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Author(s): Robert Bryant and Phillip Griffiths
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REDUCTION FOR CONSTRAINED VARIATIONAL PROBLEMS AND $\int \frac{\kappa^2}{2} ds$

By Robert Bryant* and Phillip Griffiths**

Introduction. In this paper we will study certain functionals whose domain of definition consists of integral curves of an exterior differential system. Our purpose is twofold. First, we want to extend to this general setting the Marsden-Weinstein reduction for Hamiltonian systems [4] admitting a Lie group of symmetries. Secondly, we want to use this general method to investigate the global behaviour of solution curves of the Euler-Lagrange equations associated to the functional

$$\Phi(\gamma) = \frac{1}{2} \int_\gamma \kappa^2 ds \tag{1}$$

defined on immersed curves γ in a surface S of constant curvature.

In the flat case $S = \mathbf{E}^2$ it is natural to restrict the functional (1) to curves of fixed length, and among other things we are able to give an almost complete picture of the closed solution curves to the Euler-Lagrange equations. One conclusion that may be drawn from our study is this:

Denote by $[S^1, \mathbf{R}^2]$ the isotopy classes of immersions f: $S^1 \to \mathbf{R}^2$ of fixed positive length. According to the Whitney theorem, $\pi_0([S^1, \mathbf{R}^2]) \xrightarrow{\approx} \mathbf{Z}$, where the map is given by assigning to each f the index of $\gamma = f(S^1)$. We show that the functional (1) has at least one critical value in each component of $[S^1, \mathbf{R}^2]$.

Another conclusion of our study is in the case when $S = H^2$, the hyperbolic plane. We show that the Euler-Lagrange equations associated to (1) may be represented as a linear flow on a 2-torus $T^2(\mu)$, provided that an "energy level" $\mu \cdot \mu$ lies in the interval $(-1, 0)$. By letting μ vary we conclude the existence of infinitely many periodic solution curves. This result

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also implies the existence of infinitely many closed immersed surfaces of revolution that are critical values for the Wilmore functional (cf. [5]).

In [2] there is a study of the Euler-Lagrange differential system in the general setting described above, and we will follow the notations and terminology used there. In Chapter II of [2] there is also a general discussion of how a group of symmetries gives 1st integrals. This is used to integrate the Euler equations associated to a number of variational problems in homogeneous spaces (including those associated to (1)). However, in [2] the essential final step of using the full reduction procedure is not taken, and that is what we do here in Sections 1, 2 for general variational problems. In Sections 3, 4 we show how the general method allows us to draw global conclusions about the solutions to the Euler-Lagrange equations associated to (1).

(Added in press) After this paper was finished, the paper [3a] of Langer and Singer came to the author's attention. Their paper reproduces some of our results and undertakes a study of stability of the critical pts of Φ.

1. Review of Variational Formalism.

a) *Notations from differential systems* (cf. [2]). A Pfaffian differential system (I, ω) on a manifold X is given by the following data:

i) A sub-bundle $I \subset T^*(X)$. Locally we may choose a coframe $\theta^1, \ldots, \theta^s$ for I and we think of I as giving the Pfaffian differential equations

$$\theta^1 = \cdots = \theta^s = 0. \tag{1.a.1}$$

ii) Another vector bundle $L \subset T^*(X)$ with $I \subset L \subset T^*(X)$. If rank $(L/I) = n$ then ω denotes a locally given n-form on X which induces a nonzero section of $\Lambda^n(L/I)$. We think of ω as giving the independence condition

$$\omega \neq 0. \tag{1.a.2}$$

Throughout this paper we shall restrict to the case $n = 1$. An integral manifold of (I, ω) is given by a compact 1-dimensional manifold (possibly with boundary) N together with a smooth mapping

(1.a.3) $$f\colon N \to X$$

satisfying

(1.a.4) $$\begin{cases} f^*I = 0 \\ f^*\omega \neq 0. \end{cases}$$

The 1st equation means that $f^*\theta = 0$ for all smooth 1-forms θ such that $\theta(x) \in I_x \subset T_x^*(X)$ for all $x \in X$. Thus, $f(N)$ is an immersed integral curve of the ODE system (1.a.1) satisfying the transversality condition (1.a.2). If we define an *integral element* of (I, ω) to be given by a point $x \in X$ and line $E \subset T_x(X)$ satisfying

$$\begin{cases} \theta(x)|_E = 0 \quad \text{for all} \quad \theta \in I \\ \omega(x)|_E \neq 0, \end{cases}$$

then the totality of integral elements forms a subset $V(I, \omega) \subset \mathbf{P}T(X)$ of the projectivized tangent bundle. Any immersion (1.a.3) has a canonical lift $f_*\colon N \to \mathbf{P}T(X)$, and the condition to be an integral manifold is just $f_*(N) \subseteq V(I, \omega)$. We shall denote the totality of all integral manifolds by $\mathcal{V}(I, \omega)$.

Sometimes a Pfaffian differential system is given by a $C^\infty(X)$ sub-module $\mathcal{I}$ of the 1-forms on X together with a 1-form ω, where it is *not* assumed that the subset $I = \{\theta(x) \in T_x^*(X)\colon x \in X \text{ and } \theta \in \mathcal{I}\} \subset T^*(X)$ gives a sub-bundle. This failure of I to have constant rank is caused by the following phenomenon (cf. below): In order to construct integral curves of I along which $\omega \neq 0$ we must be able to first find integral elements (as defined below). In examples it will turn out that $\omega(x) \in I_x$ at a general point of X, and then the only way there can exist integral elements is for $\dim I_x$ to jump up along a subset of X. This will be explained more fully in the construction below.

Example. Let Ψ be a 2-form on X (whose rank may vary), and define the *Cartan system* $\mathcal{C}(\Psi)$ to be the Pfaffian system given by

$$\{v \lrcorner \Psi\colon v \in C^\infty(T(X)) \text{ is a vector field on } X\}.$$

This example arises naturally in variational problems (cf. [2] and below).

Given (I, ω) on X we may define integral manifolds by (1.a.4) where the first condition is replaced by $f^*(I) = 0$. Under reasonable hypotheses (that we do not try to make precise—cf. [2]) we may associate to (I, ω) a Pfaffian system (I^*, ω), in the previous sense, on a submanifold $X^* \subset X$ such that the integral manifolds on (I, ω) and (I^*, ω) coincide, as follows:

Construction. Define integral elements of (I, ω) to be given by lines $E \subset T_x(X)$ satisfying $\theta(x)|_E = 0$ for all $\theta \in I$ and $\omega(x)|_E \neq 0$. The totality of such integral elements gives a subset $V(I, \omega) \in \mathbf{P}T(X)$ of the projectivized tangent bundle of X. Denote the projection by $\pi: \mathbf{P}T(X) \to X$ and inductively define

$$\begin{cases} X_1 = \pi(V(I, \omega)) \\ V_1(I, \omega) = \{E \in V(I, \omega) \colon E \text{ is tangent to } X_1\} \\ X_2 = \pi(V_1(I, \omega)) \\ V_2(I, \omega) = \{E \in V_1(I, \omega) \colon E \text{ is tangent to } X_2\} \\ \quad\vdots \end{cases}$$

This gives $X_1 \supset X_2 \supset \cdots$, and (under our reasonable assumptions) we will have $X_k = X_{k+1} = \cdots = X^*$ for some k. Moreover,

$$I^* = \{\theta(x) \in T_x^*(X^*) \colon \theta \in I|_{X^*}\}$$

will give a sub-bundle of $T^*(X^*)$, and by elementary reasoning we will have $\mathcal{V}(I, \omega) = \mathcal{V}(I^*, \omega)$.

We shall call (I^*, ω) the *involutive prolongation* of (I, ω). It is discussed in detail, together with numerous illustrative examples, in Chapter I of [2].

b) *The Euler-Lagrange differential system.* A variational problem $(I, \omega; \varphi)$ is the study of the functional

$$\Phi \colon \mathcal{V}(I, \omega) \to \mathbf{R} \tag{1.b.1}$$

given by

$$\Phi(N, f) = \int_N f^*\varphi \tag{1.b.2}$$

where (I, ω) is a Pfaffian differential system on a manifold X, $(N, f) \in \mathcal{V}(I, \omega)$ is a typical integral manifold (1.a.3), and where φ is a 1-form on X. Associated to $(I, \omega; \varphi)$ we shall define the Euler-Lagrange Pfaffian differential system (J, ω) on a new manifold Y. We refer the reader to [2] for a discussion of how this system is derived and of how the integral manifolds of (J, ω) give stationary values of the functional (1.b.2). Remark that *any* variational problem for curves (especially constrained and/or higher order problems) can be reduced to the setup we are considering. Remark also that it is still unknown whether or not in general the Euler-Lagrange system (I, ω) gives *necessary* as well as sufficient conditions for stationary values of Φ.

For this we define the affine sub-bundle

$$Z \subset T^*(X)$$

by $Z = \bigcup_{x \in X} Z_x$ where

$$Z_x = \{\varphi(x) + I_x \subset T^*_x(X)\}.$$

Note that Z only depends on φ modulo I. Let ψ be the restriction to Z of the tautological 1-form on $T^*(X)$ and set

$$\Psi = d\psi.$$

On Z we thus have the 2-form Ψ and 1-form ω obtained by pulling back ω on X, and by the above discussion the involutive prolongation of $(\mathcal{C}(\Psi), \omega)$ on Z gives a Pfaffian system (J, ω) on a submanifold $Y \subset Z$.

Definition. (J, ω) is the *Euler-Lagrange differential system* associated to the variational problem $(I, \omega; \varphi)$.

