\mathbf{Z}$, ω_α is of type $(p+r+1, p-r)$ with $r \geqq 0$. Since $\dim \mathbf{Z} = p+1$, we will have $\omega_\alpha = 0$ unless $r = 0$. Assume then that $\omega_\alpha = \omega$ is of type $(p+1, p)$; we can write (in many ways) $\omega = \phi_1 \wedge d\lambda + \phi_2 \wedge d\bar{\lambda}$. Analyzing types, we see that: $\phi_1 = \psi_1 + \psi_2$ where ψ_1 is type (p, p) and ψ_2 is type $(p-1, p+1)$; $\phi_2 = \eta_1 + \eta_2$ where η_1 is type $(p+1, p-1)$ and η_2 is type (p, p). Since ω is type $(p+1, p)$, $\psi_2 \wedge d\lambda + \eta_2 \wedge d\bar{\lambda} = 0$. It follows then that $\eta_2 \mid Z_\lambda = 0$.

Now $\pi(\lambda) = \int_{C_\lambda} \omega = \int_0^\lambda (\int_{Z_\zeta} \phi_1) d\zeta + \int_0^\lambda (\int_{Z_\zeta} \phi_2) \overline{d\zeta}$. But $\phi_2 \mid Z_\zeta = \eta_1 \mid Z_\zeta + \eta_2 \mid Z_\zeta = 0$. Thus $\pi(\lambda) = \int_0^\lambda (\int_{Z_\zeta} \phi_1) d\zeta$ and so $\dfrac{\partial \pi(\lambda)}{\partial \bar{\lambda}} = 0$.

The general effective family will differ from the continuous system case in that the Z_λ may have singularities; these will cause no trouble in integrating smooth forms and so may be ignored.

This completes our proof of Theorem (2.20).

Assume now that $\{Z_\lambda\}_{\lambda \in \Delta}$ is a continuous system and $Z = Z_0$, $\boldsymbol{N} = \boldsymbol{N}_0$. The differential

$$\phi_*: T_0(\Delta) \rightarrow H^{n-p-1, n-p}(V) + \cdots + H^{0, 2n-2p-1}(V),$$

and we want to give a formula for ϕ_*. Actually, we will have a mapping $\psi: H^0(Z, \mathcal{O}(\boldsymbol{N})) \rightarrow H^{n-p-1, n-p}(V) + \cdots + H^{0, 2n-2p-1}(V)$ and then $\phi_* = \psi \circ \rho$ where $\rho = \rho_0$ is the characteristic mapping (2.21).

It is easier to give the dual mapping:

$$\psi^*: H^0(\Omega_V^{2p+1}) + \cdots + H^p(\Omega_V^{p+1}) \rightarrow H^p(Z, \Omega_Z^p(\boldsymbol{N}^*)), \tag{2.22}$$

where we have used $H^{p+1+r, p-r}(V) \cong H^{p-r}(\Omega_V^{p+1+r})$. To do this, we make the following remark: Let $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$ be an exact sequence of vector spaces with $\dim A = n$, $\dim A'' = p$, $\dim A' = n - p$. Then there is a canonical exact seqence

$$\Lambda^2 A' \otimes \Lambda^{p-2} A \rightarrow \Lambda^{p+1} A \rightarrow A' \otimes \Lambda^p A'' \rightarrow 0. \tag{2.23}$$

Applied to the exact sheaf sequence along Z: $0 \rightarrow \mathcal{O}(\boldsymbol{N}^*) \rightarrow \Omega_{V|Z}{}^1 \rightarrow \Omega_Z{}^1 \rightarrow 0$, (2.23) gives a sheaf mapping $\Omega_{V|Z}{}^{p+1} \rightarrow \Omega_Z{}^p \otimes \mathcal{O}(\boldsymbol{N}^*) \rightarrow 0$ which gives in cohomology a diagram:

$$\left\{ \begin{array}{ccc} H^p(\Omega_V{}^{p+1}) & & \\ \downarrow & \searrow \psi^* & \\ H^p(\Omega_{V|Z}{}^{p+1}) & \rightarrow & H^p(\Omega_Z{}^p(\boldsymbol{N}^*)) \end{array} \right. \tag{2.24}$$

(2.25) THEOREM. *$\phi^* = \rho^* \psi^*$ where $\psi^*(H^{p-r}(\Omega_V{}^{p+r+1})) = 0$ for $r > 0$, $\psi^*: H^p(\Omega_V{}^{p+1}) \rightarrow H^p(\Omega_Z{}^p(\boldsymbol{N}^*))$ is given in (2.24) and $\rho^*: H^p(\Omega_Z{}^p(\boldsymbol{N}^*)) \rightarrow T_0(\Delta)^*$ is the dual of the characteristic map (2.21).*

Proof. Let ξ be a class in $H^{2p+1,0}+\cdots+H^{p+1,p}$. Then $\langle\xi,\phi_*(\frac{\partial}{\partial\lambda})\rangle=\frac{\partial}{\partial\lambda}\langle\xi,\phi(\lambda)\rangle]_{\lambda=0}=\frac{\partial}{\partial\lambda}(\int_{C_\lambda}\xi)]_{\lambda=0}$. From the proof of Theorem (2.20) we see that $\frac{\partial}{\partial\lambda}(\int_{C_\lambda}\xi)]_{\lambda=0}=0$ if $\xi\in H^{2p+1,0}+\cdots+H^{p+2,p-1}$ so that we may suppose that $\xi\in H^{p+1,p}$. Then we must show that:

$$\frac{\partial}{\partial\lambda}(\int_{C_\lambda}\xi)]_{\lambda=0}=\langle\psi^*(\xi),\rho(\frac{\partial}{\partial\lambda})\rangle, \tag{2.26}$$

where ψ^* is given in (2.24).

We can choose local coordinates $z^1,\cdots,z^p$; $w^1,\cdots,w^{n-p}$ on V such that Z is given by $w^1=\cdots=w^{n-p}=0$. Locally, Z_λ will be given by $w^\alpha=\phi^\alpha(z,\lambda)$ where $\phi^\alpha(z,\lambda)$ is holomorphic in both variables and $\phi^\alpha(z,0)=0$. Thus we can write $\phi^\alpha(z,\lambda)=\zeta^\alpha(z)\lambda+[2]$ where [2] are terms of order 2 in λ. The normal vector field $\rho(\frac{\partial}{\partial\lambda})=\sum_{\alpha=1}^{n-p}\zeta^\alpha(z)\frac{\partial}{\partial w^\alpha}$ (cf. [16]).

Locally, we can write

$$\xi=\sum_{\alpha=1}^{p}\xi_\alpha(z,w)\,dz^Id\bar z^Idw^\alpha+(2)$$

where $dz^I=dz^1\wedge\cdots\wedge dz^p$ and (2) are terms involving $dw^\alpha\wedge dw^\beta$. Then $\psi^*(\xi)$ is *a* (p,p)-form on Z with values in $\boldsymbol{N}^*$ given by

$$\psi^*(\xi)=\sum_{\alpha=1}^{n-p}\xi_\alpha(z)\,dz^Id\bar z^I\otimes dw^\alpha$$

where $\xi_\alpha(z)=\xi_\alpha(z,0)$. Thus we get that:

$$\langle\psi^*(\xi),\rho(\frac{\partial}{\partial\lambda})\rangle=\int_Z\{\sum_\alpha\xi_\alpha(z)\zeta^\alpha(z)\,dz^Id\bar z^I\}, \tag{2.27}$$

where the expression in $\{\cdots\}$ is a (p,p) form on Z.

On the other hand, let Δ be the polycylinder with coordinatss $(z^1,\cdots,z^p,\lambda)$ and define $\gamma:\Delta\to V$ by

$$\gamma(z,\lambda)=(z^1,\cdots,z^p;\phi^1(z,\lambda),\cdots,\phi^{n-p}(z,\lambda)).$$

Then C_λ is the union of images $\gamma(\Delta)$ ond so $\int_{C_\lambda}\xi$ is the sum of integrals $\int_{\gamma(\Delta)}\xi$. But $\int_{\gamma(\Delta)}\xi=\int_\Delta\sum\{\xi_\alpha(z,\phi(z,\lambda))\frac{\partial\phi^\alpha(z,\lambda)}{\partial\lambda}\}dzd\bar z^Id\lambda$. Then it is clear that $\frac{\partial}{\partial\lambda}(\int_{C_\lambda}\xi)]_{\lambda=0}$ is given by (2.27) since $\zeta^\alpha(z)=\frac{\partial\phi^\alpha(z,\lambda)}{\partial\lambda}]_{\lambda=0}$. This completes the proof of the equality (2.26) and, with it, Theorem (2.25).

Examples. (i) When $p=0$, $Z=(x_1,\cdots,x_k)$ and $\boldsymbol{N}=T_{x_1}(V)+\cdots+T_{x_k}(V)$. We let $\Omega_V{}^1(x_1,\cdots,x_k)$ be the sheaf of holomorphic 1-forms ω with $\omega(x_j)=0$, and let the continuous system be obtained by letting the points x_j vary freely. The diagram (2.24) becomes:

$$(2.28)\qquad \left\{\begin{array}{l} 0 \\ \downarrow \\ H^0(\Omega_V{}^1(x_1,\cdots,x_k)) \\ \downarrow \\ H^0(\Omega_V{}^1) \\ \downarrow \quad \searrow\phi^* \\ H^0(\Omega_V{}^1|_Z)=T_{x_1}(V)^*+\cdots+T_{x_k}(V)^* \\ \downarrow \\ H^1(\Omega_V{}^1(x_1,\cdots,x_k)). \end{array}\right.$$

The mapping ϕ^* is simply the restriction of forms. If we choose the points $x_1,\cdots,x_k$ ($k\geqq h^{1,0}$) in a general manner, then $H^0(\Omega_V{}^1(x_1,\cdots,x_k))=0$ and so ϕ^* is into, ϕ_* is onto. Thus, if $V^{(k)}=\underbrace{V\circ\cdots\circ V}_{k}$ is the k-fold *symmetric product* of V, the mapping $\phi\colon V^{(k)}\to T_0(V)$ given by $\phi(x_1,\cdots,x_k)=(\cdots,\sum_{j=1}^{k}\int_{x_0}^{x_j}\xi_\alpha,\cdots)/(\text{periods})$ is onto, and the dual tangent space to the fibre of ϕ through $(x_1,\cdots,x_k)$ is $H^1(\Omega^1(x_1,\cdots,x_k))$.

(ii) For $p=n-1$, we let $Z\subset V$ be a sufficiently ample prime divisor so that, in particular, $H^{n-1}(V,\Omega^n[-Z])=0=H^1(V,\mathbf{\mathcal{O}}[Z])$, $[Z]$ being the line bundle determined by the divisor Z. Let $\{Z_\lambda\}_{\lambda\in B}$ be the continuous system generated by Z (cf. [13]) and identify $T_{\lambda_0}(B)$ with $H^0(Z,\mathbf{\mathcal{O}}(\boldsymbol{N}))$. Then (2.24) becomes:

$$(2.29)\qquad \begin{array}{l} 0 \\ \downarrow \\ H^{n-1}(\Omega_V{}^n) \\ \downarrow \quad \searrow\phi^* \\ H^{n-1}(\Omega_V{}^n|_Z)=H^{n-1}(Z,\Omega_Z{}^{n-1}(\boldsymbol{N}^*)) \\ \downarrow \\ H^n(V,\Omega_V{}^n[-Z]) \\ \downarrow \\ H^n(\Omega_V{}^n) \\ \downarrow \\ 0 \end{array}$$

Dualizing (2.29) gives:

$$(2.30)\qquad 0 \to \boldsymbol{C} \to H^0(V, \mathcal{O}_V[Z]) \to H^0(Z, \mathcal{O}_Z(\boldsymbol{N})) \xrightarrow{\phi_*} H^1(V, \mathcal{O}) \to 0,$$

which is a piece of the exact cohomology sequence of $0 \to \mathcal{O}_V \to \mathcal{O}_V[Z] \to \mathcal{O}_Z(\boldsymbol{N}) \to 0$. It follows that $\phi\colon B \to T_{n-1}(V)$ is onto and the fibre passing through Z is just the complete linear system $|Z|$. Thus we are quickly led to the standard structure theorems on the Picard variety of V ([13], [28]).

When $n = 1$, these examples coincide and the tangent space of the fibre of ϕ passing through $(x_1, \cdots, x_k)$ is the complete linear system $|x_1 + \cdots + x_k|$; dualizing the sheaf sequences contains a proof of *Abel's theorem* for curves.

(iii) In general, of course, ϕ will *not* be onto; at most, we can have $\phi_*\colon T_{\lambda_0}(B) \to H^{n-p-1,n-p} \to 0$. But it seems likely that this will *not* generally be possible. For example, let V be a threefold, $Z \subset V$ a general curve, and $\Omega_V{}^2(Z)$ the sheaf of holomorphic 2-forms on V vanishing on Z. There is some evidence that we can have $H^1(\Omega_V{}^2(Z)) = 0$, and then (2.24) becomes:

$$(2.31)\qquad \left\{ \begin{tikzcd} & 0 \arrow[d] & & \\ & H^1(\Omega_V{}^2) \arrow[d] \arrow[dr, "\phi^*"] & & \\ H^1(Z, \mathcal{O}(\det \boldsymbol{N}^*)) \arrow[r, "\psi"] \arrow[dr, "\theta"'] & H^1(\Omega_V{}^2|_Z) \arrow[r] \arrow[d] & H^1(Z, \Omega_Z{}^1(\boldsymbol{N}^*)) \arrow[r] & 0 \\ & H^2(\Omega_V{}^2(Z)). & & \end{tikzcd} \right.$$

Thus ϕ^* will be injective if, and only if, $\ker \psi = \ker \theta$ which seems to not be always possible.

In general, to determine the fibres of $\phi\colon B \to T_p(V)$ passing through $Z \subset V$, we will have to know the dual space of $H^{p+1}(\Omega_V{}^{p+1}(Z))$, which points up the difficulty in finding the algebraic equivalence relation $\approx$ such that $Z \approx Z'$ if, and only if, $\phi(Z) = \phi(Z')$.

(c) We now put a polarization on the torii $T_S{}^*$ constructed in I.2.(b) above. To do this, we let Q be a skew-symmetric form on E_0 with matrix $Q = Q(e_\rho, e_\sigma)$ and such that $Q^{-1} = (q_{\rho\sigma})$ is integral. We let $B_Q \subset B$ be those subspaces $S \subset E$ which satisfy $Q(S, S) = 0$ as well as $S \cap \bar{S} = 0$.

(2.32) Proposition. *For $S \in B_Q$, there exists a holomorphic line bundle $\boldsymbol{L} \to T_S{}^*$ whose characteristic class is $\omega = \sqrt{-1} \sum_{\alpha,\beta} h_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, where the Hermitian matrix $H = (h_{\alpha\bar{\beta}})$ is given by $H = (\sqrt{-1}\, \Omega Q^t \bar{\Omega})^{-1}$.*

Proof. Let $\xi_1,\cdots,\xi_m$ be a basis for S and write $\xi_\alpha=\sum_{\rho=1}^{2m}\pi_{\alpha\rho}e_\rho$. We form the $m\times 2m$ matrix $\Omega=(\pi_{\alpha\rho})$ and write $T_S{}^*=\boldsymbol{C}^m/\Gamma_S{}^*$ where $\Gamma_S{}^*$ is the lattice in $\boldsymbol{C}^m$ generated by the $2m$ column vectors of Ω. The condition $Q(S,S)=0$ is now written $\Omega Q{}^t\Omega=0$ (cf. I.1.(a)). We let

$$H_1=-\sqrt{-1}\,\Omega Q{}^t\bar{\Omega}$$

so that $H=-(H_1)^{-1}$.

Every vector $\xi\in\boldsymbol{C}^m$ can be written as a *real* linear combination of $\xi_1,\cdots,\xi_{2m}$, and this gives an $\boldsymbol{R}$ isomorphism $\boldsymbol{C}^m\cong\boldsymbol{R}^{2m}$ such that ξ_p corresponds to the ρ-th coordinate vector of $\boldsymbol{R}^{2m}$. Letting $x^1,\cdots,x^{2m}$ be the real coordinates on $\boldsymbol{R}^{2m}$ and $z^1,\cdots,z^m$ be the complex coordinates, we have $dz^\alpha=\sum_{\rho=1}^{2m}\pi_{\alpha\rho}dx^\rho$. We remark that $dx^1,\cdots,dx^{2m}$ give a basis for $H^1(T_S{}^*,\boldsymbol{Z})$.

