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**Periods of integrals on algebraic manifolds, III (Some global differential-geometric properties of the period mapping)**

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# PERIODS OF INTEGRALS ON ALGEBRAIC MANIFOLDS, III (SOME GLOBAL DIFFERENTIAL-GEOMETRIC PROPERTIES OF THE PERIOD MAPPING)

*by* PHILLIP A. GRIFFITHS (1)

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## 0. Introduction.

*a)* In this paper we shall study some global properties of the periods of integrals in an algebraic family of algebraic varieties. Although our results are mostly in (algebraic) geometry, the proofs are purely transcendental. In fact, we may roughly describe our methods as giving various applications of the *maximum principle* to problems in algebraic geometry. For the most part these methods have only succeeded in treating the situation when the parameter space for the non-singular varieties is complete. While the results should be true in general, it appears that new methods will be required to handle the situation when singular varieties are permitted in our family. These questions are discussed from time to time as they occur in the text below.

The paper divides naturally into two parts. The first treats linear problems and is a study of the differential-geometric properties of the *Hodge bundles* as defined in § 2. The use of the maximum principle here is similar to the classical Bochner method [3], and is based on the rather remarkable *structure equations* and *curvature properties* of the Hodge bundles. The second part deals with global properties of the *period mapping* [11], and the methods are those of *hyperbolic complex analysis* which, to paraphrase Chern [7], is the philosophy that suitable curvature conditions on complex manifolds impose strong restrictions on holomorphic mappings between these manifolds.

A more detailed introduction to the two parts of the paper will now be given.

*b)* We consider an algebraic family of algebraic varieties $\{V_s\}_{s\in S}$ as defined in § 1. For the time being we may think of the parameter space S as being a (generally non-compact) algebraic curve. The algebraic varieties $V_{\bar{s}}$ corresponding to the points $\bar{s}$ at infinity in S may be thought of as the specializations of a generic $V_s$ having acquired singularities.

If we replace $V_s$ by the cohomology groups $H^n(V_s, \mathbf{C})$ and the various subspaces $H^{p,q}(V_s)\subset H^n(V_s, \mathbf{C})$ $(p+q=n)$, then we find that the algebraic family $\{V_s\}_{s\in S}$ gives rise to a whole collection of holomorphic vector bundles over S. We can abstract the data of theses bundles and arrive at what we call a *variation of Hodge structure* (§ 2).

Now the bundles which turn up in a variation of Hodge structure have intrinsic Hermitian differential geometries (§ 4), and we use more or less standard methods in differential geometry to deduce results about the variation of Hodge structure which, in case this variation of Hodge structure arises from a family of algebraic varieties, have interpretations as theorems on invariant cycles and on holomorphic cross-sections of families of intermediate Jacobians.

The only real twist is that the Hermitian vector bundles which appear generally have *indefinite* Hermitian metrics. In such a situation, the maximum principle does not usually apply. We are only able to push things through by using the so-called *infinitesimal period relation* [11] satisfied by periods of integrals and which is incorporated into the definition of variation of Hodge structure. It is perhaps worth pointing out that the maximum principle is used to show that certain differential equations are satisfied, rather than to show that a " harmonic tensor " is zero as was the classical case [3].

In § 4 we give a review of Hermitian differential geometry and, in particular, discuss the second fundamental form of a holomorphic vector bundle embedded in an Hermitian vector bundle. The main differential geometric results on variation of Hodge structure (Theorems (5.2) and (5.9)) are stated and discussed in § 5. The proofs of these theorems are given in § 6 where we derive the structure equations for a variation of Hodge structure. This section is the heart of Part I of the paper, and we have used the Cartan method of *moving frames* ([6], [8]) to expose the structure equations (6.4)-(6.8), (6.12), and (6.18) of a variation of Hodge structure. These equations are to me quite remarkable and are much richer than one might have thought from just the classical case when the $V_s$ are curves. For example, if $\{V_s\}_{s\in S}$ forms an algebraic family of algebraic surfaces with complete parameter space S and if $\gamma_s \in H_2(V_s, \mathbf{Z})$ is an invariant 2-cycle, then there is a non-negative function $\psi(s)$ whose vanishing at $s\in S$ is necessary and sufficient that $\gamma_s$ be the homology class of an algebraic curve on $V_s$. It was a pleasant surprise to me that $\psi$ turns out to be pluri-subharmonic on S. Even in case S is not complete, $\psi$ should be bounded, but this depends on the local invariant cycle conjecture (3.3).

In § 7 we give three applications of the results in section 5. The first of these are some *rigidity properties* of variation of Hodge structures with complete base space (Corollary (7.3) and (7.4)). These particular results were motivated by a question of Grothendieck [17] and have appeared previously in the preprint [12] with the same proof as given here. In this paper the rigidity theorems are given as consequences of Theorem (7.1), which was also in § 8 of [12] but was poorly stated there. The much better formulation given below is due to Deligne, whose paper [9] has several points of contact with this one, which are discussed in § 3 below. The second application is the positivity of certain bundles arising from a variation of Hodge structure (Propositions (7.7) and (7.15)). The third application is a Mordell-Weil type of theorem for cross-sections of families of intermediate Jacobians (Theorem (7.19)). Again this is a result purely about variation of Hodge structure but which is suggested by algebraic geometry. In this case the motivation comes from the study of intermediate cycles on algebraic varieties and the connection with Theorem (7.19) is explained in Appendix A.

*c)* Associated to any variation of Hodge structure, with parameter space S, there is a *period matrix domain* D [11], which is a *homogeneous complex manifold* $D=G/H$ of a non-compact simple Lie group G divided by a compact subgroup H, together with a denumerable subgroup $\Gamma$ of G and a holomorphic *period mapping* [11]:

$$\Phi : S \rightarrow \Gamma\backslash D. \tag{0.1}$$

In fact, the giving of a variation of Hodge structure over S is equivalent to giving a period mapping (0.1) satisfying an infinitesimal period relation which can be stated purely in terms of D. These period matrix domains are discussed in § 8, and the correspondence between variations of Hodge structure and period mappings is given by Proposition (9.3). In many interesting cases, such as when the variation of Hodge structure arises from an algebraic family of algebraic varieties, the *monodromy group* $\Gamma$ is a discrete subgroup of G and consequently $\Gamma\backslash D$ is a complex analytic variety. The point of view we have taken in Part II is to apply hyperbolic complex analysis to study the period mapping (0.1).

We are especially interested in the asymptotic behavior of the period mapping $\Phi$ as we go to infinity in S. In case $\dim_C S=1$, a neighborhood of S at infinity is a punctured disc $\Delta^*$, and the period mapping (0.1) may be *localized at infinity* and lifted to the universal covering of $\Delta^*$ to yield a holomorphic mapping:

$$\Phi : H \to D$$

from the upper half-plane $H=\{z=x+iy : y>0\}$ to the period matrix domain D, and which satisfies the equivariance condition:

$$\Phi(z+1)=T\cdot\Phi(z)$$

where $T\in\Gamma$ is the *Picard-Lefschetz transformation* associated to the local monodromy around the origin in the punctured disc $\Delta^*$. In case $\dim_D S>1$, we can use Hironaka's *resolution of singularities* to have a similar localization at infinity given by a holomorphic mapping:

**(0.2)**
$$\Phi : \underbrace{H\times\ldots\times H}_{d} \to D$$

which satisfies:

$$\Phi(z_1, \ldots, z_j+1, \ldots, z_d)=T_j\cdot\Phi(z_1, \ldots, z_d)$$

where the $T_j\in G$ are commuting automorphisms of D.

To use metric methods for the study of the mapping (0.2), we introduce the standard *Poincaré metric* $ds^2_H$ on $H\times\ldots\times H$ and the G-invariant metric $ds^2_D$ on D deduced from the Cartan-Killing form on the Lie algebra of G. Now the metric $ds^2_D$ does not have the (negative) curvature properties necessary to make hyperbolic complex analysis work on an arbitrary holomorphic mapping (0.2). However, if we use the infinitesimal period relation, then the necessary curvature conditions will be satisfied *relative to the mapping* $\Phi$. Using this together with a formula of Chern [7], in § 10 we prove a generalized *Schwarz lemma* (Theorem (10.1)) which says that the period mapping $\Phi$ is both *distance and volume decreasing* with respect to $ds^2_H$ and $ds^2_D$, cf. [16].

The main geometric applications of the Schwarz lemma are Theorems (9.5) and (9.6), both of whose proofs are given in § 11. The first of these is a sort of *Riemann extension theorem*, and says that a period mapping $\Phi : \Delta^* \to D$ from the punctured disc to a period matrix domain D extends holomorphically across the origin. Our proof makes essential use of an ingenious argument from [25] (Proposition (11.1)). The second result is that the period mapping (0.2) is (essentially) a proper mapping, and consequently the closure of the image of $\Phi$ is an analytic set containing $\Phi(S)$ as the complement of an analytic subvariety.

A third geometric theorem is Theorem (9.7), which says that the image $\Phi(S)$ is *canonically a projective algebraic variety* in case S is complete. Our final result (Theorem (9.8)) in this section is a theorem about the global monodromy group $\Gamma$ of a variation of Hodge structure with complete parameter space. The statements that $\Gamma$ is completely reducible, and that $\Gamma$ is finite if it is solvable, are simply adaptations of similar results of Deligne [9] in the geometric case, which are discussed in § 3 below. The characterization of the case when $\Gamma$ is a finite group was given in [12].

*d)* As mentioned above, Appendix A contains a result about algebraic cycles and intermediate Jacobians varying in an algebraic family of algebraic varieties. In Appendix B we give some examples. In Appendix C we discuss some conjectures which should be true but which we are unable to prove. Finally, in Appendix D we give an application of the results in § 9 to the global monodromy group of certain (algebraic) K3 surfaces.

*e)* This paper is a successor to [11]. However, our point of view has evolved somewhat and perhaps a more appropriate general reference is the survey article [13], which in particular discusses most of the results in this paper and takes up many related problems and conjectures. Finally, this paper is essentially self-contained, except for § 10 where we use a formula from [7] and a result from [16] about the curvature of the metric $ds_D^2$ discussed above.

It is my pleasure to thank the referee for many helpful suggestions and comments.

PART I

# DIFFERENTIAL-GEOMETRIC PROPERTIES OF VARIATION OF HODGE STRUCTURE

## 1. Algebraic families of algebraic varieties.

By an *algebraic family of algebraic varieties* we shall mean that we are given connected and smooth algebraic varieties X, S and a morphism $f: X \to S$ such that

(i) $f$ is smooth, proper, and connected, and

(ii) There is a distinguished projective embedding $X \subset \mathbf{P}_N$.

Setting $V_s = f^{-1}(s)$ $(s \in S)$ we may think of $f: X \to S$ as the algebraic family $\{V_s\}_{s \in S}$ of smooth, complete, connected, and projective algebraic manifolds parametrized by S.

The parameter space S is generally not complete, and we shall want to consider *smooth compactifications* of the situation $f: X \to S$. Such a smooth compactification is given by a diagram:

$$\begin{array}{ccc} X & \subset & \overline{X} \\ f\downarrow & & \downarrow \bar{f} \\ S & \subset & \overline{S} \end{array} \tag{1.1}$$

where $\overline{X}$, $\overline{S}$ are smooth, complete, and projective algebraic varieties which contain X, S respectively as Zariski open sets, and where $\overline{X}-X$ and $\overline{S}-S$ are each *divisors with normal crossings*. Thus, for example, $\overline{S}-S$ is locally given by:

$$s_1 \dots s_k = 0 \tag{1.2}$$

where $s_1, \dots, s_k$ are part of a local holomorphic coordinate system on $\overline{S}$. The divisors $D_j$ given locally by $s_j = 0$ in (1.2) will be called the *irreducible branches of* $\overline{S}-S$. We then have $S = \overline{S} - D$ where $D = D_1 + \dots + D_m$ is the divisor with normal crossings. As another example, if $\dim S = 1$ and if $\overline{S}-S$ is locally given by $s = 0$, then $\bar{f}: \overline{X} \to \overline{S}$ will be given locally by:

$$x_1^{a_1} \dots x_l^{a_l} = s \tag{1.3}$$

where $x_1, \dots, x_l$ is part of a local holomorphic coordinate system on $\overline{X}$.

Such smooth compactifications exist by the fundamental work of Hironaka [20].

We want now to say what it means to *localize the situation* (1.1) *at infinity*. Let

$\overline{S}-S$ be given locally by (1.2) where $s_1, \ldots, s_d$ is a holomorphic coordinate system on $\overline{S}$. Denote by P the open polycylinder given by $0 \leq |s_j| < \varepsilon$ ($j=1, \ldots, d$) and let $P^* = P \cap S$. Thus letting $\Delta$ be a disc in $\mathbf{C}$ and $\Delta^*$ the corresponding punctured disc, we have $P \cong (\Delta)^d$ and $P^* \cong (\Delta^*)^k \times (\Delta)^{d-k}$. Set $\overline{Y} = f^{-1}(P)$ and $Y = \overline{Y} \cap X$. Then the localization of (1.1) at infinity is given by:

$$\begin{array}{ccc} Y & \subset & \overline{Y} \\ \downarrow & & \downarrow \\ P^* & \subset & P \end{array} \tag{1.4}$$

We will generally refer to $P^*$ as a *punctured polycylinder*.

## 2. Variation of Hodge structure.

We shall linearize the situation (1.1). For this we now consider X, S as complex manifolds and $f: X \to S$ as an analytic fibre space and topological fibre bundle. Fix a base point $s_0 \in S$ and consider the action of the fundamental group $\pi_1(S)$ of S based at $s_0$ on the cohomology $H^n(V_{s_0}, \mathbf{C})$. If $L \in H^2(V_{s_0}, \mathbf{Q})$ is the cohomology class of the hyperplane section relative to the given projective embedding $X \subset \mathbf{P}_N$, then L is invariant under $\pi_1(S)$. Thus for $n \leq m = \dim_{\mathbf{C}} V$ we may define the *primitive cohomology* $P^n(V_{s_0}, \mathbf{C})$ to be the kernel of:

$$L^{r+1}: H^{m-r}(V_{s_0}, \mathbf{C}) \to H^{m+r+2}(V_{s_0}, \mathbf{C}) \qquad (n = m-r).$$

Because of the *Lefschetz decomposition* [22]:

$$H^n(V_{s_0}, \mathbf{C}) = \bigoplus_{k=0}^{[n/2]} L^k P^{n-2k}(V_{s_0}, \mathbf{C}), \tag{2.1}$$

which is a $\pi_1(S)$-invariant direct sum (over $\mathbf{Q}$) decomposition of $H^n(V_{s_0}, \mathbf{C})$, it will suffice to consider the primitive cohomology.

Let $E = P^n(V_{s_0}, \mathbf{C})$ and denote by $\mathbf{E} \to S$ the complex vector bundle, with constant transition functions, associated to the action of $\pi_1(S)$ on E. There is the usual *flat, holomorphic connection*:

$$\mathbf{D}: \mathcal{O}_S(\mathbf{E}) \to \Omega^1_S(\mathbf{E})$$

which one has on any such vector bundle associated to a representation of the fundamental group. In fact we have a short exact sheaf sequence:

$$0 \to \mathscr{C}(\mathbf{E}) \to \mathcal{O}_S(\mathbf{E}) \xrightarrow{\mathbf{D}} \Omega^1_S(\mathbf{E}) \tag{2.2}$$

where the *sheaf* $\mathscr{C}(\mathbf{E})$ *of locally constant sections* of $\mathbf{E}$ has the following interpretation:

Let $R^n_{f_*}(\mathbf{C})$ be the usual *Leray cohomology sheaf* of $f: X \to S$, which we recall is the sheaf arising from the presheaf:

$$U \to H^n(f^{-1}(U), \mathbf{C})$$

where U runs through the family of all open sets in S, and define the *Leray primitive cohomology sheaf* $P^n_{f_*}(\mathbf{C})$ to be the kernel of:

$$L^r : R^{m-r}_{f_*}(\mathbf{C}) \to R^{m+r+2}_{f_*}(\mathbf{C}) \qquad (n=m-r).$$

Then $\mathscr{C}(\mathbf{E})$ is just $P^n_{f_*}(\mathbf{C})$.

Now the fibre $\mathbf{E}_s$ is the vector space $P^n(V_s, \mathbf{C})$ and as such has the structure of the primitive cohomology vector space of a Kähler manifold [30]. Translating this structure into data on the flat bundle $\mathbf{E} \to S$, what we find is the following [13]:

I) A flat conjugation $e \mapsto \bar{e}$ $(e \in \mathbf{E})$.

II) A flat, non-degenerate bilinear form

$$Q : \mathbf{E} \otimes \mathbf{E} \to \mathbf{C}, \qquad Q(e, e') = (-1)^n Q(e', e) \tag{2.3}$$

called the *Hodge bilinear form*; and

III) A filtration of $\mathbf{E}$ by holomorphic sub-bundles

$$\mathbf{F}^0 \subset \mathbf{F}^1 \subset \ldots \subset \mathbf{F}^{n-1} \subset \mathbf{F}^n = \mathbf{E} \tag{2.4}$$

called the *Hodge filtration*.

*Remarks*. — (i) The conjugation on $\mathbf{E}$ is induced from the usual conjugation on $H^n(V_s, \mathbf{C}) = H^n(V_s, \mathbf{R}) \otimes \mathbf{C}$.

(ii) The bilinear form (2.3) is given by:

$$Q(e, e') = \pm \int_{V_s} L^{m-n} e e' \qquad (m = \dim_{\mathbf{C}} V_s) \tag{2.3$'$}$$

where $e, e' \in P^n(V_s, \mathbf{C}) \subset H^n(V_s, \mathbf{C})$.

(iii) Letting $P^{n-q,q}(V_s) = H^{n-q,q}(V_s) \cap P^n(V_s, \mathbf{C})$, we have for the fibre $\mathbf{F}^q_s$ that:

$$\mathbf{F}^q_s = P^{n,0}(V_s) + \ldots + P^{n-q,q}(V_s). \tag{2.4$'$}$$

We will denote the data of a flat bundle $\mathbf{E}$ with I)-III) above by $\mathscr{E} = (\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$, and as conditions on this data we have:

IV) The Hodge filtration (2.4) is *isotropic*, which means that:

$$(\mathbf{F}^q)^\perp = \mathbf{F}^{n-q-1} \tag{2.5}$$

where $(\mathbf{F}^q)^\perp = \{e \in \mathbf{E} : Q(e, \mathbf{F}^q) = 0\}$.

V) The bilinear form (2.3) is real (i.e. $Q = \bar{Q}$) and, if we let

$$\mathbf{F}^{n-q,q} = \mathbf{F}^q \cap \bar{\mathbf{F}}^{n-q} = \mathbf{F}^q \cap (\bar{\mathbf{F}}^{q-1})^\perp,$$

then we have the *Hodge decomposition*, which is a $C^\infty$ direct sum decomposition:

$$\mathbf{E} = \bigoplus_{q=0}^{n} \mathbf{F}^{n-q,q} \qquad (\bar{\mathbf{F}}^{n-q,q} = \mathbf{F}^{q,n-q}). \tag{2.6}$$

VI) *The Riemann-Hodge bilinear relations*

$$(2.7)\qquad \begin{cases} Q(\mathbf{F}^{n-q,q}, \overline{\mathbf{F}}^{n-r,r})=0 & (q\neq r) \\ (-i)^n(-1)^q Q(\mathbf{F}^{n-q,q}, \overline{\mathbf{F}}^{n-q,q})>0 \end{cases}$$

are valid; and

VII) The *infinitesimal period relation* [11]

$$(2.8)\qquad \mathbf{D}:\mathcal{O}_S(\mathbf{F}^q)\to\Omega^1_S(\mathbf{F}^{q+1})$$

holds.

*Definition.* — We will call the data $\mathcal{E}=(\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$ given by I)-III) and satisfying the conditions IV)-VII) a *variation of Hodge structure.*

It is of course not necessary that a variation of Hodge structure come from an algebraic family of algebraic varieties $f:X\to S$. In case $\mathcal{E}=(\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$ does arise from $f:X\to S$, we will say that *the variation of Hodge structure arises from a geometric situation.*

*Remarks* **(2.9)**. — (i) Let $\mathcal{E}=(\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$ be a variation of Hodge structure. Referring to (2.5), we have natural isomorphisms:

$$(2.10)\qquad (\mathbf{F}^q/\mathbf{F}^{q-1})^\vee\cong\mathbf{F}^{n-q}/\mathbf{F}^{n-q-1},$$

which are isomorphisms of holomorphic vector bundles.

(ii) We may symbolically rewrite (2.8) as

$$(2.11)\qquad Q(\mathbf{D}\cdot\mathbf{F}^q, \mathbf{F}^{n-q-2})=0.$$

(iii) Referring again to the infinitesimal period relation (2.8), we see that the connection $\mathbf{D}$ induces a linear bundle mapping of holomorphic bundles

$$(2.12)\qquad \sigma_q:\mathbf{E}^q\to\mathbf{E}^{q+1}\otimes\check{\mathbf{T}}$$

where $\mathbf{E}^q=\mathbf{F}^q/\mathbf{F}^{q-1}$. The vector bundles $\mathbf{E}^q$ will be called the *Hodge bundles*, a terminology which we shall now try to justify. In case the variation of Hodge structure $\mathcal{E}$ arises from a geometric situation $f:X\to S$, the fibre $\mathbf{E}^q_s$ is given by:

$$(2.13)\qquad \mathbf{E}^q_s=H^q(V_s, \Omega^{n-q}_{V_s})_0\qquad (s\in S;\ q=0, 1, \ldots, n),$$

where $H^q(V_s, \Omega^{n-q}_{V_s})_0$ is the kernel of the cup product:

$$L^{r+1}:H^q(V_s, \Omega^{m-r+q}_{V_s})\to H^{q+r+1}(V_s, \Omega^{m+1-q}_{V_s})\qquad (n=m-r)$$

when we consider L as a class in $H^1(V_s, \Omega^1_{V_s})$. In other words, in the geometric situation the fibre $\mathbf{E}^q_s$ is just the primitive part of the Hodge cohomology space $H^{n-q,q}(V_s)$.