Remarks. i) As noted above, (J, ω) depends only on φ modulo I. Moreover, if we add to φ an exact form df, then the obvious map $Y \to Y + df$ maps the old Euler-Lagrange system to the new one.

ii) If $\theta^1, \cdots, \theta^s$ is a local coframe for I, then a point of Z will be

$$\psi(x, \lambda) = \varphi(x) + \lambda_\alpha \theta^\alpha(x)$$

(throughout we use summation convention). Clearly

$$\Psi(x, \lambda) = d\varphi(x) + d\lambda_\alpha \wedge \theta^\alpha(x) + \lambda_\alpha d\theta^\alpha(x).$$

By contracting Ψ with tangent vectors $\partial/\partial\lambda_\alpha$ we see that integral manifolds of (J, ω) project to X to give integral manifolds of (I, ω) (briefly, we may say that *J contains I*).

iii) The variational problem $(I, \omega; \varphi)$ is defined to be *nondegenerate* in case $\dim Y = 2m + 1$ and $\Psi^m \neq 0$. In many examples the characteristic direction of Ψ then generates a global foliation

$$Y \xrightarrow{\tilde{\omega}} Q$$

where Q is a $2m$-dimensional symplectic manifold with 2-form Ω satisfying $\tilde{\omega}^*\Omega = \Psi$. We may then think of the integral manifolds of (J, ω) as themselves constituting a symplectic manifold (cf. [2] for amplification and examples).

iv) Even without the nondegeneracy condition it still makes sense to speak of a stationary path: We give $\mathcal{V}(I, \omega)$ the topology it inherits as a subspace of the full immersion space with the C^∞ topology, and then we seek critical points of Φ: $\mathcal{V}(I, \omega) \to \mathbf{R}$ with this topology. In this case the stationary paths are the projections to X of the *characteristic curves* of Ψ on Z, where γ: $N \to Z$ is defined to be characteristic if

i) $\gamma'(t) \lrcorner \Psi_{\gamma(t)} = 0$ for all $t \in N$
ii) $\gamma^*(\omega) \neq 0$.

In particular, one may have characteristic curves depending on arbitrary functions (instead of just constants as in classical cases). An example of this is the functional

$$\int_\gamma \kappa ds$$

define on immersed curves $\gamma \subset \mathbf{E}^2$.

2. Reduction for Constrained Variational Problems.

a) *The reduced system.* There is a well known reduction procedure for a Hamiltonian system with symmetries (cf. [4]). We shall give an extension of this to general variational problems. The outcome is that a symmetry group allows us to systematically reduce the dimension of the space on which the Euler-Lagrange system is defined.

Let a connected Lie group G with Lie algebra $\mathfrak{g}$ act on a manifold X, and denote the induced action on $T^*(X)$ by

$$\eta_g \colon T^*(X) \to T^*(X), \qquad g \in G.$$

Let $(I, \omega; \varphi)$ on X be a variational problem and denote by $Z \subset T^*(X)$ the affine sub-bundle constructed in Section 1b) above.

Definition. G is a *group of symmetries of the variational problem* $(I, \omega; \varphi)$ if

$$\eta_g(Z) = Z, \qquad g \in G.$$

It follows that G leaves invariant the sub-bundle $I \subset T^*(X)$ together with the 1-form φ considered modulo I, and conversely. Thus, this definition agrees with that in [2].

Given a group of symmetries of $(I, \omega; \varphi)$ we shall define the *momentum mapping*

$$m \colon Y \to \mathfrak{g}^* \tag{2.a.1}$$

where (J, ω) on Y is the Euler-Lagrange differential system (cf. Section 1b)). In fact, m will be the restriction to $Y \subset Z$ of a map

$$m \colon Z \to \mathfrak{g}^*$$

that we now construct. For $\xi \in \mathfrak{g}$ we denote by ξ the vector field on Z given by the action of the 1-parameter group $\eta_{\exp t\xi}$. We then set

$$\langle m(p), \xi \rangle = (\xi \lrcorner \psi)(p), \qquad p \in Z, \tag{2.a.2}$$

where ψ is the restriction to Z of the canonical 1-form on $T^*(X)$. Since ξ is tangent to the submanifold $Y \subset Z$ (this means that $\xi(p) \in T_p(Y) \subset$

$T_p(Z)$ for $p \in Y$), the momentum mapping (2.a.1) is given by the same formula (2.a.2) where $p \in Y$.

We denote by

$$\mathrm{Ad}^*\colon G \to \mathrm{Aut}(\mathfrak{g}^*)$$

the *coadjoint representation* (cf. [3]). From the obvious fact that $\eta_g^*\psi = \psi$ and easily verified fact that $(\eta_g)_*\xi = (\mathrm{Ad}_{g^{-1}})_*\xi$, it follows that:

The momentum mapping (2.a.1) *is* Ad**-equivariant, in the sense that*

$$m(\eta_g(p)) = (\mathrm{Ad}_{g^{-1}})^* m(p). \tag{2.a.3}$$

For $\mu \in \mathfrak{g}^*$ we set

$$G_\mu = \{g \in G\colon (\mathrm{Ad}_{g^{-1}})^*\mu = \mu\},$$

and note that G_μ acting on Y leaves $\mathrm{m}^{-1}(\mu)$ invariant. Assume that $m^{-1}(\mu)$ is a submanifold of Y and that the quotient space $Y_\mu = G_\mu \setminus m^{-1}(\mu)$ exists as a manifold; thus we have

$$\begin{array}{c} m^{-1}(\mu) \subset Y \\ \downarrow{\scriptstyle \pi} \\ Y_\mu \end{array} \tag{2.a.4}$$

We claim that:

(2.a.5) *There exists a unique 2-form* Ψ_μ *on* Y_μ *such that*

$$\pi^*\Psi_\mu = \Psi|_{m^{-1}(\mu)}.$$

Proof. Let $\mathfrak{g}_\mu$ be the Lie algebra of G_μ, so that the vertical tangent spaces to the fibering (2.a.4) are spanned by the vector fields ξ where $\xi \in \mathfrak{g}_\mu$ (note that these vector fields are tangent to $m^{-1}(\mu) \subset Y$). Next we consider the identity on Y

$$0 = \mathcal{L}_\xi\psi = \xi \lrcorner \Psi + d(\xi \lrcorner \psi). \tag{2.a.6}$$

Restricting to $m^{-1}(\mu)$, the last term is zero since $\xi \lrcorner \psi = \langle \mu, \xi \rangle$ is constant there. It follows that

$$\xi \lrcorner (\Psi|_{m^{-1}(\mu)}) = 0. \tag{2.a.7}$$

If we use the well-known criterion that a form α on $m^{-1}(\mu)$ is $\pi^*(\alpha_\mu)$ for a unique form α_μ on Y_μ if, and only if, $\xi \lrcorner \alpha = 0 = \xi \lrcorner d\alpha = 0$ for all $\xi \in \mathfrak{g}_\mu$, then (2.a.5) follows from (2.a.7) and $d\Psi = 0$.

Definitions. We shall call Y_μ the *reduced space* associated to $\mu \in \mathfrak{g}^*$, and $(Y_\mu, \mathcal{C}(\Psi_\mu))$ will be called the *reduced system.*

The importance of the reduced system lies in the following observations:

(2.a.8) (NOETHER'S THEOREM). *The momentum mapping* (2.a.1) *is constant on integral curves of* (J, ω).

This follows from (2.a.6) in the form

$$d(\xi \lrcorner \psi) = -\xi \lrcorner \Psi \in J = \mathcal{C}(\Psi).$$

Next, it is clear that

(2.a.9) *The integral curves of* $(J, \omega)|_{m^{-1}(\mu)}$ *are permuted among themselves by* G_μ.

Finally, it is also clear that

(2.a.10) *The integral curves of* $(J, \omega)|_{m^{-1}(\mu)}$ *project to integral curves of* $\mathcal{C}(\Psi_\mu)$.

In practice, (2.a.8)-(2.a.10) reduce the determination of the integral curves of (J, ω) to finding the integral curves of the reduced system plus one more integration to lift these curves to $m^{-1}(\mu)$. In fact, it is not difficult to show that:

(2.a.11) *If the variational problem* $(I, \omega; \varphi)$ *is nondegenerate, then* $(Y, \mathcal{C}(\Psi_\mu))$ *is nondegenerate in the sense that*

$$\begin{cases} \dim Y_\mu = 2k + 1 \\ \Psi_\mu{}^k \neq 0. \end{cases}$$

Through each point $q \in Y_\mu$ there is then a unique integral curve γ of $\mathcal{C}(\Psi_\mu)$ obtained by flowing along the characteristic direction field of Ψ_μ. For each $p \in \mu^{-1}(q)$, there is a unique integral curve $\tilde{\gamma}$ of (J, ω) lying over γ. In examples given below the nature of the one integration required to lift γ to $\tilde{\gamma}$ will be clear.

b) *Reduction for homogeneous variational problems.* Let G be a Lie group with Lie algebra $\mathfrak{g} = T_e(G)$. Using *left* translation we make the identification

$$T(G) \cong G \times \mathfrak{g}. \tag{2.b.1}$$

We shall consider Pfaffian systems given by a left invariant sub-bundle $I_0 \subset T^*(G)$. By (2.b.1) the distribution $I_0^\perp \subset T(G)$ is simply $G \times \mathfrak{a}$ for a unique subspace $\mathfrak{a} \subset \mathfrak{g}$. To write the differential system in the form described in Section 1a), we let ω be the $\mathfrak{g}$-valued left invariant Maurer-Cartan form on G, uniquely characterized by left invariance plus the condition that at $e \in G$

$$\omega(\xi) = \xi, \qquad \xi \in \mathfrak{g} = T_e(G). \tag{2.b.2}$$

It satisfies the Maurer-Cartan equation

$$d\omega = -\frac{1}{2}[\omega, \omega]. \tag{2.b.3}$$

On

$$X = G \times \mathfrak{a} \times \mathbf{R},$$

where $\mathbf{R}$ has coordinate t, we consider the $\mathfrak{g}$-valued 1-form

$$\theta = \omega - Pdt, \qquad P \in \mathfrak{a}. \tag{2.b.4}$$

Here, t: $X \to \mathbf{R}$ is the projection on the 3rd factor and P: $X \to \mathfrak{a}$ is the projection on the 2nd factor. Then $\theta = \omega - Pdt$ is a $\mathfrak{g}$-valued 1-form on X, and its components generate a sub-bundle I of $T^*(X)$. Clearly (I, dt) gives a G-invariant differential system whose integral curves are

$$t \to (g(t), \alpha(t), t) \in X$$

where $g(t) \in G$ is a curve whose tangent vector $g'(t)$ satisfies

$$\omega(g'(t)) = \alpha(t) \in \mathfrak{a}.$$

(Note: Throughout the remainder of this paper the independence condition will be $dt \neq 0$; this is because we want to reserve the notation ω for the Maurer-Cartan form.)

To obtain a left invariant variational problem we take

$$\varphi = L(P)dt$$

where

$$L: \mathfrak{a} \to \mathbf{R}$$

is a function. Several examples of invariant variational problems arising from Frenet liftings of curves in homogeneous spaces are given in [2], and below we shall investigate the global properties of one such system.

