THE RAMANUJAN AND DENSITY CONJECTURES

PETER SARNAK

RELATIVE LANGLANDS PROGRAM CONFERENCE ; JANUARY 2026 SINGAPORE.

CLASSICAL DENSITY CONJECTURES FOR ZETA FUNCTIONS

- LEAST PRIME IN A PROGRESSION

DIRICHLET: GIVES INFINITELY MANY PRIMES

$p \equiv a(q)$ AS LONG AS $(a,q)=1$.

HOW LARGE IS THE LEAST SUCH PRIME $:= h(a,q)$

- THE NO' OF PRIMES OF SIZE x IS $\frac{x}{\log x}$ (P.N. TH)

THESE PLACE THEMSELVES INTO $\phi(q)$ RESIDUE CLASSES MOD q (REDUCED INVERTIBLE). SO IF THE PRIMES ARE RANDOM THEN BY THE COUPON COLLECTOR WE EXPECT THAT

- ALMOST ALL CLASSES ARE HIT IF

$\frac{x}{\log x} / \phi(q)$ GOES TO INFINITY SLOWLY

- ALL CLASSES ARE HIT IF THIS GOES A BIT FASTER.

UNDER GRH (GENERALIZED RIEMANN HYPOTHESIS):

$$h(a,q) \ll_{\varepsilon} q^{2+\varepsilon} \quad \text{FOR ALL } a$$

$$h(a,q) \ll_{\varepsilon} q^{1+\varepsilon} \quad \text{FOR ALMOST ALL } a \text{ (SHARP)}$$

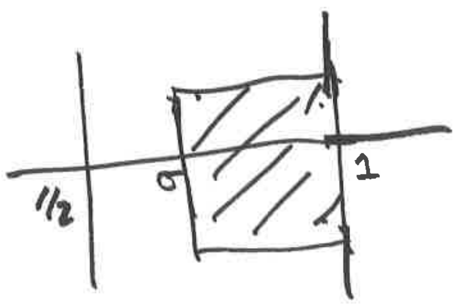
UNCONDITIONALLY: $\exists\, C < \infty$ S.T. $h(a,q) \ll q^{C}$ (LINNIK).

- FOR SOME OF THESE APPLICATIONS ONE DOES NOT NEED GRH BUT ONLY A DENSITY HYPOTHESIS TO PROVE SHARP BOUNDS.
- THE SIGNATURE FOR SUCH APPLICATIONS IS IF IN THE USE OF GRH ONE CAN ALLOW A FEW ZEROS OFF THE LINE $Re(s) = 1/2$, WITH NO LOSS.

DENSITY HYPOTHESIS: $L(s,\chi)$ A DIRICHLET L-FUNCTION WITH CONDUCTOR q

FOR $\sigma > \frac{1}{2}$;

$$\left| \{ \chi \,(\mathrm{mod}\, q) : L(\rho,\chi)=0,\ \rho=\beta+i\gamma,\ |\gamma|\leq 1,\ \beta \geq \sigma \} \right| \ll_{\varepsilon} q^{2(1-\sigma)+\varepsilon}.$$



TOTAL NUMBER OF ZEROS IN BOX.

THIS DENSITY HYPOTHESIS IMPLIES THE SHARP ALMOST ALL ~~PRIMES~~ a, $h(a,q) \ll_{\varepsilon} q^{1+\varepsilon}$!

- DENSITY THEOREMS FOR FAMILIES OF L-FUNCTIONS ARE KNOWN AND SERVE AS A SUBSTITUTE FOR GRH FOR CERTAIN PROBLEMS.
- THE CELEBRATED BOMBIERI–VINOGRADOV THEOREM IS ONE SUCH.

DIOPHANTINE ANALOGUES

LEAST LIFT IN STRONG APPROXIMATION

$n \geq 2$ FIXED:

$$\pi_q : SL_n(\mathbb{Z}) \longrightarrow SL_n(\mathbb{Z}/q\mathbb{Z})$$

REDUCTION MOD q IS ONTO.

$$|SL_n(\mathbb{Z}/q\mathbb{Z})| \approx q^{n^2-1}.$$

GIVEN $\hat{\delta} \in SL_n(\mathbb{Z}/q\mathbb{Z})$ WE SEEK THE <u>SMALLEST</u> LIFT $\delta \in \pi_q^{-1}(\hat{\delta})$ IN EUCLIDEAN NORM ($SL_n(\mathbb{Z}) \subset \mathbb{R}^{n^2}$ AS MATRICES),