Write now $dx^\rho=\sum_\alpha\psi_{\rho\alpha}dz^\alpha+\sum_\alpha\bar{\psi}_{\rho\alpha}d\bar{z}^\alpha$. It follows that

$$\sum_\alpha(\psi_{\rho\alpha}\pi_{\alpha\sigma}+\bar{\psi}_{\rho\alpha}\bar{\pi}_{\alpha\sigma})=\delta_\sigma{}^\rho$$

or, in matrix terms, $\Psi\Omega+\Psi\bar{\Omega}=I$. Thus $(\Psi\bar{\Psi})=\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}^{-1}$. From $\Omega Q{}^t\Omega=0$, we get

$$\begin{pmatrix}\Omega\\ \bar{\Omega}\end{pmatrix}Q({}^t\Omega{}^t\bar{\Omega})=\begin{pmatrix}0 & \Omega Q{}^t\bar{\Omega}\\ \bar{\Omega}Q{}^t\Omega & 0\end{pmatrix}=\begin{pmatrix}0 & \sqrt{-1}\,H_1\\ -\sqrt{-1}\,\bar{H}_1 & 0\end{pmatrix}.$$

Taking inverses in this relation gives:

$$\begin{pmatrix}{}^t\Psi\\ {}^t\bar{\Psi}\end{pmatrix}Q^{-1}(\Psi\bar{\Psi})=\begin{pmatrix}0 & \sqrt{-1}\,\bar{H}_1{}^{-1}\\ -\sqrt{-1}\,H_1{}^{-1} & 0\end{pmatrix}=\begin{pmatrix}0 & -\sqrt{-1}\,H\\ \sqrt{-1}\,\bar{H} & 0\end{pmatrix}.$$

Thus ${}^t\Psi Q^{-1}\bar{\Psi}=-\sqrt{-1}\,H$ and ${}^t\Psi Q^{-1}\Psi=0$.

Let now $\omega=\sqrt{-1}\sum_{\alpha,\beta}h_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta$. Then

$$\omega=\sqrt{-1}\sum h_{\alpha\beta}\pi_{\alpha\rho}\bar{\pi}_{\beta\sigma}dx^\rho\wedge dx^\sigma=\sqrt{-1}\{\sum_{\rho,\sigma}({}^t\Omega H\bar{\Omega}-{}^t\bar{\Omega}\bar{H}\Omega)_{\rho\sigma}dx^\rho\wedge dx^\sigma\}.$$

But

$$\begin{aligned}\sqrt{-1}\,({}^t\Omega H\bar{\Omega}-{}^t\bar{\Omega}\bar{H}\Omega)&=\sqrt{-1}(\sqrt{-1}\,{}^t\Omega{}^t\Psi Q^{-1}\Psi\bar{\Omega}+\sqrt{-1}\,{}^t\bar{\Omega}{}^t\bar{\Psi}Q^{-1}\Psi\Omega)\\ &=-({}^t\Omega{}^t\Psi Q^{-1}-{}^t\Omega{}^t\Psi Q^{-1}\Psi\Omega+{}^t\bar{\Omega}{}^t\bar{\Psi}Q^{-1}-{}^t\bar{\Omega}{}^t\bar{\Psi}Q^{-1}\bar{\Psi}\bar{\Omega})=-Q^{-1}.\end{aligned}$$

Thus $\omega=-\sum_{\rho,\sigma}q_{\rho\sigma}dx^\rho\wedge dx^\sigma=\sqrt{-1}\sum_{\alpha,\beta}h_{\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta$ and so $\omega\in H^{1,1}(T_S{}^*)\cap H^2(T_S{}^*,\boldsymbol{Z})$. The existence of $\boldsymbol{L}$ now follows from the Kodaira-Spencer version of the Lefschetz theorem [9].

Remark. The proof of Proposition (3.10) in I.3 shows that we may find a metric in $\boldsymbol{L}$ whose *curvature form* is ω. Combing this with Proposition (2.32), we have:

(2.33) COROLLARY. *Suppose the Hermitian matrix* $\sqrt{-1}\,\Omega Q^t\bar{\Omega}$ *has signature* $(q, n-q)$. *Then* $\boldsymbol{L} \to T_S{}^*$ *has a q-convex polarization* (cf. I.3(c)).

Example. We continue on with the example (2.1) (cf. (2.15) and (2.16)). Thus we let $E = H^{2n-2p-1}(V, \boldsymbol{C})$ and

$$S = H^{2n-2p-1,0}(V) + \cdots + H^{n-p,n-p-1}(V).$$

By using the *Lefschetz decomposition,* we can find a bilinear form Q on E_0 such that $Q(S,S) = 0$, Q^{-1} is integral, and $\sqrt{-1}\,Q(H^{n-p,n-p-1}, \bar{H}^{n-p,n-p-1}) > 0$. It will *not* in general be the case that $\sqrt{-1}\,Q(S,\bar{S}) > 0$ (cf. I.3.(b), equation (3.4)).

(2.34) PROPOSITION. (i) *There is a natural line bundle* $\boldsymbol{L} \to T_p(V)$ *with a q-convex polarization where* $q \geqq h^{p+1,p}$. (ii) *Let* $\phi\colon B \to T_p(V)$ *be a holomorphic mapping such that* $\phi_*\{T_\lambda(B)\}$ *always lies in a translate of the subspace* $H^{n-p-1,n-p}$ *of* E/S. *Then the line bundle* $\boldsymbol{L}$ *is positive on* $\phi(B)$.

Proof. (i) is clear from Corollary (2.33). To see (ii), we use the isomorphism $T_p(V) \cong T_{n-p-1}(V)^*$ and choose a basis $\xi_1, \cdots, \xi_m$ for $S_{n-p-1} = H^{2p+1,0}(V) + \cdots + H^{p+1,p}(V)$ such that $\xi_1, \cdots, \xi_k$ give a basis for $H^{p+1,p}$ and $\xi_{k+1}, \cdots, \xi_m$ lie in $H^{2p+1,0} + \cdots + H^{p+2,p-1}$. Since $T_{n-p-1}(V)^* = S^*_{n-p-1}/\Gamma^*_{n-p-1}$, it follows that the dual tangent space to $T_p(V)$ at the origin is S_{n-p-1} and, by assumption, $\langle H^{2p+1,0} + \cdots + H^{p+2,p-1}, \phi_* T_\lambda(B)\rangle = 0$. The result now follows from the following easily verified fact: if H_1 is a non-singular Hermitian matrix whose first $q \times q$ block is positive-definite, then the first $q \times q$ block of $H_1{}^{-1}$ is positive definite. Q.E.D.

By combining this Proposition with Theorem (2.25), we get:

(2.35) THEOREM. *Let* $\{Z_\lambda\}_{\lambda \in B}$ *be an algebraic family of p dimensional subvarieties of* V *as in Theorem* (2.20). *Then the line bundle* $\boldsymbol{L} \to T_p(V)$ *is positive on* $\phi(B)$, *where* $\phi\colon B \to T_p(V)$ *is the holomorphic mapping* (2.19).

Remark. If there is at most one $r \geqq 0$ for which $H^{p+r+1,p-r}(V) \neq 0$, it is easy to see that we may assume $\boldsymbol{L} \to T_p(V)$ is positive. Thus, for example, $T_0(V)$ (Albanese variety) and $T_{n-1}(V)$ (Picard variety) have polarizations in the usual sense. But so also does $T_1(V)$ where $V \subset P_4$ is a *cubic threefold,* since $H^{3,0}(V) = 0$ and $\dim H^{2,1}(V) = 5$ in this case.

One should compare this Theorem with Theorem (1.34) in II.1.(c) above; cf. the conclusion (1.37).

(d) We shall now discuss a relation between the family of torii $V^* \xrightarrow{\pi^*} B$ constructed in II.2.(b) above and the higher order period relations given in the conclusion (1.37). Having fixed the real vector space E_0 with basis $e_1, \cdots, e_{2m}$ and real linear coordinates $x^1, \cdots, x^{2m}$, this gives a fixed real torus E_0/Γ where $\Gamma = (e_1, \cdots, e_{2m})_{\mathbf{Z}}$. If $S \in B$ is a subspace with basis $\xi_1, \cdots, \xi_m$, then $\xi_\alpha = \sum_{\rho=1}^{2m} \pi_{\alpha\rho} e_\rho$ and we define a complex structure on T by setting $dz^\alpha = \sum_{\rho=1}^{2m} \pi_{\alpha\rho} dx^\rho$ (cf. the proof of Proposition (2.32)). If $\zeta_1, \cdots, \zeta_m$ is a new basis for S, then $\zeta_\alpha = \sum_\beta A_{\alpha\beta}\xi_\alpha = \sum_{\beta,\rho} A_{\alpha\beta}\pi_{\beta\rho}e_\rho$ and the complex structure is then given by $dw^\alpha = \sum_{\beta,\rho} A_{\alpha\beta}\pi_{\beta\rho}dx^\rho = \sum_\beta A_{\alpha\beta}dz^\beta$; i.e. the complex torus $T_S{}^*$ depends only on the subspace $S \in B$.

As before we can write $dx^\rho = \sum_\alpha (\psi_{\rho\alpha}dz^\alpha + \bar{\psi}_{\rho\alpha}d\bar{z}^\alpha)$ where $\Psi\Omega + \bar{\Psi}\bar{\Omega} = I_{2m}$ so that $(\Psi\bar{\Psi}) = \begin{pmatrix}\Omega \\ \bar{\Omega}\end{pmatrix}^{-1}$.

Now we let $\omega = \frac{1}{2}\{\sum_{\rho,\sigma} q_{\rho\sigma}dx^\rho \wedge dx^\sigma\}$ where $Q^{-1} = (q_{\rho\sigma})$ is an integral skew-symmetric matrix. We let B_Q be those $S \in B$ such that ω is of type (1,1) on $T_S{}^*$. Then we have:

$$\begin{aligned}\omega = \tfrac{1}{2}\{\sum_{\rho,\sigma} q_{\rho\sigma}dx^\rho \wedge dx^\sigma\} &= \sum_{\alpha,\beta} ({}^t\Psi Q^{-1}\Psi)_{\alpha\beta}dz^\alpha \wedge dz^\beta \\ &+ \sqrt{-1}\,\{\sum_{\alpha,\beta}(H_1)_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta\} + \sum_{\alpha,\beta}({}^t\bar{\Psi}Q^{-1}\bar{\Psi})_{\bar{\alpha}\bar{\beta}}d\bar{z}^\alpha \wedge d\bar{z}^\beta\end{aligned} \tag{2.34}$$

where

$$H_1 = -\sqrt{-1}\,({}^t\Psi Q^{-1}\bar{\Psi}). \tag{2.35}$$

If ω is of type (1,1), then $\begin{pmatrix}{}^t\Psi \\ {}^t\bar{\Psi}\end{pmatrix} Q^{-1}(\Psi\bar{\Psi}) = \begin{pmatrix} 0 & -\sqrt{-1}\,H_1 \\ \sqrt{-1}\,\bar{H}_1 & 0\end{pmatrix}$ and so, taking inverses, $\begin{pmatrix}\Omega \\ \bar{\Omega}\end{pmatrix} Q({}^t\Omega{}^t\bar{\Omega}) = \begin{pmatrix} 0 & -\sqrt{-1}\,H \\ \sqrt{-1}\,\bar{H} & 0\end{pmatrix}$ $(H = \bar{H}_1{}^{-1})$. Thus we get (cf. Proposition (2.32)):

(2.36) PROPOSITION. *B_Q consists of all S satisfying $Q(S,S) = 0$. If Ω is a period matrix for S, then $\omega = \sqrt{-1}\,\{\sum_{\alpha,\beta}({}^tH^{-1})_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta\}$ where $\sqrt{-1}\,\Omega Q{}^t\bar{\Omega} = H$.*

Now we let $W_1 = H^1(T, \boldsymbol{C})$ and define:

$$(2.37) \qquad Q_1(\xi, \eta) = \frac{(-1)^{m-1}}{(m-1)!} \int_T \xi\eta\omega^{m-1} \qquad (\xi, \eta \in H^1(T, \boldsymbol{C}));$$

we let $W_2 = \{\phi \in H^2(T, \boldsymbol{C}), \omega^{m-1}\phi = 0\}$ and define:

$$(2.38) \qquad Q_2(\phi, \psi) = \frac{(-1)^{m-2}}{(m-2)!} \int_T \phi\psi\omega^{m-2} \qquad (\phi, \psi \in W_2).$$

If $T = T_S{}^*$ for some S satisfying

$$(2.39) \qquad \begin{cases} \Omega Q^t\Omega = 0 \\ \sqrt{-1}\,\Omega Q^t\bar{\Omega} = H > 0, \end{cases}$$

then W_2 is the space of *primitive classes* and the inner products Q_1, Q_2 are the ones of the Hodge theory (cf. I.1.(c)).

We now let $B^+{}_Q$ be those $S \in B_Q$ which satisfy (2.39) and we let:

$$(2.40) \qquad \begin{cases} A(S) = H^{1,0}(T_S{}^*) \subset W_1, \\ B(S) = H^{2,0}(T_S{}^*) \subset W_2. \end{cases}$$

(2.41) PROPOSITION. (i) $Q_1(A(S), A(S)) = 0$ *and* $\sqrt{-1}\,Q_1(A(S), \overline{A(S)}) > 0$. (ii) $Q_2(B(S), B(S)) = 0$ *and* $Q_2(B(S), B(S)) > 0$.

Proof. These are the *bilinear relations* of Hodge [10]; the equations $Q_1(A(S), T(S)) = 0$ and $Q_2(B(S), B(S)) = 0$ follow simply by considerations of type. We shall verify the bilinear inequalities in a special case so as to check our signs.

Suppose then that $Q = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, $\omega = -\sum_\alpha dx^\alpha \wedge dx^{m+\alpha}$, and let $dz^\alpha = dx^\alpha + \sqrt{-1}\,dx^{m+\alpha}$ so that $\Omega = (I, \sqrt{-1}\,I)$ and (2.39) is satisfied. Then $\frac{\sqrt{-1}}{2}\, dz^\alpha \wedge d\bar{z}^\alpha = dx^\alpha \wedge dx^{m+\alpha}$. Clearly $\sqrt{-1}\,Q_1(dz^\alpha, d\bar{z}^\beta) = 0$ for $\alpha \neq \beta$ and

$$\sqrt{-1}\,Q_1(dz^\alpha, d\bar{z}^\alpha) = \frac{\sqrt{-1}}{(m-1)!} \int_T dz^\alpha \wedge d\bar{z}^\alpha \wedge (-\omega)^{m-1}$$

$$= 2\int_T \prod_{\alpha=1}^m \left(\frac{\sqrt{-1}}{2}\, dz^\alpha \wedge d\bar{z}^\alpha\right) > 0.$$

Also, $Q_2(dz^\alpha \wedge dz^\beta, d\bar{z}^\mu \wedge d\bar{z}^\lambda)$ $(\alpha < \beta, \mu < \lambda) = 0$ unless $\alpha = \mu$, $\beta = \lambda$, and

$$Q_2(dz^\alpha \wedge dz^\beta, d\bar{z}^\alpha \wedge d\bar{z}^\beta) = \frac{1}{(m-2)!} \int_T - dz^\alpha \wedge d\bar{z}^\alpha \wedge dz^\beta \wedge d\bar{z}^\beta \wedge (-\omega)^{m-2}$$
$$= 4 \int_T \prod_{\alpha=1}^{m} (\frac{\sqrt{-1}}{2} dz^\alpha \wedge d\bar{z}^\alpha) > 0.$$

This completes the proof.

We now let $D_1 \subset G(m, W_1)$ be all subspaces A which satisfy:

$$\text{(2.42)} \qquad \begin{cases} Q_1(A, A) = 0 \\ \sqrt{-1}\, Q_1(A, \bar{A}) > 0. \end{cases}$$

Because of Proposition (2.41), the mapping $S \to A(S)$ of $B^+{}_Q$ to D_1 is a complex analytic isomorphism.

We let $D_2 \subset G(\frac{m(m-1)}{2}, W_2)$ be all subspaces B which satisfy:

$$\text{(2.43)} \qquad \begin{cases} Q_2(B, B) = 0 \\ Q_2(B, \bar{B}) > 0. \end{cases}$$

Now $W_2 \subset \Lambda^2 W_1$ and, if $A \in D_1$, then by Proposition (2.41), $B(A) = \Lambda^2 A \subset W_2$ and $B(A) \in D_2$; this gives a complex analytic mapping $\Phi \colon D_1 \to D_2$.