We shall now apply the methods discussed in Section 1 to derive the Euler-Lagrange system and reduction procedure for the G-invariant variational problem (I, dt, φ). For $Z \subset T^*(X)$ as defined in Section 1b) we have an identification

$$X \times \mathfrak{g}^* \leftrightarrow Z,$$

where we identify the pair $(x, \lambda) \in X \times \mathfrak{g}^*$ with the 1-form

$$(\varphi + \langle \lambda, \theta \rangle)(x) \in T^*_x(X).$$

Under this identification, the canonical 1-form ψ on Z may be written

$$\psi = L(P)dt + \langle \lambda, \theta \rangle. \tag{2.b.5}$$

Here, we are regarding $\lambda: X \times \mathfrak{g}^* \to \mathfrak{g}^*$, the projection on the 2nd factor, as a $\mathfrak{g}^*$-valued function on $X \times \mathfrak{g}^*$.

Using (2.b.3) and (2.b.4) the 2-form $\Psi = d\psi$ is given by

$$\Psi = dL(P) \wedge dt - \langle \lambda, dP \wedge dt \rangle + \langle d\lambda, \theta \rangle - \frac{1}{2} \langle \lambda, [\omega, \omega] \rangle. \tag{2.b.6}$$

To compute the Cartan system, we consider $\alpha \in \mathfrak{a}$ as an invariant vector field on the vector space $\mathfrak{a}$, and for each $P \in \mathfrak{a}$ we consider the differential

$$dL(P) \in T_P^*(\mathfrak{a}) \cong \mathfrak{a}^*.$$

Then, since $\langle dP, \alpha \rangle = \alpha$ (here P is considered an $\mathfrak{a}$-valued function)

$$\alpha \lrcorner \Psi = \langle dL(P) - \lambda, \alpha \rangle \; dt.$$

Because the independence condition $dt \neq 0$ this gives

$$\langle dL(P) - \lambda, \alpha \rangle = 0 \quad \text{for all} \quad \alpha \in \mathfrak{a}. \tag{2.b.7}$$

(Note: In case the Legendre transform $dL\colon \mathfrak{a} \to \mathfrak{a}^*$ is one-to-one, this equation determines P as function of λ.) Next for $\xi \in \mathfrak{g}$ viewed as a left invariant vector field on G, we have, using (2.b.2) and (2.b.6), that

$$\begin{aligned} \xi \lrcorner \Psi &= -\langle d\lambda, \xi \rangle - \langle \lambda, [\xi, \omega] \rangle \\ &= -\langle d\lambda + ad_\omega^* \lambda, \xi \rangle. \end{aligned}$$

Here,

$$ad^*\colon \mathfrak{g} \to \operatorname{Hom}(\gamma^*, \mathfrak{g}^*)$$

is the differential of the coadjoint representation, and we are considering ad_ω^* as a $\operatorname{Hom}(\mathfrak{g}^*, \mathfrak{g}^*)$-valued 1-form. If we recall that the Cartan system $\mathcal{C}(\Psi)$ contains the original system $\theta = 0$, this equation may be replaced by

$$d\lambda + ad_P^* \lambda \cdot dt = 0.$$

Summarizing, the Euler-Lagrange system is generated by the equations

$$\begin{cases} \text{(i)} \;\; \theta = \omega - P dt = 0 \\ \text{(ii)} \;\; \langle dL(P) - \lambda, \alpha \rangle = 0 \qquad \alpha \in \mathfrak{a}. \\ \text{(iii)} \;\; d\lambda + ad_P^* \lambda \cdot dt = 0 \end{cases} \tag{2.b.8}$$

As in [2] we shall call (iii) the *Euler equations* associated to the invariant variational problem.

We shall now turn to the reduction procedure. For this we denote by ξ the *right* invariant vector field induced by the left multiplication by the 1-parameter subgroup $\exp t\xi$ of G. Then

$$\begin{aligned}\xi(g) &= (R_g)_*\xi(e)\\ &= (L_{g^{-1}})_*(R_g)_*\xi(g),\end{aligned}$$

where $\xi \in \mathfrak{g}$ is considered as a left invariant vector field on G, and this gives (cf. (2.a.3))

(2.b.9) $$\xi(g) = ((Ad_{g^{-1}})(\xi))(g).$$

By definition, the momentum mapping is the restriction to $Y \subset Z$ of

$$m: Z \to \mathfrak{g}^*$$

where

$$\begin{aligned}\langle m(g, \lambda, P, t), \xi\rangle &= (\xi \lrcorner \psi)(g, \lambda, P, t)\\ &= \langle \lambda, (Ad_{g^{-1}})\xi\rangle\\ &= \langle (Ad_{g^{-1}})^* \cdot \lambda, \xi\rangle\end{aligned}$$

by (2.b.5) and (2.b.9). Thus

(2.b.10) $$m(g, \lambda, P, t) = (Ad_{g^{-1}})^*\lambda.$$

Restricting to $Y \subset Z$ we have for $\mu \in \mathfrak{g}^*$

$$m^{-1}(\mu) = \{(g, \lambda, P, t) \in Y: \lambda = Ad_g^*\mu\}.$$

Following still the general theory, the isotropy group G_μ acts on $m^{-1}(\mu)$ by left multiplication on G. If we define as usual the *coadjoint orbit*

$$\mathcal{O}_{Ad^*}(\mu) = \{Ad_g{}^*\mu \colon g \in G\} \subset \mathfrak{g}^*,$$

then there is a natural identification

$$G_\mu \diagdown G \cong \mathcal{O}_{Ad^*}(\mu)$$

(this uses that $Ad^*_{g_1g_2} = Ad^*_{g_2}\, Ad^*_{g_1}$). Rather than consider the whole reduced space $Y_\mu = G_\mu \diagdown m^{-1}(\mu)$ we consider its image in $\mathfrak{g}^*$. This gives the diagram

$$\begin{array}{c} m^{-1}(\mu) \subset Y \\ \Big\downarrow{\scriptstyle \pi_\mu} \\ \mathcal{O}_{Ad^*}(\mu) \subset \mathfrak{g}^* \end{array}$$

where the equations

$$\pi_\mu(g, \lambda, P, t) = \lambda$$

$$(Ad_{g^{-1}})^*\lambda = \mu$$

are satisfied. (In case the Legendre transform is nondegenerate so that λ determines P on Y, Y_μ differs from $\mathcal{O}_{Ad^*}(\mu)$ by only the trivial t factor. This may be eliminated by applying reduction to the time shift automorphism.)

Along a solution curve $g(t) = (g(t), \lambda(t), P(t), t)$ to the Euler-Lagrange differential system, we have by (2.b.8)

$$\text{(2.b.11)} \qquad \begin{cases} \text{(i) } \omega(g(t)) = P(t)dt \\ \text{(ii) } \langle (dL)(P(t)) - \lambda(t), \alpha \rangle = 0 \quad \text{for all} \quad \alpha \in \mathfrak{a} \\ \text{(iii) } \dfrac{d\lambda(t)}{dt} + ad^*_{P(t)}\lambda(t) = 0. \end{cases}$$

Equations (i) and (iii) say that

$$\frac{d}{dt}\left((Ad_{g(t)^{-1}})^*\lambda(t)\right) = 0;$$

in particular, $\lambda(t) \in \mathfrak{g}^*$ moves on a coadjoint orbit (cf. [2]). If we solve the Euler equations (iii), then we determine $P(t)$ by (ii) (at least in the non-degenerate case), and by one more integration determine $g(t)$ by (*i*). In summary:

(2.b.12)
Given $\mu \in \mathfrak{g}^*$ *we construct the* G_μ *bundle* $B(\mu) \to \mathcal{O}_{Ad^*}(\mu)$ *where*

$$B(\mu) = \{(g, \lambda) \in G \times \mathfrak{g}^* : (Ad_{g^{-1}})^* \lambda = \mu\}.$$

Then solution curves $g(t) \in G$ *to our variational problem satisfy* $(g(t), \lambda(t)) \in B_\mu$ *where* $\lambda(t) \in \mathfrak{g}^*$ *is a solution to the Euler equations and* (*i*), (*ii*) *in* (2.b.11) *are satisfied.*

3. Study of $\frac{1}{2} \int \kappa^2 ds$ in the Euclidean Case.

a) *Unrestricted Variations.* We shall work on the manifold $\mathfrak{F}(\mathbf{E}^2)$ of frames (x, e_1, e_2) in $\mathbf{E}^2$. The structure equations of a moving frame are

$$\text{(3.a.1)} \qquad \begin{cases} dx = \omega^1 e_1 + \omega^2 e_2 \\ de_1 = \rho e_2 \\ de_2 = -\rho e_1 \end{cases}$$

where we have set $\rho = \omega_1^2$ in the usual notation. It is well known that we may identify $\mathfrak{F}(\mathbf{E}^2)$ with the group G of Euclidean motions, and then ω^1, ω^2, ρ give a basis for *left* invariant 1-forms on G. This will be made quite explicit below. From (3.a.1) we obtain the Maurer-Cartan equations

$$\text{(3.a.2)} \qquad \begin{cases} d\omega^1 = -\omega^2 \wedge \rho \\ d\omega^2 = \omega^1 \wedge \rho \\ d\rho = 0. \end{cases}$$

If ξ_1, ξ_2, ξ_3 is the basis for $\mathfrak{g}$ dual to ω^1, ω^2, ρ, then by (2.b.3)

$$\begin{cases} [\xi_1, \xi_2] = 0 \\ [\xi_2, \xi_3] = \xi_1 \\ [\xi_1, \xi_3] = -\xi_2. \end{cases}$$

From this it follows that (cf. [2])

(3.a.3) *For* $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathfrak{g}^*$ *the coadjoint orbits are given by*

$$\lambda_1^2 + \lambda_2^2 = c^2, \qquad c \geqq 0,$$

This will also be made quite explicit below.

