⓪

RAMANUJAN GRAPHS

PETER SARNAK

ADMA COLLOQUIUM SERIES

22 JUNE 2024.

- RAMANUJAN GRAPHS ARE HIGHLY CONNECTED SPARSE LARGE GRAPHS.

- THE TENSION BETWEEN SPARSE AND HIGHLY CONNECTED IS WHAT MAKES THEM SO USEFUL IN VARIED APPLICATIONS.

SEE HOORY-LINIAL-WIGDERSON BAMS 43(2006) AND D. SPIELMAN "IMPACT OF RAMANUJAN GRAPHS" IAS LECTURE JUNE/3/2024.

$\boxed{\text{EXPANDERS}}$

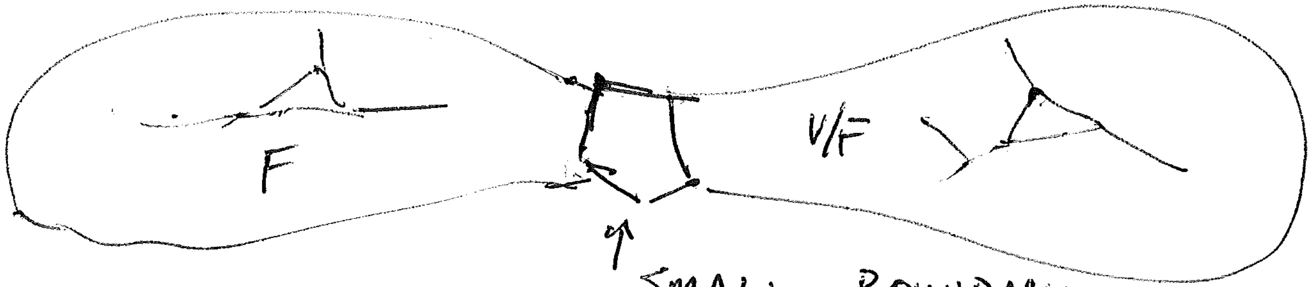
WE STICK TO d-REGULAR ($d \geq 3$ FIXED) CONNECTED GRAPHS X ON $n = |V(X)|$ VERTICES; $n \to \infty$ (THESE ARE SPARSE!)

CHEEGER'S CONSTANT:

$$h(X) := \min_{\phi \subsetneq F \subsetneq V(X)} \frac{|\partial F|}{\min(|F|, |V \setminus F|)}$$

∂F ARE ALL VERTICES ADJACENT TO F BUT NOT IN F

SO $h(X)$ NOT CLOSE TO ZERO MEANS YOU CANNOT PARTITION THE VERTICES INTO TWO LARGE SETS WITH SMALL BOUNDARY.



SMALL BOUNDARY
(BOTTLE NECK)

SMALL h.

- $X_{n,d}$ IS AN EXPANDER (FAMILY) IF

  $h(X) \geq h > 0$ FOR ALL X, $n \to \infty$.

- EXISTENCE OF EXPANDERS IS COUNTER INTUITIVE, THEY WERE FIRST CONSTRUCTED BY SHOWING THAT THE RANDOM $X_{n,d}$ IS SUCH BY BARZDIN (1967) IN ANOTHER LANGUAGE.

SPECTRAL GAP :

X d-REGULAR

A THE ADJACENCY OPERATOR (MATRIX)

$$Af(v) = \sum_{w \sim v} f(w)$$ ON FUNCTIONS

$f : V(X) \to \mathbb{C}$.

SPECTRUM (X) = SPECTRUM $(A) \subset [-d, d]$

$= \{ \lambda_0(X) \geq \lambda_1(X) \cdots \geq \lambda_{n-1}(X) \}$.

- $\lambda_0(X) = d$ AND IS SIMPLE IFF X IS CONNECTED.
- $\lambda_{n-1}(X) = -d$ IFF X IS BIPARTITE.

THERE IS A RELATION BETWEEN $\lambda_1(X)$ AND $h(X)$ (CHEEGER, BUSER, ALON-MILMAN, ..)

$$\frac{d - \lambda_1(X)}{2} \leq h(X) \leq \sqrt{2d\,(d - \lambda_1(X))}$$

SO WANT THE GAP $d - \lambda_1(X)$ TO BE LARGE FOR EXPANSION.