(2.44) THEOREM. *If $m \geqq 2$, the mapping Φ is a one-to-one embedding. Furthermore, there exists an algebraic subvariety $Z \subset G(\frac{m(m-1)}{2}, W_2)$ such that $\Phi(D_1) = D_2 \cap Z$.*

Proof. First we consider the mapping $\phi \colon G(m, W) \to G(\frac{m(m-1)}{2}, \Lambda^2 W)$ given by $\phi(A) = \Lambda^2 A$. If $m \geqq 2$, then ϕ is one-to-one: If $A_1 \neq A_2$, there exists $\lambda \in W^*$ with $\langle \lambda, A_1 \rangle \neq 0$, $\langle \lambda, A_2 \rangle = 0$. Since $m \geqq 2$, we can find $\mu \in W^*$ with $\langle \mu, A_1 \rangle \neq 0$ and $\lambda \wedge \mu \neq 0$. Then $\langle \lambda \wedge \mu, A_1 \wedge A_1 \rangle \neq 0$ but $\langle \lambda \wedge \mu, A_2 \wedge A_2 \rangle = 0$ so that $\phi(A_1) \neq \phi(A_2)$. Thus Φ is 1-1 and we have already seen (cf. (2.11)) that Φ_* is non-singular.

Now let $X_1 \subset G(m, W_1)$ be all A satisfying $Q_1(A, A) = 0$ and $X_2 \subset G(\frac{m(m-1)}{2}, W_1)$ all B satisfying $Q_2(B, B) = 0$. We claim that $\phi \colon G(m, W_1) \to G(\frac{m(m-1)}{2}, \Lambda^2 W_1)$ maps X_1 into X_2. To begin with, if $Q_1(A, A) = 0$ and $A \cap \bar{A} = 0$, then A corresponds to a complex torus and so $Q_2(\Lambda^2 A, \Lambda^2 A) = 0$ by consideration of type. Thus $Q_2(\phi(A), \phi(A))$ is an analytic function on a connected variety and which vanishes on an open set; i. e. $Q_2(\phi(A), \phi(A)) \equiv 0$. We have to show that $\phi(A) = \Lambda^2 A \subset W_2 \subset \Lambda^2 W_1$;

i. e. $\omega^{m-1}(\Lambda^2 A) = 0$. But, if $\xi, \eta \in A$, $Q_1(\xi,\eta) = \frac{(-1)^{m-1}}{(m-1)!}\int_T \xi\eta\omega^{m-1} = 0$ and so $\xi\eta\omega^{m-1} = 0$; i. e. $\omega^{m-1}(\Lambda^2 A) = 0$. This shows that $\phi\colon X_1 \to X_2$.

Suppose now that $A \in X_1$ and $\phi(A) = \Lambda^2 A \in D_2$; i. e. $Q_2(\phi(A),\overline{\phi(A)}) > 0$. We claim that $A \in D_1$; i. e. $\sqrt{-1}\, Q_1(A,\bar{A}) > 0$. In fact, this can be checked by a direct computation.

We now let $\boldsymbol{G}_1$ be all linear transformations $T\colon W_1 \to W_1$ which preserve ω; $T\omega = \omega$. Then $\det(T) = 1$ and $\boldsymbol{G}_1$ is the complex group preserving Q_1, since

$$Q_1(T\xi, T\eta) = \frac{(-1)^{m-1}}{(m-1)!}\int_T T\xi T\eta\omega^{m-1}$$
$$= \frac{(-1)^{m-1}}{(m-1)!}\int_T T\xi T\eta (T\omega)^{m-1} = Q_1(\xi,\eta).$$

If $G_1 \subset \boldsymbol{G}_1$ is the real group, then G_1 preserves D_1 and acts transitively there ($G_1 \cong Sp(m,\boldsymbol{R})$ and $D_1 \cong \boldsymbol{H}_m$).

We let $\boldsymbol{G}_2$ be the linear transformations $M\colon W_2 \to W_2$ which preserve Q_2 and $G_2 \subset \boldsymbol{G}_2$ the real subgroup. Then G_2 preserves D_2 and acts transitively there ($G_2 \cong SO(m(m-1), m^2-1)$) and D_2 is a period matrix domain for 2-forms).

If $T \in \boldsymbol{G}_1$, then T induces $\Lambda^2 T\colon \Lambda^2 W_1 \to \Lambda^2 W_1$ and $\Lambda^2 T$ preserves W_2 (since $T\omega = \omega$). Moreover, T preserves Q_2 since

$$Q_2(\Lambda^2 T(\xi\wedge\eta), \Lambda^2 T(\phi\wedge\psi)) = \frac{(-1)^{m-2}}{(m-2)!}\int_T T\xi T\eta T\phi T\psi\omega^{m-2}$$
$$= \frac{(-1)^{m-2}}{(m-2)!}\int_T \xi\eta\phi\psi\omega^{m-2} = Q_2(\xi\wedge\eta, \phi\wedge\psi).$$

Thus $\phi\colon X_1 \to X_2$ is $\boldsymbol{G}_1$-equivariant and $\phi(X_1)$ is the $\boldsymbol{G}_1$-orbit of a point in X_2 (where we consider $\boldsymbol{G}_1$ as a subgroup of $\boldsymbol{G}_2$ as above). This gives:

(2.45) THEOREM. *$\Phi(D_1)$ is the G_1-orbit of a point in D_2, where $G_1 \subset G_2$ is a subgroup which preserves $\Phi(D_1)$.*

(e) We now discuss the relationship between the complex torii $T_p(V)$ and *Weil's intermediate Jacobians* $A_p(V)$ [28]. As in II. 2. (a) we let $E_{0,p} = H^{2n-2p-1}(V,\boldsymbol{R})$, $\Gamma_p = H^{2n-2p-1}(V,\boldsymbol{Z})$ (modulo torsion), and

$$E_p = E_{0,p} \otimes_{\boldsymbol{R}} \boldsymbol{C} = H^{2n-2p-1}(V,\boldsymbol{C}).$$

The almost complex structure on V induces an automorphism

$$C\colon H^{2n-2p-1}(V,\boldsymbol{R}) \to H^{2n-2p-1}(V,\boldsymbol{R})$$

with $C^2 = -1$ ([29]). We let $J_p \subset E_p$ be the $+i$ eigenspaces of C. Then $J_p \cap \bar{J}_p = 0$ and so:

$$(2.46) \qquad A_p(V) = (E_p/J_p) \text{ modulo } \Gamma_p,$$

gives a complex torus, which is Weil's p-th Jacobian [28]. Observe that:

$$(2.47) \qquad J_p = \cdots + H^{n-p+2,n-p-3} + H^{n-p,n-p-1} + H^{n-p-2,n-p+1} + \cdots.$$

For example, if $n = 3$ and $p = 1$, $J_1 = H^{2,1} + H^{0,3}$ whereas $S_1 = H^{3,0} + H^{2,1}$ ($T_1(V) = (E_1/S_1)$ modulo Γ_1).

By *Poincaré duality,*

$$\begin{aligned}(E_p/J_p) &\cong (\cdots + H^{n-p+1,n-p-2} + H^{n-p-1,n-p} + H^{n-p-3,n-p+2} + \cdots) \\ &\cong (\cdots + H^{p-1,p+2} + H^{p+1,p} + H^{p+3,p-2} + \cdots)^* = J^*_{n-p-1}.\end{aligned}$$

This proves, as in Proposition (2.16), that:

$$(2.48) \qquad A_p(V) \cong A_{n-p-1}(V)^*,$$

where $A_{n-p-1}(V)^* = J^*_{n-p-1}/\Gamma^*_{n-p-1}$. As in Proposition (2.14), to find $A_{n-p-1}(V)^*$ explicitly, we choose a basis $\omega^1, \cdots, \omega^m$ for

$$J_{n-p-1} = (\cdots + H^{p+3,p-2} + H^{p+1,p} + H^{p-1,p+2} + \cdots)$$

and free generators $\gamma_1, \cdots, \gamma_{2m}$ for $H_{2p+1}(V, \boldsymbol{Z}) = \Gamma^*_{n-p-1}$, and form the *period matrix*:

$$(2.49) \qquad \Sigma_p(V) = (\tau_{\rho\alpha}), \text{ where } \tau_{\rho\alpha} = \int_{\gamma_\rho} \omega^\alpha.$$

Then the rows of $\Sigma_p = \Sigma_p(V)$ generate a lattice in $\boldsymbol{C}^m$, and $A_{n-p-1}(V)^*$ is $\boldsymbol{C}^m$ modulo this lattice. Using (2.48), the same method as used to define (2.18) gives a mapping:

$$(2.50) \qquad \psi : B(Z_0) \to A_p(V).$$

Here, $Z_0 \subset V$ is an algebraic p-cycle and $B(Z_0)$ parametrizes the algebraic p-cycles $Z \subset V$ with $Z \sim Z_0$.

Now

$$\begin{aligned}T_0(A_p(V))^* = (E_p/J_p)^* &\cong J_{n-p-1} \\ &= (\cdots + H^{p+3,p-2} + H^{p+1,p} + H^{p-1,p+2} + \cdots).\end{aligned}$$

The same proof as in Theorem (2.20) shows that:

$$(2.51) \qquad \psi^*(H^{p+k+1,p-k}) = 0 \text{ unless } k = 0.$$

This proves that $\psi^*\bar{T}_0(A_p(V))=0$ and $\psi^*(H^{p+2k+1,p-2k})=0$ unless $k=0$, which gives:

(2.52) PROPOSITION. *The mapping* ψ *in* (2.50) *is holomorphic. The differential* $\psi_*\colon T_0(B)\to T_0(A_p(V))$ *is determined by* $\psi^*\colon H^{p+1,p}(V)\to T_0(A_p(V))^*$ *and is given by the cohomology mapping* ψ^* *in* (2.24).

We want now to discusss a polarization on $A_p(V)$. This is the same procedure as in Proposition (2.32), because we observe that $Q(J_{n-p-1},J_{n-p-1})=0$ by type considerations. Thus there is a line bundle $\boldsymbol{L}_A\to A_{n-p-1}(V)^*\cong A_p(V)$ whose characteristic class is $\omega=\sqrt{-1}\sum_{\alpha,\beta}k_{\alpha\bar\beta}dw^\alpha\wedge d\bar w^\beta$, where the Hermitian matrix $K=(k_{\alpha\bar\beta})=(\sqrt{-1}\,{}^t\Sigma Q\bar\Sigma)^{-1}$. We claim that $K>0$. This is because:

$$J_{n-p-1}=\sum_k H^{p+1+2k,p-2k},\qquad \bar J_{n-p-1}\sum_l \mathrm{H}^{p-2l,p+1+2l},$$

and $\sqrt{-1}\,Q(H^{p+1+2k,p-2k},H^{p-2l,p+1+2l})=0$ for $k\neq l$ while

$$\sqrt{-1}\,Q(H^{p+1+2k,p-2k},H^{p-2k,p+1+2k})>0.$$

For example, when $p=1$,

$$\sqrt{-1}\,Q(H^{3,0},H^{0,3})<0,\qquad \sqrt{-1}\,Q(H^{2,1},H^{1,2})>0,$$
$$\sqrt{-1}\,Q(H^{1,2},H^{2,1})<0,\qquad \sqrt{-1}\,Q(H^{0,3},H^{3,0})>0.$$

This proves:

(2.53) PROPOSITION. *The holomorphic line bundle* $\boldsymbol{L}_A\to A_p(V)$ *has a* 0-*convex polarization, and is consequently a positive line bundle.*

The main result is:

(2.54) THEOREM. *There exists a real linear isomorphism* $\xi\colon T_p(V)\to A_p(V)$ *such that:*

$$\begin{array}{lccl} & & T_p(V) & \\ & \phi\nearrow & & \\ \text{(i)}\quad B & & \downarrow\xi & \textit{commutes;}\\ & \searrow\psi & & \\ & & A_p(V) & \end{array}$$

(ii) *if* $\boldsymbol{L}_T\to T_p(V)$ *and* $\boldsymbol{L}_A\to A_p(V)$ *are the complex line bundles associated to the polarizations on* $T_p(V)$ *and* $A_p(V)$, *then* $\xi^*(\boldsymbol{L}_A)=\boldsymbol{L}_T$;

(iii) *if* $\vartheta\in H^0(\mathcal{O}_{A_p(V)}(\boldsymbol{L}_A))$ *is a holomorphic section of* $\boldsymbol{L}_A\to A_p(V)$, *then the* C^∞ *section* $\xi^*\vartheta$ *of* $\boldsymbol{L}_T\to T_p(V)$ *is holomorphic on* $\phi(B)$.

Proof. We let $J_{n-p-1} = (\cdots + H^{p+3,p-2} + H^{p+1,p} + H^{p-1,p+2} + \cdots)$ be the $+i$ eigenspace of the operator C on $H^{2p+1}(V, \boldsymbol{C})$ and $S_{n-p-1} = H^{2p+1,0} + \cdots + H^{p+1,p}$. We may choose bases $\omega^1, \cdots, \omega^k; \omega^{k+1}, \cdots, \omega^m$ for S_{n-p-1} and $\phi^1, \cdots, \phi^k; \phi^{k+1}, \cdots, \phi^m$ for J_{n-p-1} such that: $\omega^\alpha = \phi^\alpha \in H^{p+1,p}$ for $1 \leqq \alpha \leqq k$, and $\omega^\alpha = \phi^\alpha$ or $\omega^\alpha = \bar{\phi}^\alpha$ for $k < \alpha \leqq m$. Let $\gamma_1, \cdots, \gamma_{2m}$ be a basis for $H_{2p+1}(V, \boldsymbol{Z})$ (modulo torsion) and $\Omega = (\pi_{\rho\alpha})$ where $\pi_{\rho\alpha} = \int_{\gamma_\rho} \omega^\alpha$, $\Sigma = (\tau_{\rho\alpha})$ where $\tau_{\rho\alpha} = \int_{\gamma_\rho} \phi^\alpha$. Then the rows of Ω (respectively , Σ) generate a lattice Γ_T (respectively, Γ_A) in $\boldsymbol{C}^m$, and $T_p(V) = \boldsymbol{C}^m/\Gamma_T$, $A_p(V) = \boldsymbol{C}^m/\Gamma_A$. Define $\xi\colon \boldsymbol{C}^m \to \boldsymbol{C}^m$ by $\xi(z^1, \cdots, z^m) = (w^1, \cdots, w^m)$ where $w^\alpha = z^\alpha$ if $\omega^\alpha = \phi^\alpha$, $w^\alpha = \bar{z}^\alpha$ if $\bar{\omega}^\alpha = \phi^\alpha$. This is a real linear isomorphism.

If $e_\rho = (\int_{\gamma_\rho} \omega^1, \cdots, \int_{\gamma_\rho} \omega^m)$ and $f_\rho = (\int_{\gamma_\rho} \phi^1, \cdots, \int_{\gamma_\rho} \phi^m)$, then $\Gamma_T = (e_1, \cdots, e_m)_{\boldsymbol{Z}}$ and $\Gamma_A = (f_1, \cdots, f_m)_{\boldsymbol{Z}}$. But $\xi(e_\rho) = f_\rho$ since

$$\overline{\int_{\gamma_\rho} \omega^\alpha} = \int_{\gamma_\rho} \bar{\omega}^\alpha.$$

Thus $\xi\colon \Gamma_T \to \Gamma_A$ and so $\xi\colon T_p(V) \to A_p(V)$.

By definition, if $\lambda \in B$ and $Z_\lambda - Z_0 = \partial C_\lambda$, then

$$\phi(\lambda) = (\int_{C_\lambda} \omega^1, \cdots, \int_{C_\lambda} \omega^m) \text{ and } \psi(\lambda) = (\int_{C_\lambda} \phi^1, \cdots, \int_{C_\lambda} \phi^m).$$

But, as in the proof of Theorem (2.20), $\int_{C_\lambda} \omega^\alpha = 0 = \int_{C_\lambda} \phi^\alpha$ if $k < \alpha \leqq m$ so that

$$\phi(\lambda) = (\int_{C_\lambda} \omega^1, \cdots, \int_{C_\lambda} \omega^k, 0, \cdots, 0) \text{ and}$$

$$\psi(\lambda) = (\int_{C_\lambda} \omega^1, \cdots, \int_{C_\lambda} \omega^k, 0, \cdots, 0).$$

Then it is clear that $\xi\phi(\lambda) = \psi(\lambda)$, which proves (i).

Observe that (iii) follows from (i) and (ii), so that it will suffice to prove (ii). To do this, it will be enough to show that $\xi^*(\omega_A) = \omega_L$, where $\omega_A = c_1(\boldsymbol{L}_A)$ and $\omega_L = c_1(\boldsymbol{L}_T)$.