For convenience we consider $\mathbf{E}^2$ as oriented $\mathbf{R}^2$ with coordinates (x_1, x_2) and orientation $dx_1 \wedge dx_2 > 0$. An immersed curve $\gamma \subset \mathbf{E}^2$ is the image of a mapping

$$x \colon N \to \mathbf{E}^2 \tag{3.a.4}$$

where $N = \{a \leqq t \leqq b\}$ and $x(t) = (x_1(t), x_2(t))$ is of class C^2 on the closed interval $[a, b]$. Associated to (3.a.4) are its arclength element $ds = \sigma(t)dt$ where $\sigma(t) = \|x'(t)\| > 0$, and Frenet frame $F(t) = (x(t), e_1(t), e_2(t))$ where $e_1(t) = x'(t)/\sigma(t)$ is the unit tangent and $e_1(t) \wedge e_2(t) > 0$. Under this Frenet lifting

$$\begin{tikzcd} & \mathfrak{F}(\mathbf{E}^2) \arrow[d] \\ N \arrow[ur, dashed, "F"] \arrow[r, "x"] & \mathbf{E}^2 \end{tikzcd}$$

the structure equations (3.a.1) pull back to the Frenet equations

$$\begin{cases} dx(t) = \sigma(t)dt\, e_1, & \sigma(t) > 0, \\ de_1(t) = \sigma(t)\kappa(t)dt\, e_2, & \\ de_2(t) = -\sigma(t)\kappa(t)dt\, e_1 & \end{cases} \tag{3.a.5}$$

where $\kappa(t)$ is the curvature.

To give Frenet liftings as integral manifolds of a differential system, we let $\mathfrak{F}^0(\mathbf{E}^2)$ be the oriented frames and set $X = \mathfrak{F}^0(\mathbf{E}^2) \times \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}$ where $\mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}$ has coordinates (σ, κ, t). On X we consider the Pfaffian differential system generated by the Pfaffian equations

$$\begin{cases} \omega^1 - \sigma dt = 0, & \sigma > 0, \\ \omega^2 = 0 & \\ \rho - \kappa\sigma dt = 0. & \end{cases} \tag{3.a.6}$$

Clearly, the integral manifolds of $(I,\ dt)$ are just the Frenet liftings. We also set

$$\varphi = Ldt = \frac{1}{2}\kappa^2 dt,$$

so that the variational problem $(I,\ dt;\ \varphi)$ amounts to considering the functional

$$\Phi(\gamma) = \frac{1}{2}\int_\gamma \kappa^2 ds \tag{3.a.7}$$

on immersed curves $\gamma \subset \mathbf{E}^2$. (Note: In our general formulation given in Section 2b), $\mathfrak{a} \subset \mathfrak{g}$ corresponds to span $\{e_1, e_3\} = \{\sigma e_1 + \lambda e_3 : \sigma, \lambda \in \mathbf{R}\} \subset \mathfrak{g}$. Here, we are restricting to the open subset $\mathbf{R}^+ \times \mathbf{R} \subset \mathfrak{a}$ given by $\sigma > 0$ and setting $\lambda = \kappa\sigma$ there.)

To find the Euler-Lagrange system associated to the variational problem $(I,\ dt;\ \varphi)$ we let $Z = X \times \mathbf{R}^3$ where $\mathbf{R}^3 \cong \mathfrak{g}^*$ has coordinates $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, and on Z we set (cf. Section 1b))

$$\psi = \frac{1}{2}\kappa^2\sigma dt + \lambda_1(\omega^1 - \sigma dt) + \lambda_2\omega^2 + \lambda_3(\rho - \kappa\sigma dt).$$

Using the structure equations (3.a.2) we may determine $\Psi = d\psi$, and then a straightforward calculation shows that the Cartan system $\mathcal{C}(\Psi)$ is generated by the Pfaffian equations (cf. [2])

$$\begin{cases} \text{(i)}\ \ \partial/\partial\lambda_1 \lrcorner \Psi = \omega^1 - \sigma dt = 0 \\ \text{(ii)}\ \ \partial/\partial\lambda_2 \lrcorner \Psi = \omega^2 = 0 \\ \text{(iii)}\ \ \partial/\partial\lambda_3 \lrcorner \Psi = \rho - \kappa\sigma dt = 0 \end{cases}$$

$$\begin{cases} \text{(iv)}\ \ \partial/\partial\sigma \lrcorner \Psi = \left(\frac{1}{2}\kappa^2 - \lambda_3\kappa - \lambda_1\right)dt = 0 \\ \text{(v)}\ \ \partial/\partial\kappa \lrcorner \Psi = \sigma(\kappa - \lambda_3)dt = 0 \end{cases}$$

$$\begin{cases} \text{(vi)}\ \ -\partial/\partial\rho \lrcorner \Psi = d\lambda_3 - \lambda_1\omega^2 + \lambda_2\omega^1 = 0 \\ \text{(vii)}\ \ -\partial/\partial\omega^1 \lrcorner \Psi = d\lambda_1 - \lambda_2\rho = 0 \\ \text{(viii)}\ \ -\partial/\partial\omega^2 \lrcorner \Psi = d\lambda_2 + \lambda_1\rho = 0. \end{cases}$$

Here $\partial/\partial\rho$, $\partial/\partial\omega^1$, $\partial/\partial\omega^2$, ... denotes the tangent frame on Z dual to the coframe ρ, ω^1, ω^2, ... Equations (i)–(iii) give the original system. Since $\sigma dt \neq 0$ equations (iv), (v) give

$$\begin{cases} \lambda_3 = \kappa \\ \lambda_1 = -\kappa^2/2. \end{cases} \tag{3.a.8}$$

Using the notations from 1b), it is easy to see that $Y \subset Z$ is given by (3.a.8).

Equations (vi)–(viii) are the Euler equations. Since $I \subset \mathcal{C}(\Psi)$ we may replace (vi) by

$$d\lambda_3 + \lambda_2\sigma dt = 0,$$

and then on integral curves of I this equation is

$$\lambda_2 = -\dot{\kappa} = -d\kappa/ds. \tag{3.a.9}$$

The coadjoint orbit condition (3.a.3) then gives the *conservation law*

$$\dot{\kappa}^2 + \frac{\kappa^4}{4} = c^2 \geqq 0, \qquad c \geqq 0, \tag{3.a.10}$$

where c is an "energy level" for our problem. At this juncture, it follows (cf. below) that $\kappa(s)$ is an elliptic function of the arclength parameter s. Then it follows from the general reduction procedure that the position vector $x(s) = (x_1(s), x_2(s))$ may be obtained by once integrating elliptic functions. In particular, the question of closed solution curves will reduce to one concerning periods of elliptic integrals (this will be true for all the specific problems in this paper).

It is however instructive to explicitly carry out the reduction. On $\mathfrak{F}^0(\mathbf{E}^2)$ we introduce coordinates $(x_1, x_2, \theta) \in \mathbf{R}^2 \times S^1$ where the coordinates of a frame $F = (x, e_1, e_2)$ are given by writing

$$\begin{cases} x = (x_1, x_2) \\ e_1 = (\cos\theta, \sin\theta) \\ e_2 = (-\sin\theta, \cos\theta). \end{cases}$$

Then by the structure equations (3.a.1)

$$(3.a.11)\qquad \begin{cases} \omega^1 = (dx, e_1) = \cos\theta\, dx_1 + \sin\theta\, dx_2 \\ \omega^2 = (dx, e_2) = -\sin\theta\, dx_1 + \cos\theta\, dx_2 \\ \rho = (de_1, e_2) = d\theta, \end{cases}$$

and the structure equations (3.a.2) hold. To F we associate the matrix

$$(3.a.12)\qquad g = \begin{pmatrix} 1 & 0 & 0 \\ x_1 & \cos\theta & -\sin\theta \\ x_2 & \sin\theta & \cos\theta \end{pmatrix}$$

and then

$$g^{-1}dg = \begin{pmatrix} 0 & 0 & 0 \\ \cos\theta\, dx_1 + \sin\theta\, dx_2 & 0 & -d\theta \\ -\sin\theta\, dx_1 + \cos\theta\, dx_2 & d\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ \omega^1 & 0 & -\rho \\ \omega^2 & \rho & 0 \end{pmatrix}$$

where the 2nd step follows from (3.a.11). It follows that ω^1, ω^2, ρ give a basis for the *left* invariant 1-forms on the Lie group G of matrices (3.a.12).

The Euler equations (vi)-(viii) may be written

$$d(-\lambda_3, \lambda_2, -\lambda_1) = (-\lambda_3, \lambda_2, -\lambda_1)\begin{pmatrix} 0 & 0 & 0 \\ \omega^1 & 0 & -\rho \\ \omega^2 & \rho & 0 \end{pmatrix}.$$

Letting $\hat{\lambda}$ be the row vector $(-\lambda_3, \lambda_2, -\lambda_1)$ this is

$$d\hat{\lambda} = \hat{\lambda} g^{-1} dg;$$

i.e.,

$$d(\hat{\lambda} g^{-1}) = 0. \tag{3.a.13}$$

To put this equation in a general context we remark that the linear mapping $\lambda \to \hat{\lambda}$ induces an isomorphism $\mathfrak{g}^* \cong \mathbf{R}^3$ under which the coadjoint representation is given by

$$\widehat{Ad_{g^{-1*}} \cdot \lambda} = \hat{\lambda} \cdot g^{-1}.$$

Noether's theorem (2.a.8) is

$$\hat{\lambda} = \hat{\mu} \cdot g, \qquad \mu = \text{constant}, \tag{3.a.14}$$

on solution curves to the Euler-Lagrange system. Using (3.a.8) and (3.a.9), (3.a.14) is

$$\begin{cases} \text{(i)} \ -\kappa = \mu_3 + \mu_2 x_1 - \mu_1 x_2 \\ \text{(ii)} \ -\dot{\kappa} = \mu_2 \cos\theta - \mu_1 \sin\theta \\ \text{(iii)} \ -\dfrac{\kappa^2}{2} = \mu_2 \sin\theta + \mu_1 \cos\theta \end{cases} \tag{3.a.15}$$

Equations (ii) and (iii) give (3.a.10) where

$$c^2 = \mu_1{}^2 + \mu_2{}^2.$$

If $c = 0$, then $\dot{\kappa} = \kappa = 0$ and the solution curve is a straight line.