(iv) Referring to remark (iii) just above, we shall give a homological interpretation of the maps (2.12) in case $\mathcal{E}$ arises from a geometric situation $f:X\to S$. To do this we recall the *Kodaira-Spencer infinitesimal deformation class* [24]

$$\rho_s\in H^1(V_s, \Theta_{V_s})\otimes\check{\mathbf{T}}_s\qquad (s\in S).$$

The pairing $\Theta_{V_s}\otimes\Omega_{V_s}^{n-q}\to\Omega_{V_s}^{n-q+1}$ gives:

$$\textbf{(2.14)}\qquad H^1(V_s,\Theta_{V_s})\otimes H^q(V_s,\Omega_{V_s}^{n-q})\to H^{q+1}(V_s,\Omega_{V_s}^{n-q-1}).$$

Comparing (2.14) and (2.13) we see that cup product with the Kodaira-Spencer class gives:

$$\textbf{(2.15)}\qquad \rho_s:\mathbf{E}_s^q\to\mathbf{E}_s^{q+1}\otimes\check{\mathbf{T}}_s.$$

From [11] it follows that $\rho_s$ in (2.15) is the same as $\sigma_q$ in (2.12). Summarizing :

*Proposition* **(2.16)**. — *In case* $\mathscr{E}$ *arises from a geometric situation, the linear mapping* $\sigma_q$ *in* (2.12) *is the cup product with the Kodaira-Spencer class.*

## 3. Remarks on the homology of algebraic fibre spaces.

*a)* Consider the situation (1.1) and let:

$$\begin{array}{ccc} Y & \subset & \overline{Y} \\ f\downarrow & & \downarrow\bar{f} \\ P^* & \subset & P \end{array}$$

be a localization at infinity as discussed just preceding (1.4). Since $P^*=(\Delta^*)^k\times(\Delta)^{d-k}$ where $\Delta$ is a disc and $\Delta^*$ is a punctured disc, the fundamental group $\pi_1(P^*)$ is free abelian and has as generators the paths around the deleted point in each of the factors $\Delta^*$. The corresponding automorphisms of the cohomology $H^*(V_{s_0},\mathbf{C})$ are called *Picard-Lefschetz* (P.-L.) *transformations.* In case $k=1$ we shall denote the P.-L. transformation on the primitive cohomology by $T\in\mathrm{Aut}(P^n(V_{s_0},\mathbf{C}))$.

*b)* Let $f:X\to S$ be an algebraic family of algebraic varieties as defined in § 1. We consider the Leray cohomology sheaves $R_{f_*}^q(\mathbf{C})$, and we recall the *Leray spectral sequence* $\{E_r^{p,q}\}$ which abuts to $H^*(X,\mathbf{C})$ and with $E_2^{p,q}=H^p(S,R_{f_*}^q(\mathbf{C}))$. We will prove the following result of Blanchard and Deligne (cf. [2] and Deligne's paper in *Publ. I.H.E.S.*, vol. 35, pp. 107-126):

*Proposition* **(3.1)**. — *The above spectral sequence degenerates at the* $E_2$*-term. In particular the restriction mapping*

$$\textbf{(3.2)}\qquad H^n(X,\mathbf{C})\to H^0(S,R_{f_*}^n(\mathbf{C}))\to 0$$

*is surjective.*

*Proof.* — The cohomology class L of the hyperplane section operates by cup product on the terms $E_r$ $(r\geqq 2)$ of the spectral sequence and it commutes with the differentials $d_r:E_r\to E_{r+1}$. Using this let us show that $d_2=0$, the argument for the other $d_r$ $(r\geqq 3)$ being similar. Because of the Lefschetz decomposition (2.1), which in the present situation reads as:

$$R_{f_*}^q(\mathbf{C})=P_{f_*}^q(\mathbf{C})\oplus LP_{f_*}^{q-2}(\mathbf{C})\oplus L^2P_{f_*}^{q-4}(\mathbf{C})\oplus\dots,$$

it will suffice to show that

$$d_2 : H^p(S, P^q_{f_*}(\mathbf{C})) \to H^{p+2}(S, R^{q+1}_{f_*}(\mathbf{C}))$$

is zero for $q \leqq m = \dim_{\mathbf{C}} V_s$. Writing $q = m - t$ and using that

$$L^{t+1} : R^{m-t-1}_{f_*}(\mathbf{C}) \to R^{m+t+1}_{f_*}(\mathbf{C})$$

is an isomorphism [30], we find a commutative diagram

$$\begin{array}{ccc} H^p(S, P^{m-t}_{f_*}(\mathbf{C})) & \xrightarrow{d_2} & H^{p+2}(S, R^{m-t-1}_{f_*}(\mathbf{C})) \\ \downarrow & & \| \wr \\ H^p(S, R^{m+t+2}_{f_*}(\mathbf{C})) & \xrightarrow{d_2} & H^{p+2}(S, R^{m+t+1}_{f_*}(\mathbf{C})) \end{array}$$

Since the dotted vertical arrow is zero by the definition of the primitive cohomology sheaf, we see that our desired $d_2$ is zero.

*Remark.* — Proposition (3.1) says that there is no *transgression* in the cohomology of algebraic fibre spaces. The result (3.2) was known classically in the following dual formulation [26]:

Let $\gamma \in H_n(V_{s_0}, \mathbf{C})$ be a homology class on $V_{s_0}$ invariant under the action of the fundamental group $\pi_1(S)$ on the homology of the fibres. (We will speak of $\gamma$ as an *invariant cycle.*) Then there exists a cycle $\mathscr{L}(\gamma) \in H_{n+d}(X, \mathbf{C})$ such that:

$$\mathscr{L}(\gamma) \cdot V_{s_0} = \gamma.$$

The cycle $\mathscr{L}(\gamma)$ is called the *locus of* $\gamma$, and it is thought of intuitively as the locus of the cycle $\gamma_s \in H_n(V_s, \mathbf{C})$ as $s$ varies over S. Lefschetz's proof that $\mathscr{L}(\gamma)$ exists is really a homological version of the proof of (3.1) given above.

We shall refer to (3.1) as the *locus of an invariant cycle* theorem.

*c)* There are two variants of the locus of an invariant cycle theorem (3.2). The first is a somewhat interesting *conjectural* local result around an irreducible branch of $\overline{S} - S$ (cf. §§ 8, 15 in [13] for further discussion).

*Conjecture* **(3.3)** *(local invariant cycle problem).* — Let:

$$\begin{array}{ccc} Y & \subset & \overline{Y} \\ f\downarrow & & \downarrow\bar{f} \\ P^* & \subset & P \end{array}$$

be a localization of (1.1) around infinity as discussed in *a)* above, and let $\gamma \in H^n(V_{s_0}, \mathbf{Q})$ be a cohomology class invariant under $\pi_1(P^*, s_0)$. Then there exists $\overline{\Gamma} \in H^n(\overline{Y}, \mathbf{Q})$ with $\overline{\Gamma} | V_{s_0} = \gamma$.

*Remarks.* — It is trivial that there exists $\Gamma \in H^n(Y, \mathbf{Q})$ with $\Gamma | V_{s_0} = \gamma$, so the conjecture has to do with the singular fibres of $\overline{Y}$ lying over $P - P^*$. Thus far (3.3) has proved surprisingly difficult to handle and, in particular, it does not seem to be a topological result but will most likely require some sort of Hodge theory (§ 15 in [13]) ([1]).

The second variant is the following striking result of Deligne [9]:

*Theorem* **(3.4)** *(Deligne). — Referring to* (1.1), *we have a commutative diagram*

**(3.5)**

$$\begin{array}{ccc} H^n(X) & & \\ \downarrow & \overset{r}{\searrow} & \\ & & H^n(V_{s_0}) \\ & \underset{\bar{r}}{\nearrow} & \\ H^n(\bar{X}) & & \end{array}$$

*where the arrows are all restriction mappings of cohomology, and the image of $r$ is equal to the image of $\bar{r}$ in* (3.5).

*Remark.* — This result is a global version of (3.3).

*d)* Let $f: X \to S$ be an algebraic family of algebraic varieties and $\mathscr{E} = (\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$ the resulting variation of Hodge structure (§ 2). We shall use (3.2) and (3.5) to deduce results about $\mathscr{E}$, which will then later in § 7 be proved to hold for an *arbitrary* variation of Hodge structure which has a complete base space. It *should* be possible to prove the results of § 7 with no such assumptions, and this matter is taken up in Appendix C.

The following are given in [9] by Deligne as consequences of (3.5):

**(3.6)** *Let $\varphi \in H^0(S, R^n_{f_*}(\mathbf{C}))$ be an invariant, locally constant cohomology class. Then the same is true of the Hodge $(p, q)$ components of $\varphi$ $(p+q=n)$.*

*Proof.* — This is clear since we have:

$$H^n(\bar{X}, \mathbf{C}) \to H^0(S, R^n_{f_*}(\mathbf{C})) \to 0$$

and $\bar{X}$ is a Kähler manifold.

**(3.7)** *Let $I^n_{\mathbf{Q}} = P^n(V_{s_0}, \mathbf{Q})^{\pi_1(S)}$ be the invariant part of the primitive cohomology $P^n(V_{s_0}, \mathbf{Q})$ under the monodromy group $\Gamma$. Then there is an orthogonal direct sum decomposition*

**(3.8)** $$P^n(V_{s_0}, \mathbf{Q}) = I^n_{\mathbf{Q}} \oplus E^n_{\mathbf{Q}}.$$

*Proof.* — This follows from (3.6) and the properties of the Hodge inner product.

From (3.7), Deligne has deduced:

**(3.9)** *The action of the monodromy group $\Gamma$ on $P^n(V_{s_0}, \mathbf{Q})$ is completely reducible. Furthermore, if $\Gamma$ is solvable, then it is a finite group.*

## 4. Remarks on Hermitian differential geometry.

a) *Connections in Hermitian vector bundles.* — Let $\mathbf{H} \to S$ be a holomorphic vector bundle and

$$D_{\mathbf{H}} : A^0(\mathbf{H}) \to A^1(\mathbf{H})$$

([1]) *Added in proof.* — This conjecture has now been proved for $n = 2$ by Katz, and then in the general case by Deligne, using his theory of mixed Hodge structure [9].

a $C^\infty$ connection. Then there is a decomposition

$$D_H = D'_H + D''_H$$

of $D_H$ into types (1, 0) and (0, 1), and $D_H$ is said to be *compatible with the complex structure* if $D''_H = \bar\partial$. Suppose that $\mathbf{H}$ has an Hermitian metric

$$(\ ,\ ): \ \mathbf{H}\otimes\mathbf{H} \to \mathbf{C}, \qquad (e, e') = \overline{(e', e)}.$$

We do not require that ( , ) be positive definite, but it should of course be non-singular.

*Lemma* **(4.1)**. — *There is a unique connection* $D_H$ *such that*

(i) *the Hermitian metric* ( , ) *is flat, and*

(ii) $D_H$ *is compatible with the complex structure.*

*Proof.* — Let $e_1, \dots, e_r$ be a local holomorphic frame for $\mathbf{H}$ and let $h_{\rho\sigma} = (e_\rho, e_\sigma)$ denote the Hermitian metric. Then the required connection $D_H(e_\rho) = \sum_{\sigma=1}^{r} \theta_\rho^\sigma e_\sigma$ is given by $\theta = h^{-1}\partial h$ where $\theta = (\theta_\rho^\sigma)$, $h = (h_{\rho\sigma})$ are the connection and metric matrices respectively.

Now we consider a holomorphic vector bundle $\mathbf{H}\to S$ having a connection $D_H$ which is compatible with the complex structure. Let $\mathbf{K}\subset\mathbf{H}$ be a holomorphic sub-bundle with quotient bundle $\mathbf{L}$, so that we have an exact sequence

**(4.2)** $$0 \to \mathbf{K} \to \mathbf{H} \to \mathbf{L} \to 0.$$

The connection $D_H$ induces, in the obvious way, a mapping

**(4.3)** $$b : A^0(\mathbf{K}) \to A^1(\mathbf{L})$$

which is linear over the $C^\infty$ functions and is called the *second fundamental form of* $\mathbf{K}$ *in* $\mathbf{H}$.

*Lemma* **(4.3)**. — *The second fundamental form of* $\mathbf{K}$ *in* $\mathbf{H}$ *is of type* (1, 0), *so that* $b\in A^{1,0}(\mathrm{Hom}(\mathbf{K}, \mathbf{L}))$.

*Proof.* — Given $e\in\mathbf{K}_{s_0}$, choose a $C^\infty$ section $f$ of $\mathbf{K}$ with $f(s_0) = e$. Then by definition $b(e)$ is the projection on $\mathbf{L}$ of $(D_H f)(s_0)$. Since we may choose $f$ to be holomorphic and since $D''_H = \bar\partial$, we have $D''_H f = 0$ so that $b(e)$ is of type (1, 0) as desired.

Suppose now that $\mathbf{H}\to S$ is a holomorphic Hermitian vector bundle with holomorphic sub-bundle $\mathbf{K}$ as in (4.2). Assume that the Hermitian metric ( , ) is non-singular when restricted to $\mathbf{K}$. Then there is induced a $C^\infty$ splitting of (4.2) by considering $\mathbf{L}$ as being $\{e\in\mathbf{H} : (e, \mathbf{K}) = 0\}$, and so:

**(4.4)** $$D_K = D_H - b$$

induces a connection in $\mathbf{H}$.

*Lemma* **(4.5)**. — $D_K = D_H - b$ *in* (4.4) *is the metric connection in* $\mathbf{K}$.

*Proof.* — By lemma (4.3), $D''_K = \bar\partial$. For $e, e'\in A^0(\mathbf{K})$:

$$d(e, e') = (D_H e, e') + (e, D_H e') = (D_K e, e') + (e, D_K e')$$

since $\mathbf{L} = (\mathbf{K})^\perp$ as a $C^\infty$ sub-bundle of $\mathbf{H}$. Q.E.D.

Similarly the connection $D_{\mathbf{H}}$, the holomorphic projection $\mathbf{H} \xrightarrow{\pi} \mathbf{L} \to 0$, and the $C^\infty$ injection $0 \to \mathbf{L} \xrightarrow{i} \mathbf{H}$ induce a connection $D_{\mathbf{L}}$ in $\mathbf{L}$ by

**(4.6)** $$D_{\mathbf{L}}(f) = \pi \circ D_{\mathbf{H}} \circ i(f).$$

*Lemma* **(4.7)**. — *The connection* $D_{\mathbf{L}}$ *in* (4.6) *is the metric connection in* $\mathbf{L}$.

The proof is analogous to the proof of (4.5).

b) *Curvature in Hermitian vector bundles.* — Given a connection $D_{\mathbf{H}} : A^0(\mathbf{H}) \to A^1(\mathbf{H})$, the curvature $\Theta_{\mathbf{H}}$ is defined by

**(4.8)** $$\Theta_{\mathbf{H}} \cdot e = (D_{\mathbf{H}})^2 \cdot e \qquad (e \in A^0(\mathbf{H})).$$

In case $D_{\mathbf{H}}$ is the metric connection for an Hermitian metric, $\Theta_{\mathbf{H}}$ is of type (1, 1) and satisfies the symmetry

$$(\Theta_{\mathbf{H}} e, e') + (e, \Theta_{\mathbf{H}} e') = 0 \qquad (e, e' \in \mathbf{H}).$$

For us, the main use of the curvature is as it appears in the following:

*Lemma* **(4.9)**. — *Let* $\varphi$ *and* $\varphi'$ *be two local holomorphic sections of* $\mathbf{H} \to S$ *and* $\psi = (\varphi, \varphi')$ *the inner product. Then*

**(4.10)** $$\partial\bar\partial\psi = (D'_{\mathbf{H}}\varphi, D'_{\mathbf{H}}\varphi') - (\Theta_{\mathbf{H}} \cdot \varphi, \varphi').$$

*Proof.* — $\partial\bar\partial\psi = -\bar\partial\partial(\varphi, \varphi') = -\bar\partial(D'_{\mathbf{H}}\varphi, \varphi')$ (by Lemma (4.1) and since $\varphi'$ is holomorphic) $= -(D''_{\mathbf{H}} D'_{\mathbf{H}}\varphi, \varphi') + (D'_{\mathbf{H}}\varphi, D'_{\mathbf{H}}\varphi')$ (by Lemma (4.1) again). Now $D''_{\mathbf{H}} D'_{\mathbf{H}}\varphi = (D''_{\mathbf{H}} D'_{\mathbf{H}} + D'_{\mathbf{H}} D''_{\mathbf{H}})\varphi$ (since $\varphi$ is holomorphic) $= \Theta_{\mathbf{H}} \cdot \varphi$ (by (4.8)). Q.E.D.

If $\varphi$ is a holomorphic section of $\mathbf{H} \to S$, then we may write locally:

**(4.11)** $$\begin{cases} (D'_{\mathbf{H}}\varphi, D'_{\mathbf{H}}\varphi) = \sum_{i,j} g_{i,\bar j}\, ds^i \wedge d\bar s^j \\ -(\Theta_{\mathbf{H}}\varphi, \varphi) = \sum_{i,j} h_{i,\bar j}\, ds^i \wedge d\bar s^j \end{cases}$$

where $g = (g_{i,\bar j})$ and $h = (h_{i,\bar j})$ are Hermitian matrices and $s^1, \ldots, s^d$ are local holomorphic coordinates on S. From (4.10) and the *maximum principle* for *plurisubharmonic functions* [19], we have

*Lemma* **(4.12)**. — *Let* $\varphi$ *be a holomorphic section of* $\mathbf{H} \to S$ *such that*

(i) *the* length $\psi = (\varphi, \varphi)$ *is bounded on* S, *and*

(ii) *the Hermitian matrices g and h in* (4.11) *are everywhere positive semi-definite.*

*Then* $\psi$ *is constant and we have* $(D_{\mathbf{H}}\varphi, D_{\mathbf{H}}\varphi) = 0 = (\Theta_{\mathbf{H}}\varphi, \varphi)$.

Now let $\mathbf{H} \to S$ be a holomorphic vector bundle with an Hermitian metric and metric connection $D_{\mathbf{H}}$. Suppose that $\mathbf{K} \subset \mathbf{H}$ is a holomorphic sub-bundle such that the restriction of the metric on $\mathbf{H}$ to $\mathbf{K}$ is non-singular. Then we are in the situation of Lemmas (4.5) and (4.7).

*Lemma* **(4.13)**. — *The curvature of the metric connection in* $\mathbf{K}$ *is given by* $(\Theta_{\mathbf{K}} e, e') = (\Theta_{\mathbf{H}} e, e') - (be, be')$ $(e, e' \in \mathbf{K}_{s_0})$.

*Proof.* — Choose holomorphic sections $f, f'$ of $\mathbf{K}$ such that $f(s_0)=e$, $f'(s_0)=e'$. Then by (4.10) applied to $\mathbf{H}$ we have:

$$\begin{aligned}(\Theta_{\mathbf{H}} f, f') &= (\mathrm{D}'_{\mathbf{H}} f, \mathrm{D}'_{\mathbf{H}} f') - \partial\bar{\partial}(f, f') \\ &= ((\mathrm{D}'_{\mathbf{K}} + b) f, (\mathrm{D}'_{\mathbf{K}} + b) f') - \partial\bar{\partial}(f, f') \\ &= (\mathrm{D}'_{\mathbf{K}} f, \mathrm{D}'_{\mathbf{K}} f') + (bf, bf') - \partial\bar{\partial}(f, f') \\ &= (\Theta_{\mathbf{K}} f, f') + (bf, bf'),\end{aligned}$$

where we have used Lemma (4.5) in the second step, the equation:

$$(\mathrm{D}'_{\mathbf{K}} f, bf') = 0 = (bf, \mathrm{D}'_{\mathbf{K}} f)$$

in the third step, and (4.10) applied to the bundle $\mathbf{K}$ in the last step. Q.E.D.

To give the curvature in $\mathbf{L}$, we use the conjugate linear isomorphisms:

$$\check{\mathbf{K}} \cong \mathbf{K}$$
$$\check{\mathbf{L}} \cong \mathbf{L}$$

induced by the Hermitian metrics to define

$$c \in \mathrm{A}^{0,1}(\mathrm{Hom}(\mathbf{L}, \mathbf{K}))$$

as the image of the second fundamental form $b$ under the isomorphism $\mathrm{A}^{1,0}(\check{\mathbf{K}} \otimes \mathbf{L}) \cong \mathrm{A}^{0,1}(\mathbf{K} \otimes \check{\mathbf{L}})$.

*Lemma* **(4.14)**. — *The metric curvature in* $\mathbf{L}$ *is given by:*

$$(\Theta_{\mathbf{L}} f, f') = (\Theta_{\mathbf{H}} f, f') + (cf, cf') \qquad (f, f' \in \mathbf{L} \cong (\mathbf{K})^{\perp}).$$

## 5. Statement of main differential-geometric properties of the Hodge bundles.

The results stated in this section will be proved in § 6 below.

Let $\mathscr{E} = (\mathbf{E}, \mathbf{D}, \mathrm{Q}, \{\mathbf{F}^q\})$ be a variation of Hodge structure as defined in § 2. Using the Hodge bilinear form $\mathrm{Q}$, we have an Hermitian metric ( , ) in $\mathbf{E}$ given by

**(5.1)** $$(e, e') = (-i)^n \mathrm{Q}(e, \bar{e}') \qquad (e, e' \in \mathbf{E}).$$

This Hermitian metric induces non-singular Hermitian metrics in the holomorphic sub-bundles $\mathbf{F}^q$ of $\mathbf{E}$, and $(-1)^{q-1}( \, , \, )$ subsequently induces a positive-definite Hermitian metric in the Hodge bundle $\mathbf{E}^q = \mathbf{F}^q/\mathbf{F}^{q-1}$.