CALL ITS NORM:

$$f_n(\hat{\delta}, q)$$

THE NUMBER OF $\delta \in SL_n(\mathbb{Z})$ WITH $\|\delta\| \leq T$ IS KNOWN TO BE

$$\sim C\,T^{n^2-n} \qquad \text{AS } T \to \infty.$$

SO IF REDUCTION IS RANDOM WE EXPECT THAT FOR

$$T > q^{1+\frac{1}{n}+\varepsilon}$$

WE MIGHT COVER ALL OF $SL_n(\mathbb{Z}/q\mathbb{Z})$.

A DENSITY HYPOTHESIS IN THIS SETTING WHICH WAS RECENTLY PROVED IMPLIES

- AN OPTIMAL ALMOST ALL LIFTING

$$h(\hat{\delta}, q) \ll_{\varepsilon} q^{1+\frac{1}{n}+\varepsilon}$$

FOR ALMOST ALL $\hat{\delta} \in SL_n(\mathbb{Z}/q\mathbb{Z})$

- $$h(\hat{\delta}, q) \ll_{\varepsilon} q^{2+\frac{2}{n}+\varepsilon}$$ FOR ALL $\hat{\delta}$

<u>NB</u>: IT IS <u>NOT</u> TRUE THAT <u>ALL $\hat{\delta}$'s</u> HAVE OPTIMALLY SMALL LIFTS OF SIZE $q^{1+\frac{1}{n}+\varepsilon}$; THERE ARE BIG (BUT RARE) HOLES (FIRST FOUND BY G. HARMON). THE <u>ALL $\hat{\delta}$</u> BOUND ABOVE IS NOT SHARP (A COMPLETE ANALYSIS FOR $n=2$ IS IN S 2014) (LETTER).

ON OPTIMAL UNIVERSAL QUANTUM GATES AND THE SOLOVAY-KITAEV THEOREM.

PURE NUMBER THEORY IN THE ARCHITECTURE OF (UNIVERSAL) GATES IN A QUANTUM COMPUTER:

IBM's GATES:

Operator	Gate(s)		Matrix
Pauli-X (X)	$\boxed{X}$	$\oplus$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
Pauli-Y (Y)	$\boxed{Y}$		$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$
Pauli-Z (Z)	$\boxed{Z}$		$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
Hadamard (H)	$\boxed{H}$		$\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
Phase (S, P)	$\boxed{S}$		$\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$
$\pi/8$ (T)	$\boxed{T}$		$\begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}$
Controlled Not (CNOT, CX)			$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
Controlled Z (CZ)	$\boxed{Z}$		$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$
SWAP			$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
Toffoli (CCNOT, CCX, TOFF)			$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$

- THAT THE X, Y, Z, H, S, T GENERATE OPTIMAL ONE QUBIT GATES IN TERMS OF SHORT CIRCUITS TO APPROXIMATE THE GENERAL ONE QU-BIT GATE MAKES USE OF THE (PROVEN) RAMANUJAN CONJECTURES!

RAMANUJAN AND DENSITY CONJECTURES

SETTING IS A SEMISIMPLE (OR REDUCTIVE) GROUP DEFINED OVER $\mathbb{Q}$.

$G = G(\mathbb{R})$ ITS REAL POINTS (MORE GENERALLY ITS ADELIC POINTS)

$G(\mathbb{Z})$ ITS INTEGRAL POINTS.

$\Gamma \leq G(\mathbb{Z})$ A CONGRUENCE SUBGROUP

I.E. $\Gamma(N) \leq \Gamma \leq G(\mathbb{Z})$ SOME N

$\Gamma(N) = \{ \gamma \in \Gamma : \gamma \equiv I \ (\mathrm{MOD}\ N) \}$.

WE DECOMPOSE THE REGULAR REPRESENTATION OF G ON

$$L^2(\Gamma \backslash G) \ ; \ R_g f(x) = f(xg),$$

INTO IRREDUCIBLE (UNITARY) REPRESENTATIONS OF G

$$= \bigoplus_{\pi \in \widehat{G}_{\mathrm{UNIT}}} m(\pi) H_\pi ,$$

$m(\pi)$ MULT OF π

IN GENERAL THERE IS A CONTINUOUS PART 'EISENSTEIN SERIES'

$\widehat{G}_{\mathrm{unit}}$ = THE SET (TOPOLOGICAL) OF ALL IRRED UNITARY REPRESENTATIONS OF G.