We write the period matrices:

$$\begin{cases} \Omega = (\Omega_1, \cdots, \Omega_{p+1}) \\ \Sigma = (\Sigma_1, \cdots, \Sigma_{p+1}), \end{cases}$$

where the Ω_μ correspond to the summands in $S_{n-p-1}=\sum_{k\geqq 0} H^{p+k+1,p-k}$ and the Σ_μ correspond to the summands in $J_{n-p-1}=\sum_{l} H^{p+1+2l,p-2l}$. By choosing our bases as above, we may assume that either $\Omega_\mu=\Sigma_\mu$ or $\Omega_\mu=\bar{\Sigma}_\mu$. Furthermore, ${}^t\Omega_\mu Q\Omega_\nu=0$ for $\mu\neq\nu$ and similarly for the Σ_μ. Letting $H_\mu=(i{}^t\Omega_\mu Q{}^t\bar{\Omega}_\mu)^{-1}$ and $K_\mu=(i{}^t\Sigma_\mu Q\bar{\Sigma}_\mu)^{-1}$, we have

$$H=\begin{pmatrix} H_1 & & 0\\ & \ddots & \\ 0 & & H_{p+1}\end{pmatrix},\quad K=\begin{pmatrix} K_1 & & 0\\ & \ddots & \\ 0 & & K_{p+1}\end{pmatrix}$$

where $H_\mu=K_\mu$ if $\Omega_\mu=\Sigma_\mu$, $H_\mu=-{}^tK_\mu$ if $\Omega_\mu=\bar{\Sigma}_\mu$.

Now write $\omega_A=i(\sum_{\alpha,\beta,\mu} k_{\mu,\alpha\bar{\beta}}dw_\mu{}^\alpha\wedge d\bar{w}_\mu{}^\beta)$ and $\omega_T=i(\sum_{\alpha,\beta,\mu} h_{\mu,\alpha\bar{\beta}}dz_\mu{}^\alpha\wedge d\bar{z}_\mu{}^\beta)$ where $K_\mu=(k_{\mu,\alpha\bar{\beta}})$ and $H_\mu=(h_{\mu,\alpha\bar{\beta}})$. Then $\xi^*(dw_\mu{}^\alpha)=dz_\mu{}^\alpha$ if $\Omega_\mu=\Sigma_\mu$ and $\xi^*(dw_\mu{}^\alpha)=d\bar{z}_\mu{}^\alpha$ if $\Omega_\mu=\bar{\Sigma}_\mu$. Thus

$$\begin{aligned}\xi^*\omega_A &= i(\sum_{\substack{\alpha,\beta\\ \Omega_\mu=\Sigma_\mu}} k_{\mu,\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta+\sum_{\substack{\alpha,\beta\\ \Omega_\mu=\bar{\Sigma}_\mu}} k_{\mu,\alpha\bar{\beta}}d\bar{z}^\alpha\wedge dz^\beta)\\ &= i(\sum_{\alpha,\beta,\mu} h_{\mu,\alpha\bar{\beta}}dz^\alpha\wedge d\bar{z}^\beta)=\omega_T,\end{aligned}$$

since $H_\mu=-{}^tK_\mu$ if $\Omega_\mu=\bar{\Sigma}_\mu$.

This completes the proof of Theorem (2.54).

(f) We want to give two applications of Theorem (2.54). First we observe:

(2.55) LEMMA. *Let $Z_0\subset V$ be an effective p-cycle and $\{Z_\lambda\}_{\lambda\in B}$ an irreducible algebraic family of effective p-cycles with $Z_0=Z_{\lambda_0}$. Then the mapping $\phi\colon B\to T_p(V)$ given in (2.28) is continuous, hence holomorphic everywhere.*

This implies that $\phi(B)\subset T_p(V)$ is an analytic subvariety; in fact, ϕ is a proper holomorphic mapping.

(2.56) PROPOSITION. *$\phi(B)$ is an algebraic subvariety of the analytic torus $T_p(V)$.*

Proof. By Theorem (2.54), the C^∞ sections $\xi^*(\vartheta)$ of $\boldsymbol{L}_T{}^\mu\to T_p(V)$ ($\vartheta\in H^0(\mathcal{O}_A(\boldsymbol{L}_A{}^\mu))$) will be holomorphic on $\phi(B)$; for $\mu\geqq 3$, these sections give a projective embedding of $\phi(B)$. This proves the Proposition.

Suppose now that $\{V_t\}_{t\in\Delta}$ is an analytic family of polarized alegbraic manifolds with Δ a polycylinder, $V=V_0$. Let $Z_t\subset V_t$ be an effective algebraic p-cycle, varying analytically with t, and $\{Z_{t,\lambda}\}_{\lambda\in B_t}$ an irreducible algebraic family with $Z_t = Z_{t,\lambda_0}$.

(2.57) PROPOSITION. *The algebraic varieties $\psi_t(B_t) \subset A_p(V_t)$ vary holomorphically with t, even though the torii $A_p(V_t)$ don't.*

Proof. By Theorem (2.54) $\phi(B_t) = \psi(B_t)$, and $\phi(B_t) \subset T_p(V_t)$ varies analytically with t since $T_p(V_t)$ does.

The second application is more in the nature of a remark. What we want to do is draw a parallel between the period mappings $\Phi\colon B \to D/\Gamma$ (cf. II.1) and mapping (2.18) $\phi\colon B \to T_p(V)$ associated with the p-cycles on V.

Considering the torus $T_p(V)$, the tangent space at the origin is $\sum_{k\geqq 0} H^{n-p-k-1,n-p+k}(V)$, which we write as $P_0 \oplus N_0$ where

$$(2.58)\qquad \begin{cases} P_0 = \sum_{k\geqq 0} H^{n-p-2k-1,n-p+2k} \\ N_0 = \sum_{l\geqq 0} H^{n-p-2l,n-p+2l+1}. \end{cases}$$

This gives, at each point $x \in T_p(V)$, a translation-invariant splitting of the tangent space:

$$(2.59)\qquad T_x = P_x \oplus N_x.$$

Since $P_0{}^* = \sum_{k\geqq 0} H^{p+2k+1,p-2k}$, $N_0{}^* = \sum_{l\geqq 0} H^{p+2l,p-2l+1}$, it follows that the curvature form ω_T of $\boldsymbol{L}_T \to T_p(V)$ is *positive* on P_x and *negative* on N_x. Furthermore, for $\phi\colon B \to T_p(V)$, $\phi_*\colon T_\lambda(B) \to P_{\phi(\lambda)}$. This is the analogue of Theorem (1.34); it says that the *period-like* mapping ϕ satisfies infinitesimal (but *not* finite) period relations, and that $\boldsymbol{L}_T \mid \phi(B)$ is positive.

Now Theorem (2.54) gives us holomorphic sections $\xi^*(\vartheta)$ of $\boldsymbol{L}_T \mid \phi(B)$. In fact, we have:

$$(2.60)\qquad \bar\partial \xi^*(\vartheta) \mid P_x = 0.$$

Unfortunately, this is misleading as regards the period mapping $\phi\colon B \to D/\Gamma$. Let $\{V_t\}_{t\in B}$ be an algebraic family of polarized algebraic manifolds; here B may be complete or affine. Then $\boldsymbol{L} \mid \Phi(B)$ is positive (Theorem (1.34)) and we may look for C^∞ sections θ of $\boldsymbol{L} \to D/\Gamma$ such that $\bar\partial\theta \mid \Phi_*(T_t(B)) = 0$. Since $\Phi_*(T_t(B)) \subset H_{\Phi(t)}$, by analogy with (2.60), we might look for C^∞ sections θ with

$$(2.61)\qquad \bar\partial\theta \mid H_\Omega = 0.$$

But the distribution $x \to P_x$ on $T_p(V)$ is *integrable*, so that (2.60) implies no additional equations; whereas $\Omega \rightarrow H_\Omega$ is *not* integrable and (2.61) gives $\bar{\partial}\theta \mid H + [H,H] + [H,[H,H]] + \cdots = 0$. If D is the period matrix space for holomorphic 2-forms, then $H_\Omega + [H,H]_\Omega = T_\Omega(D)$ is the whole tangent space, so that θ satisfying (2.61) would be a holomorphic section of $\boldsymbol{L} \rightarrow D/\Gamma$. Generally there are no such sections.

A final application of Theorem (2.54) concerns the cohomology groups $H^r(\mathcal{O}(\boldsymbol{L}_T))$. Suppose that $\boldsymbol{L}_T \rightarrow T_p(V)$ has a q-convex polarization and let δ be the *Pffafian* of Q^{-1} (if $\omega = \frac{1}{2}\sum_{\rho,\sigma=1}^{2m} q_{\rho\sigma}dx^\rho \wedge dx^\sigma$, then $\omega^m = \pm\frac{\delta}{m!}dx^1 \wedge \cdots \wedge dx^{2m}$). Then we have proved (I.3.(d), Proposition (3.23)) that $H^r(\mathcal{O}(\boldsymbol{L}_T)) = 0$ for $r \neq q$ and $\dim H^q(\mathcal{O}(\boldsymbol{L}_T)) = \delta$. Similarly, $H^r(\mathcal{O}(\boldsymbol{L}_A))$ for $r > 0$ and $\dim H^0(\mathcal{O}(\boldsymbol{L}_A)) = \delta$. Let $\omega^1, \cdots, \omega^m$ on $T_p(V)$ and $\phi^1, \cdots, \phi^m$ on $A_p(V)$ have the same meaning as in the proof of Theorem (2.54), and let $\eta = \bar{\omega}^{\alpha_1} \wedge \cdots \wedge \bar{\omega}^{\alpha_q}$ where $\xi^*(\phi^{\alpha_j}) = \bar{\omega}^{\alpha_j}$. We define $\xi^*: H^0(\mathcal{O}(\boldsymbol{L}_A)) \rightarrow H^q(\mathcal{O}(\boldsymbol{L}_T))$ by:

$$\xi: \vartheta \rightarrow \xi^*(\vartheta)\eta. \tag{2.62}$$

This makes sense, since $\omega^{\alpha_1}, \cdots, \omega^{\alpha_q}$ give a basis for N_0^* (2.58) and $\bar{\partial}\xi^*(\vartheta) \mid P_x = 0$. Thus $\bar{\partial}\xi^*(\vartheta) \equiv 0(\bar{\omega}^{\alpha_1}, \cdots, \bar{\omega}^{\alpha_q})$ so that

$$\bar{\partial}[\xi^*(\vartheta)\bar{\omega}^{\alpha_1} \wedge \cdots \wedge \bar{\omega}^{\alpha_q}] = 0.$$

It can be shown that ξ^* in (2.62) is an isomorphism.

II. 3. Examples of the local period mapping. We want to discuss now the question of when the *periods give local moduli.* For analytic fibre spaces $\boldsymbol{V} \xrightarrow{\pi} \Delta$ which are *regular*; i.e. $\dim \rho_t(T_t(\Delta))$ is constant, the period mapping Ω will locally distinguish inequivalent subvarieties if either:

(3.1) the cup product (1.30) is onto (cf. Theorem (1.29)); or

(3.2) the cup product:

$$H^1(V,\Theta)_\omega \otimes H_0^{q-r+1,r-1} \rightarrow H_0^{q-r,r}$$

is non-degenerate in the first factor (i.e., if $\theta\phi = 0$ for all $\phi \in H_0^{q-r+1,r-1}$, then $\theta = 0$ in $H^1(V,\Theta)_\omega$).

(a) *Riemann surfaces.* Let V be a compact Riemann surface of genus $p > 1$. Then it is well known that there exists an *effectively parametrized, locally complete family* $\boldsymbol{V} \xrightarrow{\pi} \Delta$ with $V = V_0$, Δ a polycylinder in $H^1(V,\Theta)$, and with the Kodaira-Spencer mapping ρ_0 being the identity [20]. The cup product (1.30) then becomes $(n=q=1, r=0)$:

$$H^0(V,\Omega^1)\otimes H^0(V,\Omega^1)\xrightarrow{\ \mu\ } H^0(V,\Omega^2). \tag{3.3}$$

Thus μ is onto if, and only if, the *quadratic differentials* are generated by *Abelian differentials,* and we have:

(3.4) NOETHER'S THEOREM. *The mapping μ is onto if $p=2$ or if $p>2$ and V is non-hyperelliptic.*

Combining this with (3.1), we get:

(3.5) PROPOSITION [27]. *The periods give local coordinates in the moduli space if $p=2$ or if $p>2$ and V is non-hyperelliptic.*

Remarks. If V is hyperelliptic, then it is given by an affine equation $y^2=\prod_{i=1}^{2p+2}(x-a_i)$ in $\boldsymbol{C}^2$ with coordinates x,y. The abelian differentials are generated by $\frac{dx}{y},\cdots,x^{p-1}\frac{dx}{y}$, and so these differentials generate a space of quadratic differentials with basis $\omega_\alpha=x^\alpha(\frac{dx}{y})^2$ $(0\leqq\alpha\leqq 2p-2)$. Thus, if $p>2$, $2p-1<3p-3$, μ is *not* onto, and the differential Ω_* of the period mapping is singular at V (cf. [27]). If $p=2$, μ is onto and Ω_* is injective.

We now outline a proof of (3.4) in the non-hyperelliptic case. Let $\boldsymbol{K}\to V$ be the *canonical bundle* and $|\boldsymbol{K}|=\boldsymbol{P}\{H^0(V,\mathcal{O}(\boldsymbol{K}))^*\}$ the associated *complete linear system.* Thus $|\boldsymbol{K}|$ is a P_{p-1} and the hyperplane sections of the rational mapping $\psi\colon V\to|\boldsymbol{K}|$ are all of the form (ω) where $\omega\in H^0(V,\mathcal{O}(\boldsymbol{K}))$ is an Abelian differential. From the theory of algebraic curves [5], we recall:

(i) $\psi\colon V\to P_{p-1}$ is a regular embedding;

(ii) the general hyperplane section (ω) meets V in $2p-2$ points, *any* $p-1$ of which are linearly independent in P_{p-1}.

Let now ω be a general Abelian differential with $(\omega)=A_1+\cdots+A_{2p-2}$ $(A_i\neq A_j$ for $i\neq j)$. Since any $p-1$ points from (ω) are independent, given $A_{i_1}+\cdots+A_{i_{p-1}}$ contained in (ω), we can find an abelian differential ϕ with $\phi(A_{i_1})=0,\cdots,\phi(A_{i_{p-2}})=0$, $\phi(A_{i_{p-1}})\neq 0$. Consider the exact sheaf sequence:

$$0\to\mathcal{O}(\boldsymbol{K})\xrightarrow{\ \omega\ }\mathcal{O}(\boldsymbol{K}^2)\to\boldsymbol{K}^2{}_{A_1}\oplus\cdots\oplus\boldsymbol{K}^2{}_{A_{2p-2}}\to 0,$$

which gives the cohomology diagram:

$$
\begin{array}{c}
0 \to H^0(\boldsymbol{K}) \xrightarrow{\ \omega\ } H^0(\boldsymbol{K}^2) \to \boldsymbol{K}^2{}_{A_1} \oplus \cdots \oplus \boldsymbol{K}^2{}_{A_{2p-2}} \to H^1(\boldsymbol{K}) \to 0 \\
\downarrow\mu \quad \nearrow\xi \\
H^0(\boldsymbol{K}) \otimes H^0(\boldsymbol{K}).
\end{array}
$$

Since Image $\omega \subset$ Image μ, we must prove: $\dim(\text{Image}\,\xi) = 2p - 3$.

Set $Q = A_{2p-2}$ and, for any j with $1 \leqq j \leqq 2p - 3$, write $A_1 \cdots A_{2p-2} = P_1 \cdots P_{p-2} A_j R_1 \cdots R_{p-2} Q$. We may choose ϕ_j, η_j with

$$
\begin{gathered}
\phi_j(P_1) = \cdots = \phi_j(P_{p-2}) = 0, \ \phi_j(A_j) \neq 0; \\
\eta_j(R_1) = \cdots = \eta_j(R_{p-2}) = 0, \ \eta_j(A_j) \neq 0.
\end{gathered}
$$

Obviously then the elements $\xi(\phi_j \eta_j)$ are linearly independent. This proves that $\dim(\text{Image}\,\xi) \geqq 2p - 3$, and Noether's theorem follows.

(b) *Special complex manifolds.* A *special complex manifold* is a compact, complex Kähler manifold V whose canonical bundle $\boldsymbol{K}$ is trivial; thus there exists an everywhere non-zero holomorphic n-form ϕ on V. For reasons stemming from duality, these manifolds are frequently amendable to computation. Examples include Abelian varieties, hypersurfaces of degree $n + 2$ in P_{n+1}, and K3 surfaces [17].

(3.6) PROPOSITION. *The periods give local coordinates in the local moduli space of any special complex manifold V.*

Proof. If we are ignoring polarizations, this follows from the isomorphism (cf. (1.32)):

$$
H^0(\Omega_V{}^n) \otimes H^{n-1}(\Omega_V{}^1) \xrightarrow[\approx]{\ \mu\ } H^{n-1}(\Omega_V{}^1).
$$

If we have a polarized family, this follows from the isomorphism (cf. (1.31)):

$$
H^0(\Omega_V{}^n) \otimes H^{n-1}(\Omega_V{}^1)_0 \xrightarrow[\approx]{\ \mu\ } H^{n-1}(\Omega_V{}^1)_0.
$$

Remarks. We have actually shown that the periods of the holomorphic n-forms ϕ give local coordinates in the moduli space. For K3 surfaces, this is due to Andreotti and Weil (cf. [6]). If $\dim H^2(V, \Theta) = \dim H^{n-2}(V, \Omega^1) = 0$, then V has $\dim H^1(V, \Theta) = \dim H^{n-1}(V, \Omega^1)$ local moduli (cf. [20]).