Referring to (2.12), we have linear bundle maps $\sigma_q : \mathbf{E}^q \to \mathbf{E}^{q+1} \otimes \check{\mathbf{T}}$ and ${}^t\sigma_{q-1} : \mathbf{E}^q \to \mathbf{E}^{q-1} \otimes \check{\mathbf{T}}$ induced by the flat holomorphic connection $\mathbf{D}$.

*Theorem* **(5.2)** *(Curvature of Hodge bundles).* — *The curvature of the metric connection* $\mathrm{D}_{\mathbf{E}^q}$ *is given by:*

$$(\Theta_{\mathbf{E}^q} e, e') = (\sigma_q e, \sigma_q e') - ({}^t\sigma_{q-1} e, {}^t\sigma_{q-1} e') \qquad (e, e' \in \mathbf{E}^q)$$

*where we agree that* $\sigma_{-1} = 0 = \sigma_n$.

*Remark.* — If we choose local frames for all of the Hodge bundles $\mathbf{E}^q$, then Theorem (5.2) gives for the *curvature matrix* that

$$\Theta_{\mathbf{E}^q} = A_q \wedge {}^t\bar{A}_q - B_q \wedge {}^t\bar{B}_q, \tag{5.3}$$

where $A_q$, $B_q$ are matrices of (1, 0) forms with $B_0 = 0$, $A_n = 0$.

*Remark.* — From (2.10) we have isomorphisms

$$\check{\mathbf{E}}^q \cong \mathbf{E}^{n-q}. \tag{5.4}$$

From (5.1) it follows that the isomorphisms in (5.4) are all *isometries*. Using the isomorphism

$$\check{\mathbf{E}}^q \otimes \mathbf{E}^{q+1} \otimes \check{\mathbf{T}} \cong \check{\mathbf{E}}^{n-q-1} \otimes \mathbf{E}^q \otimes \check{\mathbf{T}},$$

we see that $\sigma_q$ corresponds to $\sigma_{n-q-1}$, and so

$$\Theta_{\mathbf{E}^q} = (\sigma_q, \sigma_q) - ({}^t\sigma_{q-1}, {}^t\sigma_{q-1}) = -{}^t\Theta_{\check{\mathbf{E}}^{n-q}},$$

which is the correct relation between the curvature of an Hermitian vector bundle and the curvature of its dual.

Our second main application of the structure equations of variation of Hodge structure is

*Proposition* **(5.5)**. — *Let $\Phi$ be a holomorphic section of $\mathbf{F}^q$ over an open set $U \subset S$ and assume that the projection of $D\Phi$ in $\mathbf{E}/\mathbf{F}^q$ is zero. Then $\Phi$ induces a section $\varphi$ of $\mathbf{E}/\mathbf{F}^{q-1} \cong \check{\mathbf{F}}^{n-q+1}$, and the differential forms*

$$\begin{cases} (-1)^{n-q}(D'_{\check{\mathbf{F}}^{n-q+1}}\varphi, D'_{\check{\mathbf{F}}^{n-q+1}}\varphi) \\ (-1)^{n-q+1}(\Theta_{\check{\mathbf{F}}^{n-q+1}}\varphi, \varphi) \end{cases} \tag{5.6}$$

*are* positive, *in the sense that the Hermitian matrices defined as in* (4.11) *are positive (semi-definite).*

*Corollary* **(5.7)**. — *Let $\Phi$ and $\varphi$ be as in Proposition* (5.5) *above and assume that*

(i) U *is all of* S, *and*

(ii) *the length of the section $\varphi$ of $\mathbf{E}^q \subset \mathbf{E}/\mathbf{F}^{q-1}$ is bounded.*

*Then* $D_{\mathbf{E}/\mathbf{F}^{q-1}}\varphi = 0 = D_{\mathbf{E}^q}\varphi$.

This Proposition and Corollary will be proved together in § 6 below.

*Proposition* **(5.8)**. — *Let $\varphi$ be a holomorphic section of $\mathbf{E}^q \subset \mathbf{E}/\mathbf{F}^{q-1}$ over an open set* $U \subset S$, *and assume that* $D_{\mathbf{E}/\mathbf{F}^{q-1}}\varphi = 0$. *Then there exists a unique section $\Psi$ of $\mathbf{F}^q$ satisfying*

(i) $\mathbf{D}\Psi = 0$;

(ii) $\Psi$ *projects onto* $\varphi$; *and*

(iii) *the inner product* $(\Psi, \mathbf{F}^{q-1}) = 0$.

Combining (5.7) and (5.8) we find:

*Theorem* **(5.9)** *(Theorem on global sections of Hodge bundles).* — *Let $\Phi$ be a holomorphic section of $\mathbf{F}^q \to S$ such that*

a) *the projection of $\mathbf{D}\Phi$ in $\mathbf{E}/\mathbf{F}^q$ is zero, and*

b) *the length of the induced section $\varphi$ of $\mathbf{E}^q = \mathbf{F}^q/\mathbf{F}^{q-1}$ is bounded.*

*Then there exists a section* $\Psi$ *of* $\mathbf{F}^q$ *satisfying*

(i) $\mathbf{D}\Psi = 0$;

(ii) $\Psi - \Phi$ *is a section of* $\mathbf{F}^{q-1}$; *and*

(iii) *the inner product* $(\Psi, \mathbf{F}^{q-1}) = 0$.

## 6. Structure equations for variation of Hodge structure.

We want to prove (5.2) and (5.5)-(5.8). Our method of proof is to use the *calculus of frames* [6], [8], where by definition a *frame* is a $C^\infty$ basis, over an open set, of the vector bundle in question.

Given a variation of Hodge structure $\mathscr{E} = (\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$, we shall consider *unitary frames adapted to the Hodge filtration* (2.4). By definition this is a frame

$$e_1, \ldots, e_{h_1}; e_{h_1+1}, \ldots, e_{h_2}; \ldots; e_{h_{n-1}+1}, \ldots, e_{h_n} \tag{6.1}$$

where $h_q = \dim \mathbf{F}^q_s$ and where the following conditions are satisfied:

(i) referring to the Hermitian inner product (5.1) we have

$$(e_i, e_j) = (-1)^{q-1}\delta^i_j \quad (h_{q-1} \leqq i, j \leqq h_q) \tag{6.2}$$

and all other inner products are zero;

(ii) the vectors

$$e_1, \ldots, e_{h_q}$$

give a basis for $\mathbf{F}^q_s$ for all points $s \in S$ where the frame is defined; and

(iii) under the conjugate linear isomorphism

$$\mathbf{E}^q \cong \mathbf{E}^{n-q}$$

given by (5.4) and the metric (5.1), we have

$$\bar{e}_{h_{q-1}+j} = e_{h_{n-q-1}+j} \quad (1 \leqq j \leqq h_q - h_{q-1}). \tag{6.3}$$

*Remarks.* — We first observe that (6.2) and (6.3) are compatible:

$$\begin{aligned}(-1)^{q-1}\delta^i_j = (e_{h_{q-1}+i}, e_{h_{q-1}+j}) &= (-i)^n Q(e_{h_{q-1}+i}, \bar{e}_{h_{q-1}+j}) \\ &= (-1)^n(-i)^n Q(e_{h_{n-q-1}+j}, \bar{e}_{h_{n-q-1}+i}) = (-1)^n(-1)^{n-q-1}\delta^j_i.\end{aligned}$$

Secondly, I should like to comment that the *alternation of signs* in (6.2) is of extreme importance — it is this plus the *infinitesimal bilinear relation* (2.8) which makes everything go through.

The flat holomorphic connection $\mathbf{D}$, which by Lemma (4.1) is the metric connection for the metric (5.1) in $\mathbf{E}$, is given by

$$\mathbf{D}e_i = \sum_{j=1}^{h_n} \theta^j_i e_j, \tag{6.4}$$

where the differential 1-forms $\theta_i^j$ satisfy the *integrability condition*

$$d\theta_j^i + \sum_{k=1}^{h_n} (\theta_k^i \wedge \theta_j^k) = 0. \tag{6.5}$$

From (6.2) and the flatness of the metric we find

$$\theta_{h_{q-1}+j}^{h_{p-1}+i} + (-1)^{p+q}\bar{\theta}_{h_{p-1}+i}^{h_{q-1}+j} = 0 \qquad (1 \leqq i \leqq h_p - h_{p-1};\ 1 \leqq j \leqq h_q - h_{q-1}). \tag{6.6}$$

From (6.3) we have

$$\bar{\theta}_{h_{q-1}+j}^{h_{p-1}+i} = (-1)^n \theta_{h_{n-q-1}+j}^{h_{n-p-1}+i}. \tag{6.7}$$

As remarked below (6.3), it is unfortunately the case that the signs are quite important and so must be kept track of carefully.

The infinitesimal period relation (2.8) gives

$$\theta_{h_{q-1}+j}^{h_{p-1}+k} = 0 \quad \text{for } p \geqq q+1,\ 1 \leqq j \leqq h_q - h_{q-1},\ k \geqq 0. \tag{6.8}$$

At this point we have used all of the information in I)-VII) of § 2.

From Lemmas (4.3) and (4.5) we have

*Lemma* **(6.9)**. — *The second fundamental form of* $\mathbf{F}^q$ *in* $\mathbf{E}$ *is given by*

$$b_q = \sum_{\substack{1 \leq i \leq h_q \\ h_q+1 \leq j \leq h_n}} \theta_i^j \check{e}_i \otimes e_j, \qquad \theta_i^{j''} = 0; \tag{6.10}$$

*and the metric connection in* $\mathbf{F}^q$ *is given by*

$$D_{\mathbf{F}^q} e_i = \sum_{j=1}^{h_q} \theta_i^j e_j, \qquad 1 \leqq i \leqq h_q. \tag{6.11}$$

Using the *Cartan structure equation* [6], the curvature of the metric connection in $\mathbf{F}^q$ is given by

$$(\Theta_{\mathbf{F}^q})_j^i = d\theta_j^i + \sum_{k=1}^{h_q} (\theta_k^i \wedge \theta_j^k) \qquad (1 \leqq i, j \leqq h_q).$$

From (6.5) and (6.8) we obtain

$$(\Theta_{\mathbf{F}^q})_j^i = -\sum_{k=1}^{h_{q+1}-h_q} (\theta_{h_q+k}^i \wedge \theta_j^{h_q+k}) \qquad (1 \leqq i, j \leqq h_q). \tag{6.12}$$

We shall now prove Proposition (5.5) by proving the following three lemmas.

*Lemma* **(6.13)**. — *Let* $\check{\varphi}$ *be a vector in* $\check{\mathbf{F}}^q$ *such that* $\check{\varphi}$ *projects to zero in* $\check{\mathbf{F}}^q \to \check{\mathbf{F}}^{q-1} \to 0$. *Then the differential form*

$$(-1)^q (\Theta_{\check{\mathbf{F}}^q} \cdot \check{\varphi}, \check{\varphi})$$

*is positive in the sense of Proposition* (5.5).

*Proof.* — Writing $\check{\varphi}=\sum_{l=1}^{h_q-h_{q-1}}\varphi^l\check{e}_{h_{q-1}+l}$, we have from (6.12) that

$$\begin{aligned}(\Theta_{\check{\mathbf{F}}^q}\check{\varphi},\check{\varphi})&=\sum_{i,j,k,l}\theta^{h_{q-1}+j}_{h_q+k}\wedge\theta^{h_q+k}_{h_{q-1}+i}(\check{e}_{h_{q-1}+i},\check{e}_{h_{q-1}+k})\varphi^j\bar{\varphi}^l\\&=(-1)^q(\sum_{i,j,k}\theta^{h_q+k}_{h_{q-1}+i}\wedge\theta^{h_{q-1}+j}_{h_q+k}\varphi^j\bar{\varphi}^i)\\&=(-1)^q(\sum_k((\sum_i\theta^{h_q+k}_{h_{q-1}+i}\bar{\varphi}^i)\wedge\overline{(\sum_j\theta^{h_q+k}_{h_{q-1}+j}\bar{\varphi}^j)}))\qquad\text{(by (6.6))}\\&=(-1)^q\sum_k(\psi^k\wedge\bar{\psi}^k)\end{aligned}$$

where $\psi^k=\sum_i\theta^{h_q+k}_{h_{q-1}+i}\bar{\varphi}^i$ is of type (1, 0) (by (6.10)). This proves the Lemma.

*Lemma* **(6.14)**. — *Let* $\check{\varphi}$ *be as in Lemma* (6.13) *and assume that* $D'_{\check{\mathbf{F}}^q}\check{\varphi}$ *projects to zero in* $\check{\mathbf{F}}^q\to\check{\mathbf{F}}^{q-1}\to 0$. *Then the differential form*

$$(-1)^{q-1}(D'_{\check{\mathbf{F}}^q}\check{\varphi},D'_{\check{\mathbf{F}}^q}\check{\varphi})$$

*is positive.*

*Proof.* — By assumption $D'_{\check{\mathbf{F}}^q}\check{\varphi}=\sum_{i=1}^{h_q-h_{q-1}}\psi^{h_{q-1}+i}\check{e}_{h_{q-1}+i}$ where $\psi^{h_{q-1}+i}$ is of type (1, 0). Then

$$(-1)^{q-1}(D'_{\check{\mathbf{F}}^q}\check{\varphi},D'_{\check{\mathbf{F}}^q}\check{\varphi})=\sum_i(\psi^{h_{q-1}+i}\wedge\bar{\psi}^{h_{q-1}+i})$$

as required.

*Lemma* **(6.15)**. — *Let* $\Psi$ *be a holomorphic section of* $\mathbf{F}^p$ *such that* $\mathbf{D}\Psi$ *projects to zero in* $\mathbf{E}/\mathbf{F}^p$. *Denote by* $\psi$ *the section of* $\mathbf{E}/\mathbf{F}^{p-1}$ *induced by* $\Psi$, *and let* $\check{\psi}$ *be the holomorphic section of* $\check{\mathbf{F}}^{n-p+1}$ *which corresponds to* $\psi$ *under the isomorphism* $\mathbf{E}/\mathbf{F}^{p-1}\cong\check{\mathbf{F}}^{n-p+1}$. *Then* $D_{\check{\mathbf{F}}^{n-p+1}}\check{\psi}$ *projects to zero in* $\check{\mathbf{F}}^{n-p+1}\to\check{\mathbf{F}}^{n-p}\to 0$.

*Proof.* — What we must prove is that $D_{\mathbf{E}/\mathbf{F}^{p-1}}\psi$ projects to zero in $\mathbf{E}/\mathbf{F}^{p-1}\to\mathbf{E}/\mathbf{F}^p\to 0$. Now $\Psi=\sum_{j=1}^{h_p}\psi^je_j$ and by assumption we have

$$\textbf{(6.16)}\qquad\sum_{j=1}^{h_p}\psi^j\theta_j^{h_p+k}=0\qquad(1\leq k\leq h_{p+1}-h_p).$$

The section $\psi$ of $\mathbf{E}/\mathbf{F}^{p-1}$ is $\sum_{j=1}^{h_p-h_{p-1}}\psi^{h_{p-1}+j}e_{h_{p-1}+j}$ and the Lemma follows from (6.8), (4.7) and (6.16).

It is clear that Proposition (5.5) follows from Lemmas (6.13), (6.14) and (6.15). We now prove Corollary (5.7).

We use the notation of Proposition (5.5) and Corollary (5.7). From Lemma (4.12) it follows that the length $(\varphi,\varphi)$ of $\varphi$ is constant and $(D_{\mathbf{E}/\mathbf{F}^{q-1}}\varphi,D_{\mathbf{E}/\mathbf{F}^{q-1}}\varphi)=0$. By Lemma (6.15) we have that the projection of $D_{\mathbf{E}/\mathbf{F}^{q-1}}\varphi$ on $\mathbf{E}/\mathbf{F}^q$ is zero, from which it follows that $D_{\mathbf{E}/\mathbf{F}^{q-1}}\varphi=0$. It now follows from the exact sequence:

$$0\to\mathbf{E}^q\to\mathbf{E}/\mathbf{F}^{q-1}\to\mathbf{E}/\mathbf{F}^q\to 0$$

and Lemma (4.5) that $D_{\mathbf{E}^q}\varphi=0$. This proves the Corollary.

We now prove Proposition (5.8). We may assume that $\varphi$ is a unit vector, i.e. that the length $(\varphi, \varphi)=(-1)^{q-1}$, and may then take:

$$\varphi = e_{h_{q-1}+1}$$

in our frames (6.1). From (6.4) and Lemma (4.7), we have $\sum_{j=h_{q-1}+1}^{h_n} \theta^j_{h_{q-1}+1} e_j = 0$ which gives:

$$\textbf{(6.17)} \qquad \theta^j_{h_{q-1}+1} = 0 \quad \text{for } j \geqq h_{q-1}+1.$$

Differentiating (6.17) and using (6.5) gives:

$$\begin{aligned} 0 = -d\theta^{h_{q-1}+1}_{h_{q-1}+1} &= \sum_{j=1}^{h_n} (\theta^{h_{q-1}+1}_j \wedge \theta^j_{h_{q-1}+1}) \\ &= \sum_{j \leqq h_{q-1}} (\theta^{h_{q-1}+1}_j \wedge \theta^j_{h_{q-1}+1}) \qquad \text{(by (6.17))} \\ &= \sum_{j=1}^{h_{q-1}-h_{q-2}} (\theta^{h_{q-1}+1}_{h_{q-2}+j} \wedge \bar{\theta}^{h_{q-1}+1}_{h_{q-2}+j}) \qquad \text{(by (6.8) and (6.6)).} \end{aligned}$$

Since $\theta^{h_{q-1}+1}_{h_{q-2}+j}$ is of type (1, 0), we must have:

$$\bar{\theta}^{h_{q-1}+1}_{h_{q-2}+j} = 0 = \theta^{h_{q-2}+j}_{h_{q-1}+1} \quad \text{for } j=1, \ldots, h_{q-1}-h_{q-2}.$$

This gives $\mathbf{D}e_{h_{q-1}+1}=0$, from which Proposition (5.8) follows.

We now prove Theorem (5.2). Taking the frame:

$$e_{h_{q-1}+1}, \ldots, e_{h_q}$$

in $\mathbf{E}^q$, we see from Lemmas (4.5) and (4.7) that:

$$\mathrm{D}_{\mathbf{E}^q} e_{h_{q-1}+j} = \sum_{i=1}^{h_q-h_{q-1}} \theta^{h_{q-1}+i}_{h_{q-1}+j} e_i.$$

From the Cartan structure equation:

$$(\Theta_{\mathbf{E}^q})^{h_{q-1}+i}_{h_{q-1}+j} = d\theta^{h_{q-1}+i}_{h_{q-1}+j} + \sum_k (\theta^{h_{q-1}+i}_{h_{q-1}+k} \wedge \theta^{h_{q-1}+k}_{h_{q-1}+j})$$

and from (6.5), (6.6), (6.8) it follows that:

$$\textbf{(6.18)} \qquad (\Theta_{\mathbf{E}^q})^{h_{q-1}+i}_{h_{q-1}+j} = -\sum_{l=1}^{h_{q-1}-h_{q-2}} (\theta^{h_{q-1}+i}_{h_{q-2}+l} \wedge \bar{\theta}^{h_{q-1}+j}_{h_{q-2}+l}) + \sum_{m=1}^{h_{q+1}-h_q} (\theta^{h_q+m}_{h_{q-1}+j} \wedge \bar{\theta}^{h_q+m}_{h_{q-1}+i}).$$

Now Theorem (5.2) follows from (6.18) and the equation

$$\sigma_q(e_{h_{q-1}+j}) = \sum_{m=1}^{h_{q+1}-h_q} \theta^{h_q+m}_{h_{q-1}+j} e_{h_q+m},$$

which says that $\sigma_q$ given by (2.12) is just the second fundamental form of $\mathbf{E}^q$ in $\mathbf{E}/\mathbf{F}^{q-1}$.

Finally, we shall prove a Lemma for use in the proof of Theorem (7.19) below.

*Lemma* **(6.19)**. — *Assume that* S *is complete and let* $\varphi$ *be a global holomorphic section of* $\mathbf{E}/\mathbf{F}^{q-1}$ *such that*

(i) $\varphi$ *projects to zero in* $\mathbf{E}/\mathbf{F}^{q-1}\rightarrow\mathbf{E}/\mathbf{F}^{q}$, *and*

(ii) *the projection of* $\mathbf{D}\varphi$ *to* $\mathbf{E}/\mathbf{F}^{q}$ *is zero (this makes sense since* $\mathbf{D}\cdot\mathcal{O}_{S}(\mathbf{F}^{q-1})\subset\Omega^{1}_{S}(\mathbf{F}^{q})$).

*Then there exists a constant section* $\Phi$ *of* $\mathbf{E}$ *such that*

a) $\Phi$ *projects to* $\varphi$ *in* $\mathbf{E}\rightarrow\mathbf{E}/\mathbf{F}^{q-1}$ *and*

b) *the inner product* $(\Phi, \mathbf{F}^{q-1})=0$.

*Proof.* — Referring to Theorem (5.2), we have:

**(6.20)** $$(\Theta_{\mathbf{E}^{q}}\varphi, \varphi)=-({}^{t}\sigma_{q-1}\cdot\varphi, {}^{t}\sigma_{q-1}\cdot\varphi)$$

when we consider $\varphi$ as a holomorphic section of $\mathbf{E}^{q}$ (by (i)) and when we use $\sigma_{q}\cdot\varphi=0$ (by (ii)). From (6.20) and Lemma (4.12) we have $D_{\mathbf{E}^{q}}\cdot\varphi=0$. It follows that $D_{\mathbf{E}/\mathbf{F}^{q-1}}\varphi=0$ and then our result follows from Proposition (5.8).

## 7. Applications.

a) *Invariant cycle and rigidity theorems.*

*Theorem* **(7.1)** *(Invariant cycle theorem).* — *Let* $\mathscr{E}=(\mathbf{E}, \mathbf{D}, \mathrm{Q}, \{\mathbf{F}^{q}\})$ *be a variation of Hodge structure and assume that*

(i) S *is complete, or*

(ii) *the Picard-Lefschetz transformations around the irreducible branches of* $\overline{\mathrm{S}}-\mathrm{S}$ *are trivial.*

*Let* $\Phi$ *be a flat section of* $\mathbf{E}\rightarrow\mathrm{S}$. *Then the Hodge* $(p, q)$ *components of* $\Phi$ *are flat sections of* E.