FOR $\pi \in \widehat{G_{\text{unit}}}$, DEFINE $p(\pi) \geq 2$ THE L^p-SIZE OF ITS MATRIX COEFFICIENTS

$$p(\pi) = \inf_{p \geq 2} \left\{ \langle \pi(g)v, v \rangle \in L^p(G) \text{ FOR GENERIC } v \in H_\pi \right\}$$

- FOR π TRIVIAL (OR 1-DIMENSIONAL) $p(\pi) = \infty$.

<u>DEFINITION:</u> π IS <u>TEMPERED</u> IF $p(\pi) = 2$.

- THE TEMPERED π'S ARE THE ONES THAT OCCUR IN THE DECOMPOSITION OF $L^2(G)$ ON ITSELF (THE SPECTRUM OF THE 'UNIVERSAL COVER') AND IN THE CORRESPONDING PLANCHAREL FORMULA OF HARISH-CHANDRA.

- THE INTERPRETATION OF THE CLASSICAL RAMANUJAN CONJECTURE IN TERMS THAT THE LOCAL CONSTITUENTS OF THE DECOMPOSITION OF

$$L^2_{\text{cusp}}(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A})),$$

ARE <u>TEMPERED</u>; IS DUE TO SATAKE.

• THE NAIVE EXTENSION : THAT THE NONTRIVIAL REPRESENTATIONS IN THE DECOMPOSITION OF $L^2(\Gamma \backslash G)$ MUST BE TEMPERED, IS FALSE IN ANY GENERALITY.
(SAITO-KUROKAWA, HOWE-PIATETSKI-SHAPIRO USING THETA CORRESPONDENCES, BURGER-LI-S "RAMANUJAN DUALS")

• BASED ON HIS TRACE FORMULA <u>ARTHUR</u> HAS FORMULATED GENERAL CONJECTURES (NECESSARILY VERY COMPLICATED) WHICH GIVE RESTRICTIONS ON WHICH NON-TEMPERED π'S CAN OCCUR IN SUCH DECOMPOSITIONS. IN PRINCIPLE HIS CONJECTURES / THEOREMS REDUCE THE PROBLEM TO THE CASE OF GL_n.

• FOR GL_n ITSELF IT IS VERY LIKELY THAT SATAKE'S FORMULATION FOR

$$L^2_{cusp}(GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{A}))$$

IS VALID (LANGLANDS CONJECTURE)

DENSITY CONJECTURE (XUE-S, S)

FIX $B \subset \widehat{G}_{unit}$ COMPACT; FOR $2 \leq \sigma \leq \infty$

$$\#\{\pi \in L^2(\Gamma\backslash G) : p(\pi) \geq \sigma,\ \pi \in B\} \ll_{\varepsilon} \mathrm{Vol}(\Gamma\backslash G)^{\frac{2}{\sigma}+\varepsilon}$$

COMMENTS:

- THE EXPONENT LINEARY INTERPOLATES IN $1/\sigma$ BETWEEN THE TWO KNOWN EXTREMES $\sigma = \infty$, $\sigma = 2$.
- THE DENSITY IS CONJECTURED TO HOLD IN GENERAL AND FOR OTHER FAMILIES $\mathcal{F}$ OF AUTOMORPHIC FORMS (SHIN-TEMPLIER-S)
- IN ALL CASES ONE CAN PROVE A FULL GAP AT THE TOP: THERE IS A $\sigma_0 = \sigma_0(\mathcal{F}) < \infty$ EXPLICIT SUCH THAT THE L.H.S. ABOVE IS 0 (EXCEPT FOR THE TRIVIAL REPRESENTATION) FOR $\sigma > \sigma_0$. (BURGER-S, CLOZEL)

THE DENSITY CONJECTURE YIELDS OPTIMAL EQUIDISTRIBUTION ON SMALL SETS IN VARIOUS CONTEXTS:

(a) THE $SL_n(\mathbb{Z})$ LIFTING PROBLEM.

(b) OPTIMALLY SHORT CIRCUITS FOR ARITHMETICALLY DEFINED UNIVERSAL QUANTUM GATES (PARZANCHEVSKI – S FOR PU(2), EVRA PU(4)).