Assume $c > 0$. Then we may find $g_0 \in G$ with $\hat{\mu} \cdot g_0^{-1} = (0, -c, 0)$. Applying the rigid motion g_0 to $\mathbf{E}^2$, we may assume that $\hat{\mu} = (0, -c, 0)$ in which case (3.a.15) becomes

$$\begin{aligned} \text{(i)} \quad x_1 &= \kappa/c \\ \text{(ii)} \quad \dot{\kappa}_2 &= c\cos\theta \\ \text{(iii)} \quad \frac{\kappa^2}{2} &= c\sin\theta. \end{aligned}$$

From $\dot{x}_2(s) = \sin\theta(s)$ we obtain

$$x_2(s) = x_2(0) + \int_0^s \frac{\kappa(\xi)^2}{2c} d\xi$$

$$> x_2(0).$$

Thus, even though $\kappa(s)$ and therefore $x_1(s)$ may be periodic functions of s, $x_2(s)$ cannot be and there are no closed solution curves in $\mathbf{E}^2$ to the Euler-Lagrange equations. Of course, this is clear *a priori*, since dilation about a point in $\mathbf{E}^2$ induces a variation of closed curves along which $\frac{1}{2}\int \kappa^2 ds$ is clearly not stationary.

b) *Fixed length variations.* We shall now study the functional (3.a.7) defined on immersed curves of fixed length. It turns out (cf. [2]) that the Euler equations here are the same as the Euler equations for the functional (3.a.7) on curves of variable length in a surface of constant curvature. Careful study of the reduction procedure will lead to the closed solution curves mentioned in the introduction.

We give an immersed $\gamma \subset \mathbf{E}^2$ of fixed length ℓ by a mapping

$$x\colon N \to \mathbf{E}^2$$

where $N = \{0 \leqq s \leqq \ell\}$ and $\|x'(s)\| = 1$. The Frenet equations (3.a.5) are then valid with s replacing t and $\sigma(t) \equiv 1$; we write this as

$$\begin{tikzcd} & \mathfrak{F}(\mathbf{E}^2) \arrow[d] \\ N \arrow[ur, dashed, "F"] \arrow[r, "x"] & \mathbf{E}^2, \end{tikzcd} \qquad x(N) = \gamma.$$

A length-preserving variation of γ is given by a 1-parameter family of Frenet liftings $F_t\colon N \to \mathfrak{F}(\mathbf{E}^2)(0 \leqq t \leqq \epsilon)$ such that $F_0 = F$ and $F_t^*\omega^1 = ds$. Thus, to set up the fixed length variational problem we let $X = \mathfrak{F}(\mathbf{E}^2) \times \mathbf{R} \times \mathbf{R}$ where $\mathbf{R} \times \mathbf{R}$ has coordinates (s, κ), and on X we consider the G-invariant Pfaffian equations

$$
(3.b.1) \qquad \begin{cases} \omega^1 - ds = 0 \\ \omega^2 = 0 \\ \rho - \kappa ds = 0 \end{cases}
$$

together with

$$
\varphi = L(\kappa)ds = \frac{1}{2}\kappa^2 ds.
$$

The variational problem $(I, ds; \varphi)$ then corresponds to fixed length variations of the functional (3.a.7).

On $Z = X \times \mathbf{R}^3$ where $\mathbf{R}^3 \cong \mathfrak{g}^*$ has coordinates $(\lambda_1, \lambda_2, \lambda_3)$ we consider

$$
\psi = \frac{1}{2}\kappa^2 ds + \lambda_1(\omega^1 - ds) + \lambda_2\omega^2 + \lambda_3(\rho - \kappa ds).
$$

Following our usual prescription, the Euler-Lagrange system is generated by the Pfaffian equations

$$
\begin{cases} \text{(i)} \ -\partial/\partial s \lrcorner \Psi = d\left(\frac{1}{2}\kappa^2 - \lambda_3\kappa - \lambda_1\right) = 0 \\ \text{(ii)} \quad \partial/\partial\kappa \lrcorner \Psi = (\kappa - \lambda_3)ds = 0 \end{cases}
$$

$$
\begin{cases} \text{(iii)} \ \partial/\partial\lambda_1 \lrcorner \Psi = \omega^1 - ds = 0 \\ \text{(iv)} \ \partial/\partial\lambda_3 \lrcorner \Psi = \omega^2 = 0 \\ \text{(v)} \ \partial/\partial\lambda_3 \lrcorner \Psi = \rho - \kappa ds = 0 \end{cases}
$$

$$
\begin{cases} \text{(vi)} \ -\partial/\partial\rho \lrcorner \Psi = d\lambda_3 - \lambda_1\omega^2 + \lambda_2\omega^1 \\ \text{(vii)} \ -\partial/\partial\omega^1 \lrcorner \Psi = d\lambda_1 - \lambda_2\rho = 0 \\ \text{(viii)} \ -\partial/\partial\omega^2 \lrcorner \Psi = d\lambda_2 + \lambda_1\rho = 0. \end{cases}
$$

Equations (iii)–(v) are the original Pfaffian system (3.b.1), and since $ds \neq 0$ (ii) gives

$$
\lambda_3 = \kappa.
$$

Then (i) implies that, on any solution curve to the Euler-Lagrange system,

$$\lambda_1 = -R - \frac{1}{2}\kappa^2$$

where R is a constant to be determined by the condition that the solution curves have length ℓ. As before, (vi) gives on solution curves that $\lambda_2 = -\dot{\kappa} = -d\kappa/ds$. We collect these equations together as

$$\lambda = (\lambda_1, \lambda_2, \lambda_3) = \left(-R - \frac{1}{2}\kappa^2, -\dot{\kappa}, \kappa\right) \tag{3.b.2}$$

on solution curves.

The coadjoint orbit condition (3.a.3) is

$$\dot{\kappa}^2 + \frac{\kappa^4}{4} + R\kappa^2 + R^2 = b^2, \qquad b \geqq 0. \tag{3.b.3}$$

As before this equation may be integrated by elliptic functions (cf. below). To look for closed solution curves in $\mathbf{E}^2$ we apply the reduction procedure. For

$$\hat{\lambda} = (-\lambda_3, \lambda_2, -\lambda_1) = \left(-\kappa, -\dot{\kappa}, R + \frac{1}{2}\kappa^2\right)$$

we have equations (3.a.13) and (3.a.14) as before. Writing

$$g(s) = \begin{pmatrix} 1 & 0 & 0 \\ x_1(s) & \cos\theta(s) & -\sin\theta(s) \\ x_2(s) & \sin\theta(s) & \cos\theta(s) \end{pmatrix},$$

(3.a.15) becomes

$$\begin{cases} \text{(i)} & \kappa = -\mu_3 - \mu_2 x_1 + \mu_1 x_2 \\ \text{(ii)} & \dot{\kappa} = -\mu_2\cos\theta + \mu_1\sin\theta \\ \text{(iii)} & \frac{1}{2}\kappa^2 + R = -\mu_2\sin\theta - \mu_1\cos\theta \end{cases} \tag{3.b.4}$$

where $\mu_1^2 + \mu_2^2 = b^2$. If $b = 0$, then $\mu_1 = \mu_2 = 0$ and the solution curve is part of an arc of a circle with constant curvature $\kappa = \sqrt{-R/2}$ (thus we must have $R \leqq 0$). In particular, circles give closed solution curves. If $b \neq 0$ then as in the previous section, we may reduce to the case $\hat{\mu} = (0, -b, 0)$ and (3.b.4) becomes

$$\text{(3.b.5)} \qquad \begin{cases} \text{(i)} \quad x_1(s) = \kappa(s)/b \\ \text{(ii)} \ \cos\theta(s) = \dot{\kappa}(s)/b. \\ \text{(iii)} \ \sin\theta(s) = \left(\dfrac{1}{2}\kappa(s)^2 + R\right)/b. \end{cases}$$

As before we cannot have a closed solution curve to the Euler-Lagrange equations if $R \geqq 0$. Thus we assume that $R < 0$, and by a dilation we may reduce to the case $R = -1/2$. Then (3.b.3) becomes

$$\text{(3.b.6)} \qquad \dot{\kappa}^2 = \frac{1}{4}(c^2 - (\kappa^2 - 1)^2)$$

where $c = 2b$.

To study (3.b.6) by the method of phase portraits we consider the algebraic curves given in $\mathbf{C}^2$ by the equation

$$\text{(3.b.7)} \qquad y^2 = \frac{1}{4}(c^2 - (x^2 - 1)^2) = -\frac{x^4}{4} + \frac{x^2}{2} + \frac{c^2 - 1}{4}$$

depending on a parameter $c \geqq 0$. The *real* points on these curves may be plotted as follows

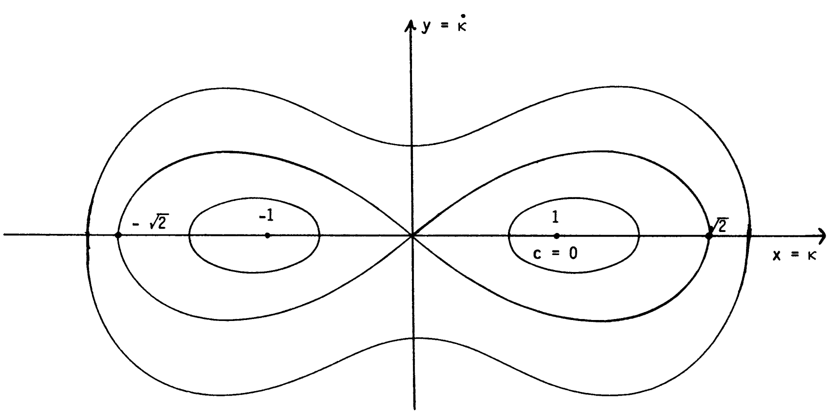

Figure 1

Let E_c denote the complex elliptic curve determined by (3.b.7). Then E_c is smooth for $c \neq 0, 1, \infty$. The mapping $f\colon E_c \to \mathbf{P}^1$ given by $(x, y) \to x$ is a 2-sheeted covering with branch points at

$$x = \pm\sqrt{1 \pm c}.$$

On E_c the holomorphic 1-form may be taken to be

$$\eta = \frac{dx}{2y} = \frac{dx}{\sqrt{c^2 - (x^2 - 1)^2}} \tag{3.b.8}$$

The 1-form

$$\sigma = 2(x^2 - 1)\eta = \frac{2(x^2 - 1)dx}{\sqrt{c^2 - (x^2 - 1)^2}}$$

is a 1-form of the 2nd kind on E_c with double poles at $f^{-1}(\infty)$, and the integrals

$$\begin{cases} \pi_1(c) = 2\displaystyle\int_{\sqrt{1-c}}^{\sqrt{1+c}} \sigma, & 0 < c < 1, \\ \pi_2(c) = 2\displaystyle\int_{-\sqrt{1+c}}^{\sqrt{1+c}} \sigma, & 1 < c < \infty, \end{cases} \tag{3.b.9}$$

are periods of σ. For reasons to appear below we shall need the following

(3.b.10) LEMMA. (i) *For* $0 < c < 1$, $\pi_1(c) < 0$.