*Proof.* — We shall prove in § 11 below that, with the assumption (ii) above, the variation of Hodge structure $\mathscr{E}$ and section $\Phi$ of $\mathbf{E}$ both extend to $\overline{\mathrm{S}}$. Thus we may assume that S is complete.

Referring to the theorem (5.9) on global sections of Hodge bundles, we may find a section $\Psi_{n}$ of $\mathbf{E}$ satisfying $\mathbf{D}\Psi_{n}=0$, the projection of $\Phi-\Psi_{n}$ in $\mathbf{E}/\mathbf{F}^{n-1}$ is zero, and the inner product $(\Psi_{n}, \mathbf{F}^{n-1})=0$. Writing $\Phi=\Phi_{n-1}+\Psi_{n}$, we may apply the same reasoning to find a flat section $\Psi_{n-1}$ of $\mathbf{F}^{n-1}$ such that $(\Psi_{n-1}, \mathbf{F}^{n-2})=0$ and $\Phi=\Phi_{n-2}+\Psi_{n-1}+\Psi_{n}$ where $\Phi_{n-2}$ is a flat section of $\mathbf{F}^{n-2}$. Continuing in this way we find our theorem.

*Remarks.* — (i) In [12] we gave the above proof of Theorem (7.1) but formulated the result in a clumsy way. The above formulation was given by Deligne [9], who, as remarked in § 3, has proved

*Theorem* **(7.2)** *(Deligne).* — *With no assumptions on* S *but with the assumption that* $\mathscr{E}$ *arises from a geometric situation, the same conclusion as in Theorem* (7.1) *is valid.*

In fact Theorem (7.2) follows immediately from Deligne's result (3.5).

(ii) Of course, we would conjecture that (7.1) is true with no assumptions on S.

*Corollary* **(7.3)**. — *With the notations and assumptions of Theorem* (7.1), *we suppose further that* $n=2m$ *is even and that* $\Phi$ *is a flat section of* $\mathbf{E}\to S$ *which is of type* $(m, m)$ *at one point* (i.e. *the Hodge components* $\Phi^{p,q}=0$ *for* $(p, q)\neq(m, m)$). *Then* $\Phi$ *is everywhere of type* $(m, m)$.

*Remark.* — In [18] Grothendieck, as a (non-trivial) consequence of the Tate conjectures, was led to suggest that, if $f: X\to S$ is an algebraic family of algebraic varieties and $\Phi$ a section in $H^0(S, R^{2m}_{f_*}\mathbf{Q})$ which is an algebraic cycle at one point $s_0\in S$, then $\Phi(s)\in H^{2m}(V_s, \mathbf{Q})$ is everywhere an algebraic cycle. It was this problem which initially started me looking into sections of Hodge bundles.

*Corollary* **(7.4)**. — *Let* $\mathscr{E}$ *and* $\mathscr{E}'$ *be two variations of Hodge structure which satisfy the assumptions of Theorem* (7.1). *Suppose there is a linear isomorphism* $\sigma: \mathbf{E}_{s_0}\to\mathbf{E}'_{s_0}$ $(s_0\in S)$ *which is equivariant with respect to the action of* $\pi_1(S)$ *on* $\mathbf{E}_{s_0}$ *and* $\mathbf{E}'_{s_0}$, *and which commutes with the Hodge decompositions of* $\mathbf{E}_{s_0}$ *and* $\mathbf{E}'_{s_0}$. *Then there is a global isomorphism* $\mathscr{E}\cong\mathscr{E}'$ *of the variations of Hodge structure which induces* $\sigma$ *at* $s_0\in S$.

*Proof.* — Because of $\pi_1(S)$-equivariance, we may consider $\sigma$ as a global flat section of $\check{\mathbf{E}}\otimes\mathbf{E}'$. Also $\sigma$ is of type $(n, n)$ at $s_0$ since it commutes with Hodge decompositions. The result now follows from (7.3).

*Remark.* — This corollary, which should be thought of as a *rigidity theorem*, was proved for $n=1$ by Grothendieck [17] in the geometric case and by Borel-R. Narasimhan [5] in the general case. Because of Deligne's theorem (7.2), the corollary is true in general when $\mathscr{E}$ and $\mathscr{E}'$ both arise from geometric situations.

We can formulate an analogous result about homomorphisms (and not just isomorphisms) between variations of Hodge structure. As we see no applications for such, we shall not discuss the matter further.

b) *Negative bundles and variation of Hodge structure.* — Let $\mathbf{H}\to S$ be a holomorphic vector bundle. We say that $\mathbf{H}$ is *negative (semi-definite)* if there exists a (positive-definite) Hermitian metric ( , ) in $\mathbf{H}$ whose metric curvature $\Theta_{\mathbf{H}}$ (cf. Lemma (4.1)) has the property that the differential forms:

$$(\Theta_{\mathbf{H}}e, e)=\sum_{i,j}h_{i,\bar{j}}\,ds^i\wedge d\bar{s}^j \qquad (h_{ij}=\bar{h}_{j,\bar{i}}) \tag{7.5}$$

are negative in the sense that the Hermitian matrix $(h_{i,\bar{j}})$ is negative. Observe that $\mathbf{H}$ is negative if the matrix of the metric curvature has a local expression:

$$\Theta_{\mathbf{H}}=-A\wedge{}^t\bar{A} \tag{7.6}$$

where A is a matrix of (1, 0) forms. From theorem (5.2) we have

*Proposition* **(7.7)**. — *Let* $\mathscr{E}=(\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$ *be a variation of Hodge structure. Then the Hodge bundle* $\mathbf{E}^n$ *is negative.*

We say that a locally free coherent analytic sheaf is negative if this is true of the corresponding holomorphic vector bundle.

*Corollary* **(7.8)**. — *Let $f: X \to S$ be an algebraic family of algebraic varieties. Then the direct image sheaf $R^n_{f_*}(\mathcal{O}_X)$ is negative. More generally, if $q$ is the least integer with $H^{n-q}(V_s, \Omega^q_{V_s}) \neq 0$, then the direct image sheaf $R^{n-q}_{f_*}(\Omega^q_{X/S})$ is negative.*

Recall that a *cohomology class* $\omega \in H^{2k}(X, \mathbf{R})$ *is negative* if we have

$$\int_Z \omega \leqq 0 \tag{7.9}$$

for all compact $k$-dimensional algebraic subvarieties Z of S.

*Corollary* **(7.10)**. — *The* Chern monomials $c_I = c_{i_1} \dots c_{i_l}$ *of the Hodge bundle* $\mathbf{E}^n \to S$ *are negative. In particular the 1st Chern class $c_1(R^n_{f_*}(\mathcal{O}_X))$ is negative (semi-definite), and we furthermore have*

$$\int_S c_1(R^n_{f_*}(\mathcal{O}_X)) < 0 \tag{7.11}$$

*in case* S *is a complete curve,* $n=1$ *or* 2, *and* $\mathcal{E}$ *is not trivial.*

*Proof.* — It is well known that the Chern classes of a holomorphic vector bundle $\mathbf{H} \to S$ can be computed from the curvature $\Theta_{\mathbf{H}}$ of a metric connexion. In particular, if (7.6) holds, then it follows that locally

$$c_I = \left(\frac{i}{2\pi}\right)^{|I|} \left(\sum_\alpha (\Psi_\alpha \wedge \overline{\Psi}_\alpha)\right) \tag{7.12}$$

where $|I| = i_1 + \dots + i_l$ is the degree of $c_I$ and $\Psi_\alpha$ are $(|I|, 0)$ forms. The first two statements of (7.10) follow from (7.12), and (7.11) follows the fact (cf. Theorem (5.2)) that, for $n=1$ or 2:

$$c_1(R^n_{f_*}(\mathcal{O}_X)) = 0 \Rightarrow \sigma_{n-1} = 0.$$

To give our final application of Theorem (5.2), we define the *canonical bundle* $\mathbf{K}(\mathcal{E})$ *of the variation of Hodge structure* $\mathcal{E} = (\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$ by:

$$\mathbf{K}(\mathcal{E}) = (\det \mathbf{E}^0)^n \otimes (\det \mathbf{E}^1)^{n-1} \otimes \dots \otimes (\det \mathbf{E}^{n-1}). \tag{7.13}$$

The first Chern class of the line bundle $\mathbf{K}(\mathcal{E})$ is given by the differential form $\dfrac{i}{2\pi}(\omega(\mathcal{E}))$ where

$$\omega(\mathcal{E}) = n(\operatorname{Trace} \Theta_{\mathbf{E}^0}) + (n-1)(\operatorname{Trace} \Theta_{\mathbf{E}^1}) + \dots + \operatorname{Trace} \Theta_{\mathbf{E}^{n-1}}. \tag{7.14}$$

*Proposition* **(7.15)**. — *For a tangent vector $\eta$ to* S, *we have $\langle \omega, \eta \wedge \overline{\eta} \rangle \geqq 0$ with equality if, and only if, $\sigma_q(\eta) = 0$ for $q = 0, \dots, n-1$.*

*Proof.* — This follows by direct computation from the formula

$$\langle \omega, \eta \wedge \overline{\eta} \rangle = \sum_{q=0}^{n-1} |\sigma_q(\eta)|^2$$

which results from (7.14) and Theorem (5.2).

c) *A Mordell-Weil theorem for intermediate Jacobians.* — Let $\mathcal{E} = (\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$ be a variation of Hodge structure where we assume that $n = 2m+1$ is odd. Then

$$\mathbf{E} = \mathbf{F}^m \oplus \overline{\mathbf{F}}^m, \quad \mathbf{F}^m \cap \overline{\mathbf{F}}^m = 0. \tag{7.16}$$

We let $\mathbf{E}_{\mathbf{R}}=\{e\in\mathbf{E} : e=\bar{e}\}$ be the set of real points in $\mathbf{E}$ and assume given a *flat lattice* $\Lambda\subset\mathbf{E}_{\mathbf{R}}$. Equivalently, we are given a $\pi_1(\mathrm{S})$-invariant lattice $\Lambda_{s_0}$ in $(\mathbf{E}_{\mathbf{R}})_{s_0}$. Letting $\mathbf{E}_+=\mathbf{E}/\mathbf{F}^m$ and $\mathbf{J}=\mathbf{E}_+/\Lambda$, we obtain an analytic fibre space $\pi:\mathbf{J}\to\mathrm{S}$, $\pi^{-1}(s)=\mathbf{J}_s$ of complex tori $\mathbf{J}_s=\mathbf{F}_s^m\backslash\mathbf{E}_s/\Lambda_s$ which we shall call the *family of intermediate Jacobians associated to* $\mathscr{E}$ and $\Lambda$. We note that the tori $\mathbf{J}_s$ are abelian varieties if $m=0$, but not (in general) otherwise.

Let $\mathbf{J}\to\mathrm{S}$ be a family of intermediate Jacobians as above and $\mathscr{J}$ the sheaf of holomorphic sections of this fibre space of complex tori. There is an obvious exact sequence

$$0\to\mathscr{C}(\Lambda)\to\mathscr{O}_{\mathrm{S}}(\mathbf{E}_+)\to\mathscr{J}\to 0 \tag{7.17}$$

where $\mathscr{C}(\Lambda)$ is the (locally-constant) sheaf of sections of the lattice $\Lambda$ over S. From (7.17) and the relations

$$\mathbf{D}\mathscr{C}(\Lambda)=0$$
$$\mathbf{D}\cdot\mathscr{O}_{\mathrm{S}}(\mathbf{F}^m)\subset\Omega^1_{\mathrm{S}}(\mathbf{F}^{m+1}),$$

we obtain a sheaf mapping

$$\mathrm{D}_{\mathbf{J}}:\mathscr{J}\to\Omega^1_{\mathrm{S}}(\mathbf{E}/\mathbf{F}^{m+1}). \tag{7.18}$$

The algebro-geometric significance of (7.18) will be discussed in Appendix A below (cf. Theorem (A.8)). We denote by $\mathscr{H}om(\mathrm{S},\mathbf{J})$ the sub-sheaf of sections $\nu\in\mathscr{J}$ which satisfy $\mathrm{D}_{\mathbf{J}}\nu=0$, and shall refer to sections in $\mathscr{H}om(\mathrm{S},\mathbf{J})$ as being *integrable*. In the abelian variety case ($m=0$), all holomorphic sections are integrable.

Suppose now that S is complete. Referring to Theorem (7.1) we see that:

$$\mathrm{H}^0(\mathrm{S},\mathscr{C}(\mathbf{E}))=\mathrm{H}^0(\mathrm{S},\mathscr{C}(\mathbf{F}^m))\oplus\overline{\mathrm{H}^0(\mathrm{S},\mathscr{C}(\mathbf{F}^m))}$$

and it follows that:

$$\mathrm{H}^0(\mathrm{S},\mathscr{C}(\mathbf{F}^m))\backslash\mathrm{H}^0(\mathrm{S},\mathscr{C}(\mathbf{E}))/\mathrm{H}^0(\mathrm{S},\mathscr{C}(\Lambda))=\mathrm{J}(\mathscr{E})$$

is a complex torus which we call the *trace* or *fixed part* of $\mathbf{J}\to\mathrm{S}$ (cf. Proposition (A.7) and the succeeding remark for an algebro-geometric interpretation of this fixed part).

*Theorem* **(7.19)** *(Mordell-Weil for families of intermediate Jacobians). — Let* $\mathbf{J}\to\mathrm{S}$ *be a family of intermediate Jacobians associated to a variation of Hodge structure* $\mathscr{E}$ *and lattice* $\Lambda$ *over a complete base space* S. *Then the group* $\mathrm{Hom}(\mathrm{S},\mathbf{J})=\mathrm{H}^0(\mathrm{S},\mathscr{H}om(\mathrm{S},\mathbf{J}))$ *of global, integrable cross-sections of* $\mathbf{J}\to\mathrm{S}$ *is an extension of the fixed part* $\mathrm{J}(\mathscr{E})$ *by a finitely generated abelian group.*

*Remark.* — The integrability condition $\mathrm{D}_{\mathbf{J}}\nu=0$ will be satisfied for any cross-section $\nu$ of $\mathbf{J}\to\mathrm{S}$ " which arises from algebraic cycles in case $\mathbf{J}\to\mathrm{S}$ comes from a geometric situation " where we refer to Appendix A for an explanation of the phrase in quotation marks (cf. Theorem (A.8)).

*Proof.* — From the exact cohomology sequence of (7.17) we have:

$$0\to\mathscr{K}\to\mathrm{Hom}(\mathrm{S},\mathbf{J})\to\mathrm{H}^1(\mathrm{S},\mathscr{C}(\Lambda))$$

where $\mathscr{H}$ is the vector space of holomorphic sections $\varphi$ of $\mathbf{E}_+$ which satisfy $\mathbf{D}\varphi \in \Omega^1_S(\mathbf{F}^{m+1})$. Since $H^1(S, \mathscr{C}(\Lambda))$ is finitely generated, our theorem follows from Lemma (6.19) in the same way that Theorem (7.1) followed from (5.9).

*Remark.* — The extension of this theorem to arbitrary base S is discussed in Appendix C (cf. (C.3)). In particular Theorem (C.12) in this appendix gives such an extension to arbitrary base in case $n = 1$, which is just the usual Mordell-Weil theorem (over function fields).

PART II

# DIFFERENTIAL-GEOMETRIC PROPERTIES OF THE PERIOD MAPPING

## 8. Classifying spaces for Hodge structures.

Let E be a complex vector space and

$$0 < h_0 \leqq h_1 \leqq \ldots \leqq h_{n-1} < h_n = \dim E$$

an increasing sequence of integers which is *self-dual* in the sense that

$$h_{n-q-1} = h_n - h_q \quad \text{for } 0 \leqq q \leqq n.$$

We also assume given a non-singular bilinear form

$$Q : E \otimes E \to \mathbf{C}, \qquad Q(e, e') = (-1)^n Q(e', e),$$

and consider the set $\check{D}$ of all filtrations

$$F^0 \subset F^1 \subset \ldots \subset F^{n-1} \subset F^n = E, \qquad \dim F^q = h_q,$$

which satisfy the *first Riemann bilinear relation*

$$(8.1) \qquad \begin{cases} (F^q)^{\perp} = F^{n-q-1}, & \text{or equivalently} \\ Q(F^q, F^{n-q-1}) = 0. \end{cases}$$

We will say that such filtrations are *isotropic* or *self-dual.*

*Proposition* (**8.2**). — *$\check{D}$ is, in a natural way, a projective and smooth complete algebraic variety which is a homogeneous space*

$$\check{D} = \mathbf{G}/\mathbf{B}$$

*of a complex simple Lie group* **G** *divided by a parabolic subgroup* **B**.

*Proof.* — Let $G(h, E)$ be the Grassman variety of $h$-planes through the origin in E. Observe that the filtration:

$$F^0 \subset \ldots \subset F^m, \qquad m = \left[\frac{n-1}{2}\right],$$

determines $F^0 \subset \ldots \subset F^n$ by using the first bilinear relation (8.1). From this we have an obvious projective embedding:

$$(8.3) \qquad \check{D} \to G(h_0, E) \times \ldots \times G(h_m, E)$$

which exhibits $\check{D}$ as a complete and projective algebraic variety.

We shall prove that $\check{D}$ is smooth by exhibiting the tangent space $\mathbf{T}_F(\check{D})$ to $\check{D}$ at a given point $F=(F^0, \ldots, F^n)$. First we recall the natural identification

$$\mathbf{T}_S(G(h, E)) \cong \mathrm{Hom}(S, E/S) \qquad (S \in G(h, E)).$$

The tangent space to $F \in G(h_0, E) \times \ldots \times G(h_m, E)$ is

$$\bigoplus_{q=0}^{m} \mathrm{Hom}(F^q, E/F^q), \qquad m=\left[\frac{n-1}{2}\right], \tag{8.4}$$

from which we see that $\mathbf{T}_F(\check{D})$ is given by all $f=\bigoplus_{q=0}^{m} f_q$ ($f_q \in \mathrm{Hom}(F^q, E/F^q)$) in (8.4) which satisfy the conditions that the diagrams

$$\begin{array}{ccc} F^q & \xrightarrow{f_q} & E/F^q \\ \downarrow & & \downarrow \\ F^{q+1} & \xrightarrow[f_{q+1}]{} & E/F^{q+1} \end{array} \qquad (q=0, 1, \ldots, m-1) \tag{8.5}$$

are commutative, and that we have

$$Q(f_m(e), e')+Q(e, f_m(e'))=0 \quad \text{for } e, e' \in F^m. \tag{8.6}$$

Now let $\mathbf{G} \subset GL(E)$ be the complex orthogonal group of the bilinear form $Q$; thus $\mathbf{G}$ is the complex simple Lie group of all linear transformations $T: E \to E$ which satisfy:

$$Q(Te, Te')=Q(e, e') \qquad (e, e' \in E).$$

Each $T \in \mathbf{G}$ induces an automorphism $T: \check{D} \to \check{D}$ by

$$T \cdot (F^0 \subset \ldots \subset F^n)=(TF^0) \subset \ldots \subset (TF^n).$$

This action of $\mathbf{G}$ on $\check{D}$ is transitive and the isotropy group $\mathbf{B}$ of a given point $F_0 \in \check{D}$ is a *parabolic subgroup* of $\mathbf{G}$. This gives the desired representation $\check{D}=\mathbf{G}/\mathbf{B}$.

*Remark* **(8.7)**. — We define an important holomorphic sub-bundle $\mathbf{I}_F(\check{D})$ of the complex tangent bundle $\mathbf{I}(\check{D})$ as follows: $\mathbf{I}_F(\check{D})$ consists of all $f=\bigoplus_{q=0}^{m} f_q$ in (8.4) which satisfy the *infinitesimal bilinear relation* (cf. (2.11))

$$Q(f_q e, e')=0 \qquad (e \in F^q,\ e' \in F^{n-q-2}). \tag{8.8}$$

We now assume given a *conjugation* $e \mapsto \bar{e}$ of E such that

$$Q(\bar{e}, \bar{e}')=\overline{Q(e, e')}.$$

In other words we are given that $E=E_{\mathbf{R}} \otimes \mathbf{C}$ where $Q$ is real on the real subspace $E_{\mathbf{R}}$ of vectors $e \in E$ which satisfy $e=\bar{e}$. Define the Hermitian inner product ( , ) in E by

$$(e, e')=(-i)^n Q(e, \bar{e}') \qquad (e, e' \in E). \tag{8.9}$$

We define the *period matrix domain* $D \subset \check{D}$ to consist of all isotropic filtrations $F^0 \subset \ldots \subset F^n$ which satisfy the *second Riemann bilinear relation*:

$$\textbf{(8.10)} \qquad \begin{cases} (\ ,\ ) : F^q \otimes F^q \to \mathbf{C} & \text{is non-singular and} \\ (-1)^q (\ ,\ ) : E^q \otimes E^q \to \mathbf{C} & \text{is positive definite} \\ \text{where } E^q = \{e \in F^q : (e, F^{q-1}) = 0\}. \end{cases}$$

We may combine (8.1) and (8.10) by saying that D is the set of filtrations $F^0 \subset \ldots \subset F^n$, with $\dim F^q = h_q$, which satisfy:

$$\textbf{(8.11)} \qquad \begin{cases} Q(F^q, F^{n-q-1}) = 0 \\ (-i)^n Q(F^q, \overline{F}^q) \text{ is non-singular} \\ (-1)^q (-i)^n Q(E^q, \overline{E}^q) > 0. \end{cases}$$

*Proposition* **(8.12)**. — D *is an open complex submanifold of* $\check{D}$ *which is a* homogeneous complex manifold

$$D = G/H$$

*of a real, simple, non-compact Lie group* G *divided by a compact subgroup* H.