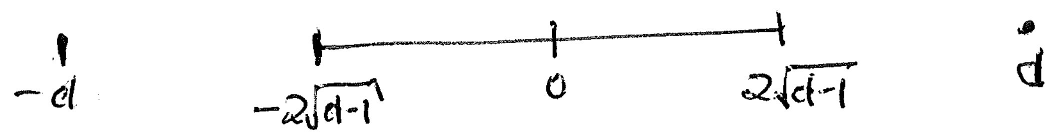
NB IN THE INAUGURAL LECTURE OF THIS LECTURE SERIES PETER CAMERON EXPLAINED THE FEATURES OF MAKING OPTIMAL SPECTRAL GAPS AT THE BOTTOM; "HOFFMAN" GRAPHS AND LINE GRAPHS.

HERE WE WANT OPTIMAL LARGE GAP AT THE TOP.

- THERE IS A LIMIT TO HOW LARGE THIS CAN BE

ALON-BOPPANA (FIX $d \geq 3$)

$$\underline{\lim}_{n \to \infty} \lambda_1 (X_{n,d}) \geq 2\sqrt{d-1}$$



- THE SPECTRUM OF A ON $\ell^2(T_d)$ THE T_d — REGULAR TREE IS $[-2\sqrt{d-1}, 2\sqrt{d-1}]$ (KESTEN).

DEFINITION (OPTIMAL EXPANDERS)

A GRAPH (SEQUENCE WITH $n \to \infty$) IS

a) **RAMANUJAN** IF FOR $j = 1, \ldots, n-1$

$$|\lambda_j (X_{n,d})| \leq 2\sqrt{d-1}.$$

b). **BIPARTITE RAMANUJAN** IF IT IS BIPARTITE $\lambda_{n-1}(X) = -d$ AND

$$|\lambda_j (X_{n,d})| \leq 2\sqrt{d-1} \quad \text{FOR} \quad j = 1, \ldots, n-2.$$

"LINEAR PROGRAM" FOR CUBIC SATURATES:

$$\text{TRACE}\,(\Delta(X)) = 3n = \lambda_1 + \lambda_2 \cdots + \lambda_{n-1}$$

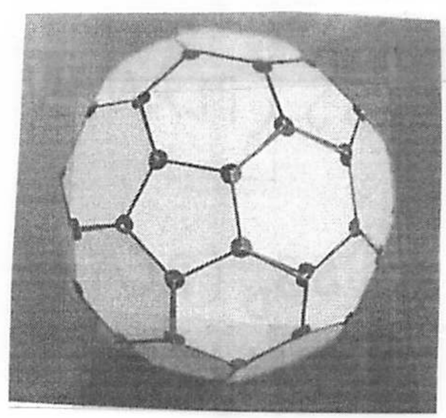
SO $$\mu_\Delta(X) = \min_{j \geq 1} \lambda_j \leq \frac{\lambda_1 + \cdots + \lambda_{n-1}}{n-1} \leq \frac{3n}{n-1} \leq 4 \quad (n \geq 4)$$

FOR $n=4$ THIS IS ACHIEVED AND IS UNIQUE!

$X = K_4$  ; $\mu = 4$

$X = K_{3,3}$  ; $\mu = 3$

$X =$ PETERSON  ; $\mu = 2$

$X =$ FOOTBALL  ; $\mu = 0.234..$

$$\alpha = 3 - 2\sqrt{2} = 0.1715..$$

THEY EXIST !

• [LUBOTZKY–PHILLIPS–S], [MARGULIS] 1986 (*)

CONSTRUCT SUCH GRAPHS USING NUMBER THEORY AND SPECIFICALLY PROVEN CASES OF THE RAMANUJAN CONJECTURES (EICHLER).

HENCE THE NAME.

THE RAMANUJAN CONJECTURE (MODULAR FORMS)