(c) *Continuous systems and the period mapping.* To determine the rank of the (local) period mapping Ω, we come up against a *multiplicative* problem in cohomology (cf. Theorem (1.29)), which is generally difficult. However, many families $\{V_t\}_{t\in\Delta}$ of algebraic manifolds are given in nature as a family of submanifolds of a fixed algebraic manifold W; e.g., using *Chow varieties* or subfamilies thereof. The proper notion here is that of a *continuous system* $[V_t]_{t\in\Delta}$ of submanifolds of an algebraic manifold W (cf. [16] and section II.2.(b), the proof of Theorem (2.20)). In this case, the multiplicative problem can be rephrased as a problem on *linear systems* which, in certain cases, can be solved. We shall now carry out this reduction.

Let $[V_t]_{t\in\Delta}$ be a continuous system of submanifolds $V_t \subset W$ and let $V = V_{t_0}$. If $\boldsymbol{N} \to V$ is the *normal bundle* of $V \subset W$ and $T = T(W) \mid V$, we have the exact sheaf sequence

$$0 \to \Theta \to \mathcal{O}(T) \to \mathcal{O}(\boldsymbol{N}) \to 0. \tag{3.7}$$

Assuming that $H^0(V, \Theta) = 0$, we have in cohomology

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{O}(T)) \arrow[r] & H^0(\mathcal{O}(\boldsymbol{N})) \arrow[rr, "\delta"] & & H^1(\Theta) \\ & & T_0(\Delta) \arrow[u, "\chi"] \arrow[urr, "\rho"'] & & \end{tikzcd} \tag{3.8}$$

Here $\chi\colon T_0(\Delta) \to H^0(\mathcal{O}(\boldsymbol{N}))$ is the *characteristic map* (cf. (2.21)) or *infinitesimal displacement mapping.* The continuous system $[V_t]_{t\in\Delta}$ gives rise to an analytic fibre space $\{V_t\}_{t\in\Delta}$ (cf. the proof of Theorem (2.20)) and $\rho\colon T_0(\Delta) \to H^1(\Theta)$ is the *Kodaira-Spencer mapping* (1.5).

(3.9) PROPOSITION. *The differential Ω_* of the period mapping is non-singular if the product:*

$$H^0(\boldsymbol{N})/H^0(T) \otimes H^0(\Omega_V{}^n) \to H^0(\Omega_V{}^n(\boldsymbol{N}))/H^0(\Omega_V{}^n(T)), \tag{3.10}$$

is non-degenerate in the first factor.

Proof. From (3.8), $H^0(\boldsymbol{N})/H^0(T)$ is a subspace $S \subset H^1(V, \Theta)$ and we have to prove that

$$S \otimes H^0(\Omega_V{}^n) \to H^1(\Omega_V{}^{n-1}) \tag{3.11}$$

is non-degenerate in the first factor.

Dualizing the sheaf sequence (3.7) gives $0 \to \mathcal{O}(\boldsymbol{N}^*) \to \mathcal{O}(T^*) \to \Omega_V{}^1 \to 0$ and, in cohomology,

$$H^{n-1}(\Omega_V{}^1) \to H^n(\boldsymbol{N}^*) \to H^n(T^*). \tag{3.12}$$

Applying *Serre duality* to (3.12) gives

$$(3.13)\qquad 0 \to H^0(\Omega_V{}^n(\boldsymbol{N}))/H^0(\Omega_V{}^n(T)) \xrightarrow{\ \delta\ } H^1(\Omega_V{}^{n-1}).$$

Now let $\eta \in H^0(\boldsymbol{N})$, $\omega \in H^0(\Omega_V{}^n)$. Then $\eta\cdot\omega \in H^0(\Omega_V{}^n(\boldsymbol{N}))$ and, from (3.8), $\delta(\eta)\cdot\omega \in H^1(\Omega_V{}^{n-1})$. From this and (3.13) we observe: $\delta(\eta\cdot\omega) = \delta(\eta)\cdot\omega$.

It follows that, if $\eta \in H^0(\boldsymbol{N})$, $\omega \in H^0(\Omega_V{}^n)$, and $\eta\cdot\omega \neq 0$ in

$$H^0(\Omega_V{}^n(\boldsymbol{N}))/H^0(\Omega_V{}^n(T)),$$

then $\delta(\eta\cdot\omega) = \delta(\eta)\cdot\omega \neq 0$ in $H^1(\Omega_V{}^{n-1})$ and so (3.11) is non-degenerate in the first factor. This proves Proposition (3.9).

(3.14) COROLLARY. *If, in* (3.8), χ *is onto* (*the characteristic system is complete*) *and* δ *is onto, and if* (3.10) *is non-degenerate in the first factor, then the periods give local coordinates in the local moduli space for* V.

Remark. If χ and δ are onto, then V has the *postulated number* $\dim H^1(V,\Theta)$ of moduli (locally).

(d) *Surfaces in* P_3. What we shall prove is:

(3.15) THEOREM. *Let* $V \subset P_3$ *be a non-singular surface of degree* $n \geqq 5$. *Then the periods of the holomorphic 2-forms give local moduli for* V.

Proof. Let $P = P_3$ and $\boldsymbol{E} \to P$ be the hyperplane line bundle. If $\xi = [\xi_0, \xi_1, \xi_2, \xi_3]$ are homogeneous coordinates on P, then $\xi_0, \xi_1, \xi_2, \xi_3$ give a basis for $H^0(P, \mathcal{O}(\boldsymbol{E}))$. Setting $\mathcal{O}(\boldsymbol{E})^4 = \underbrace{\mathcal{O}(\boldsymbol{E}) \oplus \mathcal{O}(\boldsymbol{E}) \oplus \mathcal{O}(\boldsymbol{E}) \oplus \mathcal{O}(\boldsymbol{E})}_{4}$, we recall the exact sequence

$$(3.16)\qquad 0 \to \mathcal{O} \xrightarrow{\ \nu\ } \mathcal{O}(\boldsymbol{E})^4 \xrightarrow{\ \pi\ } \mathcal{O}(T(P)) \to 0,$$

where $\nu(f) = (f\xi_0, f\xi_1, f\xi_2, f\xi_3)$ and $\pi(\theta_0,\theta_1,\theta_2,\theta_3) = \sum_{j=0}^{3} \theta_j \dfrac{\partial}{\partial \xi_j}$. We remark that the exactness of (3.16) is essentially *Euler's theorem*; $\pi\nu(f) = f(\sum_{j=0}^{3} \xi_j \dfrac{\partial}{\partial \xi_j})$ and $\sum_{j=0}^{3} \xi_j \dfrac{\partial g}{\partial \xi_j} = 0$ if $g(\lambda\xi) = g(\xi)$ for λ a non-zero complex number.

Suppose now that $V \subset P$ is a non-singular surface given by $F(\xi_0,\xi_1,\xi_2,\xi_3) = 0$ where $F \in H^0(P, \mathcal{O}(\boldsymbol{E}^n))$ is a homogeneous polynomial of degree n. Set $T = T(P) \mid V$, $\boldsymbol{H} = \boldsymbol{E} \mid V$, $\boldsymbol{N} = \boldsymbol{H}^n =$ normal bundle of $V \subset P$, and $\boldsymbol{K} = \boldsymbol{H}^{n-4} =$ canonical bundle of V (*Proof.* From (3.7), $\boldsymbol{K} = (\det \boldsymbol{N})(\det \boldsymbol{T}^*) = \boldsymbol{H}^n\boldsymbol{H}^{-4}$ by (3.16)). We record the usual exact sheaf sequences (cf. (3.7)):

$$
(3.17)\qquad \begin{cases} 0 \to \Theta_V \to \mathcal{O}_V(T) \to \mathcal{O}_V(\boldsymbol{H}^n) \to 0\,; \\ 0 \to \mathcal{O}_P \xrightarrow{\ F\ } \mathcal{O}_P(\boldsymbol{E}^n) \xrightarrow{\ r\ } \mathcal{O}_V(\boldsymbol{H}^n) \to 0 \\ \qquad\qquad\qquad\qquad (r = \text{restriction to } V)\,; \end{cases}
$$

we combine (3.16) and (3.17) into:

$$
(3.18)\qquad \begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{O}_V \arrow[d] & \mathcal{O}_P \arrow[d, "F"] & \\
 & \mathcal{O}_V(\boldsymbol{H})^4 \arrow[d] & \mathcal{O}_V(\boldsymbol{E}^n) \arrow[d] & \\
0 \arrow[r] \Theta_V \arrow[r] & \mathcal{O}_V(T) \arrow[r] \arrow[d] & \mathcal{O}_V(\boldsymbol{H}^n) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 &
\end{tikzcd}
$$

(3.19) Lemma. If $n > 4$, the exact cohomology diagram of (3.18) is:

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
 & \boldsymbol{C} \arrow[d, "\nu"] & \boldsymbol{C} \arrow[d, "F"] & & \\
 & H^0(\mathcal{O}_P(\boldsymbol{E})^4) = H^0(\mathcal{O}_V(\boldsymbol{H})^4) \arrow[d] \arrow[dr, "\tau"] & H^0(\mathcal{O}_P(\boldsymbol{E}^n)) \arrow[d] & & \\
0 \arrow[r] & H^0(\mathcal{O}_V(T)) \arrow[r] \arrow[d] & H^0(\mathcal{O}_V(\boldsymbol{H}^n)) \arrow[r, "\delta"] \arrow[d] & H^1(\Theta_V) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}
$$

Proof. We have to show: (i) $H^1(\mathcal{O}_V) = 0 = H^1(\mathcal{O}_P(\boldsymbol{E}^{-3}))$; (ii) $H^0(\Theta_V) = 0$; and (iii) δ is onto. Since $H^1(\mathcal{O}_P(\boldsymbol{E}^{-k})) = 0$ for $k > 0$, (i) follows from the exact cohomology sequence of $0 \to \mathcal{O}_P(\boldsymbol{E}^{-4}) \xrightarrow{\ F\ } \mathcal{O}_P \xrightarrow{\ r\ } \mathcal{O}_V \to 0$. As for (ii), $\boldsymbol{K} = \boldsymbol{H}^{n-4}$ is *positive* and so

$$
\begin{aligned}
\dim H^0(\Theta) = \dim H^2(\Omega_V{}^2 \otimes \Omega_V{}^1) &= \dim H^{1,2}(\mathcal{O}_V(\boldsymbol{K})) \\
&= \dim H^{1,0}(\mathcal{O}_V(\boldsymbol{K}^*)) = 0
\end{aligned}
$$

(cf. [1]).

We now prove (iii). Using $H^1(\mathcal{O}_V(\boldsymbol{H})) = 0$, we get

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ H^0(\mathcal{O}_V(\boldsymbol{H}^n)) & \to & H^1(\Theta) & \to & H^1(\mathcal{O}(T)) \\ & & & & \downarrow \\ & & & & H^2(\mathcal{O}_V) \\ & & & & \downarrow \nu \\ & & & & H^2(\mathcal{O}_V(\boldsymbol{H}^4)). \end{array}$$

It will suffice to show that $H^2(\mathcal{O}_V) \xrightarrow{\nu} H^2(\mathcal{O}_V(\boldsymbol{H})^4)$ is injective. Dualizing, we have

$$\begin{array}{ccc} 0 & & 0 \\ \uparrow & & \uparrow \\ H^0(\mathcal{O}_V(\boldsymbol{H}^{n-4})) & \xleftarrow{\nu^*} & H^0(\mathcal{O}_V(\boldsymbol{H}^{n-5})^4) \\ \uparrow & & \uparrow \\ H^0(\mathcal{O}_P(\boldsymbol{E}^{n-4})) & \xleftarrow{\psi} & H^0(\mathcal{O}_P(\boldsymbol{E}^{n-5})^4) \end{array}$$

and, to show that ν^* is onto, we will show that ψ is onto. Now

$$\psi(F_0, F_1, F_2, F_3) = \xi_0 F_0 + \xi_1 F_1 + \xi_2 F_2 + \xi_3 F_3$$

where F_0, F_1, F_2, F_3 are forms of degree $n-5$. From this it is clear that ψ is onto, and the Lemma is proved.

Remark. If $[V_t]_{t\in\Delta}$ is the continuous system generated by $V \subset P$, then Δ is a polycylinder in $H^0(V, \mathcal{O}_V(\boldsymbol{N}))$ and the V_t are obtained by perturbing the equation $F(\xi)=0$ of V. In particular, the characteristic system is complete [16]. The statement that δ is onto in Lemma (3.19) then implies that the analytic family $\{V_t\}_{t\in\Delta}$ contains all of the local moduli of V (cf. [21]).

To prove Theorem (3.15), by Corollary (3.14) we must show that the product (cf. (3.10)):

$$\begin{aligned}(3.20)\quad & H^0(\mathcal{O}_V(\boldsymbol{H}^n))/H^0(\mathcal{O}_V(T)) \otimes H^0(\mathcal{O}_V(\boldsymbol{H}^{n-4})) \\ & \to H^0(\mathcal{O}_V(\boldsymbol{H}^{2n-4}))/H^0(\mathcal{O}_V(T\boldsymbol{H}^{n-4}));\end{aligned}$$

is non-degenerate in the first factor.

Let $G(\xi) = G(\xi_0\,\xi_1, \xi_2, \xi_3) \in H^0(\mathcal{O}_V(\boldsymbol{H}^{n+k}))$ be a form of degree $n+k$. Then $G(\xi)$ lies in $H^0(\mathcal{O}_V(T\boldsymbol{H}^k))$ if, and only if, $G = \sum_{j=0}^{3} \lambda_j \dfrac{\partial F}{\partial \xi_j}$ where the $\lambda_j(\xi)$

are forms of degree $k+1$. Thus, to prove that (3.20) is non-degenerate in the first factor, we must show the following algebraic lemma:

(3.21) LEMMA. *If $G(\xi)$ is a form of degree n such that*

$$GQ=\sum_{j=0}^{3}\lambda_j\frac{\partial F}{\partial\xi_j}\quad (\lambda_j(\xi)\ \textit{homogeneous of degree}\ n-3)$$

for all forms Q of degree $n-4$, then $G=\sum_{j=0}^{3}\phi_j\frac{\partial F}{\partial\xi_j}$ ($\phi_j(\xi)$ homogeneous of degree 1).

Originally we had proved (3.21) if $G(\xi)=\xi_0^n-P(\xi_1,\xi_2,\xi_3)$, so that Theorem (3.15) held for "almost all" V. However, David Mumford pointed out that the following result of Macauley would give Lemma (3.21), and hence Theorem (3.15).

(3.22) THEOREM (Macauley [26]). *Let $Q_0,\cdots,Q_m$ be homogeneous polynomials in $\xi_0,\cdots,\xi_m$ of degrees $r_1,\cdots,r_m$ such that $\sqrt{(Q_0,\cdots,Q_m)}=(\xi_0,\cdots,\xi_m)=\mathfrak{m}$. Then those $Q(\xi)$ such that $Q\cdot\mathfrak{m}^l\subset(Q_0,\cdots,Q_m)$ are precisely $(Q_0,\cdots,Q_m)+\mathfrak{m}^{\rho-l}$ where $\rho=\sum_{j=0}^{m}(r_j-1)-1$.*

Proof of Lemma (3.21). We take $Q_j=\frac{\partial F}{\partial\xi_j}$ so that $r_j=n-1$. Then $\sqrt{(Q_0,Q_1,Q_2,Q_3)}=(\xi_0,\xi_1,\xi_2,\xi_3)$ since $F(\xi)=0$ is non-singular. The assumption in Lemma (3.21) is that $G\cdot\mathfrak{m}^{n-4}\subset(Q_0,Q_1,Q_2,Q_3)$, and, by Macauley's theorem, $G\in(Q_0,Q_1,Q_2,Q_3)+\mathfrak{m}^{3n-5}$. Since $\deg G=n<3n-5$, it follows that $G\in(Q_0,Q_1,Q_2,Q_3)$; i.e., $G(\xi)=\sum_{j=0}^{3}\phi_j\frac{\partial F}{\partial\xi_j}$.