*Proof.* — Let $G \subset \mathbf{G}$ be the *real form* of all real linear transformations $T : E \to E$ which preserve Q. Then, under the natural action of $\mathbf{G}$ on $\check{D}$, G leaves invariant, and acts transitively on, D. It is clear that the isotropy group $H = G \cap \mathbf{B}$ of a point $F_0 \in D$ is a compact subgroup of G.

*Definition.* — As mentioned D is called a period matrix domain and $\check{D}$ will be termed the *compact dual* of D.

It is clear that D parametrizes the *universal family of Hodge structures* determined by E, Q, the conjugation $e \mapsto \bar{e}$, and the numbers $h_q$. However, this universal family of Hodge structures over D is generally not a variation of Hodge structure in the sense of § 2 because of the infinitesimal bilinear relation (2.8).

We refer to [16] for a discussion of the group-theoretic properties of D and $\check{D}$, especially as regards the equivariant embedding:

$$\begin{array}{ccc} D & \longrightarrow & \check{D} \\ \| & & \| \\ G/H & \longrightarrow & \mathbf{G}/\mathbf{B} \end{array}$$

In the following examples we let G and H be as above, K will denote the *maximal compact subgroup* of G and $R = G/K$ the corresponding *Riemannian symmetric space*, and $h^q = h_q - h_{q-1}$ the *Hodge numbers*.

*Example* $\textbf{(8.13)}_e$. — When $n = 2m$ is even,

$$G = SO(a, b; \mathbf{R}) \qquad (a = h^0 + h^2 + \ldots + h^{2m},\ b = h^1 + h^3 + \ldots + h^{2m-1})$$

is the orthogonal group of the quadratic form $\sum_{i=1}^{a}(x_i)^2 - \sum_{j=1}^{b}(y_j)^2$, the compact isotropy group is

$$H = U(h^0) \times \ldots \times U(h^{m-1}) \times SO(h^m),$$

and the maximal compact subgroup of G is

$$K = SO(a; \mathbf{R}) \times SO(b; \mathbf{R}).$$

We may identify the Riemannian symmetric space $R = G/K$ with the set of real $a$-planes $S \subset E_{\mathbf{R}}$ such that $Q(S, S) > 0$. The equivariant fibering

$$\begin{array}{ccc} \varpi : D & \longrightarrow & R \\ \| & & \| \\ G/H & \longrightarrow & G/K \end{array}$$

is given by

**(8.14)$_e$** $$\varpi(F^0 \subset \ldots \subset F^{2m}) = E^0 \oplus E^2 \oplus \ldots \oplus E^{2m}.$$

*Example.* — When $n = 2m+1$ is odd,

$$G = Sp(2a; \mathbf{R}) \qquad (a = h^0 + \ldots + h^m)$$

is the group leaving the skew-form $\sum_{j=1}^{a}(x_j \wedge x_{a+j})$ invariant, the compact isotropy group is

$$H = U(h^0) \times \ldots \times U(h^m),$$

and the maximal compact subgroup of G is

$$K = U(a).$$

We may identify R with the set of complex $a$-planes $S \subset E$ which satisfy

$$\begin{cases} Q(S, S) = 0 \\ iQ(S, \overline{S}) > 0, \end{cases}$$

and the equivariant fibering $\varpi : D \rightarrow R$ is given by

**(8.14)$_o$** $$\varpi(F^0 \subset \ldots \subset F^{2m+1}) = E^0 \oplus E^2 \oplus \ldots \oplus E^{2m}.$$

Now according to (8.4), (8.5), (8.6), we may identify the tangent bundle to D as

**(8.15)** $$\mathbf{T}_F(D) = \bigoplus_{q=0}^{m} \bigoplus_{p=1}^{m-q} \mathrm{Hom}_Q(E^q, E^{p+q}) \qquad \left(F \in D,\ m = \left[\frac{n-1}{2}\right]\right).$$

The identification (8.15) is G-invariant, and the positive definite metrics $(-1)^q(\ ,\ )$ on $E^q$ induce a G-invariant Hermitian metric $ds^2_D$ on D. Group theoretically, $ds^2_D$ is the metric induced by the *Cartan-Killing form* on the Lie algebra of G [16].

*Proposition* **(8.16)**. — *In the equivariant fibering*

$$\begin{array}{ccc} \varpi : D & \longrightarrow & R \\ \| & & \| \\ G/H & \longrightarrow & G/K \end{array}$$

*of the period matrix domain* D *over the Riemannian symmetric space* R, *the fibre* $Z_{F_0}$ *through each point* $F_0 \in D$ *is a compact complex submanifold, and we have*

$$\mathbf{I}_{F_0}(D) \subset (\mathbf{T}_{F_0}(Z_{F_0}))^{\perp} \tag{8.17}$$

*where* $\mathbf{I}_{F_0}(D) \subset \mathbf{T}_{F_0}(D)$ *is given by* (8.8).

*Proof.* — We treat first the case when $n=2m$ is even. The point $\varpi(F_0)$ is the same as giving an orthogonal direct sum decomposition

$$E_{\mathbf{R}} = S_{\mathbf{R}} \oplus S_{\mathbf{R}}^{\perp}$$

if $E_{\mathbf{R}}$ such that Q is positive on $S_{\mathbf{R}}$ and negative on $S_{\mathbf{R}}^{\perp}$. We shall discuss the case when $m=2l$ is even — the other case is similar. The fibre $\varpi^{-1}(\varpi(F_0))$ is the homogeneous space

$$Z_{F_0} = \left(\frac{SO(a;\mathbf{R})}{U(h^0)\times\ldots\times U(h^{2l-2})\times SO(h^{2l})}\right) \times \left(\frac{SO(b;\mathbf{R})}{U(h^1)\times\ldots\times U(h^{2l-1})}\right)$$

and has the following geometric description:

$Z_{F_0}$ consists of all pairs of filtrations

$$\begin{cases} T^0 \subset T^2 \subset \ldots \subset T^{2l-2} \subset S, & \dim_{\mathbf{C}} T^{2p} = h^0 + \ldots + h^{2p} \\ T^1 \subset T^3 \subset \ldots \subset T^{2l-1} \subset S^{\perp}, & \dim_{\mathbf{C}} T^{2p+1} = h^1 + \ldots + h^{2p+1} \end{cases}$$

which satisfy $Q(T^{2l-2}, T^{2l-2})=0$ or $Q(T^{2l-1}, T^{2l-1})=0$ as the case may be. These filtrations define a point $F^0 \subset \ldots \subset F^{2m}$ in $\varpi^{-1}(\varpi(F_0)) \subset D$ by letting $F^0=T^0$, $F^1=T^0+T^1$, $F^2=T^2+T^3$, ..., on up through $F^{m-1}=T^{2l-2}+T^{2l-1}$. Then we let $F^m=(F^{m-1})^{\perp}$, $F^{m+1}=(F^{m-2})^{\perp}$, etc. It is clear that $Z_{F_0}$ is a compact, complex analytic submanifold passing through $F_0$ and that

$$\mathbf{T}_{F_0}(Z_{F_0}) = \bigoplus_{q=0}^{m-1} \mathrm{Hom}(E^q, E^{q+2}) \tag{8.18$_e$}$$

under the identification (8.15). From this, Proposition (8.16) and (8.17) are clear.

In case $n=2m+1$ is odd, the point $\varpi(F_0)$ is given by a subspace $S \subset E$, $\dim_{\mathbf{C}} S = a = \frac{1}{2}\dim_{\mathbf{C}} E$, which satisfies

$$\begin{cases} Q(S,S)=0 \\ iQ(S,\overline{S})>0. \end{cases}$$

The fibre $\tilde{\omega}^{-1}(\tilde{\omega}(F_0))$ is the homogeneous space

$$Z_{F_0} = \frac{U(a)}{U(h^0) \times \ldots \times U(h^m)},$$

and may be described as all filtrations:

$$T^0 \subset T^2 \subset \ldots \subset T^{2m} \subset S, \qquad \text{with} \qquad \dim_{\mathbf{C}} T^{2q} = h^0 + \ldots + h^{2q}.$$

These filtrations define a point $F^0 \subset \ldots \subset F^{2m+1}$ in $Z_{F_0} \subset D$ by letting $F^0 = T^0$, $F^1 = T^0 + (T^0)^{\perp} \cap \overline{S}$, etc., on up to $F^m$. Then $F^{m+1} = (F^m)^{\perp}, \ldots, F^{2m} = (F^0)^{\perp}$. Clearly

$$\textbf{(8.18)}_0 \qquad \mathbf{T}_{F_0}(Z_{F_0}) = \bigoplus_{q=0}^{m} \operatorname{Hom}(E^q, E^{q+2}),$$

and Propositions (8.16) and (8.17) follow.

*Remark.* — It may be noted that, except for the cases $n=1$ or $n=2$ and $h^0=1$, the fibres of $\tilde{\omega}$ are non-trivial, so that D is not a bounded domain in $\mathbf{C}^N$. Also, except for the case $n=2$, the inclusion (8.17) is strict, so that there are additional conditions on a variation of Hodge structure other than transversality to the fibres of $\tilde{\omega}$.

## 9. Statement of results on variation of Hodge structure and period mappings.

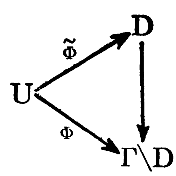
*a)* Let $\mathscr{E} = (\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$ be a variation of Hodge structure, with base space S, as defined in § 2. Letting E be the complex vector space $\mathbf{E}_{s_0}$ and taking the conjugation, bilinear form, and Hodge numbers $h^q = \dim_{\mathbf{C}} \mathbf{F}^q_{s_0}/\mathbf{F}^{q-1}_{s_0}$ induced on E by $\mathscr{E}$, we may define a period matrix domain D as in § 8.

We now recall that the *holonomy group* of the flat connection **D** induces, by parallel displacement of a flat frame, the *monodromy representation*

$$\rho : \pi_1(S) \to G$$

of the fundamental group of S (based at $s_0$) in the automorphism group G of D as defined in § 8. The image Γ of $\pi_1(S)$ in G will be called the *monodromy group* of $\mathscr{E}$.

A continuous mapping $\Phi : S \to \Gamma\backslash D$ will be said to be *locally liftable* if, given $s \in S$, there exists a neighborhood U of $s$ and a continuous mapping $\tilde{\Phi} : U \to D$ such that the diagram




is commutative. A locally liftable mapping Φ is holomorphic if the local liftings $\tilde{\Phi}$ are holomorphic, and a locally liftable holomorphic mapping Φ is said to satisfy the *infinitesimal period relation* if the local liftings $\tilde{\Phi}$ satisfy

$$\textbf{(9.1)} \qquad \tilde{\Phi}_*(\tau) \in \mathbf{I}_{\tilde{\Phi}(u)} \qquad (u \in U,\ \tau \in \mathbf{T}_u(U)),$$

where $\widetilde{\Phi}_*$ is the differential of $\widetilde{\Phi}$ and $\mathbf{I}\subset\mathbf{T}(\mathrm{D})$ is defined by (8.8). We may symbolically rewrite (9.1) as:

$$(9.2) \qquad \Phi_*: \mathbf{T}(\mathrm{S}) \to \mathbf{I}(\mathrm{D}).$$

*Proposition* **(9.3)**. — *The giving of a variation of Hodge structure $\mathscr{E}$ with monodromy group $\Gamma$ is equivalent to giving a locally liftable holomorphic mapping*

$$(9.4) \qquad \Phi: \mathrm{S} \to \Gamma\backslash \mathrm{D}$$

*which satisfies the infinitesimal period relation* (9.2).

*Definition.* — We call $\Phi$ in (9.4) the *period mapping* associated to $\mathscr{E}$. This terminology is explained in [11].

*b)* Our first result on variation of Hodge structure as interpreted by the period mapping is

*Theorem* **(9.5)** *(Extension of period mapping around branches of finite order).* — *Let $\mathscr{E}$ be a variation of Hodge structure over* S *and let* D *be an irreducible branch of* $\overline{\mathrm{S}}-\mathrm{S}$ *such that the associated Picard-Lefschetz transformation* T *is of finite order* (cf. § 3). *Localize the period mapping* (9.4) *around a simple point* $\bar{s}\in\mathrm{D}$ *to obtain* $\Phi: \mathrm{P}^* \to \Gamma\backslash\mathrm{D}$ (cf. § 1). *Then*

$$\mathrm{P}^* \cong \Delta^* \times \Delta^{d-1}$$

*is the product of a punctured disc with a polycylinder, and there exists a finite covering*

$$\widetilde{\mathrm{P}}^* \to \mathrm{P}^* \qquad (\widetilde{\mathrm{P}}^* \cong \Delta^* \times \Delta^{d-1})$$

*and a lifting* $\widetilde{\Phi}: \widetilde{\mathrm{P}}^* \to \mathrm{D}$ *of the period mapping* $\Phi$ *such that* $\widetilde{\Phi}$ *extends holomorphically to the closed polycylinder* $\widetilde{\mathrm{P}} \cong \Delta \times \Delta^{d-1}$.

To state our second main result, we assume that the monodromy group $\Gamma$ is a *discrete subgroup* of G. This is the case if $\mathscr{E}$ arises from a geometric situation $f: \mathrm{X} \to \mathrm{S}$. With this assumption, the quotient space $\Gamma\backslash\mathrm{D}$ is a *complex space* or *analytic space* in the sense of [19]. In fact, the projection $\Gamma\backslash\mathrm{G} \to \Gamma\backslash\mathrm{G}/\mathrm{H}$ is a proper mapping and so $\Gamma$ acts properly discontinuously on D. Thus $\Gamma\backslash\mathrm{D}$ is a separated topological space which is locally the quotient of a polycylinder by a finite group.

Let $\Phi: \mathrm{S} \to \Gamma\backslash\mathrm{D}$ be the period mapping (9.4). By Theorem (9.5) we may extend this period mapping to a holomorphic mapping $\Phi: \mathrm{S}' \to \Gamma\backslash\mathrm{D}$, where S′ is the union of S with those points at infinity around which the Picard-Lefschetz transformations are of finite order.

*Theorem* **(9.6)** *(Analyticity of the image of the period mapping).* — *The image* $\Phi(\mathrm{S}')$ *is a closed analytic subvariety of* $\Gamma\backslash\mathrm{D}$ *which contains* $\Phi(\mathrm{S})$ *as the complement of an analytic subvariety. Furthermore, the volume* $\mu_{\Gamma\backslash\mathrm{D}}(\Phi(\mathrm{S}'))$ *of* $\Phi(\mathrm{S}')$, *computed with respect to the invariant metric on* D, *is finite.*

Our third result is

*Theorem* **(9.7)**. — *Let $\mathscr{E}$ be a variation of Hodge structure whose monodromy group* $\Gamma$ *is discrete and whose base* S *is complete. Then the image* $\Phi(S) \subset \Gamma \backslash D$ *is a complete projective algebraic variety. In fact the canonical bundle* $\mathbf{K}(\mathscr{E})$ (cf. (7.13)) *of* $\mathscr{E}$ *is* ample *over* $\Phi(S)$.

*Remark.* — Using Theorem (9.6), we see that (9.7) remains true if we only assume that all the P.-L. transformations around the branches of S at infinity are of finite order.

This theorem follows from (7.15) and the results of Grauert [10]. We refer also to [13], § 10 for a discussion of this theorem together with some related open questions.

As our final result we give a theorem about the monodromy group $\Gamma$ of a variation of Hodge structure $\mathscr{E} = (\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$.

*Theorem* **(9.8)** *(theorem about the monodromy group of a variation of Hodge structure).* — *Assume that either the Picard-Lefschetz transformations are all of finite order or that* $\mathscr{E}$ *arises from a geometric situation. Then*

(i) *the global monodromy group* $\Gamma$ *is completely reducible*;

(ii) $\Gamma$ *is finite if, and only if, the variation of Hodge structure is trivial*; *and*

(iii) *if* $\Gamma$ *is solvable, then it is finite.*

*Proof.* — The first statement follows from Theorem (7.1) in the same way as (3.7) followed from (3.6). The third statement follows from the first by Grothendieck's argument given in Deligne [9]. Finally, the second statement follows from (9.5) and the fact that a horizontal, holomorphic mapping $\Phi : Z \rightarrow D$ from a compact, complex manifold Z to a period matrix domain D is constant [11].

*c)* We will give some local statements which will imply (9.5) and (9.6). For the first we let

$$\mathbf{H}(D) \subset \mathbf{T}(D)$$

be the *horizontal sub-bundle* defined by

$$\textbf{(9.8)} \qquad \mathbf{H}_{F_0}(D) = (\mathbf{T}_{F_0}(Z_{F_0}))^{\perp} \quad \text{(cf. (8.18))}.$$

The word " horizontal " follows from the fact that $\mathbf{H}(D)$ is the complement to the bundle along the fibres in $\tilde{\omega} : D \rightarrow R$. A locally liftable holomorphic mapping $\Phi : S \rightarrow \Gamma \backslash D$ is *horizontal* if we have

$$\textbf{(9.9)} \qquad \Phi_* : \mathbf{T}(S) \rightarrow \mathbf{H}(D)$$

in the same sense as (9.2).

Theorem (9.5) follows from Proposition (9.3) and

*Proposition* **(9.10)**. — *Let* $P^* = \Delta^* \times \Delta^{d-1}$ *be the product of a punctured disc with a polycylinder, and let* $\Phi : P^* \rightarrow D$ *be a horizontal, holomorphic mapping. Then* $\Phi$ *extends to a holomorphic mapping* $\Phi : \Delta^d \rightarrow D$.

We now claim that Theorem (9.6) follows from the *proper mapping theorem* [19] together with the following

*Proposition* **(9.11)**. — *Let* $\Delta(\rho)$ *be the disc* $0 \leqq |z| < \rho$, $\Delta^*(\rho)$ *the corresponding punctured disc, and* $P(k, l; \rho) = (\Delta^*(\rho))^k \times (\Delta(\rho))^l$ *the product. We shall refer to* $P(k, l; \rho)$ *as a* punctured polycylinder. *Let* $\Phi : P(k, l; \rho) \to \Gamma\backslash D$ *be a locally liftable holomorphic mapping.*

(i) *Let* $\gamma_1, \ldots, \gamma_k$ *be the canonical generators of* $\pi_1(P(k, l; \rho))$ *and* $T_j = \Phi_*(\gamma_j) \in \Gamma$ *the corresponding Picard-Lefschetz transformations. Assume that* $T_1, \ldots, T_j$ *are of infinite order and* $T_{j+1}, \ldots, T_k$ *are of finite order. Let* $\{z_n\} = \{(z_n^1, \ldots, z_n^{k+l})\} \in P(k, l; \rho)$ *be a sequence of points with* $\inf_{1 \leqq \alpha \leqq j} |z_n^\alpha| \to 0$ *as* $n \to \infty$. *Then the sequence* $\{\Phi(z_n)\} \in \Gamma\backslash D$ *does not converge.*

(ii) *The volume* $\mu_{\Gamma\backslash D}(\Phi(P(k, l; \rho/2)))$ *is finite.*

*Proof of* (9.6) *from* (9.11). — We claim that $\Phi : S' \to \Gamma\backslash D$ is a proper mapping. If not, there is a divergent sequence $\{s_n\} \in S'$ such that $\Phi(s_n)$ converges in $\Gamma\backslash D$. We may assume that $\{s_n\}$ converges to some point $\bar{s} \in \bar{S} - S'$. By localizing around $\bar{s}$ and using (i) in Proposition (9.11), we arrive at a contradiction.

The proof that the volume $\mu_{\Gamma\backslash D}(\Phi(S'))$ is finite follows from (ii) in (9.11) by localizing around $\bar{S} - S'$ and an obvious compactness argument.

## 10. The generalized Schwarz lemma.

Let $P \subset \mathbf{C}^d$ be the polycylinder $\{(z_1, \ldots, z_d) : 0 \leqq |z_j| < 1\}$ of unit radius, and denote by $ds_P^2$ the standard *Poincaré metric* given by:

$$ds_P^2 = \sum_{i=1}^{d} \frac{dz_i\, d\bar{z}_i}{(1 - |z_i|^2)^2}.$$

Denote by $\omega_P$ the *associated 2-form* $\dfrac{\sqrt{-1}}{2}\left\{\sum_{i=1}^{d} \dfrac{dz_i \wedge d\bar{z}_i}{(1-|z_i|^2)^2}\right\}$ so that $(\omega_P)^d$ is the non-Euclidean volume of P.

Let D be a period matrix domain with (suitably normalized) invariant metric $ds_D^2$ and associated 2-form $\omega_D$. We want to prove

*Theorem* **(10.1)** *(generalized Schwarz lemma).* — *Let* $\Phi : P \to D$ *be a horizontal, holomorphic mapping. Then we have*

$$\textbf{(10.2)} \qquad \begin{cases} \Phi^*(ds_D^2) \leqq ds_P^2 \\ \Phi^*(\omega_D)^d \leqq (\omega_P)^d. \end{cases}$$

*Proof.* — We first show

*Lemma* **(10.3)**. — *If the volume estimate* $\Phi^*(\omega_D)^d \leqq (\omega_P)^d$ *holds for* $d = 1$, *then we have the distance estimate* $\Phi^*(ds_D^2) \leqq ds_P$.