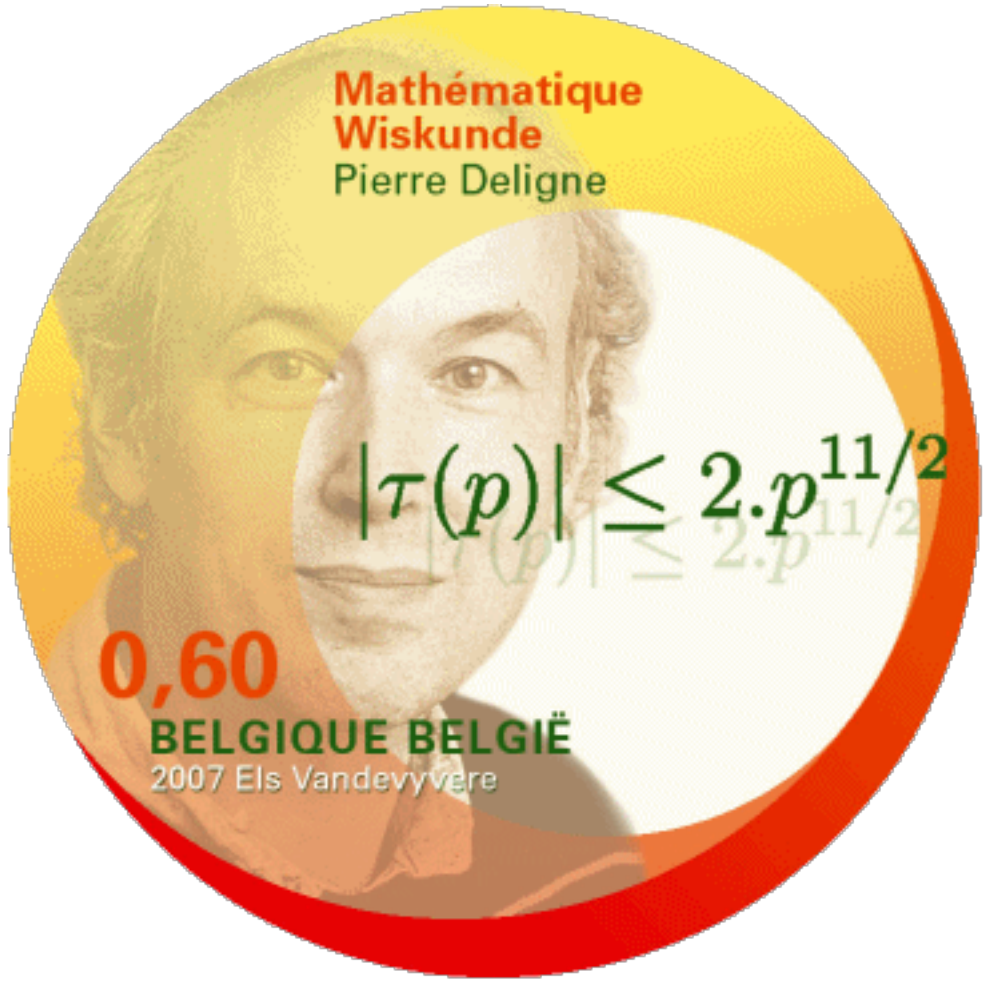
$$\Delta(q) = q\prod_{n=1}^{\infty}(1-q^n)^{24} := \sum_{m=1}^{\infty}\tau(m)\,q^m$$

$$= q - 24q^2 + 252q^3 - 1472q^4 \cdots$$

- $\tau(mn) = \tau(m)\tau(n)$ IF $(m,n)=1$ (PROVED BY MORDELL)
- $|\tau(p)| \leq 2p^{11/2}$ (ONLY RAMANUJAN COULD CONJECTURE)

THE LAST WAS PROVEN BY P. DELIGNE AS PART OF HIS PROOF OF THE WEIL CONJECTURES.

(*) ONE CAN INTERPRET THE BRANDT MATRIX AND WORK OF Y. IHARA AS ANTECEDENTS OF THESE.


Mathématique
Wiskunde
Pierre Deligne
$|\tau(p)| \leq 2.p^{11/2}$
0,60
BELGIQUE BELGIË
2007 Els Vandevyvere

L-P-5 RAMANUJAN CAYLEY GRAPHS (CONCRETE, EXPLICIT)

q A PRIME $q \equiv 1(4)$, $i^2 \equiv -1 \pmod{q}$

LET S BE THE 6 MATRICES IN $GL_2(\mathbb{F}_q)$

$$S = \left\{ \begin{bmatrix} 1 \pm 2i & 0 \\ 0 & 1 \mp 2i \end{bmatrix}, \begin{bmatrix} 1 & \pm 2 \\ \mp 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & \pm 2i \\ \pm 2i & 1 \end{bmatrix} \right\}$$

THESE GENERATE

(i) $G = PGL_2(\mathbb{F}_q)$ IF $q \equiv 2$ OR $3 \pmod{5}$

(ii) $G = PSL_2(\mathbb{F}_q)$ IF $q \equiv 1$ or $4 \pmod{5}$

THE CAYLEY GRAPHS OF G WITH THESE GENERATORS; THAT IS THE GRAPH WHOSE VERTICES ARE THE ELEMENTS OF G AND $g \sim g'$ IF $gs = g'$ FOR SOME $s \in S$;

ARE 6-REGULAR BIPARTITE IN CASE (i) AND NONBIPARTITE IN CASE (ii)

RAMANUJAN GRAPHS.