(c) OPTIMAL S-ARITHMETIC DIOPHANTINE APPROXIMATION FOR ALMOST ALL POINTS (NEVO-GORODNIK-GHOSH / JANA-KAMBER) GOLOBEV/KAMBER

(d.) VIA MATSUSHIMA'S FORMULA ALMOST SHARP UPPER BOUND FOR DIMENSIONS OF THE COHOMOLOGIES OF $\Gamma\backslash G/K$.

APPROACHES TO DENSITY

LOW ROAD (XUE–S)

SET UP A COUNTING PROBLEM FOR THE ELEMENTS OF $\Gamma(N)$ WHICH CAN BE EXPRESSED IN TERMS OF THE SPECTRA WITH POSITIVE WEIGHTS WHICH ARE EXPONENTIAL ON NON TEMPERED REPRESENTATIONS, AND FOR WHICH SHARP UPPER BOUNDS CAN BE DEDUCED ELEMENTARILY. SUCCEEDS FOR SL_2:

$$ad-bc=1, \quad |a|,|b|,|c|,|d| \le T$$

$$a \equiv d \equiv 1 (p), \quad b \equiv c \equiv 0 (p), \quad a,b,c,d \in \mathbb{Z}$$

$$\Rightarrow a+d \equiv 2 (p^2) \quad \text{(SAY } p \text{ IS PRIME)}$$

$$\# \ll_\varepsilon \frac{T^{2+\varepsilon}}{p^3} + 1 ; \quad \text{SHARP!}$$

- THIS METHOD EXTENDS TO "THIN GROUPS Γ" AND IS A KEY INGREDIENT IN THE PROOF OF EXPANSION FOR THESE.

- ONE CAN ALSO USE THE <u>TRACE FORMULA</u> (HUXLEY FOR $SL_2(\mathbb{Z})$) AND WAS USED IN FRACZYK–HARCOS–MAGA–MILICEVIC: <u>UNIFORMLY</u> FOR CONGRUENCE LATTICES (IRREDUCIBLE) OF <u>ARITHMETIC</u> LATTICES IN $G(\mathbb{R}) = SL_2(\mathbb{R})^a \times SL_2(\mathbb{C})^b$, DENSITY HOLDS!

HIGH ROAD

THERE ARE CELEBRATED CASES OF RAMANUJAN KNOWN FOR GL_n AND SPECIAL π'S

(1) $GL_2/\mathbb{Q}$ AND π DISCRETE SERIES (HOLOMORPHIC FORMS; DELIGNE)

(2) GL_n AND π SPECIAL COHOMOLOGICAL TYPES
HARRIS–TAYLOR; SHIN

NB: IN ALL CASES ONE REDUCES AND INVOKES THE WEIL CONJECTURES PROVED BY DELIGNE EXCEPT FOR THE RECENT (2023) PAPER BY ALLEN, CALEGARI, CARAIANI, GEE, HELM, LE HUNG, NEWTON, SCHOLZE, TAYLOR, THORNE WHERE THE PROOF FOLLOWS LANGLANDS 68 PRESCRIPTION!

• ARTHUR'S CONJECTURES GIVE AN EXPLICIT (IF VERY COMPLICATED) DESCRIPTION OF THE DISCRETE SPECTRUM OF $\Gamma\backslash G$ IN TERMS OF THE FULL SPECTRA OF GL_n.

MOREOVER ARTHUR'S CONJECTURES ARE ESSENTIALLY PROVEN FOR G A CLASSICAL GROUP.

SO IN CASES THAT THE FAMILY $\mathcal{F}$ OF π'S THAT ENTER IN THE DENSITY CONJECTURE ARE ONES FOR WHICH RAMANUJAN ON GL_n IS KNOWN ONE CAN APPROACH THE DENSITY PROBLEM

- ONE NEEDS TO ESTIMATE THE NUMBER OF NON-TEMPERED OCCURENCES IN TERMS OF HOW NON-TEMBERED THEY ARE AND TO SHOW THAT THE EXPONENTS LIE BELOW THE LINEAR INTERPOLATION.

- THIS HAS BEEN CARRIED OUT IN A NUMBER OF CASES. IN SOME THE EXPONENTS ARE WELL BELOW IN OTHERS BARELY SO. IN ALL THESE CASES THE DENSITY CONJECTURE IS PROVEN.

RAGOWSKI ($U(3)$), MARSHALL ($U(3)$), EVRA $U(N)$; COMPACT $G(\mathbb{R})$ - (NOTHING TRANSCENDETAL!)