*Proof.* — Let $\psi : \Delta \to P$ be the embedding of the unit $z$-disc into P given by

$$\psi(z) = (\alpha_1 z, \ldots, \alpha_d z), \qquad \text{with} \qquad \sum_{i=1}^{d} |\alpha_i|^2 = 1.$$

Then $\psi^*(ds_P^2) = \left(\sum_{i=1}^{d} \dfrac{|\alpha_i|^2}{(1 - |\alpha_i|^2 |z|^2)^2}\right) dz\, d\bar{z}$, so that at the origin $z = 0$ we find

$$\textbf{(10.4)} \qquad \psi^*(ds_P^2)_0 = dz\, d\bar{z} = (ds_\Delta^2)_0.$$

If $\tau$ is a tangent vector to P at $(0, \ldots, 0)$, then we can find $\alpha_j$ as above such that $\psi_*\left(\lambda\frac{\partial}{\partial z}\right)=\tau$ for a suitable $\lambda \neq 0$. Then the length $\left\|\lambda\frac{\partial}{\partial z}\right\|_\Delta = \|\tau\|_P$ by the isometry property (10.4). Using the volume decreasing assumption applied to the holomorphic curve $\Phi\circ\psi : \Delta \to D$, we have $\|\Phi_*\tau\|_D = \left\|\Phi_*\psi_*\lambda\frac{\partial}{\partial z}\right\|_D \leq \left\|\lambda\frac{\partial}{\partial z}\right\|_\Delta = \|\tau\|_P$, or $\|\Phi_*\tau\|_D \leq \|\tau\|_P$ which is what we want to prove.

We will base our proof of the volume estimate on a formula of Chern [7]. To explain his formula we let M and N be $d$-dimensional complex Hermitian manifolds and $f: M \to N$ a holomorphic mapping. Using unitary frames as in [7], we write

$$\textbf{(10.5)} \qquad \begin{cases} ds_M^2 = \sum_{i=1}^{d} \omega_i \bar{\omega}_i \\ ds_N^2 = \sum_{j=1}^{d} \theta_j \bar{\theta}_j . \end{cases}$$

Then $f^*(\theta_j) = \sum_{i=1}^{d} a_j^i \omega_i$ and we have

$$\textbf{(10.6)} \qquad f^*(\theta_N)^d = |\det(a_j^i)|^2 (\omega_M)^d$$

where $\omega_M$ and $\theta_N$ are the respective 2-forms associated to the metrics (10.5). The non-negative function $u = |\det(a_j^i)|^2$ is the ratio of the volume elements, and we are looking for a formula for the *Laplacian* $\Delta \log u$ near a point $m_0 \in M$ where $u(m_0) > 0$. (Recall that the Laplacian $\Delta f$ of a function $f$ is defined by $i\partial\bar{\partial}f = (\Delta f)\cdot\omega_M$.)

The desired formula involves the *Ricci form* $\mathrm{Ric}_N$ of N and *scalar curvature* $R_M$ of M. To explain these terms, we recall that the metrics (10.5) induce intrinsic Hermitian geometries (cf. Lemma (4.1)) on N and M and we let

$$\textbf{(10.7)} \qquad \begin{cases} \Omega_{ij} = \frac{1}{2}\sum_{k,l} R_{ijkl}\, \omega_k \wedge \bar{\omega}_l & (R_{ijkl} = \bar{R}_{jilk}) \\ \Theta_{ij} = \frac{1}{2}\sum_{k,l} S_{ijkl}\, \theta_k \wedge \bar{\theta}_l & (S_{ijkl} = \bar{S}_{jilk}) \end{cases}$$

be the *curvature forms* of the metrics in M and N respectively (cf. (4.8) and [7]). The Ricci form is defined by:

$$\textbf{(10.8)} \qquad \mathrm{Ric}_N = \sum_{i=1}^{d} \Omega_{ii} = \frac{1}{2}\left(\sum_{i,k,l} R_{iikl}\, \omega_k \wedge \bar{\omega}_l\right),$$

and the scalar curvature is given by:

$$\textbf{(10.9)} \qquad R_M = \frac{1}{2}\left(\sum_{i,j} R_{iijj}\right) = \mathrm{Trace}(\mathrm{Ric}_M)$$

*Theorem* **(10.10)** *(Chern [7]). — In a neighborhood of a point $m_0 \in M$ where $u(m_0) > 0$, we have*

$$\frac{1}{2}\Delta \log u = R_M - \mathrm{Trace}(f^*\mathrm{Ric}_N).$$

Return now to our period mapping $\Phi : P \to D$. Given $z_0 \in P$, either $\Phi^*(\omega_D)^d(z_0) = 0$ and (10.2) is trivially true, or else $\Phi^*(\omega_D)(z_0) \neq 0$ in which case the differential $\Phi_*$ of $\Phi$ is injective at $z_0$ and the image $W = \Phi(U)$ of a small neighborhood $U$ of $z_0$ is a $d$-dimensional complex manifold with Hermitian metric $ds^2_W$ induced from $ds^2_D$. Denote by $\omega_W$ the associated 2-form.

*Lemma* **(10.11)**. — *Let* $\mathrm{Ric}_{\mathbf{H}}$ *and* $\omega_{\mathbf{H}}$ *be the restrictions of the Ricci form* $\mathrm{Ric}_D$ *and 2-form* $\omega_D$ *to the horizontal sub-bundle* $\mathbf{H}(D) \subset \mathbf{T}(D)$. *Suppose that we have*

$$\mathrm{Ric}_{\mathbf{H}} \leqq -c(\omega_{\mathbf{H}}) \qquad (c > 0).$$

*Then, keeping the situation and notation from just above this lemma, we have the estimate:*

$$\mathrm{Ric}_W \leqq -c(\omega_W).$$

*Proof.* — By the definition (10.8), $\mathrm{Ric}_D = \mathrm{Trace}(\Theta_D)$ where $\Theta_D$ is the metric connexion of the given Hermitian metric in the tangent bundle of D. We have then an inclusion of bundles:

$$\mathbf{T}(W) \subset \mathbf{H}(D) \subset \mathbf{T}(D) \qquad (\text{over } W),$$

and we need to compare $\mathrm{Trace}(\Theta_D)$ and $\mathrm{Trace}(\Theta_W)$. The comparison of the curvatures $\Theta_W$, $\Theta_{\mathbf{H}}$, $\Theta_D$ is given by Lemma (4.13), from which it follows that:

$$\mathrm{Trace}(\Theta_W) \leqq \mathrm{Trace}(\Theta_{\mathbf{H}}) \leqq \mathrm{Trace}(\Theta_D) \qquad (\text{in } \mathbf{T}(W)).$$

Since $\omega_W = \omega_D$ restricted to $\mathbf{T}(W)$, our lemma is proved.

We now use the computation given in [16], § 7 to prove:

*Lemma* **(10.12)**. — *In the notation of Lemma* (10.11), *we have* $\mathrm{Ric}_{\mathbf{H}} \leqq -\omega_{\mathbf{H}}$. *This gives that*:

$$\mathrm{Ric}_W \leqq -\omega_W$$

*for the image manifold* $W = \Phi(U)$ *as described just above Lemma* (10.11).

We are now ready to prove the generalized Schwarz lemma (10.1). Let $P(\rho)$ be the polycylinder of radius $\rho$ given as usual by $\{z = (z_1, \ldots, z_d) : 0 \leqq |z_i| < \rho\}$ and $ds^2_P(\rho) = 4\left(\sum_{i=1}^{d} \frac{\rho^2 dz_i d\bar{z}_i}{(\rho^2 - |z_i|^2)^2}\right)$ the Poincaré metric on $P(\rho)$. We have made a slight change of scale from our original definition. With this change of scale, the scalar curvature $R_P$ of the metric $ds^2_P(\rho)$ is the constant $-d$. When $\rho = 1$ we write $ds^2_P$ for $ds^2_P(1)$ and let $\omega_P$ be the associated 2-form.

Define the non-negative function $u(z)$ on P by $\Phi^*(\omega_D)^d = u \cdot (\omega_P)^d$. We want to show that $u \leqq 1$. The idea, which is originally due to Ahlfors, is to use the maximum principle.

We first show that it suffices to consider the case when $u$ attains its maximum at some point in the interior of P. Let $z_0 \in P$. Then $z_0 \in P(\rho)$ for some $\rho < 1$ and we may define $u_\rho(z)$ in $P(\rho)$ by $\Phi^*(\omega_D)^d = u_\rho \cdot (\omega_P(\rho))^d$. Then $\lim_{\rho \to 1} u_\rho(z_0) = u(z_0)$ because of $\lim_{\rho \to 1} ds^2_P(\rho)(z_0) = ds^2_P(z_0)$. Thus it suffices to prove that $u_\rho(z_0) \leqq 1$ for $\rho < 1$. Now, for $\rho<1$, $\Phi^*(ds_D^2)$ is bounded on the closed polycylinder $\overline{P(\rho)}$, while clearly $(\omega_P(\rho))^d$ goes to infinity at the boundary $\overline{P}-P$. Thus $u_\rho(z)$ goes to zero as $z$ goes to the boundary, and so $u_\rho$ has its maximum at an interior point.

We now assume that $u$ has a maximum at $z_0 \in P$. Then by Chern's formula (10.10)

**(10.13)** $$0 \geqq \frac{1}{2}\Delta \log u = -d - \mathrm{Trace}(f^*\mathrm{Ric}_W).$$

Now using orthonormal co-frames (10.5) in the situation when $M=P$ and $N=W=\Phi(U)$ with U a neighborhood of $z_0$ in P, we have from Lemma (10.12) that

$$-\mathrm{Ric}_W = -(\sum_{i,j} S_{ij}\theta_i \wedge \overline{\theta}_j) \geqq \sum_i \theta_i \wedge \overline{\theta}_i$$

and so

**(10.14)** $$-f^*(\mathrm{Ric}_W) \geqq \sum_{i,j,k} a_i^j \overline{a}_i^k \omega_j \wedge \overline{\omega}_k.$$

Letting A be the matrix $(a_j^i)$, from (10.14) we find that

**(10.15)** $$-\mathrm{Trace}(f^*\mathrm{Ric}_W) \geqq \mathrm{Trace}(A.{}^t\overline{A}).$$

Now use the *Hadamard inequality* $\mathrm{Trace}(A.{}^t\overline{A}) \geqq d|\det A|^{2/d} = du^{1/d}$ together with (10.15) and (10.13) to find $1 \geqq u^{1/d}$, which is what we wanted to prove.

*Remark* **(10.16)**. — As in Proposition (9.11) we let $P(k,l;\rho)$ be the product $(\Delta_\rho^*)^k \times (\Delta_\rho)^l$ where $\Delta_\rho^*$ is the punctured disc $0<|z|<\rho$ and $\Delta_\rho$ is the usual disc $0 \leqq |z| < \rho$. We set $P^* = P(k,l;1)$ and $P = P(k+l, 0; 1)$. Then $P \to P^*$ is, in the usual way, the universal covering and so the Poincaré metric $ds_P^2$ induces a metric $ds_{P^*}^2$ on $P^*$. Letting $z_i = r_i \exp \theta_i$ be polar coordinates, we have explicitly that

**(10.17)** $$ds_{P^*}^2 = \left(\sum_{i=1}^{k} \frac{dr_i^2 + r_i^2 d\theta_i^2}{r_i^2(\log r_i)^2}\right) + \left(\sum_{j=k+1}^{k+l} \frac{dz_k d\overline{z}_k}{(1-|z_k|^2)^2}\right),$$

and for the volume element

**(10.18)** $$(\omega_{P^*})^{k+l} = \prod_{i=1}^{k} \frac{dr_i d\theta_i}{r_i(\log r_i)^2} \prod_{j=k+1}^{l} \frac{dr_j d\theta_j}{(1-r_j^2)^2}.$$

From (10.18) we have

*Lemma* **(10.19)**. — *For* $\rho<1$, *the volume* $\mu_{P^*}(P(k,l;\rho)) < \infty$ *of the sub-polycylinder* $P(k,l;\rho) \subset P^*$ *is finite.*

The use of the following lemma was first demonstrated by Mrs. Kwack [25]:

*Lemma* **(10.20)**. — *For* $0<\rho<1$, *let* $\sigma_\rho$ *be the circle* $|z|=\rho$ *in the punctured disc* $\Delta^*$ *given by* $0<|z|<1$. *Then the length* $l_{\Delta^*}(\sigma_\rho)$ *of* $\sigma_\rho$, *computed using the Poincaré metric* $ds_{\Delta^*}^2$ *on* $\Delta^*$, *is given by* $l_{\Delta^*}(\sigma_\rho) = \dfrac{2\pi}{\log\left(\frac{1}{\rho}\right)}$. *In particular,* $l(\sigma_\rho) \to 0$ *as* $\rho \to 0$.

*Proof.* — This follows immediately from (10.17), which gives that $ds_{\Delta^*}^2 = \dfrac{dr^2 + r^2 d\theta^2}{r^2(\log r)^2}$ in the situation at hand.

As a Corollary of Theorem (10.1) and Lemmas (10.19) and (10.20), we have:

*Corollary* **(10.21)**. — *Let* $\Phi : P^* \to \Gamma\backslash D$ *be a horizontal, holomorphic mapping. Then the volume* $\mu_{\Gamma\backslash D}(\Phi(P(k, l; \rho)))$ *of the image of the concentric punctured polycylinder* $P(k, l; \rho) \subset P^*$ *is finite. In particular,* (ii) *of Proposition* (9.11) *is valid.*

*Corollary* **(10.22)**. — *Let* $P^* = \Delta^* \times \Delta^{d-1}$ *and let* $\Phi : P^* \to D$ *be a horizontal holomorphic mapping. Suppose that* $\gamma_\rho$ *is a curve in* $P^*$ *given parametrically by* $\theta \mapsto (\rho e^{i\theta}, z_2(e^{i\theta}), \ldots, z_d(e^{i\theta}))$ *where the* $z_j(e^{i\theta})$ *are smooth curves in the unit disc* $0 \leqq |z_j| < 1$. *Then the length* $l_D(\Phi(\gamma_\rho))$ *of the image curve tends to zero as* $\rho \to 0$.

## 11. Proof of Propositions (9.10) and (9.11.)

*a)* We first prove (i) in Proposition (9.11). For simplicity we will consider the case $k=1$, $l=0$. The general situation will be done by the exact same argument.

Thus we have a locally liftable, horizontal, holomorphic mapping $\Phi : \Delta^* \to \Gamma\backslash D$ such that the Picard-Lefschetz transformation $T \in \Gamma$ is of infinite order. We assume given a sequence $\{z_n\} \in \Delta^*$ with $|z_n| \to 0$ and such that $\{\Phi(z_n)\}$ converges in $\Gamma\backslash D$. We want to show that this leads to a contradiction.

Let $\sigma_n$ be the circle $|z| = |z_n|$ and set $w_n = \Phi(z_n) \in \Gamma\backslash D$. We may assume that $w_n$ tends to a point $w \in \Gamma\backslash D$.

Choose a point $\overline{w} \in D$ lying over $w$ in the projection $\pi : D \to \Gamma\backslash D$. The stabilizer $\Gamma_{\overline{w}} = \{g \in \Gamma : g \cdot \overline{w} = \overline{w}\}$ of $\overline{w}$ is a *finite* group, and we may choose neighborhoods U of $w$ in $\Gamma\backslash D$ and $\overline{U}$ of $\overline{w}$ in D such that $\Gamma_{\overline{w}} \cdot \overline{U} = \overline{U}$ and $\pi^{-1}(U)$ is the disjoint union $\bigcup_{g \in \Gamma/\Gamma_{\overline{w}}} g \cdot \overline{U}$. We may assume also that the distance $d_D(\overline{U}, g\overline{U}) \geqq \varepsilon$ from $\overline{U}$ to its translates is bounded below for $g \in \Gamma - \Gamma_{\overline{w}}$. Finally, since $T \in \Gamma$ is of infinite order, we may assume that the intersection $T\overline{U} \cap \overline{U}$ is empty.

Choose $n$ so large that the non-Euclidean length $l_{\Delta^*}(\sigma_n)$ is less than $\varepsilon$. This is possible by Lemma (10.20). By Corollary (10.22) the length $l_D(\Phi(\sigma_n))$ of the image curve may also be assumed to be less than $\varepsilon$. Finally we may assume that the image $w_n$ of $z_n$ under $\Phi$ lies in U.

Choose $\overline{w}_n \in \overline{U}$ which projects onto $w_n$. Now take a local lifting $\widetilde{\Phi}$ of $\Phi$ in a neighborhood of $z_n$ such that $\widetilde{\Phi}(z_n) = \overline{w}_n$. Analytic continuation of $\widetilde{\Phi}$ around the circle $\sigma_n$ passing thru $z_n$ leads to the new local lifting $T \cdot \widetilde{\Phi}$ around $z_n$. This is a contradiction since the length of the image curve $\widetilde{\Phi}(\sigma_n)$ is less than $\varepsilon$, which implies that $d_D(\widetilde{\Phi}(z_n), T\widetilde{\Phi}(z_n)) < \varepsilon$, while we have $d_D(\overline{U}, T \cdot \overline{U}) > \varepsilon$.

Because of this contradiction we have proved (i) in Proposition (9.11), and the other part of this proposition has been given in Corollary (10.21).

*b)* We want to prove Proposition (9.10). For simplicity we assume that $d=1$; the general case is done by essentially the same argument.

Thus we assume given a horizontal, holomorphic mapping $\Phi : \Delta^* \to D$. We want to show that $\Phi$ extends to a continuous mapping of $\Delta$ into D. Our proof is based on the following result of Mrs. Kwack [25]:

*Proposition* (**11.1**) *(Mrs. Kwack). — Let* M *be a* compact *complex manifold with Hermitian metric* $ds_M^2$. *Let* $f: \Delta^* \rightarrow M$ *be a holomorphic mapping with the property that if* $\{\sigma_n\} \subset \Delta^*$ *is any sequence of circles* $|z| = \rho_n$ *whose radii* $\rho_n$ *tend to zero, then the lengths* $l_M(f(\sigma_n))$ *of the image circles tend to zero. Then f extends to a continuous mapping* $f: \Delta \rightarrow M$.

*Remark.* — The interesting thing about this result is that it is not at all a topological statement. The fact that M is a complex manifold seems to be quite essential. For completeness we shall give a proof of (11.1) below.

We now use (11.1) to prove our extension theorem for $\Phi: \Delta^* \rightarrow D$. Recalling that $D = G/H$ is a homogeneous complex manifold of a real simple Lie group G by a compact subgroup H, we select a discrete subgroup $\Lambda$ of G such that the quotient $\Lambda \backslash G$ is compact and such that $\Lambda$ acts without fixed points on G/H. The existence of such a *uniform subgroup* $\Lambda$ follows from a general result of Borel and Harish-Chandra [4]. Or in our case we could use the theorem in [28] to write down such a $\Lambda$.

The quotient $M = \Lambda \backslash D$ is now a compact, complex manifold M with an Hermitian metric $ds_M^2$ induced from the G-invariant $ds_D^2$ on D. From Corollary (10.22) it follows that the conditions of (11.1) are satisfied by the mapping $f: \Delta^* \rightarrow \Lambda \backslash D$ obtained by composing $\Phi$ with the projection $D \rightarrow \Lambda \backslash D$. Thus $f$ extends to give a continuous mapping $f: \Delta \rightarrow \Lambda \backslash D$. From this it follows that $\Phi$ extends to give our desired continuous mapping $\Phi: \Delta \rightarrow D$.

*Remark.* — The use of the uniform subgroup $\Lambda$ in the above proof is not as absurd as it might at first appear. To explain what I mean, we recall the embedding $D \subset \check{D}$ of D as an open domain in its compact dual. It is not too hard to show that our mapping $\Phi: \Delta^* \rightarrow D$ extends to a continuous mapping $\Phi: \Delta \rightarrow \bar{D}$. The trouble is that the image $\Phi(0)$ of the origin might lie in the boundary $\partial D = \bar{D} - D$ of D in $\check{D}$. So our extension theorem is really a question of the *pseudo-convexity* of D. Now for a bounded domain B in $\mathbf{C}^n$, it is a theorem of Siegel [29] that the existence of a properly discontinuous group $\Psi$ of automorphisms of B such that $\Psi \backslash B$ is compact already implies that B is a domain of holomorphy. Thus, if $B \subset \mathbf{C}^n$ is a bounded domain such that we have a holomorphic mapping $\Phi: \Delta^* \rightarrow B$, and if there exists a uniform subgroup $\Psi \subset \mathrm{Aut}(B)$, then $\Phi$ extends to $\Phi: \Delta \rightarrow B$ because of the usual Riemann extension theorem plus Siegel's theorem. Our proof of Proposition (9.10) is essentially a similar argument.

*c)* We now give a proof, which is essentially that of [25], of Proposition (11.1). We use the notation $\sigma(z_0)$ for the circle $|z| = |z_0|$ passing thru $z_0 \in \Delta^*$.

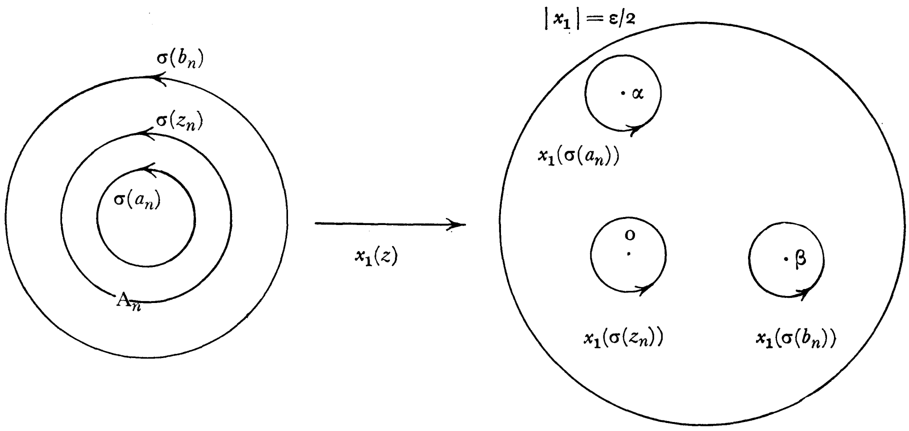
Let $\{z_n\} \subset \Delta^*$ be a sequence of points with $|z_n| \rightarrow 0$. If we set $w_n = f(z_n)$, then by the compactness of M we may assume that $w_n \rightarrow w \in M$. Let $x_1, \ldots, x_m$ be local holomorphic coordinates centered at $w \in M$ and denote by $U(\rho)$ the polycylinder $|x_j| < \rho$

around $w$. We have to show that, given $\varepsilon > 0$, there exists $\delta$ such that $f(z) \in U(\varepsilon)$ if $0 < |z| < \delta$.