EICHLER'S PROOF AND HENCE OUR CONSTRUCTION RELIES ON THE "RIEMANN HYPOTHESIS" FOR CURVES OVER FINITE FIELDS (DUE TO ANDRE WEIL).

DETOUR:

$\mathbb{F}_q(t)$ RATIONAL FUNCTIONS IN t WITH COEFF IN THE FINITE FIELD $\mathbb{F}_q$.

K A FINITE EXTENSION OF $\mathbb{F}_q(t)$

$f \in K$, $N(f)$ ITS NORM, $T = q^{-s}$

$$\zeta_K(s) = \sum_{(f) \text{ OF } \mathcal{O}_K} N(f)^{-s} = \sum_{(f) \text{ OF } \mathcal{O}_K} T^{\deg f} = \frac{P_K(T)}{(1-T)(1-qT)}$$

P_K IS A POLYNOMIAL OF DEGREE g (THE GENUS OF K)

ALL THE $2g$ ZEROS OF $P_K(T)$ ARE ON THE CIRCLE $|z| = q^{-1/2}$ (RIEMANN-HYPOTHESIS)

NB THE CONSTRUCTIONS ABOVE CAN MADE AS LONG AS $d = q+1$ WHERE q IS p^a THE CARDINALITY OF A FINITE FIELD.

REMARKABLY (FOR ME AT LEAST) THE EXISTENCE OR BIPARTITE RAMANUJAN GRAPHS FOR ALL $d \geq 3$ WAS ESTABLISHED BY

A. MARCUS - N. SRIVASTAVA - D. SPIELMAN (2015)

USING COMBINATORIAL AND STATISTICAL PHYSICS TECHNIQUES.

- THEY ESTABLISH THE BIPARTITE CASE OF THE CONJECTURE:

CONJECTURE (BILU - LINIAL)

EVERY DEGREE d RAMANUJAN GRAPH X HAS A DOUBLE COVER (WITH $2|V(X)|$ VERTICES AND DEGREE d) WHICH IS RAMANUJAN.

(THERE ARE $2^{|E(X)|}$ DOUBLE COVERS)

THE M-S-S PROOF RELIES ON REAL ROOTEDNESS PROPERTIES OF POLYNOMIALS FROM STATISTICAL PHYSICS.

- LEE-YANG THEOREM FOR THE ROOTS OF THE PARTITION FUNCTION FOR THE ISING MODEL.

CLOSELY RELATED

- HEILMAN-LIEB THEOREM FOR DIMER MATCHINGS

G ANY (FINITE) GRAPH;

$$\mu_G(\lambda) = \sum_{i \leq n/2} (-1)^i m_i \lambda^{n-2i}$$

WHERE m_i IS THE NUMBER OF MATCHINGS OF SIZE i (A MATCHING IS A SUBSET OF THE EDGES NO TWO SHARING A VERTEX)

THE μ_G IS REAL ROOTED!

(PROOF BY INDUCTION AND SEVERAL VARIABLES!)

- $\Longrightarrow$ GODSILL - GUTMAN

IF X IS d-REGULAR WITH m EDGES

$$\underset{s \in \{\pm 1\}^m}{\mathrm{EXP}} \left[\det(\lambda I - A_s) \right] = \mu_X(\lambda)$$

HAS ALL ITS ROOTS IN $\left[-2\sqrt{d-1}, 2\sqrt{d-1}\right]$.

I showed this proof for the special case to Yang and Lee. A couple of weeks later they produced their proof of the general theorem.[1] I recall Professor Yang telling me at the time that Hilfsatz II of Pólya, in the form discussed above, was one essential ingredient in their proof. I have shown Professor Yang these comments, and I would like to include his recollections here.

"When Lee arrived at Princeton in the fall of 1951, I was just recovering from my computation of the magnetization of the Ising model. I realized that the Ising model is equivalent to the concept of a lattice gas. So, we worked on that and finally produced our paper I. In the process of doing that, we discovered, by working on a number of examples, the conjectured unit circle theorem.

"We then formulated a physicist's 'proof' based on no double roots when the strength of the couplings were varied. Very soon we recognized this was incorrect; and for, I would guess, at least six weeks we were frustrated in trying to prove the conjecture. I remember our checking into Hardy's book on Inequalities, our talking to Von Neumann and Selberg. We were, of course, in constant contact with you all along (and I remember with pleasure your later help in showing us Wintner's work, which we acknowledged in our paper). Sometime in early December, I believe, you showed us the proof of the special case when all the couplings are there and are of equal strength, the case that you are now writing about in connection with Pólya's collected works. The proof was fine, but we were still stuck on the general problem. Then one evening around December 20, working at home, I suddenly recognized that by making $z_1, z_2, \ldots$ independent variables and studying their motions relative to the unit circle one could, through an induction procedure, bring to bear a reasoning similar to the one used in your argument and produce the complete proof. Once this idea was there, it took only a few minutes to tighten up all the details of the argument.