DISCRETE SERIES SHIN-TEMPLIER

$U(p,q)$ COHOMOLOGICAL AT INFINITY DALAL / GERBELLI-GAUTHIER.

- EVRA – GERBELLI-GAUTHIER – GUSTAFSSON
  COHOMOLOGICAL DENSITY FOR $SO(5)$

- MULTI-QUBIT GOLDEN GATES
  DALAL – EVRA – PARZANCHEVSKI
  FOR $PU(4)$ AND $PU(8)$

  + OPTIMAL NAVIGATION.

(THE DENSITY IS USED
AS BEFORE TO ENSURE THE
EXISTENCE OF OPTIMALLY SHORT
CIRCUITS TO ALMOST ALL
UNITARIES ; AND HERE
UNLIKE $PU(2)$ RAMANUJAN
ITSELF FAILS)

MIDDLE ROAD:

THE CASE OF GL_n (AND ALL π'S) IS CENTRAL (AGAIN BECAUSE OF FUNCTORIALITY).

- WHILE THE ELEMENTARY LOW ROAD IS STUCK; ONE CAN TRY USE THE ARTHUR-SELBERG TRACE FORMULA DIRECTLY TO ESTIMATE THE NONTEMPERED SPECTRUM. THE GEOMETRIC SIDE INVOLVES ORBITAL INTEGRALS WHICH CAN BE EXPRESSED IN TERMS OF CLASS NUMBERS AND REGULATORS OF ORDERS IN DEGREE n NUMBER FIELDS. EFFORTS TO ESTIMATE THESE SUFFICIENTLY WELL SO AS TO PROVE THE DENSITY BOUNDS HAVE SO FAR FAILED.

- THERE IS A 'FRIENDLIER' SPECTRAL FORMULA WHICH IS A GENERALIZATION OF THE PETERSSON - KUZNETSOV FORMULA FOR GL_2 THAT HAS ADVANTAGES.

- IN $\underline{GL_2}$ IT WAS EXPLOITED BY <u>IWANIEC</u> TO GIVE EVEN STRONGER DENSITY THEOREMS THAN THE GENERAL CONJ.

- IT WAS FIRST INTRODUCED FOR GL_n, $n \geq 3$ BY BUMP - FRIEDBERG - GOLDFELD.

- THE ADVANTAGE IS THAT CLASS NUMBERS AND REGULATORS ARE REPLACED BY HIGHER DIMENSIONAL KLOOSTERMAN-SUMS (GENERALIZATIONS TO FINITE FIELDS OF BESSEL FUNCTIONS) AND WHITTAKER FUNCTIONS. THE STRUCTURE AND FACTORIZATIONS OF KLOOSTERMAN ~~IS MUCH~~ AND THEIR ESTIMATION IS MUCH BETTER UNDERSTOOD THAN CLASS NUMBERS.

- WHAT IS A CHALLENGE IN THIS FORMULA IS THE ANALYSIS OF TEST FUNCTIONS WHICH ARE FLEXIBLE ENOUGH (AND POSITIVITY) TO ISOLATE THE NON-TEMPERED SPECTRUM AND GIVE SHARP BOUNDS.

THIS TOOK TIME AND IS HARD EARNED BUT THERE HAVE BEEN A SERIES BREAKTHROUGHS RECENTLY:

- THE DENSITY CONJECTURE FOR THE CUSPIDAL SPECTRUM FOR SL_n, $n \geq 3$

FOR $\Gamma_0(q)$ "HECKE GROUPS" $\left\{ \equiv \begin{bmatrix} * & * & * \\ * & * & * \\ \vdots & & * \\ 0 & 0 \cdots 0 & * \end{bmatrix} \bmod q \right\}$

BLOMER (2019) IN STRONGER FORM (IE BETTER THAN LINEAR INTERPOLATION IN THE EXPONENT)

USES KUZNETZOV FORMULA + BOUNDS FOR HYPERKLOOSTERMAN SUMS.

BLOMER + ASSING (2023) FOR $\Gamma(q)$. THE PRINCIPAL CONGRUENCE SUBGROUP q SQ.-FREE, USES THE STRUCTURE OF THE KLOOSTERMAN SUMS AND WHITTAKER FUNCTIONS AND GROUP THEORY.