Let $\varepsilon > 0$ be given. Since the lengths $l_M(f(\sigma(z_n)))$ of the images of the circles $\sigma(z_n)$ tend to zero, and since $w_n \in f(\sigma_n)$ tends to $w$, we may assume that $f(\sigma(z_n)) \subset U(\varepsilon/2)$ for all $n$.

If we cannot find the required $\delta$, then, by renumbering if necessary, we may find a sequence $\{y_n\} \in \Delta^*$ with $|z_{n+1}| < |y_n| < |z_n|$ such that $f(y_n)$ does not lie in $U(\varepsilon)$. Let $A_n$ be a maximal annulus $\alpha_n < |z| < \beta_n$ around $\sigma(z_n)$ such that $f(A_n) \subset U(\varepsilon/2)$. Then the $A_n$ are all disjoint, and we may choose $a_n \in \sigma(\alpha_n)$ and $b_n \in \sigma(\beta_n)$ with $f(a_n)$ and $f(b_n)$ lying in the boundary $\partial U(\varepsilon/2)$ of the polycylinder $U(\varepsilon/2)$. Passing to subsequences, we may assume that $f(a_n) \to a \in \partial U(\varepsilon/2)$ and $f(b_n) \to b \in \partial U(\varepsilon/2)$. Then $f(\sigma(a_n)) \to a$ and $f(\sigma(b_n)) \to b$ by the argument using lengths of circles.

Write $f(z) = (x_1(z), \ldots, x_m(z))$ and let $\alpha = x_1(a)$, $\beta = x_1(b)$. We may assume that $\alpha \neq 0$, $\beta \neq 0$. Using the $x_1$-coordinates, we have a picture




For $n$ sufficiently large, we find from the argument principle that:

$$\int_{|z| \in \sigma(a_n)} \frac{x_1(z)\,dz}{x_1(z) - x_1(z_n)} = 0 = \int_{|z| \in \sigma(b_n)} \frac{x_1(z)\,dz}{x_1(z) - x_1(z_n)}.$$

This is a contradiction, since the difference of these two integrals is the integral:

$$\int_{\partial A_n} \frac{x_1(z)\,dz}{x_1(z) - x_1(z_n)} \neq 0.$$

## APPENDIX A

### A result on algebraic cycles and intermediate Jacobians

*a)* Let V be a smooth, complete, and projective algebraic variety, and consider the odd degree cohomology

$$H^{2m+1}(V, \mathbf{C}).$$

For simplicity we will discuss the case when $H^{2m+1}(V, \mathbf{C})$ is all primitive — the general situation is essentially a " direct sum " of such cases. We set

$$H_{+}^{2m+1}(V, \mathbf{C}) = H^{2m+1,0}(V) + \ldots + H^{m+1,m}(V)$$

and define the ($m^{th}$) *intermediate Jacobian* J(V) by

$$(A.1) \qquad J(V) = H_{+}^{2m+1}(V, \mathbf{C}) \backslash H^{2m+1}(V, \mathbf{C}) / H^{2m+1}(V, \mathbf{Z}).$$

As references on the theory of intermediate Jacobians we mention [14], [27] and [23].

Denote by $\Theta(V)$ the group of algebraic cycles (modulo *rational equivalence*) on V which are of pure codimension $m+1$ and which are homologous to zero. We will define an *Abel-Jacobi homomorphism*

$$(A.2) \qquad \psi : \Theta(V) \rightarrow J(V),$$

which generalizes the usual mapping for divisors on curves ($m=0$ and $\dim_{\mathbf{C}} V = 1$). Before defining $\psi$ we need a result of Dolbeault about Hodge filtrations (cf. the appendix to [14] and the references given there). Let $A^{n,q}$ be the $C^{\infty}$ forms of type $(n, 0) + \ldots + (n-q, q)$ on V and $Z^{n,q}$ the $d$-closed forms in $A^{n,q}$. Observe that $d(A^{m,p}) \subset A^{m+1,p+1}$ and set

$$(A.3) \qquad F^{n,q} = Z^{n,q}/dA^{n-1,q-1}.$$

*Proposition* (*A*.4). — *The natural mapping*

$$F^{n,q} \rightarrow H^{n}(V, \mathbf{C})$$

*is injective with image* $H^{n,0}(V) + \ldots + H^{n-q,q}(V)$.

Suppose now that $\dim_{\mathbf{C}} V = d$ $\left(\text{then } m \leqq \frac{d-1}{2}\right)$ and let $\omega_1, \ldots, \omega_l$ be a basis for

$$(A.5) \qquad F^{2d-2m-1,d-m} \cong H^{d,d-2m-1}(V) + \ldots + H^{d-m,d-m-1}(V).$$

Observe that $F^{2d-2m-1,d-m}$ is the dual space to the tangent space

$$H^{m,m+1}(V) + \ldots + H^{0,2m+1}(V)$$

of J(V), and so we may think of $F^{2d-2m-1,d-m}$ as the space of holomorphic differentials on J(V). Letting $Z \in \Theta(V)$, we define the Abel-Jacobi map (A.2) by

$$(A.6) \qquad \psi(Z) = \begin{pmatrix} \vdots \\ \int_C \omega_\alpha \\ \vdots \end{pmatrix} \quad \text{(modulo periods)}$$

where C is a chain on V with $\partial C = Z$.

We recall from [14], [27], [23] that $\psi$ is holomorphic (in a suitable sense), and $\psi$ has nice functorial properties. In particular, suppose that X is a smooth, complete and projective algebraic variety which contains V as a smoothly embedded subvariety. The restriction map $H^{2m+1}(X, \mathbf{C}) \to H^{2m+1}(V, \mathbf{C})$ of cohomology induces a homomorphism of intermediate Jacobians

$$r: J(X) \to J(V)$$

with the following interpretation:

*Proposition* (*A*.**7**). — *Intersecting cycles on* X *with* V *induces a homomorphism* $\iota: \Theta(X) \to \Theta(V)$ *such that the diagram:*

$$\begin{array}{ccc} \Theta(X) & \xrightarrow{\psi} & J(X) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle r} \\ \Theta(V) & \xrightarrow{\psi} & J(V) \end{array}$$

*is commutative.*

*Remark.* — Let $f: X \to S$ be an algebraic family of algebraic varieties and assume that S is complete. Suppose that V is a fixed fibre of $f: X \to S$, and let $\mathcal{E} = (\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$ be the variation of Hodge structure whose fibre corresponding to V is $H^{2m+1}(V, \mathbf{C})$. Then the image in the mapping

$$J(X) \to J(V)$$

is precisely the *fixed part* $J(\mathcal{E})$ as defined in § 7, *c)*. Proposition (A.7) gives an algebro-geometric interpretation of this fixed part.

*b)* Let $f: X \to S$ be an algebraic family of algebraic varieties with fibres $V_s = f^{-1}(s)$ $(s \in S)$. We shall think of V, as just discussed in *a)* above, as being a typical fibre. For simplicity we shall continue to assume that all of $H^{2m+1}(V, \mathbf{C})$ is primitive.

Let $\mathcal{E} = (\mathbf{E}, \mathbf{D}, Q, \{\mathbf{F}^q\})$ be the variation of Hodge structure associated to $f: X \to S$ and $H^{2m+1}(V, \mathbf{C})$, and let $\Lambda \subset \mathbf{E}_{\mathbf{R}}$ be the flat lattice given by the images of

$$H^{2m+1}(V_s, \mathbf{Z}) \to H^{2m+1}(V_s, \mathbf{R}).$$

Referring to § 7, *c)*, the corresponding family of intermediate Jacobians

$$\pi: \mathbf{J} \to S, \quad J_s = J(V_s)$$

will be said to *arise from a geometric situation.*

Let now $\Theta(X/S)$ be the sheaf which associates to each open set $U \subset S$ the group of analytic cycles Z (modulo rational equivalence) of codimension $m+1$ in $f^{-1}(U)$ such that the cohomology class of Z is zero in $H^{2m+1}(f^{-1}(U), \mathbf{Z})$ (cf. [23]).

*Theorem* (*A*.**8**) *(Integrability theorem for Abel-Jacobi maps). — The Abel-Jacobi maps* (A.6) *induce a sheaf mapping*

$$\Psi : \Theta(X/S) \to \mathcal{J}$$

*which satisfies the integrability condition* $D_J \Psi = 0$.

*Proof.* — Our proof of the existence of $\Psi$ is based on [23]. We may assume that $\dim_{\mathbf{C}} S = 1$ and we let $f : Y \to \Delta$ be the situation $f : X \to S$ localized over a small disc $\Delta$ on S which has holomorphic coordinate $s$. We let $Z \subset Y$ be an algebraic cycle of codimension $m+1$ and in general position with respect to the fibres $V_s$. Then the intersections $Z_s = Z \cdot V_s$ are algebraic cycles of codimension $m+1$ on $V_s$ which are homologous to zero there. In fact, we have:

$$Z \equiv \partial C \quad (\text{modulo } \partial Y)$$

for a suitable chain C on Y, and we may put things in general position so that

$$Z_s = \partial C_s \text{ where } C_s = C \cdot V_s \quad (s \in \Delta);$$

(cf. [23] for a complete discussion of the foundational points involved here).

Let $\psi(Z_s) \in J(V_s)$ be the point defined by the Abel-Jacobi map (A.2). We want to prove that $\psi(Z_s)$ depends holomorphically on $s$. For this we choose $C^\infty$ differential forms $\omega_1, \ldots, \omega_l$ on Y such that

(i) each $\omega_\alpha$ is of type $(d, d-2m-1) + \ldots + (d-m, d-m-1)$;

(ii) $d\omega_\alpha \wedge ds = 0$; and

(iii) the restrictions $\omega_\alpha | V_s = \omega_\alpha(s)$ give a basis of $F^{2d-2m-1, d-m-1}(V_s)$ (cf. Proposition (A.4)).

The existence of such forms is proved in [23], where it is also proved that the integrals $\int_{C_s} \omega_\alpha(s)$ may be assumed to depend continuously on $s$.

Let $\omega$ be any linear combination of $\omega_1, \ldots, \omega_l$. We want to show that the integral $\int_{C_s} \omega$ depends holomorphically on $s$. Let $\gamma$ be a simple, positively oriented, closed curve in the disc $\Delta$. It will suffice to show that

$$\int_\gamma \left( \int_{C_s} \omega \right) ds = 0.$$

Let $C_\gamma = C \cap f^{-1}(\gamma)$ and $Z_\gamma$ be the intersection of Z with the part of Y lying over the region inside $\gamma$. Then by Stokes' theorem:

$$\int_\gamma \left( \int_{C_s} \omega \right) ds = \int_{C_\gamma} \omega \wedge ds = -\int_{Z_\gamma} \omega \wedge ds$$

since $d\omega \wedge ds = 0$. But $\int_{Z_\gamma} \omega \wedge ds = 0$ since $\omega \wedge ds$ is of type

$$(d+1, d-2m-1) + \ldots + (d-m+1, d-m-1)$$

whereas $Z_Y$ is an analytic set of complex dimension $d-m$. This proves the existence of the sheaf homomorphism $\Psi : \Theta(X/S) \to \mathscr{J}$ obtained by fitting together the Abel-Jacobi maps along the fibres of $f : X \to S$.

We want now to prove the integrability condition $D_J \Psi = 0$. For this we first localize to have $f : Y \to \Delta$ as before, and then choose $C^\infty$ differential forms $\omega_1, \ldots, \omega_{2l}$ on Y such that

(i) $d\omega_j \wedge ds = 0$ $(j = 1, \ldots, 2l)$;

(ii) the restrictions $\omega_j | V_s$ give a basis of $H^{2d-2m-1}(V_s, \mathbf{C})$;

(iii) $\omega_1, \ldots, \omega_l$ are of type $(d, d-2m-1) + \ldots + (d-m, d-m-1)$ and restrict to a basis of $F^{2d-2m-1, d-m-1}(V_s)$; and

(iv) $\omega_1, \ldots, \omega_k$ are of type $(d, d-2m-1) + \ldots + (d-m+1, d-m-2)$ and give a basis of $F^{2d-2m-1, d-m-2}(V_s)$.

We may think of $\omega_1, \ldots, \omega_{2l}$ as a holomorphic frame for the flat bundle $\check{\mathbf{E}}$ with fibres $\check{\mathbf{E}}_s = H^{2d-2m-1}(V, \mathbf{C})$ and which is adapted to the filtration

$$\mathbf{F}^{d-m-2} \subset \mathbf{F}^{d-m-1} \subset \check{\mathbf{E}}.$$

We let

$$D\omega_j = \sum_{i=1}^{2l} \theta_j^i \omega_i \tag{A.9}$$

be the connection in $\check{\mathbf{E}}$. Then $\theta_j^i = \sigma_j^i(s) ds$ where the functions $\sigma_j^i(s)$ on $\Delta$ have the following interpretation: Write

$$d\omega_i = \eta_i \wedge ds$$

where the $\eta_i$ are $C^\infty$ forms on Y. This is possible by the first property of the $\omega_i'$s. Then $d\eta_i \wedge ds = 0$ so that $\eta_i | V_s$ is closed and gives a cohomology class in $H^{2d-2m-1}(V_s, \mathbf{C})$, and we have

$$\eta_i = \sum_{j=1}^{2l} \sigma_i^j \omega_j \quad \text{in} \quad H^{2d-2m-1}(V_s, \mathbf{C}). \tag{A.10}$$

We can even assume that

$$\eta_i = \sum_{j=1}^{l} \sigma_i^j \omega_j \quad \text{in} \quad F^{2d-2m-1, d-m-1}(V_s) \quad (1 \leq i \leq k). \tag{A.11}$$

*Lemma* (*A*.12). — *Let* $\Gamma_s \in H_{2d-2m-1}(V_s, \mathbf{Z})$ *be a cycle varying smoothly with* $s \in \Delta$. *Then*

$$d\left(\int_{\Gamma_s} \omega_j\right) = \sum_{i=1}^{2l} \left(\int_{\Gamma_s} \omega_i\right) \theta_j^i.$$

*Proof.* — We have:

$$\frac{\partial}{\partial s}\left(\int_{\Gamma_s} \omega_j\right) = \lim_{t \to 0} \frac{1}{t}\left(\int_{\Gamma_{s+t}} \omega_j - \int_{\Gamma_s} \omega_j\right)$$

where we may restrict $t$ to be real and positive. Let $\Gamma(s, s+t)$ be the union of the cycles $\Gamma_\tau$ for $s \leqq \tau \leqq s+t$. By Stokes' theorem,

$$\begin{aligned}\frac{\partial}{\partial s}\left(\int_{\Gamma_s}\omega_j\right) &= \lim_{t\to 0}\frac{1}{t}\left(\int_{\Gamma(s,\, s+t)} d\omega_j\right)\\ &= \lim_{t\to 0}\frac{1}{t}\left(\int_{\Gamma(s,\, s+t)} \eta_j \wedge ds\right)\\ &= \int_{\Gamma_s}\eta_j\\ &= \sum_{i=1}^{2l}\left(\int_{\Gamma_s}\omega_i\right)\sigma_j^i(s) \qquad \text{(by (A.10)).}\end{aligned}$$

This completes the proof of Lemma (A.12).

*Lemma* (*A*.**13**). — *Using the notation established above, we have*

$$d\left(\int_{C_s}\omega_j\right) = \sum_{i=1}^{l}\left(\int_{C_s}\omega_i\right)\theta_j^i \quad \textit{for } 1 \leqq j \leqq k.$$

*Proof.* — As in the proof of Lemma (A.12) we let $C(s, s+t)$ the union of the $C_\tau$ for $s \leqq \tau \leqq s+t$. Then

$$\partial C(s, s+t) = C_{s+t} - C_s - Z(s, s+t)$$

where $Z(s, s+t)$ is union of the cycles $Z_\tau$ for $s \leqq \tau \leqq s+t$. Now $\int_{Z(s,\, s+t)}\omega_j = 0$ since $\omega_j$ is of type $(d, d-2m-1)+\ldots+(d-m+1, d-m-2)$. Using Stokes' theorem we then have

$$\begin{aligned}\frac{\partial}{\partial s}\left(\int_{C_s}\omega_j\right) &= \lim_{t\to 0}\frac{1}{t}\left(\int_{C(s,\, s+t)} \eta_j \wedge ds\right)\\ &= \int_{C_s}\eta_j\\ &= \sum_{i=1}^{l}\left(\int_{C_s}\omega_i\right)\sigma_j^i \qquad \text{(by (A.11)).}\end{aligned}$$

This completes the proof of Lemma (A.13).

We now choose a frame $e_1, \ldots, e_{2l}$ for the dual bundle $\mathbf{E}$ such that

$$\langle \omega_j, e_{2l+1-i}\rangle = \delta_j^i \qquad (1 \leqq i, j \leqq 2l).$$

Observe that the fibre

$$\mathbf{E}_s = H^{2m+1}(V_s, \mathbf{C})$$

and that we have

$$\begin{cases}(\omega_1, \ldots, \omega_l)^\perp = (e_1, \ldots, e_l)\\ (\omega_1, \ldots, \omega_k)^\perp = (e_1, \ldots, e_{2l-k})\end{cases}$$

so that $e_1, \ldots, e_l$ is a basis for $F^{2m+1,m}(V_s)$ and $e_1, \ldots, e_{2l-k}$ is a basis for $F^{2m+1,m+1}(V_s)$. Writing

$$\mathbf{D}e_i = \sum_{j=1}^{2l}\varphi_i^j e_j,$$

we have the relation

(*A*.**14**) $$\theta_i^j = -\varphi_{2l+1-j}^{2l+1-i}.$$

We may now prove that $D_J \Psi = 0$. The vector

$$\Omega = \sum_{j=1}^{l} \left( \int_{C_s} \omega_j \right) e_{2l+1-j}$$

is a section of **E** which projects onto $\Psi$ in the mapping $\mathbf{E} \to \mathbf{J}$. We want to compute $\mathbf{D}\Omega$ and then show that

(*A*.**15**) $$\mathbf{D}\Omega \equiv 0 \quad \text{modulo } e_1, \ldots, e_{2l-k}.$$

Using the notation " $\equiv$ " for " congruent modulo $e_1, \ldots, e_{2l-k}$ ", we have

$$\begin{aligned} D\Omega &\equiv \sum_{j=1}^{k} d\left( \int_{C_s} \omega_j \right) e_{2l+1-j} + \sum_{j=1}^{l} \left( \int_{C_s} \omega_j \right) \mathbf{D} e_{2l+1-j} \\ &\equiv \sum_{j=1}^{l} \sum_{i=1}^{l} \left( \int_{C_s} \omega_i \right) \theta_j^i e_{2l+1-j} + \sum_{j=1}^{l} \sum_{i=1}^{k} \left( \int_{C_s} \omega_j \right) \varphi_{2l+1-j}^{2l+1-i} e_{2l+1-i} \qquad \text{(by Lemma (A.13))} \\ &\equiv 0 \qquad \text{(by (A.14))}. \end{aligned}$$

This completes our proof.

## APPENDIX B

### Two Examples

a) *A family of curves.* — We shall construct and then discuss an example due to Atiyah [1] of an algebraic family of curves $f: X \to S$ where the parameter space S is itself a *complete* curve.

To construct the example, we take a smooth, complete curve C having a fixed-point-free involution $j: C \to C$. Such curves exist whenever the genus $p(C) \geqq 3$. Now we take S to be the finite unramified abelian covering of C given by the composite homomorphism $\pi_1(C) \to H_1(C, \mathbf{Z}) \to H_1(C, \mathbf{Z}_2)$. Let $\pi: S \to C$ be the covering map, $Y = S \times C$ the product variety, and $D = \Gamma_\pi + \Gamma_{j \circ \pi}$ the (non-singular) curve on Y which is the sum of the graph of $\pi$ and the graph of $j \circ \pi$. Atiyah shows that there is a non-singular algebraic surface X which is a 2-sheeted covering of Y with branch curve D. The projection $f: X \to S$ then gives X as an algebraic family of algebraic curves $\{V_s\}_{s \in S}$ where $V_s$ is a 2-sheeted covering of C with branch points at $\pi(s)$ and $j \circ \pi(s)$.

Now for us the main important thing is the existence of a non-trivial family of non-singular curves with a complete parameter space. Let $f: X \to S$ be one such family where the corresponding variation of Hodge structure is non-trivial (i.e., the fibres $V_s$ are not all birationally equivalent). The sheaf $R^1_{f_*}(\mathcal{O}_X)$ is the sheaf $\mathcal{O}(\check{\mathbf{E}}^0)$ and from (7.11) we have

$$(B.1) \qquad c_1(\mathbf{E}^0)[S] > 0.$$

We want to compute the *signature* sign(X), and to do this we use the Hirzebruch index formula for X and the Grothendieck-Riemann-Roch formula for $f: X \to S$ and $\mathcal{O}_X$ as in Atiyah [1] to obtain

$$(B.2) \qquad \begin{cases} \operatorname{sign}(X) = \dfrac{d^2}{3}[X] \\ \operatorname{ch}(1 - R^1_{f_*}(\mathcal{O}_X)) = f_*\left(1 + d + \dfrac{d^2}{12}\right), \end{cases}$$

where $d \in H^2(X, \mathbf{Z})$ is the first Chern class of the tangent bundle along the fibres of $f: X \to S$. From (B.1) and (B.2) we have

$$(B.3) \qquad \operatorname{sign}(X) = c_1(\mathbf{E}^0)[S] > 0,$$

so that the signature is not multiplicative in the fibration $f: X \to S$.

The exact same proof will give the

*General principle* (*B*.4). — Let $f: X \to S$ be an algebraic family of algebraic varieties with complete parameter space S. Then the Hirzebruch $\chi_y$-genus [21] is generally *not* multiplicative for the fibration $f: X \to S$.