(e) *Surfaces on Abelian varieties.* Let A be an Abelian variety of dimension 3 and let $R=\sum_{j=0}^{\infty}R_j$ be the *graded ring of theta functions.* To be explicit, we suppose that A has principal matrix $Q=\begin{pmatrix}0 & I_3\\ -I_3 & 0\end{pmatrix}$ and period matrix $\Omega=(I,Z_0)$ (${}^tZ_0=Z_0$, $\operatorname{Im}Z_0>0$). Then $A=\boldsymbol{C}^3/\Gamma$ where Γ is the lattice generated by the columns $e_1,\cdots,e_6$ of Ω. If w^1, w^2, w^3 are coordinates on $\boldsymbol{C}^3$, a *theta-function* $\vartheta(w)$ of *degree* n is given by an entire function $\vartheta(w)$ on $\boldsymbol{C}^3$ which satisfies:

$$(3.23)\qquad \begin{cases}\vartheta(w+e_\alpha)=\vartheta(w) & (1\leqq\alpha\leqq3),\\ \vartheta(w+e_{3+\alpha})=e^{-2\pi i n w^\alpha}\vartheta(w) & (1\leqq\alpha\leqq3).\end{cases}$$

Of course, $R_n=H^0(A,\mathcal{O}_A(\boldsymbol{E}^n))$ where $\boldsymbol{E}\to A$ is a suitable line bundle, and it is known that $\dim R_n=n^3$ (cf. [4]).

We remark now that if ϕ, ϑ are theta functions of degree n, then the expression $(\phi \frac{\partial \vartheta}{\partial w^\alpha} - \vartheta \frac{\partial \phi}{\partial w^\alpha})$ is a theta-function of degree $2n$. Given $\vartheta \in R_n$, we define $I_\vartheta \subset R_{2n}$ to be the linear span of all theta functions of the form $\phi \frac{\partial \vartheta}{\partial w^\alpha} - \vartheta \frac{\partial \phi}{\partial w^\alpha} + \eta\vartheta$ $(\phi, \eta \in R_n)$, and we let:

(3.24) $\quad I_\vartheta : R_n/(\vartheta) = \{\phi \in R_n/(\vartheta) \text{ such that } \phi \cdot R_n \subset I_\vartheta\}.$

(3.25) THEOREM. *Let $n \geqq 3$ and ϑ be a theta function of degree n such that $\vartheta(w) = 0$ defines a non-singular surface $V \subset A$. Then there exists an analytic family $\{V_t\}_{t \in \Delta}$ where ρ_0 is an isomorphism* $(\dim \Delta = n^3 + 2)$, *and we let $\Omega : \Delta \to D$ be the period matrix mapping. Then* $\dim(\ker \Omega_*) = \dim(I_\vartheta : R_n/(\vartheta))$. *In particular, if $I_\vartheta : R_n/(\vartheta) = 0$, then the periods give local coordinates in the moduli space; and this is the case if A and ϑ are both general.*

Proof. We set $\boldsymbol{L} = \boldsymbol{E}^n$ so that $\boldsymbol{L} \to A$ has characteristic class

$$\omega = n\sqrt{-1}\,\{\sum_{\alpha,\beta} (\operatorname{Im} Z_0)_{\alpha\bar\beta}^{-1} dw^\alpha \wedge d\bar w^\beta\} = \sqrt{-1}\,\{\sum_{\alpha,\beta} h_{\alpha\bar\beta} dw^\alpha \wedge d\bar w^\beta\}$$

Below we shall use several results about the cohomology of V in A, mostly dealing with the residue calculus, and which will be proved in a general context in Part III. The first of these (essentially the *Lefschetz theorem*) is that we have a diagram:

$$\begin{array}{ccccc}
 & & 0 & & 0 \\
 & & \downarrow & & \downarrow \\
0 & \to & H^1(A, \boldsymbol{Z}) & \to & H^1(A, \mathcal{O}_A) \\
 & & \downarrow & & \downarrow \\
0 & \to & H^1(V, \boldsymbol{Z}) & \to & H^1(V, \mathcal{O}_V), \\
 & & \downarrow & & \downarrow \\
 & & 0 & & 0
\end{array}$$

so that $\operatorname{Pic}(A) \cong \operatorname{Pic}(V)$. It follows (cf. II. 2.(b)) that the periods of the holomorphic 1-forms on V specify A uniquely. Having fixed A, what we have to measure is to what extent the periods of the 2-forms determine V (up to translation) in A. First we shall locate precisely the moduli of V.

Let $\Delta_1 \subset \boldsymbol{H}_3$ (= *Siegel's upper half-space* in genus 3) be a neighborhood of Z_0 (= period matrix of A) and choose a basis $\vartheta_1, \cdots, \vartheta_{N+1}$ for $R_n = R_n(A)$ such that $\vartheta = \vartheta_{N+1}$. Then each $\vartheta_j = \vartheta_j(w; Z)$ is a function on $\boldsymbol{C}^3 \times \boldsymbol{H}_3$

and we let $\Delta_2 \subset \boldsymbol{C}^N$ be a polycylinder around the origin with coordinates $s = (s^1, \cdots, s^N)$. Set $\Delta = \Delta_1 \times \Delta_2$ and write $t \in \Delta$ as $t = (Z, s)$. We define a family of surfaces $\{V_t\}_{t \in \Delta}$ as follows: For $t = (Z, s)$, we let A_Z be the Abelian variety with period matrix (I, Z) and

$$\vartheta_s = s^1\vartheta_1 + \cdots + s^N\vartheta_N + \vartheta.$$

Then $V_t \subset A_Z$ is defined by $\vartheta_s(Z, w) = 0$.

(3.26) PROPOSITION. *The above family* $\{V_t\}_{t \in \Delta}$ *gives the local moduli of* V.

Proof. It will suffice to show that $\rho_0 \colon T_0(\Delta) \to H^1(\Theta_V)$ is an isomorphism (cf. [18], [21]).

Let $T =$ tangent bundle of A, $\boldsymbol{N} = \boldsymbol{L} \mid V$ the normal bundle of $V \subset A$, and consider the diagram:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{O}_A(T\cdot \boldsymbol{L}^*) & & \mathcal{O}_A & & \\ & & & & \downarrow\vartheta & & \downarrow\vartheta & & \\ (3.27) & & & & \mathcal{O}_A(T) & & \mathcal{O}_A(\boldsymbol{L}) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \to & \Theta_V & \to & \mathcal{O}_V(T) & \to & \mathcal{O}_V(\boldsymbol{N}) & \to & 0. \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

(3.28) LEMMA. *The composite mapping* $H^q(\mathcal{O}_A(T)) \to H^{q+1}(\mathcal{O}_A)$ *given by*

$$\begin{array}{ccc} H^q(\mathcal{O}_A(T)) & & \\ \downarrow & & \\ H^q(\mathcal{O}_V(T)) & \to & H^q(\mathcal{O}_V(\boldsymbol{N})) \\ & & \downarrow\delta \\ & & H^{q+1}(\mathcal{O}_A) \end{array}$$

is the cup product with $\omega \in H^1(\Omega_A{}^1)$, *where* ω *is the characteristic class of* $\boldsymbol{L}$.

Proof. This is a general result, depending only on the fact that $V \subset A$ is a hypersurface. Let $\{U_i\}$ be an open covering of A such that $V \cap A$ is given by $f_i = 0$. If $f_{ij} = f_i/f_j$, then $d \log f_{ij} = df_{ij}/f_{ij}$ is the Čech cocycle giving $\omega \in H^1(\Omega_A{}^1)$. Note that $\dfrac{df_{ij}}{f_{ij}} = \dfrac{df_i}{f_i} - \dfrac{df_j}{f_j}$. Choose C^∞ $(1,0)$ forms ξ_i such that $d \log f_{ij} = \xi_i - \xi_j$. Then $\{\bar{\partial}\xi_i\}$ gives the Dolbeault class of ω. Note that, if $\eta = \xi_i - \dfrac{df_i}{f_i}$, then η is a global $(1,0)$ form with $\bar{\partial}\eta = \omega$ and such that η has a first order pole along V.

Now we note that if θ is a vector field on A along V, then $\theta \cdot \vartheta$ is a section of $\boldsymbol{N}$ since $\theta \cdot f_i = \theta \cdot (f_{ij} \cdot f_j) = f_{ij}(\theta \cdot f_j)$. It follows that, if $\phi \in H^q(\mathcal{O}_A(T))$ is given by a T-valued $(0,q)$ form, then the image of ϕ in $H^q(\mathcal{O}_V(\boldsymbol{N}))$ is given by $\phi \cdot \vartheta = \vartheta(\langle \phi, \dfrac{d\vartheta}{\vartheta} \rangle) = \vartheta\langle \phi, \eta \rangle$. But $\vartheta\langle\phi,\eta\rangle$ is a global $\boldsymbol{L}$-valued $(0,q)$ form and $\bar{\partial}(\vartheta\langle\phi,\eta\rangle) = \vartheta\langle\phi,\bar{\partial}\eta\rangle = \vartheta(\phi\cdot\omega)$. It is now clear that $\delta(\phi\cdot\vartheta) = \phi\cdot\omega$ in $H^{q+1}(\mathcal{O}_A)$. Q. E. D.

If we now form the cohomology diagram of (3.27) and use that $H^q(\mathcal{O}_A(\boldsymbol{L})) = 0$ for $q = 1, 2$, we get:

(3.29)
$$
\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & 0 \arrow[d] & \boldsymbol{C} \arrow[d, "\vartheta"] & & 0 \arrow[d] & \\
 & H^0(\mathcal{O}_A(T)) \arrow[d, "\xi_0"] & H^0(\mathcal{O}_A(\boldsymbol{L})) \arrow[d] \arrow[dr, "\lambda"] & & H^1(\mathcal{O}_A(T)) \arrow[d, "\xi_1"] & 0 \arrow[d] \\
0 \arrow[r] & H^0(\mathcal{O}_V(T)) \arrow[r] & H^0(\mathcal{O}_V(\boldsymbol{N})) \arrow[r, "\delta"] \arrow[dd] & H^1(\Theta_V) \arrow[r, "\mu"] & H^1(\mathcal{O}_V(T)) \arrow[r] & H^1(\mathcal{O}_V(\boldsymbol{N})) \arrow[dd] \\
 & 0 \arrow[d] \arrow[dr, "\psi_0"] & & & 0 \arrow[d] \arrow[dr, "\psi_1"] & \\
 & 0 & H^1(\mathcal{O}_A) \arrow[d] & & 0 & H^2(\mathcal{O}_A) \arrow[d] \\
 & & 0 & & & 0
\end{tikzcd}
$$

For $\theta = \sum_{\alpha=1}^{3} \theta^\alpha \dfrac{\partial}{\partial w^\alpha} \in H^0(\mathcal{O}_A(T))$, $\psi_0\xi_0\theta = \sum_{\alpha,\beta} h_{\alpha\bar{\beta}}\theta^\alpha d\bar{w}^\beta$ by Lemma (3.28), and so $\psi_0\xi_0$ is an isomorphism (this is a well-known result in abelian varieties). It follows that:

(3.30) $H^0(\mathcal{O}_V(\boldsymbol{N}))/H^0(\mathcal{O}_V(T)) \cong H^0(\mathcal{O}_A(\boldsymbol{L}))/(\vartheta)$ (via λ in (3.29)).

For $\phi = \sum \phi_{\bar{\beta}}{}^\alpha \dfrac{\partial}{\partial w^\alpha} \otimes d\bar{w}^\beta \in H^1(\mathcal{O}_A(T))$, $\psi_1\xi_1\phi = \phi\cdot\omega$ in the diagram (3.29). It follows that

$$\mu H^1(\Theta_V) = \{\phi \in H^1(\mathcal{O}_A(T)) : \phi\cdot\omega = 0\};$$

i. e. $\mu H^1(\Theta_V) \subset H^1(\mathcal{O}_A(T))$ is precisely the space of tangents to the deformations of A which leave the polarization $\boldsymbol{L} \to A$ fixed. Since:

$$H^1(\Theta_V) \cong H^0(\mathcal{O}_V(\boldsymbol{N}))/H^0(\mathcal{O}_V(T)) + \mu H^1(\Theta_V),$$

using (3. 30) and the above description of $\mu H^1(\Theta_V)$, we get Proposition (3. 26).

If $\Omega_1\colon \Delta \to D_1$ and $\Omega_2\colon \Delta \to D_2$ are the period matrix mappings for the 1-forms and 2-forms respectively, then $(\Omega_1)_*$ is non-singular on $T_{Z_0}(\Delta_1)$, $(\Omega_1)_*$ is zero on $T_0(\Delta_2)$ $(\Delta = \Delta_1 \times \Delta_2)$, and so $\dim(\ker \Omega_*) = \dim(\ker(\Omega_2)_*)$ on $T_0(\Delta_2)$; i. e., to get $\ker \Omega_*$, we hold A fixed, let V vary in the complete linear system $|\boldsymbol{L}|$ (cf. (3. 30)), and see how the mapping Ω_2 behaves. In particular, we should examine the cup product (3. 10); by (3. 10) and (3. 30), there is a linear mapping:

$$(3.31) \qquad \psi\colon H^0(\mathcal{O}_A(\boldsymbol{L}))/(\vartheta) \to \operatorname{Hom}(H^0(\Omega_V{}^2), H^0(\Omega_V{}^2(\boldsymbol{N}))/H^0(\Omega_V{}^2(T))),$$

and $\ker \psi \cong \ker(\Omega_2)_*$. To prove all but the statement about "general A and ϑ" in Theorem (3. 25), it will suffice to show:

(3. 32) PROPOSITION. *Let*

$$(3.33) \qquad \eta\colon H^0(\mathcal{O}_A(\boldsymbol{L}))/(\vartheta) \to \operatorname{Hom}(H^0(\mathcal{O}_A(\boldsymbol{L})), \boldsymbol{H}^0(\mathcal{O}_A(\boldsymbol{L}^2))/I_\vartheta)$$

be given by cup product ($\eta(\phi)\xi = \phi\cdot\xi$ for $\phi, \xi \in H^0(\mathcal{O}_A(\boldsymbol{L}))$). *Then* $\ker \eta \cong \ker \psi$ *where* ψ *is given by* (3. 31).

Proof. Dualizing the exact cohomology sequence of

$$0 \to \mathcal{O}_A(\boldsymbol{L}^*) \xrightarrow{\vartheta} \mathcal{O}_A \to \mathcal{O}_V \to 0,$$

we get:

$$(3.34) \qquad \begin{array}{l} 0 \leftarrow H^1(\Omega_A{}^3) \leftarrow H^0(\Omega_V{}^2) \xleftarrow{\ R\ } H^0(\Omega_A{}^3(\boldsymbol{L})) \xleftarrow{\ \vartheta\ } H^0(\Omega_A{}^3) \leftarrow 0 \\ \qquad\qquad\quad \nwarrow^{\omega} \quad\ \ | \\ \qquad\qquad\qquad\quad H^0(\Omega_A{}^2). \end{array}$$

Here $H^0(\Omega_A{}^2) \xrightarrow{\ \omega\ } H^1(\Omega_A{}^3)$ is cup product with $\omega \in H^1(\Omega_A{}^1)$; this is an isomorphism ([29]). Also, if $\phi \in H^0(\Omega_A{}^3(\boldsymbol{L}))$, then $\dfrac{\phi}{\vartheta}$ is a rational 3-form on A with poles on V and $R(\phi)$ is the *residue* of ϕ. These facts will be discussed in Part III. Thus $H^0(\Omega_V{}^2) \cong H^0(\Omega_A{}^2) \oplus R\{H^0(\Omega_A{}^3(\boldsymbol{L}))\}$; the periods of $H^0(\Omega_A{}^2)$ are obviously constant, and so in (3. 31) all that is essential is:

$$\begin{aligned}(3.35)\qquad &\psi\colon H^0(\mathcal{O}_A(\boldsymbol{L}))/(\vartheta)\\ &\qquad \to \operatorname{Hom}(R\{H^0(\Omega_A{}^3(\boldsymbol{L}))\}, H^0(\Omega_V{}^2(\boldsymbol{N})/H^0(\Omega_V{}^2(T))).\end{aligned}$$

Now if we dualize the cohomology diagram arising from

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{O}_A(\boldsymbol{L}^{*2}) \arrow[d, "\vartheta"] & \Omega_A{}^1(\boldsymbol{L}^*) \arrow[d, "\vartheta"] & \\
 & \mathcal{O}_A(\boldsymbol{L}^*) \arrow[d] & \Omega_A{}^1 \arrow[d] & \\
0 \arrow[r] & \mathcal{O}_V(\boldsymbol{N}^*) \arrow[r] \arrow[d] & \Omega_A|_V{}^1 \arrow[r] \arrow[d] & \Omega_V{}^1 \arrow[r] & 0, \\
 & 0 & 0 &
\end{tikzcd}$$

we get:

$$(3.36)\qquad \begin{tikzcd}
 & 0 & H^1(\Omega_A{}^2) & H^0(\Omega_A{}^1) \arrow[l, "\omega"'] \arrow[d] \\
H^1(\Omega_V{}^1) & H^0(\Omega_V{}^2(\boldsymbol{N})) \arrow[l] \arrow[u] & H^0(\Omega_V{}^2(T)) \arrow[l] \arrow[u] & H^0(\Omega_V{}^1) \arrow[l] \\
 & H^0(\Omega_A{}^3(\boldsymbol{L}^2)) \arrow[u] & H^0(\Omega_A{}^2(\boldsymbol{L})) \arrow[l, "d"'] \arrow[u] & \\
 & H^0(\Omega_A{}^3(\boldsymbol{L})) \arrow[u, "\vartheta"] & & \\
 & 0 \arrow[u] & &
\end{tikzcd}$$