(ii) *There exists at least one value of* c *with* $1 < c < \infty$ *such that* $\pi_2(c) = 0$.

The proof will be given at the end of this section.

Returning to our discussion of solution curves to the Euler-Lagrange system we have from (3.b.6) that $(\kappa(s), \dot{\kappa}(s))$ varies on one of the curves in Fig. 1, and from (3.b.5) that

$$x_1(s) = 2\kappa(s)/c$$

$$x_2(s) = x_2(0) + \frac{1}{c}\int_0^s (\kappa(\xi)^2 - 1)d\xi. \tag{3.b.11}$$

For $c \neq 0, 1, \infty$ the elliptic curve E_c is representable as $\mathbf{C}/\Lambda_c$ where $\mathbf{C}$ is the complex s-plane and $\Lambda_c \subset \mathbf{C}$ is a lattice depending on c (the same is true for $c = 0, 1, \infty$ but where Λ_c is only generated by one vector). Then $\kappa(s)$, $\dot{\kappa}(s)$ are doubly periodic meromorphic functions (elliptic functions) such that the homogeneous coordinate

$$[1, \kappa(s), \dot{\kappa}(s)]: E_c \to \mathbf{P}^2$$

gives a holomorphic mapping sending E_c to the plane algebraic curve whose affine equation is (3.b.7).

We now break our study into three cases.

Case 1. $(0 < c < 1)$. The real points of E_c consist of the two ovals in Fig. 1, and thus $\kappa(s)$ and $x_1(s)$ are periodic functions of s. To see if $(x_1(s), x_2(s))$ gives a closed curve in $\mathbf{E}^2$, we restrict to the oval where $\kappa > 0$

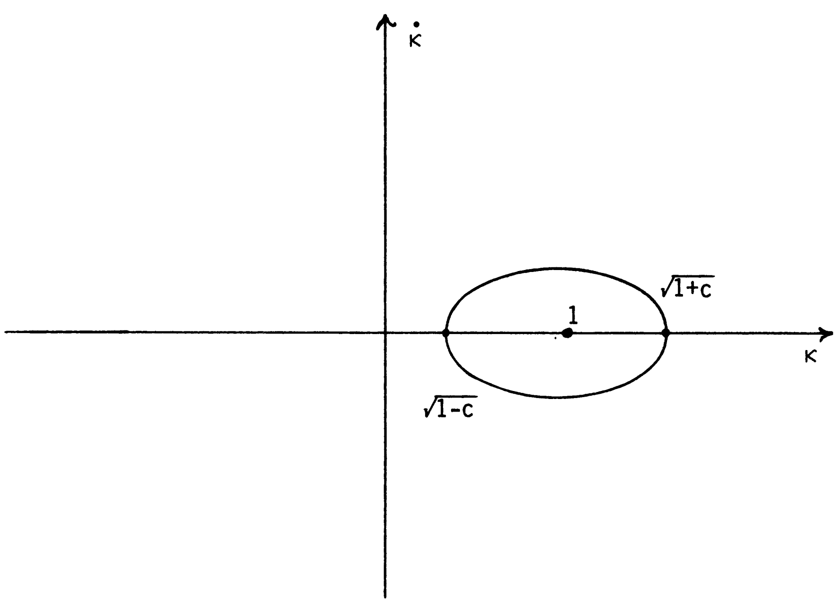

Figure 2

As $\kappa(s)$ traverses this oval once we infer from (3.b.11) that $x_2(s)$ undergoes the "phase shift"

$$\pi_1(c) = 2\int_{\sqrt{1-c}}^{\sqrt{1+c}} \frac{2(\kappa^2 - 1)d\kappa}{\sqrt{c^2 - (\kappa^2 - 1)^2}}.$$

Here we have used (3.b.5) to write

$$ds = \frac{2d\kappa}{c\cos\theta} = \frac{2d\kappa}{\sqrt{c^2 - (\kappa^2 - 1)^2}}.$$

Thus, the phase shift is just the period $\pi_1(c)$ in (3.b.9) of the elliptic integral discussed above. It follows from (i) in the lemma that $\pi_1(c) < 0$ and so there can be no closed solution curve when $0 < c < 1$.

(*Note.* It follows from the 4 vertex theorem that there can be no closed *embedded* solution curve in this case.) The picture of the integral curve is shown in Figure 3.

Case 2. $(1 < c < \infty)$. Now $\kappa(s)$ travels around the closed curve (Figure 4) and as above $x_2(s)$ undergoes the phase shift given by $\pi_2(c)$ in (3.b.9). By the lemma there exists a value of c with $\pi_2(c) = 0$. The corresponding closed curve is a figure eight (Figure 5).

If we consider circles traversed n times (where $n \neq 0$ but may be negative) as solution curves of index n, then since the figure 8 curve gives a solution curve of index 0 we conclude from Whitney's theorem that there is in each isotopy class of immersions $x\colon S^1 \to \mathbf{E}^2$ at least one solution curve to the Euler-Lagrange equations for the functional (3.a.7) with fixed length curves.

Case 3 $(c = 1)$. The solutions of (3.b.6) are seen to be

$$\kappa = \sqrt{2}\,\operatorname{sech}(s + s_0)/\sqrt{2}).$$

The solution curve is shown in Figure 6.

Note. Equations (3.b.12) are consistent with the fact that, when an elliptic curve degenerates to a rational curve, elliptic functions specialize to circular functions.