JANA-KAMBER (2024) ~~DE~~ SHOW THAT THE RESIDUAL AND CTS SPECTRUM SATISFY THE DENSITY CONJECTURE.

ABOVE $\Rightarrow$ FULL DENSITY CONJ FOR $\Gamma(q)$ q SQ. FREE.

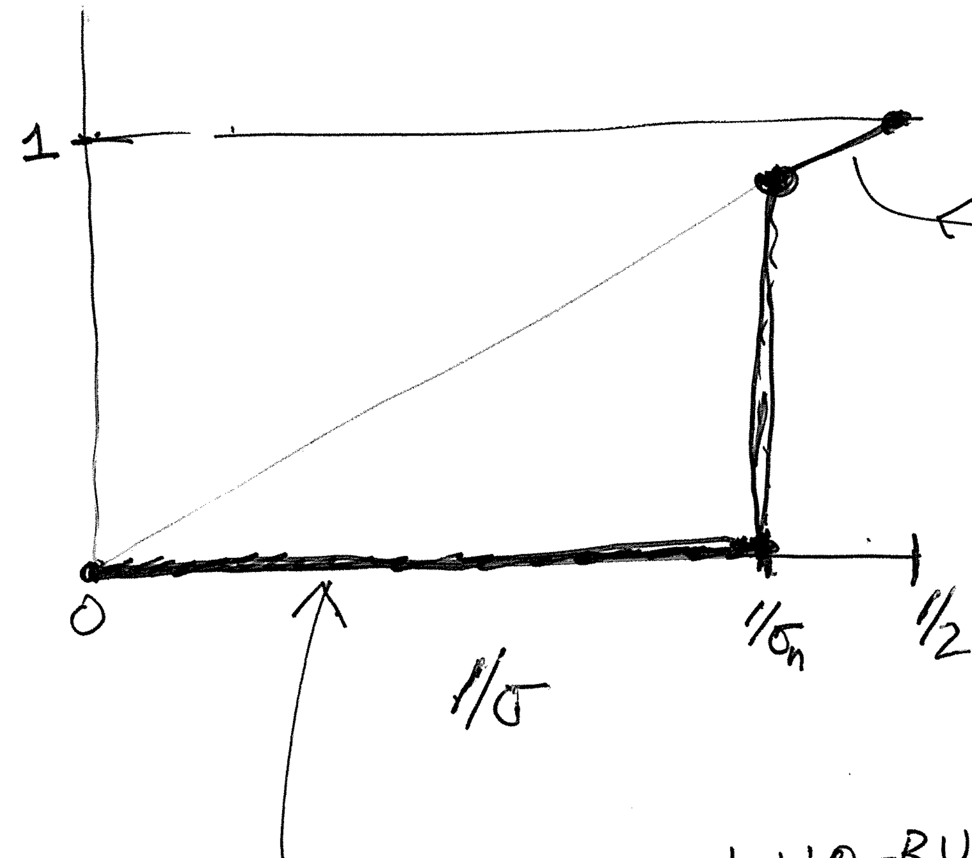
- WITH IT THE ALMOST ALL LOCAL OPTIMAL LIFTING FROM $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/q\mathbb{Z})$.

WITH THESE ADVANCES ON THE DENSITY CONJECTURE FOR GL_n USING THE KUZNETOV (RELATIVE "TRACE" FORMULA) THERE IS A PATH TO ESTABLISHING THE DENSITY CONJECTURES — AT LEAST FOR CLASSICAL GROUPS:

USING THE HIGH ROAD ARTHUR (PLUS THE REQUISITE WEYL LAW TYPE COUNTING ~~OF~~ SIZES OF FAMILIES) BRINGS ONE TO THE CENTRAL CASE OF GL_n.

WE KNOW QUITE STRONG BOUNDS TOWARDS THE FULL RAMANUJAN CONJECTURES ~~ON~~ FOR GL_n ON THE CUSPIDAL PART:

DENSITY EXPONENT FOR GL_n



BLOMER ET AL

$p(\pi) \leq \sigma_n$; LUO-RUNDICK-S BOUND TOWARDS RAMANUJAN.

FOR $n=2$ STRONGER BOUNDS ARE KNOWN
KIM-S ; BLOMER-BRUMLEY.

FOR $n=3$ THE RECENT ADVANCE BY WEE TECK GAN ESTABLISHING THE AD LIFT FROM GL_3 TO GL_8 HAS A NEW BOUND FOR σ_3 AS A COROLLARY !