In (3.36), $H^0(\Omega_A{}^2(\boldsymbol{L})) \cong$ 2-forms on A with 1-st order pole along V, $H^0(\Omega_A{}^3(\boldsymbol{L}^2)) \cong$ 3-forms on A with 2-nd order poles along V, and d is the exterior derivative. Since ω is an isomorphism, we see from (3.36) that:

$$\begin{aligned}(3.37)\qquad &H^0(\Omega_V{}^2(\boldsymbol{N}))/H^0(\Omega_V{}^2(T))\\ &\qquad \cong H^0(\Omega_A{}^3(\boldsymbol{L}^2))/\vartheta H^0(\Omega_A{}^3(\boldsymbol{L})) + d\{H^0(\Omega_A{}^2(\boldsymbol{L}))\}.\end{aligned}$$

Let $\phi = \dfrac{\xi_1 dw^2dw^3 - \xi_2 dw^1dw^3 + \xi_3 dw^2dw^3}{\vartheta} \in H^0(\Omega^2(\boldsymbol{L}))$. Then

$$d\phi = \left(\frac{\vartheta\partial\xi_1}{\partial w^1} - \frac{\xi_1\partial\vartheta}{\partial w^1} + \frac{\vartheta\partial\xi_2}{\partial w^2} - \frac{\xi_2\partial\vartheta}{\partial w^2} + \frac{\vartheta\partial\xi_3}{\partial w^3} - \frac{\xi_3\partial\vartheta}{\partial w^3}\right)\frac{dw^1dw^2dw^3}{2}.$$

It follows that, under the isomorphism

$$H^0(\Omega_A{}^3(\mathbf{L}^k)) \cong H^0(\mathbf{\mathcal{O}}_A(\mathbf{L}^k)), \quad \vartheta H^0(\Omega_A{}^3(\mathbf{L})) + d\{H^0(\Omega_A{}^2(\mathbf{L}))\} = I_\vartheta.$$

Using this in (3.37), the mapping ψ in (3.35) becomes precisely η in (3.33). In other words, by using suitable isomorphisms, the mapping ψ in (3.31) goes into η in (3.33), and this proves Proposition (3.32).

Now the proof of Proposition (3.32) gives a natural isomorphism:

$$\text{(3.38)} \qquad \ker \Omega_* \cong I_\vartheta : R_n/(\vartheta).$$

If for general ϑ, Z there is $\phi \in I_\vartheta : R_n/(\vartheta) \subset R_n/(\vartheta)$, $\phi \neq 0$; then for special ϑ, Z we would have $\phi \in I_\vartheta : R_n/(\vartheta)$, $\phi \neq 0$.

Specialize Z to $\begin{pmatrix} z^1 & 0 & 0 \\ 0 & z^2 & 0 \\ 0 & 0 & z^3 \end{pmatrix}$, $\operatorname{Im} z^\alpha > 0$, so that $A = C_1 \times C_2 \times C_3$ is a product of elliptic curves. Let 0_α be the origin on C_α and θ_α the 1-st order theta function with a simple zero at 0_α. Let $\vartheta = \theta_1{}^n\theta_2{}^n\theta_3{}^n$ and $\phi \in \operatorname{Hom}(R_n, I_\vartheta)$. We want to show that $\phi = c\vartheta$ is a multiple of ϑ. This would show that, for general Z, ϑ, $\ker \Omega_* = 0$. If $\phi \in I_\vartheta : R_n$, we have:

$$\text{(3.39)} \qquad \phi \cdot \eta \equiv (\theta_1\theta_2\theta_3)^{n-1}\{\xi_1 \frac{\partial \theta_1}{\partial w^1}\theta_2\theta_3 + \xi_2\theta_1 \frac{\partial \theta_2}{\partial w^2}\theta_3 + \xi_3\theta_1\theta_2 \frac{\partial \theta_3}{\partial w^3}\}(\vartheta),$$

for all $\eta \in R_n$. Now ϕ is a sum of terms $\tau_1\tau_2\tau_3$ where τ_α is an n-th order theta function on C_α. Let $\zeta_1\zeta_2\zeta_3$ be a term in the sum for which ζ_1 has the lowest order zero at 0_1. For all $\eta = \eta_1\eta_2\eta_3$, it follows from (3.39) that $\zeta_1\eta_1$ has a zero of order $n-1$ at 0_1. Since $n-1 \geqq 2$, we can find η_1 which doesn't vanish at 0_1 and so ζ_1 has a zero of order $n-1$ at 0_1. But then $\dfrac{\theta_1{}^n}{\zeta_1}$ is an elliptic function with a single pole; i.e. $\zeta_1 = c_1\theta_1{}^n$. Continuing, we find that $\zeta_1\zeta_2\zeta_3 = c\vartheta$ and this gives that $\phi \equiv 0(\vartheta)$; in other words, $I_\vartheta : R_n/(\vartheta) = 0$.

(f) *Periods of 3-forms; cubic threefolds.* We consider non-singular hypersurfaces in $P = P_4$. Let V be one such of degree n. Then all of $H^3(V, \mathbf{C})$ is *primitive* and we let $S_0 = H^{3,0}(V)$, $S_1 = H^{3,0}(V) + H^{2,1}(V)$, $W = H^3(V, \mathbf{C})$, so that the period matrix of V is given by $S_0 \subset S_1 \subset W$.

If $n \leqq 2$, $H^3(V) = 0$ and V is rational. For $n = 3$, $H^{3,0}(V) = 0$ and $\dim H^{2,1}(V) = 5$.

(3.40) THEOREM. *There is a family $\{V_t\}_{t \in \Delta}$ with $V = V_0$ and such that $\rho: T_0(\Delta) \to H^1(\Theta_V)$ is an isomorphism* ($\dim \Delta = 10$). *The differential of the period mapping $\Omega_*: T_0(\Delta) \to D$ is injective.*

Proof. The exact cohomology diagram of (3.18) is now:

(3.41)
$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & \\
 & C \arrow[d] & C \arrow[d] & \\
 & H^0(\mathcal{O}_P(\boldsymbol{H}))^5 \arrow[r] \arrow[d] & H^0(\mathcal{O}_P(\boldsymbol{H}^3)) \arrow[d] \arrow[dr, dashed] & \\
0 \arrow[r] & H^0(\mathcal{O}_V(T)) \arrow[r, "\lambda"] \arrow[d] & H^0(\mathcal{O}_V(\boldsymbol{N})) \arrow[r, "\delta"] \arrow[d] & H^1(\Theta_V) \to 0, \\
 & 0 & 0 &
\end{tikzcd}
$$

wher $\lambda(Q_0, \cdots, Q_4) = \sum_{\alpha=0}^{4} Q_\alpha \dfrac{\partial F}{\partial \xi_\alpha}$, $Q_\alpha(\xi)$ are linear forms, and $F(\xi) = 0$ defines $V \subset P$. Thus we have $\delta\colon H^0(\mathcal{O}_V(\boldsymbol{N}))/H^0(\mathcal{O}_V(T)) \cong H^1(\Theta_V)$ and so we can give the local moduli of V by perturbing the equation of $V \subset P$. In order to apply Proposition (1.20), we should examine the cup product

(3.42) $$H^0(\mathcal{O}_V(\boldsymbol{N}))/H^0(\mathcal{O}_V(T)) \otimes H^1(\Omega_V{}^2) \to H^2(\Omega_V{}^1),$$

and show:

(3.43) PROPOSITION. *The cup product* (3.42) *is non-degenerate in the first factor.*

Proof. From the exact diagram

$$
\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \Omega_P{}^1(\boldsymbol{H}^{-3}) \arrow[d] & & \\
 & & \Omega_P{}^1 \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_V(\boldsymbol{N}^*) \arrow[r] & \Omega_{P|V}{}^1 \arrow[r] \arrow[d] & \Omega_V{}^1 \arrow[r] & 0, \\
 & & 0 & &
\end{tikzcd}
$$

and $H^q(\Omega_P{}^1) = 0 = H^{q+1}(\Omega_P{}^1(\boldsymbol{H}^{-3}))$ $(q = 2, 3)$, we get

$$0 \to H^2(\Omega_V{}^1) \to H^3(\mathcal{O}_V(\boldsymbol{N}^*)) \to 0.$$

Dualizing this and using $\Omega_V{}^3 \cong \mathcal{O}_V(\mathbf{H}^{-2})$ gives:

(3.44) $$0 \to H^0(\mathcal{O}_V(\boldsymbol{H})) \to H^1(\Omega_V{}^2) \to 0.$$

Now $H^0(\mathcal{O}_V(\boldsymbol{H})) \cong H^0(\mathcal{O}_P(\boldsymbol{H}))$ and, if $\phi \in H^0(\mathcal{O}_V(\boldsymbol{H}))$, then $\frac{\phi\omega}{F^2}$ is a rational form on P with a 2-nd order pole along V where

$$\omega = \sum_{\alpha=0}^{4} (-1)^{\alpha} \xi_\alpha (d\xi_0 \cdots \widehat{d\xi_\alpha} \cdots d\xi_4).$$

The mapping (3.44) is given by sending ϕ into $R(\frac{\phi\omega}{F^2})$ where R is the *residue operator* (cf. Part III).

Following the pattern of (d) and (e) above, and using (3.41) and (3.44), the cup product (3.42) becomes:

$$H^0(\mathcal{O}_P(\boldsymbol{H}^3))/\Sigma_1 \otimes H^0(\mathcal{O}_P(\boldsymbol{H})) \to H^0(\mathcal{O}_P(\boldsymbol{H}^4))/\Sigma_4, \tag{3.45}$$

where $\Sigma_1 = \{ \sum_{\alpha=0}^{4} Q_\alpha \frac{\partial F}{\partial \xi_\alpha} \}$ and Σ_4 is to be determined.

The exact cohomology sequences of

$$\begin{array}{c} 0 \\ \downarrow \\ \Omega_P{}^2(\boldsymbol{L}^*) \\ \downarrow \\ \Omega_P{}^2 \\ \downarrow \\ 0 \to \Omega_V{}^1(\boldsymbol{N}^*) \to \Omega_{P|V}{}^2 \to \Omega_V{}^2 \to 0 \quad (\boldsymbol{L} = \boldsymbol{H}^3), \\ \downarrow \\ 0 \end{array}$$

$$\begin{array}{c} 0 \\ \downarrow \\ \Omega_P{}^1(\boldsymbol{L}^{*2}) \\ \downarrow \\ \Omega_P{}^1(\boldsymbol{L}^*) \\ \downarrow \\ 0 \to \mathcal{O}_V(\boldsymbol{N}^{*2}) \to \Omega_{P|V}{}^1(\boldsymbol{N}^*) \to \Omega_V{}^1(\boldsymbol{N}^*) \to 0, \\ \downarrow \\ 0 \end{array}$$

give:

$$(3.46) \qquad 0 \to H^1(\Omega_V^2) \to H^2(\Omega_V^1(\boldsymbol{N}^*)) \to 0,$$

$$(3.47) \qquad \begin{array}{ccccccc} & & & 0 & & 0 & \\ & & & \downarrow & & \downarrow & \\ 0 \to & H^2(\Omega_V^1(\boldsymbol{N}^*)) & \to & H^3(\boldsymbol{\mathcal{O}}_V(\boldsymbol{N}^{*2})) & \to & H^3(\Omega_{P|V}^1(\boldsymbol{N}^*)) & \to 0 \\ & & & \downarrow & & \downarrow & \\ & & & H^4(\boldsymbol{\mathcal{O}}_P(\boldsymbol{N}^{*3})) & & H^4(\Omega_P^1(\boldsymbol{L}^{*2})). & \end{array}$$

Dualizing (3.46) and (3.47) gives

$$(3.48) \qquad \begin{array}{ccccc} & & 0 & & 0 \\ & & \uparrow & & \uparrow \\ 0 \leftarrow H^2(\Omega_V^1) = H^1(\Omega_V^2(\boldsymbol{N})) & \leftarrow & H^0(\Omega_V^3(\boldsymbol{N}^2)) & \longleftarrow & H^3(\Omega_{P|V}^1(\boldsymbol{N}^*))^* \leftarrow 0. \\ & & \uparrow & d & \uparrow \\ & & H^0(\Omega_P^4(\boldsymbol{L}^3)) & \longleftarrow & H^0(\Omega_P^3(\boldsymbol{L}^2)). \end{array}$$

It follows that $H^2(\Omega_V^1) \cong H^0(\Omega_P^4(\boldsymbol{L}^3))/dH^0(\Omega_P^3(\boldsymbol{L}^2))$. Now $H^0(\Omega_P^4(\boldsymbol{L}^3)) \cong H^0(\boldsymbol{\mathcal{O}}_P(\boldsymbol{H}^4))$, and the mapping $H^0(\boldsymbol{\mathcal{O}}_P(\boldsymbol{H}^4)) \to H^2(\Omega_V^1)$ is given by sending ϕ into $R(\frac{\phi\omega}{F^3})$, where R is the residue operator. Also, $H^0(\Omega_P^3(\boldsymbol{L}^2)) \cong$ 3-forms with 2-nd order pole along V, and d is the exterior derivative. Thus, by (3.48), $H^2(\Omega_V^1) \cong H^0(\boldsymbol{\mathcal{O}}_P(\boldsymbol{H}^4))/\boldsymbol{\Sigma}_4$ where:

$$(3.49) \qquad \boldsymbol{\Sigma}_4 = \{Q = \sum_{\alpha=0}^{4} Q_\alpha \frac{\partial F}{\partial \xi_\alpha}, \quad \deg Q_\alpha(\xi) = 2\}.$$

Combining (3.45) and (3.49), to prove Proposition (3.43) we must show:

(3.50) If $Q(\xi)$ is a cubic form such that $QR = \sum_{\alpha=0}^{4} S_\alpha \frac{\partial F}{\partial \xi_\alpha}$ $(\deg S_\alpha = 2)$ for all linear forms R, then $Q = \sum_{\alpha=0}^{4} Q_\alpha \frac{\partial F}{\partial \xi_\alpha}$.

This follows from Theorem (3.22) where $Q_\alpha = \frac{\partial F}{\partial \xi_\alpha}$, $l = 1$, $\rho = 5$. This completes the proof of Proposition (3.43).

Remark. The torus $T_1(V) = W_S/H^3(V, \boldsymbol{Z})$ where $W_S = H^3(V, \boldsymbol{C})/H^{3,0} + H^{2,1}$ (cf. II. 2.(a)) varies holomorphically with V. Furthermore, $T_1(V)$ has a natural polarization (Proposition (2.34)) and, by Theorem (3.40), $T_1(V)$ locally determines V.

(g) *Examples where the period mapping is degenerate.* From Theorem (2.4) in I. 2.(c) it is fairly clear that, if $\{V_t\}_{t\in\Delta}$ is a family of *birationally equivalent* but *biregularly distinct* algebraic surfaces, then the period mapping $\Omega\colon \Delta \to D$ is constant. For instance:

(3.51) *Example.* Let $V \subset P_3$ be a non-singular *cubic surface.* Then $h^{2,0}(V) = 0$, $h^{1,1}(V) = 7$ and V is rational. The biregular moduli arise by perturbing the equation of V; there is a family $\{V_t\}_{t\in\Delta}$ with $V = V_0$, $\rho_0\colon T_0(\Delta) \to H^1(\Theta_V)$ an isomorphism, and with $\dim \Delta = 4$. In this family there are no periods.

Less trivially we have:

(3.52) *Example.* We now give a surface V with the following properties:

(i) V is non-singular and $H^{1,0}(V) = 0 = H^{2,0}(V)$ (thus V has no modular variety);

(ii) V is *non-ruled* and is a *minimal model,* and so biregular moduli give birational moduli; and

(iii) there is a family $\{V_t\}_{t\in\Delta}$ with $\dim \Delta = 6$ and $\rho\colon T_0(\Delta) \to H^1(\Theta_V)$ an injection.

After doing this we shall show that V has *transcendental moduli* given (roughly) by periods of a *Prym differential* on V.