Another point we are trying to illustrate is that there are interesting examples of algebraic families of algebraic varieties with a complete parameter space, although the most interesting case is certainly when the fibres are allowed to have arbitrary singularities.

b) *Lefschetz pencils of algebraic surfaces.* — In order to illustrate the existence of algebraic families of algebraic varieties $f: X \to S$ whose parameter space need not be complete but where the Picard-Lefschetz transformations are of finite order, we consider a smooth, complete, and projective threefold $W \subset \mathbf{P}_N$. A generic pencil $|\mathbf{P}_{N-1}(\lambda)|_{\lambda \in \mathbf{P}_1}$ of linear hyperplanes in $\mathbf{P}_N$ traces out on W a pencil $|V_\lambda|_{\lambda \in \mathbf{P}_1}$ of surfaces with critical points $\lambda_1, \ldots, \lambda_N$. Letting $S = \mathbf{P}_1 - \{\lambda_1, \ldots, \lambda_N\}$, the $V_\lambda$ $(\lambda \in S)$ are non-singular surfaces while the $V_{\lambda_\alpha}$ are surfaces having one isolated ordinary double point. In the obvious way we may construct an algebraic family of algebraic varieties $f: X \to S$ with $f^{-1}(\lambda) = V_\lambda$ for $\lambda \in S$. This family has the property that there is a smooth compactification

$$\begin{array}{ccc} X & \subset & \bar{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ S & \subset & \bar{S} \end{array}$$

such that $\bar{f}$ has one of the local forms

$$\begin{cases} \bar{f}(x_1, x_2, x_3) = x_1 \\ \bar{f}(x_1, x_2, x_3) = (x_1)^2 + (x_2)^2 + (x_3)^2 \end{cases}$$

where $x_1, x_2, x_3$ are suitably chosen local holomorphic coordinates on $\bar{X}$.

We want to use the theorems in Lefschetz [26] to amplify two of our results above. Before doing this, we fix a base point $\lambda_0 \in S$ and paths $l_\alpha$ from $\lambda_0$ to each critical point $\lambda_\alpha$ $(\alpha = 1, \ldots, N)$. We let $\gamma_\alpha \in \pi_1(S)$ be the closed curve obtained by going out $l_\alpha$, turning around $\lambda_\alpha$, and then returning to $\lambda_0$ along $l_\alpha$. Associated to each path $l_\alpha$, there is a *vanishing cycle* $\delta_\alpha \in H^2(V_{\lambda_0}, \mathbf{Z})$ such that

$$(B.5) \qquad T_\alpha \varphi = \varphi \pm (\delta_\alpha, \varphi)\delta_\alpha$$

where $T_\alpha$ is the automorphism of $H^2(V_{\lambda_0}, \mathbf{Z})$ corresponding to $\gamma_\alpha \in \pi_1(S)$ ([26], p. 93). We have, furthermore, that (*loc. cit.*, p. 93):

$$(B.6) \qquad (\delta_\alpha, \delta_\alpha) = -2,$$

so that $(T_\alpha)^2 = I$ and the Picard-Lefschetz transformations in our family of surfaces $f: X \to S$ are all of finite order.

*Proposition* (*B*.7). — *There is a* $\pi_1(S)$*-invariant orthogonal direct sum decomposition*

$$H^2(V_{\lambda_0}, \mathbf{Q}) = I \oplus E \tag{B.8}$$

*where* $I = H^2(V_{\lambda_0}, \mathbf{Q})^{\pi_1(S)}$ *are the invariant cycles and where* $\pi_1(S)$ *acts irreducibly on* $E = (I)^{\perp}$.

*Proof*. — Let $E' \subset E$ be a non-trivial $\pi_1(S)$-invariant subspace and $\varphi \neq 0$ a vector in $E'$. From (B.5) we have that

$$(\delta_\alpha, \varphi)\delta_\alpha \in E' \qquad (\alpha = 1, \ldots, N)$$

while some $(\delta_{\alpha_0}, \varphi) \neq 0$ since $\varphi$ is orthogonal to the invariant cycles. Thus $\delta_{\alpha_0} \in E'$ and it follows that all $\delta_\alpha \in E'$ since $\pi_1(S)$ acts transitively on the set $\{\delta_1, \ldots, \delta_N\}$ of vanishing cycles (*loc. cit.*, p. 107). Thus $E = E'$ since E is the span of $\delta_1, \ldots, \delta_N$ (*loc. cit.*, p. 93).

Our second observation is

*Proposition* (*B*.9). — *Let* $f: X \to S$ *be the family of surfaces constructed above and let* $\Gamma \subset \mathrm{Aut}(H^2(V_{s_0}, \mathbf{C}))$ *be the monodromy group. Then* $\Gamma$ *is a finite group if, and only if, the subspace* $H^{2,0}(V_{\lambda_0})$ *of* $H^2(V_{\lambda_0}, \mathbf{C})$ *is elementwise invariant.*

*Proof*. — If $H^{2,0}(V_{\lambda_0})$ is elementwise invariant, then we have $E \subset H^{1,1}(V_{\lambda_0})_0$ in (B.8). Since the intersection form is negative definite on E, we see that $\Gamma$ is a finite group. Conversely, if $\Gamma$ is a finite group, then the subspace $H^{2,0}(V_\lambda)$ of $H^2(V_\lambda, \mathbf{C})$ is locally constant. In particular $H^{2,0}(V_{\lambda_0})$ is a $\pi_1(S)$-invariant subspace of $H^2(V_{\lambda_0}, \mathbf{C})$. Let $\varphi \in H^{2,0}(V_{\lambda_0})$. Then from (B.5) we have that $(\varphi, \delta_\alpha)\delta_\alpha \in H^{2,0}(V_{\lambda_0})$ for $\alpha = 1, \ldots, N$. If some $(\varphi, \delta_{\alpha_0}) \neq 0$, then $\delta_{\alpha_0} \in H^{2,0}(V_{\lambda_0})$ which is impossible by (B.6). Thus all $(\varphi, \delta_\alpha) = 0$ and so $\varphi$ is an invariant cycle.

## APPENDIX C

### Discussion of some open questions

a) *Statement of conjectures.* — Many of our results about a variation of Hodge structure had restrictions imposed concerning the Picard-Lefschetz transformations around the branches of $\bar{S}-S$. *We should like to suggest that these theorems should be valid under much more general circumstances.*

To state things precisely, we first need a few comments about the monodromy group $\Gamma$ of the variation of Hodge structure $\mathcal{E}$, especially with regard to the Picard-Lefschetz transformations (§ 3) of $\Gamma$. Recall the *monodromy theorem* (cf. § 3 in [13] for discussion and references), which says that in case $\mathcal{E}$ arises from a geometric situation $f: X \to S$, the Picard-Lefschetz transformations T are *essentially unipotent of index n* (same $n$ as in § 2), which means that viewed as automorphisms $T: E \to E$ they satisfy the equation

$$(C.1) \qquad (T^N - I)^{n+1} = 0 \quad \text{for some } N > 0.$$

We also recall that in the geometric case the monodromy group is a *discrete subgroup* of the automorphism group G of the variation of Hodge structure. Finally we recall the theorem of Borel (§ 3 in [13]) which says that in case $\Gamma$ is an *arithmetic subgroup* of G, then the monodromy theorem holds, but without the estimate on the index of unipotency being established as yet.

Our precise conjectures are (cf. the remark added in proof at the end of Appendix C):

**(*C*.2)** The invariant cycle theorem (7.1) is true if we assume only that the monodromy theorem (C.1) is valid.

*Remark.* — We shall prove this conjecture for $n=1$ in a little while. If this conjecture is true, then the theorem (9.8) about the monodromy group would also hold.

**(*C*.3)** The Mordell-Weil type theorem (7.19) is true if we only assume the monodromy theorem (C.1), but where we need to say what it means for a holomorphic section of $\mathbf{J} \to S$ to remain holomorphic at infinity.

*Remark.* — We shall also prove this conjecture for $n=1$ below.

**(*C*.4)** Theorem (9.7), which says that the image $\Phi(S) \subset \Gamma \backslash D$ under the period mapping is canonically a projective algebraic variety in case S is complete, is true if we only assume that $\Gamma$ is a discrete subgroup of G.

*Remark.* — This conjecture is discussed in §§ 10, 11 of [3]. We also refer to § 6 of [13] where it is pointed out that this conjecture (C.4) is valid in case D is a *bounded, symmetric domain* and Γ is an arithmetic subgroup of G (Borel).

Now how is one supposed to prove the above conjectures? My own (obvious) feeling is that it should be possible to prove (C.2) and (C.3) by sufficiently ingenious use of hyperbolic complex analysis so as to be able to give a good asymptotic form of the period mapping as we go to infinity. We shall give an illustration of this below. To prove (C.4), one will need the estimates from hyperbolic complex analysis as well as a *reduction theory* for discrete subgroups of G. It also seems to me that " sufficiently ingenious use of hyperbolic complex analysis " will involve a detailed study of the geodesics of the metric $ds^2_D$ on the period matrix domain D as well as a more refined Schwarz lemma (10.1) which will give suitable estimates both ways in (10.2).

b) *Proof of the invariant cycle theorem* (7.1) *for* $n=1$. — Thus let $\mathscr{E}$ be a variation of Hodge structure, with base space S, and let Φ be a flat section of $\mathbf{E}\rightarrow S$. The Hodge filtration in this case is $\mathbf{F}^0_s\subset\mathbf{F}^1_s=\mathbf{E}_s$, and we let φ be the projection of Φ in $\mathbf{E}/\mathbf{F}^0=\mathbf{E}^1$. Using theorem (5.9), we want to prove that the length $|\varphi|^2$ of φ is uniformly bounded on S.

We may assume that $\dim_{\mathbf{C}} S=1$. We then localize over a punctured disc $\Delta^*$ at infinity given by $\Delta^*=\{s:0<|s|<1\}$. Choose a base point $s_0\in\Delta^*$ and let $\omega_1, \ldots, \omega_{2m}$ be a flat frame for $\mathbf{E}$ in a neighborhood of $s_0$. Parallel displacement of this frame around the origin induces an automorphism

$$\omega_j\rightarrow\sum_{k=1}^{2m}T^k_j\omega_k$$

where $T=(T^k_j)$ is the Picard-Lefschetz transformation around $s=0$. The monodromy theorem (C.1) is $(T^N-I)^2=0$, and by replacing $s$ with $s^N$, we may as well assume that $(T-I)^2=0$.

Now we may choose over $\Delta^*$ a holomorphic frame $\varphi_1(s), \ldots, \varphi_m(s)$ for the sub-bundle $\mathbf{F}^0\subset\mathbf{E}$. Then we define the *period matrix* $\Omega(s)=(\pi_{\alpha j}(s))$ by

$$\varphi_\alpha(s)=\sum_{j=1}^{2m}\pi_{\alpha j}(s)\omega_j \qquad (\alpha=1, \ldots, m).$$

The bilinear relations (2.7) now become the usual Riemann relations

$$(C.5)\qquad \begin{cases}\Omega Q\,{}^t\Omega=0\\ i\Omega Q\,{}^t\bar{\Omega}=J>0\end{cases}$$

where ${}^tQ^{-1}=(Q(\omega_i, \omega_j))$. The matrix $\Omega(s)$ is a multi-valued holomorphic matrix on $\Delta^*$ such that analytic continuation around $s=0$ changes Ω into ΩT.

Let γ be the flat section of the dual bundle $\mathbf{E}^*$ defined by

$$\langle\gamma, e\rangle=Q(\Phi, e) \qquad (e\in\mathbf{E}).$$

Let $\xi$ be the column vector given by

$$ {}^t\xi=(\xi_1, \ldots, \xi_m) \quad \text{where} \quad \xi_\alpha=\langle \gamma, \varphi_\alpha(s)\rangle. $$

Then, if we set $H=J^{-1}$ in (C.5), ${}^t\xi H\xi$ is a well-defined function on $\Delta^*$ and gives the length $|\varphi|^2$ of the projection $\varphi$ of $\Phi$ onto $\mathbf{E}/\mathbf{F}^0$. Thus we want to show that

$$ (C.6) \qquad {}^t\xi H\xi \text{ is bounded as } s\to 0. $$

We will now use the results of [13], § 13 to put $\Omega$ in canonical form. Accordingly we can choose the frames $\omega_1, \ldots, \omega_{2m}$ and $\varphi_1, \ldots, \varphi_m$ such that the matrices Q, T, and $\Omega$ are given by

$$ (C.7) \qquad Q=\begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}; $$

$$ (C.8) \qquad T=\begin{pmatrix} I_k & 0 & 0 & 0 \\ 0 & I_{m-k} & 0 & \Lambda \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & I_{m-k} \end{pmatrix} \qquad \Lambda={}^t\Lambda,\ \Lambda>0; $$

$$ (C.9) \qquad \Omega=(I_m, Z) $$

where $Z={}^tZ$ has the form

$$ (C.10) \qquad Z=\begin{pmatrix} Z_{11} & Z_{12} \\ {}^tZ_{12} & Z_{22} \end{pmatrix}+\left(\frac{\log s}{2\pi i}\right)\begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix} $$

where the submatrices $Z_{\alpha\beta}$ are holomorphic in the whole disc $|s|<1$. Write $Z_{\alpha\beta}=X_{\alpha\beta}+iY_{\alpha\beta}$. Then

$$ (C.11) \qquad J=i\Omega Q{}^t\bar{\Omega}=\begin{pmatrix} Y_{11} & Y_{12} \\ {}^tY_{12} & Y_{22} \end{pmatrix}-\frac{\log|s|}{2\pi}\begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix} $$

where the $Y_{\alpha\beta}$ are continuous and $Y_{11}>0$ throughout the disc $\Delta$ given by $|s|<1$.

From (C.9) and (C.10) it follows that the vector $\xi$ is continuous on $\Delta$. Using (C.6) we will be done if we prove that $J^{-1}$ is continuous on $\Delta$. Now

$$ J^{-1}=\frac{J^*}{(\det J)} $$

where $J^*$ is the usual matrix of minors of J. From (C.11) we see that each entry in $J^*$ has the form

$$ (-\log|s|)^{m-k}\cdot(\text{continuous function of } s), $$

while from (C.8) and (C.11) we have

$$ \det J=\left(\frac{-\log|s|}{2\pi}\right)^{m+k}(\det Y_{11}\cdot\det\Lambda)+(\text{lower order terms}). $$

Since $\det Y_{11}\cdot\det\Lambda>0$ throughout $\Delta$, we are done.

c) *Proof of the usual Mordell-Weil over function fields.* — We will give a transcendental proof of the usual Mordell-Weil theorem for abelian varieties defined over function fields (char. 0 of course).

*Theorem* **(C.12)**. — *Let* $\mathscr{E}$, $\Lambda$, *and* $\mathbf{J}\to S$ *be as in the statement of theorem* (7.19) *with* $n=1$. *Then there exists an extension of* $\mathbf{J}\to S$ *to an analytic fibre space* $\overline{\mathbf{J}}\to\overline{S}$ *of abelian complex Lie groups such that the group* $\mathrm{Hom}(S, \mathbf{J})$ *of holomorphic cross-sections of* $\overline{\mathbf{J}}\to\overline{S}$ *is an extension of the fixed part* $\mathbf{J}(\mathscr{E})$ *by a finitely generated abelian group.*

*Remark.* — The integrability condition $D_J\nu=0$ is vacuous in this case since $n=1$.

*Proof.* — Let $\overline{\mathbf{E}}_+\to\overline{S}$ be the holomorphic vector bundle over $\overline{S}$ whose fibre at each point $s\in\overline{S}$ is the complex Lie algebra of $\overline{\mathbf{J}}_s$. Then $\overline{\mathbf{E}}_+|S$ is what was denoted by $\mathbf{E}_+$ in the proof of theorem (7.19), and just as was the case in that proof, we want to show that any holomorphic section of $\overline{\mathbf{E}}_+\to\overline{S}$ comes from a constant section of $\mathbf{E}\to S$.

Of course this presumes that we have already defined $\overline{\mathbf{J}}\to\overline{S}$, which we now shall do. Let $\mathscr{C}(\Lambda)$ be the sheaf over S of sections of the lattice $\Lambda\subset\mathbf{E}$. We extend $\mathscr{C}(\Lambda)$ to a sheaf over $\overline{S}$ by saying that the sections of $\mathscr{C}(\Lambda)$ over an open set $U\subset\overline{S}$ are just the usual sections of $\mathscr{C}(\Lambda)$ over $U\cap S$. To define $\overline{\mathbf{E}}_+$, we will say what the sheaf $\mathcal{O}_{\overline{S}}(\check{\overline{\mathbf{E}}}_+)$ of holomorphic sections of the dual bundle is. Thus a section of $\mathcal{O}_{\overline{S}}(\check{\overline{\mathbf{E}}}_+)$ over an open set $U\subset\overline{S}$ is given by a holomorphic section $\varphi$ of $\check{\overline{\mathbf{E}}}_+$ over $U\cap S$ such that, for any section $\gamma$ of $\mathscr{C}(\Lambda)$ over U, the contraction $\langle\varphi(s),\gamma\rangle$ is a holomorphic function on all of U. There is an obvious injection $\mathscr{C}(\Lambda)\to\overline{\mathbf{E}}_+$ and $\overline{\mathbf{J}}$ is defined to be the quotient $\overline{\mathbf{E}}_+/\mathscr{C}(\Lambda)$.

We must prove that $\mathcal{O}_{\overline{S}}(\check{\overline{\mathbf{E}}}_+)$, as defined just above, is a locally free sheaf on all of $\overline{S}$, that the image $\mathscr{C}(\Lambda)\to\overline{\mathbf{E}}_+$ is discrete, and finally that the holomorphic sections of $\overline{\mathbf{E}}_+\to\overline{S}$ come from constant sections of $\mathbf{E}\to S$. This is all done using the formulae (C.7)-(C.11) above together with the observations that

(i) the flat frame $\omega_1, \ldots, \omega_{2m}$ of $\mathbf{E}\to\Delta^*$ may be chosen to be *commensurable* with the lattice $\Lambda$, and

(ii) the holomorphic sections of $\check{\overline{\mathbf{E}}}_+\to\Delta$ are just the linear combinations of $\varphi_1, \ldots, \varphi_m$ with coefficients which are analytic functions in the whole disc $\Delta$.

*Remark added in proof.* — Recent results of W. Schmid seem to show that the monodromy theorem (C.1) is true for an arbitrary variation of Hodge structure. It may be hoped that his methods will also have a bearing on (C.2) and (C.3).

## APPENDIX D

### A result on the monodromy of K3 surfaces

In $\mathbf{P}_3$ with homogeneous coordinates $\xi=[\xi_0, \xi_1, \xi_2, \xi_3]$, we consider quartic surfaces defined by an equation

$$\sum_{(i_0, \ldots, i_3)} s_{i_0 i_1 i_2 i_3} \xi_{i_0} \xi_{i_1} \xi_{i_2} \xi_{i_3} = 0.$$

The set of all such surfaces is parametrized by points $s=[\ldots, s_{i_0 i_1 i_2 i_3}, \ldots]$ in a big $\mathbf{P}_N$. We denote by S′ the Zariski open set in $\mathbf{P}_N$ of points such that the corresponding surface $V_s$ is non-singular. Such surfaces $V_s$ are among the *K3 surfaces*; i.e. they are simply-connected algebraic surfaces with trivial canonical bundle. We let $s_0 \in S'$ be a fixed point and $V=V_{s_0}$ the corresponding K3 surface. Denote by $E=\mathbf{P}^2(V, \mathbf{Q})$ the primitive part of the 2nd cohomology of V, and let $\Gamma_a \subset \mathrm{Aut}(E)$ be the arithmetic group induced from the automorphisms of $H^2(V, \mathbf{Z})$ which preserve the bilinear cup-product form and polarizing cohomology class. We denote by $\Gamma \subset \Gamma_a$ the global monodromy group; i.e. the image of $\pi_1(S', s_0)$ acting on E.

*Theorem* (*D*.1). — $\Gamma$ *is of finite index in* $\Gamma_a$.

*Proof.* — The period matrix domain D is, in this case, a bounded domain in $\mathbf{C}^{19}$ and, by the local Torelli theorem [11], the period mapping

$$\varphi: S' \rightarrow D/\Gamma$$

contains an open set in its image.

We now choose a 19-dimensional smooth algebraic subvariety $S \subset S'$ such that the restricted period mapping

$$\varphi: S \rightarrow D/\Gamma$$

contains also an open set in its image. By Theorem (9.6) above, $\varphi(S)$ is the complement of an analytic subvariety in $D/\Gamma$. Furthermore, because of the finite volume statement, it follows that $D/\Gamma$ has finite volume with respect to the canonical invariant measure on D. Now it follows that the index of $\Gamma$ in $\Gamma_a$ is given by

$$[\Gamma; \Gamma_a] = \frac{\mu(D/\Gamma)}{\mu(D/\Gamma_a)} < \infty.$$

*Remarks.* — From Theorem (9.8) it follows that $\Gamma$ is irreducible. Observe that from (B.5) we may deduce that $\Gamma$ is generated by elements of order 2.

In general, for an algebraic family of algebraic varieties as in § 1, the position of the monodromy group $\Gamma$ in the arithmetic group $\Gamma_a$ is extremely interesting. *I know of no example where $\Gamma$ is not of finite index in its Zariski closure.* In relation to this, we close by observing that the proof of Theorem (D.1) in general gives the following:

*Theorem* (*D*.**2**). — *Let $f: X \to S$ be an algebraic family of algebraic varieties, and designate by $\Gamma$ the global monodromy group. Denote by $\tilde{S}$ the universal covering of* S *and let*

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\tilde{\varphi}} & D \\ {\scriptstyle \pi_1(S)}\Big\downarrow & & \Big\downarrow{\scriptstyle \Gamma} \\ S & \xrightarrow[\varphi]{} & D/\Gamma \end{array}$$

*be the period mapping. Let $\Gamma'$ be any discrete subgroup of $G = \mathrm{Aut}(D)$ such that $\Gamma \subset \Gamma'$ and such that $\Gamma'$ leaves the closure of $\tilde{\varphi}(\tilde{S})$ invariant. Then $\Gamma$ is of finite index in $\Gamma'$.*

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