"The next morning I drove Lee to pick up some Christmas trees, and I told him the proof in the car. Later on, we went to the Institute; and I remember telling you about the proof at a blackboard.

"I remember these quite distinctly because I'm quite proud of both the conjecture and the proof. It is not such a great contribution, but I fondly consider it a minor gem."

Mark Kac

Comments on

[94] Sopra una equazione trascendente trattata da Eulero

Here Pólya establishes a previously unproved statement of Euler's on the location of the zeros of the function displayed at the beginning of the paper;

1. T. D. Lee and C. N. Yang, *Statistical theory of equations of state and phase transitions. II. Lattice gas and Ising model,* Phys. Rev. **87** (1952), pp. 410–419, especially Appendix II.

HOW ABOUT THE RANDOM d-REGULAR GRAPH, $n \to \infty$?

I HAVE A BET WITH NOGA ALON

- HE SAYS THAT ALMOST ALL SUCH ARE RAMANUJAN
- I SAY THAT ALMOST NONE ARE.

WHILE IT IS NOT HARD TO SHOW THAT THE RANDOM $X_{n,d}$ IS AN EXPANDER, I IS MUCH MORE DIFFICULT TO SHOW THAT THE RANDOM SUCH IS ALMOST RAMANUJAN:

FOR $\varepsilon > 0$ WITH PROBABILITY TENDING TO 1 THE SPECTRUM OF $X_{n,d}$ IS IN $\left[-2(d-1)^{1/2}-\varepsilon, 2(d-1)^{1/2}+\varepsilon\right]$;
(PROVED BY JOEL FRIEDMAN).

- J. HUANG – T. McKENZIE – H. T. YAU (2024) SHOW THAT WITH PROBABILITY TENDING TO 1

$$\lambda_1(X_{n,d}) \leq 2\sqrt{d-1} + O(n^{-2/3}).$$

THE EXPONENT $\frac{2}{3}$ BRINGS US WITHIN A HAIR'S BREADTH OF SHOWING THAT $\lambda_1(X_{n,d}) - 2\sqrt{d-1}$ SCALED IS "TRACY-WIDOM".

IF SO THEN 51% OF THE BIPARTITE $X'_{n,d}$'s ARE RAMANUJAN AND 27% OF THE NON BIPARTITE ARE RAMANUJAN

NUMERICAL EXPERIMENTS BY S. J. MILLER, T. NOVIKOFF, A. SABELI CONFIRM THIS!

REFERENCES:

S. HOORY – N. LINIAL – A. WIGDERSON BAMS 43(2006) 439–561.

Y. BARZDIN – A. KOLMOGOROFF "ON THE REALIZATION OF NETWORKS IN THREE-DIMENSIONAL SPACE" KOLMOGOROFF COLLECTED WORKS VOL 3.

N. ALON "EIGENVALUES AND EXPANDERS" COMBINATORICA 6 (1986) 83–96.

A. LUBOTZKY – R. PHILLIPS – P. SARNAK "RAMANUJAN GRAPHS" COMBINATORICA 8 (1988) 261–277

G. MARGULIS IN PROB. INF TRANSM. 24 (1988) 51–60.

S. RAMANUJAN "ON CERTAIN ARITHMETIC FUNCTIONS" TRANS. CAMB. PHIL. SOC 22, 159-184 (1916)

A. WEIL "ON THE RIEMANN HYPOTHESIS IN FUNCTION FIELDS" PROC. NAT. ACAD. SCI USA 27(1941) 592-594.

A. MARCUS – D. SPIELMAN – N. SRIVASTAVA ANNALS OF MATH 182 (2015) 307–325

Y. BILU – N. LINIAL COMBINATORICA 26 (2006) 495–519

T. LEE – C.N. YANG "STATISTICAL THEORY OF EQUATIONS OF STATE AND PHASE TRANSITIONS II" PHYS. REVIEW 87, 3 (1952)

J. HUANG – T. McKENZIE – H.T. YAU arXiv 2405.12161 (2024)

S.J. MILLER – T. NOVIKOFF – A. SABELLI EXP MATH 17; 231–244 (2008)