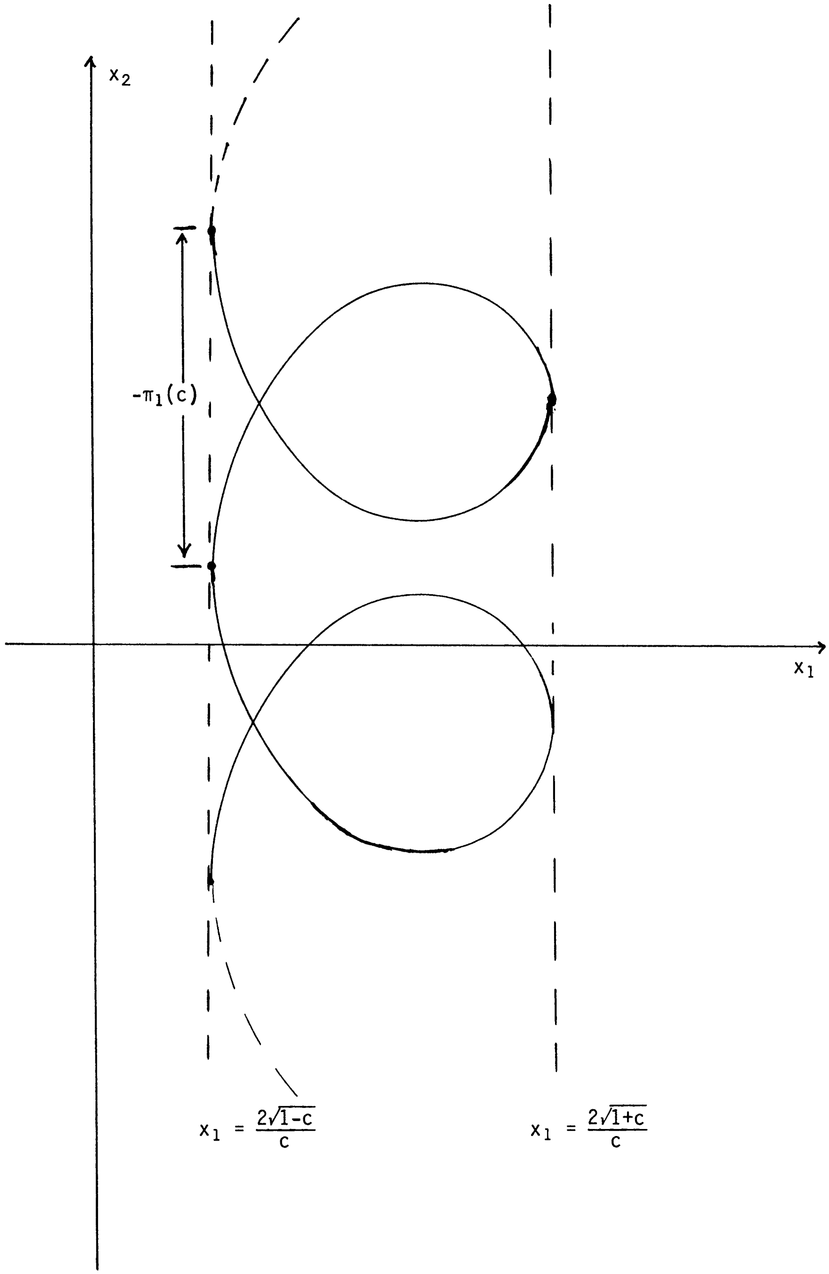

Figure 3

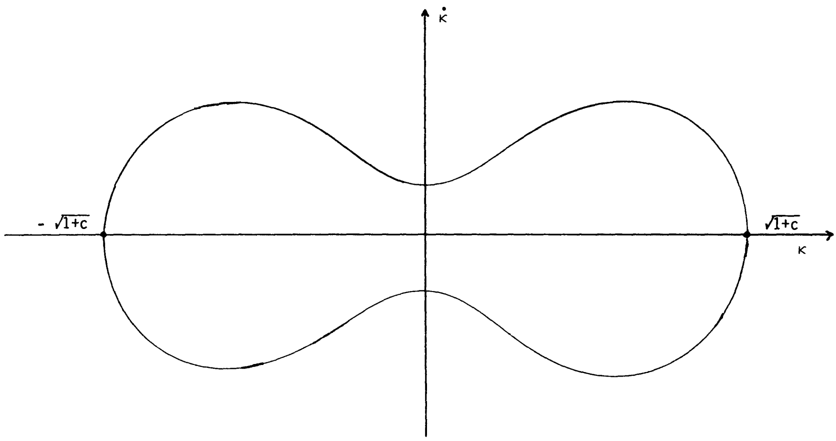

Figure 4

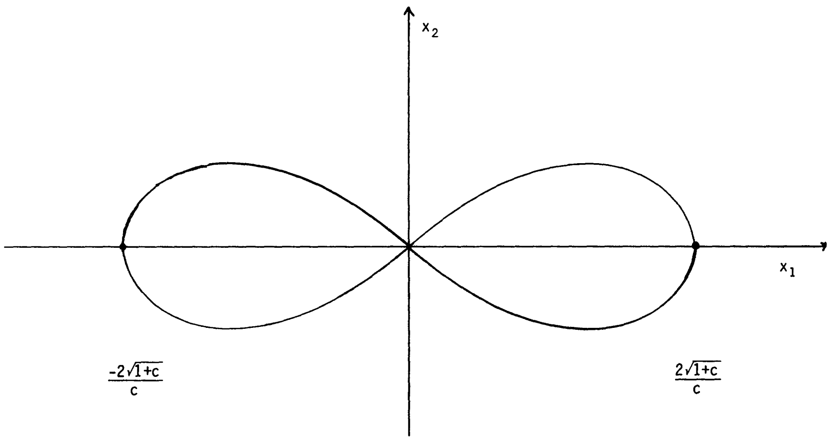

Figure 5

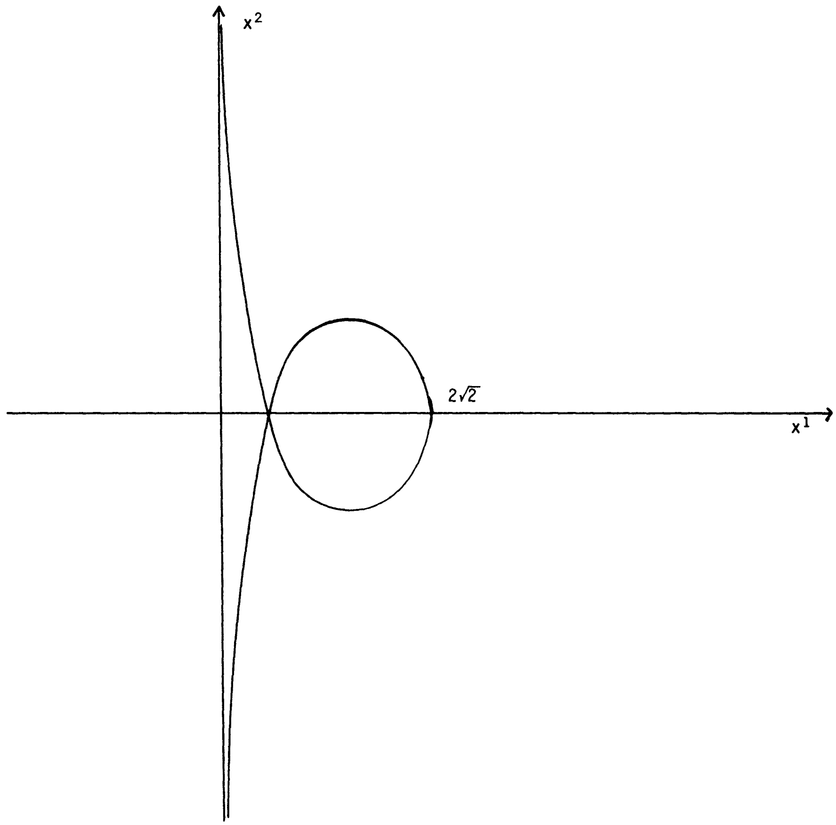

Figure 6

Translating, we may assume that $s_0 = 0$ and then

$$\text{(3.b.12)} \qquad \begin{cases} x_1(s) = 2\sqrt{2}\,\operatorname{sech}(s/\sqrt{2}) \\ x_2(s) = 2\sqrt{2}\,\tanh(s/\sqrt{2}) - s. \end{cases}$$

Proof of lemma (3.b.10). Setting $u = (\kappa^2 - 1)/c$ we get

$$\pi_1(c) = 2\int_{-1}^{1} \frac{u\,du}{\sqrt{1-u^2}\sqrt{1+cu}}$$

$$= 2\int_0^1 \frac{(\sqrt{1-cu} - \sqrt{1+cu})u\,du}{\sqrt{1-c^2u^2}\sqrt{1-u^2}}$$

$$< 0$$

where $0 < c < 1$. This proves (i) in the lemma.

Turning to (ii), the same substitution gives

$$\pi_2(c) = 2\int_{-1/c}^1 \frac{u\,du}{\sqrt{1-u^2}\sqrt{1+cu}}$$

Setting $u = (-1+cv)/(c-v)$ we obtain

$$\pi_2(c) = 2\int_0^1 \frac{(cv-1)dv}{\sqrt{v(1-v^2)(c-v)^3}}$$

(3.b.13)
$$= \frac{2}{\sqrt{c}}\int_0^1 \frac{(v-1/c)dv}{\sqrt{v(1-v^2)(1-v/c)^3}}.$$

We will show that $\pi_2(c)$ is negative for c close to 1 and positive as $c \to \infty$.

For the first we write

$$\pi_2(c) = 2(A(c) - B(c))$$

where

$$A(c) = \int_0^1 \frac{u\,du}{\sqrt{1-u^2}\sqrt{1+cu}} \geqq 0 \quad \text{and}$$

$$B(c) = \int_0^{1/c} \frac{u\,du}{\sqrt{1-u^2}\sqrt{1-cu}} \geqq 0.$$

Since $1 \leqq 1 + cu \leqq 1 + c$ for $0 \leqq u \leqq 1$ we have

$$A(c) \leqq \int_0^1 \frac{u\,du}{\sqrt{1-u^2}} = 1.$$

It will then suffice to show that $B(c) \to \infty$ as $c \to 1$. Clearly this follows from

$$(*) \qquad \lim_{c\to 1} \int_{1/\sqrt{2}}^{1/c} \frac{du}{\sqrt{1-u^2}\sqrt{1/c-u}} = \infty.$$

Setting $u = \sin\theta$ this integral is

$$\int_{\arcsin 1/\sqrt{2}}^{\arcsin 1/c} \frac{d\theta}{\sqrt{1/c - \sin\theta}}.$$

Taking $\varphi = \pi - \theta$ our claim (*) follows from

$$\lim_{c\to 1} \int_{\arccos 1/c}^{\arccos 1/\sqrt{2}} \frac{d\varphi}{\sqrt{1/c - \cos\varphi}} = \infty.$$

Setting $1/c = 1 - \epsilon$ it will suffice to show that

$$(**) \qquad \lim_{\epsilon\to 0} \int_{\arccos(1-\epsilon)}^{\arccos(1-b(\epsilon))} \frac{d\varphi}{\sqrt{(1-\cos\varphi)-\epsilon}}$$

where $\epsilon < b(\epsilon)$ and $b(\epsilon) \to 0$ as $\epsilon \to 0$. The point of this is that in a neighborhood of $\varphi = 0$

$$1 - \cos\varphi = \eta^2 \quad \text{and}$$

$$d\varphi = f d\eta, \qquad f(0) > 0.$$

For ϵ sufficiently small the integral (**) lies in an interval where $f > 0$, so it will suffice to show that for $\nu < 1$

$$\lim_{\epsilon\to 0} \int_{\sqrt{\epsilon}}^{\epsilon^{\nu/2}} \frac{d\eta}{\sqrt{\eta^2 - \epsilon}} = \infty.$$

Setting $\eta = \sqrt{\epsilon}\xi$ this integral is

$$\int_1^{\epsilon^{(\nu-1)/2}} \frac{d\xi}{\sqrt{\xi^2 - 1}},$$

which clearly diverges as $\epsilon \to 0$.

Finally, we want to show that $\pi_2(c) > 0$ for large c. From (3.b.13) this follows from the positivity of

$$\int_0^1 \frac{(v - \delta)dv}{\sqrt{v(1 - v^2)}\sqrt{(1 - \delta v)^3}}$$

for small δ. But this integral is a continuous function of δ as $\delta \downarrow 0$, with limit

$$\int_0^1 \frac{v\,dv}{\sqrt{v(1 - v^2)}} > 0.$$

Q.E.D. for the lemma.

4. Study of $\frac{1}{2}\int \kappa^2 ds$ in the Hyperbolic Case.

a. *Setting up the problem.* Let $\mathbf{R}^3$ have points $x = {}^t(x_1, x_2, x_3)$ and quadratic form

$$x \cdot x = x_1^2 + x_2^2 - x_3^2.$$

Denote by G the identity component of

$$SO(2, 1) = \left\{ g\colon {}^tg\,Q\,g = Q \quad \text{where} \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right\}.$$

Then the Lie algebra $\mathfrak{g}$ consists of all matrices

$$\xi = \begin{pmatrix} 0 & \xi_3 & \xi_1 \\ -\xi_3 & 0 & \xi_2 \\ \xi_1 & \xi_2 & 0 \end{pmatrix}. \tag{4.a.1}$$

We define $\mathbf{R}^3$-valued maps $e_1, e_2, x\colon G \to \mathbf{R}^3$ to be the column vectors of a variable matrix $g \in G$; i.e.,

$$(e_1, e_2, x) = g.$$

Then we have

$$d(e_1, e_2, x) = (e_1, e_2, x) \cdot \omega$$

where

$$\omega = g^{-1} dg = \begin{pmatrix} 0 & \rho & \omega^1 \\ -\rho & 0 & \omega^2 \\ \omega^1 & \omega^2 & 0 \end{pmatrix}$$

is the left invariant Maurer-Cartan form on G. The Maurer-Cartan equations are

$$\begin{cases} d\omega^1 = -\rho \wedge \omega^2 \\ d\omega^2 = \rho \wedge \omega^1 \\ d\rho = -\omega^1 \wedge \omega^2. \end{cases} \tag{4.a.2}$$

G is a simple Lie group, and if we use the isomorphism $\mathfrak{g} \cong \mathbf{R}^3$ given by (4.a.1) then the Cartan Killing form is

$$B(\xi, \xi) = \xi_1^2 + \xi_2^2 - \xi_3^2.$$

The set $H^2 = \{x \in \mathbf{R}^3: x \cdot x = -1 \text{ and } x_3 > 0\}$ with the metric $ds^2 = (\omega^1)^2 + (\omega^2)^2$ is the Minkowski model of the hyperbolic disk. The map x: $G \to H^2$ realizes G as the oriented frame bundle of H^2.