To begin with, let $T\colon P_3 \to P_3$ be the automorphism with matrix

$$\begin{pmatrix} 1 & & & 0 \\ & i & & \\ & & -1 & \\ 0 & & & -i \end{pmatrix}; \text{ thus } T[\xi_0, \xi_1, \xi_2, \xi_3] = [\xi_0, i\xi_1, -\xi_3, -i\xi_4]. \text{ Now } T^4 = I$$

and T has the four fixed points $[1, 0, 0, 0]$, $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, and $[0, 0, 0, 1]$ (this checks the Lefschetz fixed point formula). However $T^2[\xi_0, \xi_1, \xi_2, \xi_3] = [-\xi_0, -\xi_1, \xi_2, \xi_3]$ and so T^2 has the 2 fixed lines $C_1 = [\xi_0\, \xi_1, 0, 0]$ $C_2 = [0, 0, \xi_1, \xi_2]$.

Let W be the standard quartic $\xi_0^4 = \xi_1^4 + \xi_2^4 + \xi_3^4$; $T(W) = W$ and T has on W no fixed points. On W, T^2 has the eight fixed points $C_1 \cdot W + C_2 \cdot W$. To study them we let $\xi_0 \neq 0$, $x = \dfrac{\xi_1}{\xi_0}$, $y = \dfrac{\xi_3}{\xi_0}$, $z = \dfrac{\xi_2}{\xi_0}$ so that $x^4 + y^4 + z^4 = 1$. Then

$$T(x,y,z)=(ix,-iy,-z) \text{ and } T^2(x,y,z)=(-x,-y,z).$$

Thus the four finite fixed points of T^2 on W are given by $x=0$, $y=0$, $z^4=1$. Because of obvious symmetry, it will suffice to examine the single fixed point $P=(0,0,1)$. On a neighborhood of P in W we introduce local coordinates by the parametrization $(u,v)\to(u,v,\sqrt[4]{1-(u^4+v^4)})$. Then $T^2(u,v)=(-u,-v)$ so that, in order to desingularize $W/\{I,T,T^2,T^3\}$, we must remove the eight isolated singular points arising from an identification $(u,v)\sim(-u,-v)$. This is done by a simple dilation.

To be explicit, we parametrize the singular point by $(u,v)\xrightarrow{\psi}(u^2,uv,v^2)$. Then ψ is one-to-one on equivalence classes $(u,v)\sim(-u,-v)$, and so a neighborhood of p is isomorphic to a neighborhood of the origin on the quadric $Q=\{(p,q,r): pr=q^2\}$.

Now we cover P_1 with open sets U_0, U_1 with coordinates ζ in U_0 and $\eta=1/\zeta$ in U_1; and we let $\boldsymbol{H}\to P_1$ be the standard line bundle formed from $(U_0\times\boldsymbol{C})\cup(U_1\times\boldsymbol{C})$ by the identification: $(\zeta,\lambda)\sim(\eta,\phi)$ if, and only if, $\zeta\eta=1$ and $\lambda=\zeta\phi$. Thus $\boldsymbol{L}=\boldsymbol{H}^{-2}$ is formed from $(U_0\times\boldsymbol{C})\cup(U_1\times\boldsymbol{C})$ by the equivalence relation:

$$(\zeta,\lambda)\sim(\eta,\phi) \text{ if, and only if, } \zeta\eta=1,\ \lambda=\zeta^{-2}\phi. \tag{3.53}$$

Using (3.53), we define holomorphic functions f_0, f_1, f_2 on $\boldsymbol{L}$ by:

$$\begin{cases} f_0(\zeta,\lambda)=\lambda & f_0(\eta,\phi)=\eta^2\phi \\ f_1(\zeta,\lambda)=\lambda\zeta & f_1(\eta,\phi)=\eta\phi \\ f_2(\zeta,\lambda)=\lambda\zeta^2 & f_2(\eta,\phi)=\phi. \end{cases}$$

Then $f=(f_0,f_1,f_2)$ gives a mapping $f\colon\boldsymbol{L}\to Q$ which is biholomorphic outside zero and with f (zero section) $=(0,0,0)$. Using f we may replace a neighborhood of the singular point on Q by a tubular neighborhood of the zero section in $\boldsymbol{L}$ and, in this way, uniformize the singularity $(u,v)\sim(-u,-v)$.

We let V be the surface obtained from $Z=W/\{I,T,T^2,T^3\}$ by removing the singular points as above. Since $H^{1,0}(W)=0$, $H^{1,0}(V)=0$. We assert that $H^{2,0}(V)=0$. To see this, we observe that, on W, there is a non-vanishing regular 2-form $\omega=\dfrac{dxdy}{z^3}$. Since $T\omega=\dfrac{(idx)(-idy)}{(-z)^3}=-\omega$, it follows that there is no holomorphic 2-form on Z or V.

We claim that, on V, $\boldsymbol{K}^2 = \boldsymbol{1}$ where $\boldsymbol{K}$ is the *canonical bundle*. Since $T(\omega^2) = \omega^2$, it will suffice to show that ω^2 is non-singular along the exceptional curves which have replaced the singular points on Z. This is straightforward to verify.

Let now E be the vector space generated by the monomials $\mu = \xi_0{}^{\alpha_0}\xi_1{}^{\alpha_1}\xi_2{}^{\alpha_2}\xi_3{}^{\alpha_3}$ with:

$$\begin{cases} \alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 4 \\ \alpha_1 + 2\alpha_2 + 3\alpha_3 \equiv 0 \pmod 4). \end{cases} \tag{3.54}$$

These are the monomials with $T\mu = \mu$, and there are 10 solutions to (3.54). The number of *effective* parameters is 6, since the commutator group of T is all diagonal matrices and has dimension 4. By perturbing the equation of W with elements close to zero in E, and by desingularizing the factor surfaces Z as above, we construct a family $\{V_t\}_{t \in \Delta}$ ($\dim \Delta = 6$) of surfaces for which (i) and (iii) above are satisfied.

Since $\boldsymbol{K} \neq 1$, $\boldsymbol{K}^2 = 1$, V is not ruled. Furthermore, if $C \subset V$ is an exceptional curve, the genus $p(C)$ is given by: $p(C) = \frac{1}{2}(C^2 + CK) - 1 = \frac{1}{2}(C^2) - 1$, so that C is not of the first kind. Hence V is a *minimal model* and this proves (ii).

Remark. It has been pointed out to me by F. Gherardelli that it is possible to give the moduli in the above example by periods of "generalized" integrals, and we now give this construction.

Quite generally, over an algebraic manifold V, we consider a representation $\rho\colon \pi_1(V) \to \boldsymbol{C}^*$ and let $\boldsymbol{L} = \tilde{V} \times_{\pi_1(V)} \boldsymbol{C}$ be the associated line bundle, $\tilde{V}$ being the universal covering of V. We may speak of the sheaf $\Phi(\boldsymbol{L})$ of *locally constant* sections of $\boldsymbol{L}$, as well as the space $A^q(\boldsymbol{L})$ of C^∞ q-forms with values in $\boldsymbol{L}$. Since $\boldsymbol{L}$ has constant transition functions, the exterior derivative d operates on $A^q(\boldsymbol{L})$ and, if $H_d{}^q(\boldsymbol{L})$ are the resulting cohomology groups, we have *de Rham's theorem*:

$$H^q(\Phi(\boldsymbol{L})) \cong H_d{}^q(\boldsymbol{L}). \tag{3.55}$$

On the other hand, we may consider the sheaf $\Omega^p(\boldsymbol{L})$ of holomorphic p-forms with values in $\boldsymbol{L}$, and also the space $A^{p,q}(\boldsymbol{L})$ of C^∞ (p,q) forms with values in $\boldsymbol{L}$. The operator $\bar{\partial}$ maps $\bar{\partial}\colon A^{p,q}(\boldsymbol{L}) \to A^{p,q+1}(\boldsymbol{L})$ and, if $H_{\bar{\partial}}{}^{p,q}(\boldsymbol{L})$ are the resulting cohomology groups, we have *Dolbeault's theorem*:

$$H^q(\Omega^p(\boldsymbol{L})) \cong H_{\bar\partial}^{p,q}(\boldsymbol{L}). \tag{3.56}$$

A relation between (3.55) and (3.56) arises when we have a Kähler metric on V, as well as a locally constant metric in $\boldsymbol{L}$. Then the theory of harmonic forms on Kähler manifolds carries over verbatim (cf. [29]). Thus we have

$$H_d{}^q(\boldsymbol{L}) \cong \sum_{r+s=q} H_{\bar\partial}^{r,s}(\boldsymbol{L})\,; \tag{3.57}$$

and also the whole theory of *primitive cohomology classes*, etc. (cf. I.1.(c)) goes through in this case.

If, furthermore, ρ has integral values, then we may consider the sheaf $\psi(\boldsymbol{L})$ of integral sections of $\boldsymbol{L}$, and $H^q(\psi(\boldsymbol{L}))$ is a (complex) lattice in $H^q(\Phi(\boldsymbol{L}))$. In case the Kähler metric is a Hodge metric, then the primitive cohomology space $H^q(\Phi(\boldsymbol{L}))_0$ is defined rationally.

The special case we are interested in is $q=2$; then

$$H^2(\Phi(\boldsymbol{L}))_0 = H^{2,0}(\boldsymbol{L}) \oplus H^{0,2}(\boldsymbol{L}) \oplus H^{1,1}(\boldsymbol{L})_0; \tag{3.58}$$

and so the complex structures define a subspace $H^{2,0}(\boldsymbol{L}) \subset H^2(\Phi(\boldsymbol{L}))_0$—that is, a point in $G(h,W)$, the Grassmann variety of h-planes ($h=\dim H^{2,0}(\boldsymbol{L})$) in $W=H^2(\Phi(\boldsymbol{L}))_0$. Suppose that V is a surface and that $\boldsymbol{L}\cong\mathbf{L}^*$; this is the case of the surface above where $\boldsymbol{L}=\boldsymbol{K}$ is the canonical bundle. (There $\pi_1(V)=\boldsymbol{Z}_2$, $\boldsymbol{K}$ is associated to an integral representation of $\pi_1(V)$, and $\boldsymbol{K}=\boldsymbol{K}^*$ since $\boldsymbol{K}^2=\boldsymbol{1}$.) Then the cup product $H^2(\Phi(\boldsymbol{L}))_0\otimes H^2(\Phi(\boldsymbol{L}))_0 \to H^4(V,\boldsymbol{C})$ defines a rational, non-degenerate bilinear form Q on $H^2(\Phi(\boldsymbol{L}))_0$. This allows us to define, just as in I.1.(c), the period matrix space $D_2(\boldsymbol{L})$, and the general structure theory goes through. In this way we may now speak of the "generalized" periods, which are associated to the polarized surface V and the representation ρ of $\pi_1(V)$.

If now $\{V_t\}_{t\in\Delta}$ is the local moduli space (assumed complete), then there is defined the *period mapping* $\Omega\colon \Delta\to D_2(\boldsymbol{L})$, which is holomorphic, and we may ask when Ω gives local coordinates in Δ.

Assuming $H^1(\Phi(\boldsymbol{L}))=0$, this will be the case if the cup product:

$$H^0(\boldsymbol{K}\cdot\boldsymbol{L}) \otimes H^1(\Omega^1(\boldsymbol{L})) \to H^1(\Omega^1(\boldsymbol{K})) \tag{3.59}$$

is onto. What we claim is that, if V is the surface above and $\boldsymbol{L}=\boldsymbol{K}$ is the canonical bundle, then the cup product in (3.59) is an isomorphism. In this case,

$$H^1(\Phi(\boldsymbol{L})) = H^{1,0}(\boldsymbol{K}) + H^{0,1}(\boldsymbol{K}) \cong H^0(\Omega^1(\boldsymbol{K}^*)) + H^{2,1} \cong H^0(\Theta_V) = 0.$$

In fact,

$$H^{2,0}(\boldsymbol{K}) \otimes H^{1,1}(\boldsymbol{K}) \cong H^{2,0}(\boldsymbol{K}^*) \otimes H^{1,1}(\boldsymbol{K}) \cong H^1(\Omega^1(\boldsymbol{K})),$$

so that the generalized periods do, in fact, give local moduli in this case.

We close with the following remarks concerning V:

(1) $\dim H^2(\Theta_V) = \dim H^0(\Omega^1(\boldsymbol{K})) = \dim H^0(\Omega^1(\boldsymbol{K}^*)) = \dim H^0(\Theta_V) = 0$ and so, by the Riemann-Roch theorem, $\dim H^1(\Theta_V) = 10 =$ number of moduli of V.

(2) From the classical theory of surfaces, it is known that V is biregularly equivalent to an *Enriques surface*; i.e. a surface in P_3 with the equation

$$x^2y^2z^2 + w^2x^2y^2 + w^2y^2z^2 + w^2z^2x^2 = wxyz\ q(x,y,z,w)$$

where $q(x,y,z,w)$ is a general quadratic form. The moduli of V are obtained by perturbing q, and there are the correct number (10) of parameters ([15]).

REFERENCES.

[1] Y. Akizuki and S. Nakano, "Note on Kodaira-Spencer's proof of Lefschetz theorem," *Proceedings of Japan Academy*, vol. 30 (1954), pp. 266-272.

[2] A. Blanchard, "Sur les variétés analytiques complexes," *Ann. Sci. École Norm. Sup.*, Paris, vol. 73 (1956), pp. 157-202.

[3] H. Cartan, "Familles d'espaces complexes," E. N. S., Paris (1960-1961).

[4] F. Conforto, "Abelsche funktionen und algebraische geometrie," Springer (1956), Berlin.

[5] F. Enriques and O. Chisini, "Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche," Bologna (1915-1924).

[6] H. Grauert, "On the number of moduli of complex structures," Cont. to function theory, Tata Institute of Fundamental Research (1960).

[7] P. Griffiths, "Periods of integrals on algebraic manifolds, I (Construction and properties of the modular varieties)," *American Journal of Mathematics*, vol. 90 (1968), pp. 568-626.

[8] ———, "The extension problem in complex analysis, II; embeddings with positive normal bundle," *American Journal of Mathematics*, vol. 88 (1966), pp. 366-446.

[9] F. Hirzebruch, "Neue topologische methoden in der algebraischen geometrie," *Ergebnisse der Mathematik*, vol. 9 (1956), Springer.

[10] W. V. D. Hodge, "The theory and applications of harmonic integrals," Cambridge University Press (1959).

[11] K. Kodaira, "On a differential-geometric method in the theory of analytic stacks," *Proceedings of the National Academy of Sciences, U. S. A.*, vol. 39 (1953), pp. 1268-1273.

[12] ———, "On Kähler varieties of restricted type," *Annals of Mathematics*, vol. 60 (1954), pp. 28-48.

[13] ———, "Characteristic linear systems of complete continuous systems," *American Journal of Mathematics*, vol. 78 (1956), pp. 716-744.

[14] ———, "On compact complex analytic surfaces, I," *Annals of Mathematics*, vol. 71 (1960), pp. 111-152.

[15] ———, "On characteristic systems of families of surfaces with ordinary singularities in projective space," *American Journal of Mathematics*, vol. 87 (1965), pp. 227-256.

[16] ———, "A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds," *Annals of Mathematics*, vol. 75 (1962), pp. 146-162.

[17] ———, "On compact analytic surfaces, III," *Annals of Mathematics*, vol. 78 (1963), pp. 1-40.

[18] K. Kodaira and D. C. Spencer, "On deformations of complex analytic structures, I-II," *Annals of Mathematics*, vol. 67 (1958), pp. 328-466.

[19] ———, "On deformations of complex analytic structures, III," *Annals of Mathematics*, vol. 71 (1960), pp. 43-76.

[20] K. Kodaira, D. C. Spencer and L. Nirenberg, "On the existence of deformations of complex analytic structures," *Annals of Mathematics*, vol. 68 (1958), pp. 450-459.

[21] K. Kodaira and D. C. Spencer, "A theorem of completeness for complex analytic fibre spaces," *Acta Mathematica*, vol. 100 (1958), pp. 281-294.

[22] ———, "A theorem of completeness of characteristic systems of complete continuous systems," *American Journal of Mathematics*, vol. 81 (1959), pp. 477-500.

[23] B. O. Koopman and A. B. Brown, "On the covering of analytic loci by complexes," *Transactions of the American Mathematical Society*, vol. 34 (1932), pp. 231-251.

[24] M. Kuranishi, "On the locally complete families of complex analytic structures," *Annals of Mathematics*, vol. 75 (1962), pp. 536-577.

[25] ———, "New proof for the existence of locally complete families of complex structures," Proceedings of Conference on Complex Analysis, Minneapolis; Springer (1965), pp. 142-154.

[26] F. S. Macauley, "Algebraic theory of modular systems," Cambridge tract 19 (1916).

[27] H. E. Rauch, "On the modulii of Riemann surfaces," *Proceedings of the National Academy of Sciences, U. S. A.*, vol. 41 (1955), pp. 236-238.

[28] A. Weil, "On Picard varieties," *American Journal of Mathematics*, vol. 74 (1962), pp. 865-894.

[29] ———, "Varietes Kahleriennes," Hermann 6 (1958), Paris.