To set up the variational problem given by the functional

$$\Phi(\gamma) = \frac{1}{2}\int_\gamma \kappa^2 ds \tag{4.a.3}$$

defined on immersed curves $\gamma \subset H^2$ we follow the procedure of Section 2a). Thus, on $X = G \times \mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}$, where $\mathbf{R}^+ \times \mathbf{R} \times \mathbf{R}$ has coordinates (σ, κ, t), we consider the differential system (I, dt) generated by the Pfaffian equations

$$\begin{cases} \omega^1 - \sigma dt = 0 \\ \omega^2 = 0 \\ \rho - \kappa\sigma dt = 0. \end{cases} \tag{4.a.4}$$

The integral curves of (I, dt) are Frenet liftings of immersed curves $\gamma \subset H^2$, and the functional (4.a.3) is associated to the variational problem $(I, dt; \varphi)$ where

$$\varphi = \frac{1}{2}\kappa^2\sigma dt.$$

To compute the Euler-Lagrange equations, we identify $\mathfrak{g}$ with $\mathfrak{g}^*$ using the Killing form. Then for $\lambda, \xi \in \mathfrak{g}$

$$B(\lambda, \xi) = \frac{1}{2}\operatorname{Tr}(\lambda\xi).$$

We set

$$P = \begin{pmatrix} 0 & \kappa\sigma & \sigma \\ -\kappa\sigma & 0 & 0 \\ \sigma & 0 & 0 \end{pmatrix},$$

and on $Z = X \times \mathfrak{g}$ we consider the 1-form

$$\psi = \frac{1}{2}\kappa^2 \sigma dt + \frac{1}{2}\mathrm{Tr}(\lambda(\omega - Pdt)).$$

Computing $\Psi = d\psi$ using (4.a.2) we find that the Cartan system is generated by the Pfaffian equations (cf. [2])

$$\begin{cases} \text{(i)}\ \partial/\partial t \lrcorner \Psi = d\left(\frac{1}{2}\kappa^2\sigma + \lambda_3\kappa\sigma - \lambda_1\sigma\right) = 0 \\ \text{(ii)}\ \partial/\partial\kappa \lrcorner \Psi = \sigma(\kappa + \lambda_3)dt = 0 \\ \text{(iii)}\ \partial/\partial\sigma \lrcorner \Psi = \left(\frac{1}{2}\kappa^2 + \kappa\lambda_3 - \lambda_1\right)dt = 0 \end{cases}$$

$$\begin{cases} \text{(iv)}\ \partial/\partial\lambda_3 \lrcorner \Psi = -(\rho - \kappa\sigma)dt = 0 \\ \text{(v)}\ \partial/\partial\lambda_1 \lrcorner \Psi = \omega^1 - \sigma dt = 0 \\ \text{(vi)}\ \partial/\partial\lambda_2 \lrcorner \Psi = \omega^2 = 0 \end{cases}$$

$$\begin{cases} \text{(vii)}\quad \partial/\partial\rho \lrcorner \Psi = d\lambda_3 - \lambda_1\omega^2 + \lambda_2\omega^1 = 0 \\ \text{(viii)}\ -\partial/\partial\omega^1 \lrcorner \Psi = d\lambda_1 - \lambda_3\omega^2 + \lambda_2\rho = 0 \\ \text{(ix)}\ -\partial/\partial\omega^2 \lrcorner \Psi = d\lambda_2 + \lambda_3\omega^1 - \lambda_1\rho = 0. \end{cases}$$

Equations (iv)-(vi) are the original differential system, and (vii)-(ix) are the Euler equations. Since, under the identification $\mathfrak{g} \cong \mathfrak{g}^*$ given by the Killing form the coadjoint representation goes over to the adjoint representation, they may be written as

$$d(g\,\lambda\,g^{-1}) = 0.$$

This means that

$$g\,\lambda\,g^{-1} = \mu \tag{4.a.5}$$

is constant on solution curves to the Euler-Lagrange equations.

Equations (ii), (iii), and (vii) give on solution curves (cf. Section 2a))

$$(4.a.6)\qquad \begin{cases} \lambda_3 = -\kappa \\ \lambda_1 = -\dfrac{1}{2}\kappa^2 \\ \lambda_2 = \dot{\kappa} = d\kappa/ds. \end{cases}$$

Since $B(\lambda, \lambda) = B(\mu, \mu)$ is constant we obtain the 1st integral

$$(4.a.7)\qquad \dot{\kappa}^2 + \frac{\kappa^4}{4} - \kappa^2 = B(\mu, \mu) = \text{constant}.$$

Comparing with (3.b.3), this is formally the same equation as in the Euclidean constrained length case. In particular, the ODE has phase portrait given by Fig. 1 in Section 2b) where $a = (c^2 - 1)/4$, and as explained there this equation is integrable by elliptic functions.

b. *Investigation of the solution curves.* Given $\mu \in \mathfrak{g}$ we denote by $\Gamma_\mu \subset \mathcal{O}_{\mathrm{Ad}}(\mu) = \{g^{-1}\mu g : g \in G\}$ the closed curve traced out by

$$\lambda(s) = \begin{pmatrix} 0 & -\kappa(s) & -\dfrac{1}{2}\kappa(s)^2 \\ \kappa(s) & 0 & \dot{\kappa}(s) \\ -\dfrac{1}{2}\kappa(s)^2 & \dot{\kappa}(s) & 0 \end{pmatrix}$$

where $\kappa(s)$ is a solution to (4.a.7). Over Γ_μ we have the G_μ-bundle

$$B_\mu \to \Gamma_\mu$$

as discussed in Section 1b) (here, $G_\mu \subset G$ is the stabilizer of μ under the adjoint representation). Momentarily leaving aside degenerate orbits, G_μ is a Cartan subgroup and is therefore either S^1 (compact case) or $\mathbf{R}^*$ (noncompact case). In the compact case, B_μ is diffeomorphism to $S^1 \times S^1$ and is consequently a 2-torus. Our main result is that the flow on B_μ given by the Euler-Lagrange system is *linear*. We now proceed to carry out the relevant computations.

$$\text{For} \quad \lambda = \begin{pmatrix} 0 & \lambda_3 & \lambda_1 \\ -\lambda_3 & 0 & \lambda_2 \\ \lambda_1 & \lambda_2 & 0 \end{pmatrix} \in \mathfrak{g} \quad \text{we define}$$

$$\hat{\lambda} = \begin{pmatrix} \lambda_2 \\ -\lambda_1 \\ \lambda_3 \end{pmatrix} \in \mathbf{R}^3.$$

Then $B(\lambda, \lambda) = \hat{\lambda} \cdot \hat{\lambda}$ and (4.a.5) becomes

$$g\hat{\lambda} = \hat{\mu}. \tag{4.b.1}$$

As in Section 2a), it will suffice to study this equation when $\hat{\mu}$ runs through a complete set of orbits of G acting on $\mathbf{R}^3$. We proceed to examine cases.

Case 1. ($\hat{\mu} = 0$). Then $\hat{\lambda} = 0$ and $\dot{\kappa}(s)$ and $\kappa(s) = 0$. The solution curves are the geodesics on H^2.

Case 2. ($\hat{\mu} \cdot \hat{\mu} = -1$). By (4.8.7) this equation is

$$\frac{\kappa^4}{4} + \dot{\kappa}^2 - \kappa^2 = -1.$$

By the Cauchy-Schwarz inequality we must have

$$\begin{cases} \kappa(s) \equiv +\sqrt{2} \\ \dot{\kappa}(s) = 0. \end{cases}$$

The solution curves are closed circles of curvature $\pm\sqrt{2}$ in H^2.

Case 3. ($0 > \hat{\mu} \cdot \hat{\mu} > -1$). This turns out to be the case when G_μ is compact. Set $\hat{\mu} \cdot \hat{\mu} = -c^2$ where $0 < c < 1$. For $\Lambda = {}^t(0, 0, 1)$ we may conjugate $\hat{\mu}$ to Λ, and therefore it suffices to study the torus $B_c \subset G \times \mathbf{R}^3$ given by

$$g\hat{\lambda} = c\Lambda. \tag{4.b.2}$$

We want to introduce a suitable parametrization of B_c.

For this we restrict to the case $\lambda_3 > 0$ (i.e., $\kappa < 0$ by (4.a.6)) and denote the coordinate frame by f_1, f_2, y, so that

$$(f_1, f_2, y) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then using (4.a.6)

$$\begin{aligned} e_1\dot{\kappa} + e_2\left(\frac{1}{2}\kappa^2\right) + x(-\kappa) &= (e_1, e_2, x)\hat{\lambda} \\ &= (f_1, f_2, y)g\hat{\lambda} \\ &= (f_1, f_2, y)\Lambda \\ &= cy. \end{aligned} \tag{4.b.3}$$

Dotting both sides with x we get $\kappa = cy \cdot x$. Thus

$$(x + (\kappa/c)y)\cdot y = 0,$$

and we let (u_1, u_2, y) be a frame such that

$$x = (-\kappa/c)y - mu_2, \qquad m > 0.$$

From $x \cdot x = -1$ it follows that

$$m = \frac{1}{c}\sqrt{\kappa^2 - c^2},$$

and we may define a frame field v_1, v_2, x by

$$(v_1, v_2, x) = (u_1, u_2, y)\begin{pmatrix} 1 & 0 & 0 \\ 0 & -\kappa/c & -\dfrac{1}{c}\sqrt{\kappa^2 - c^2} \\ 0 & -\dfrac{1}{c}\sqrt{\kappa^2 - c^2} & -\kappa/c \end{pmatrix}$$

$$= (u_1, u_2, y)h_1(s).$$

Equation (4.b.3) is

$$\dot{\kappa} e_1 + \left(\frac{1}{2}\kappa^2\right)e_2 = \sqrt{\kappa^2 - c^2}\, v_2$$

or

$$(e_1, e_2) = (v_1, v_2)\begin{pmatrix} \dfrac{1}{2}\kappa^2/\sqrt{\kappa^2 - c^2} & -\dot{\kappa}/\sqrt{\kappa^2 - c^2} \\ \dot{\kappa}/\sqrt{\kappa^2 - c^2} & \dfrac{1}{2}\kappa^2/\sqrt{\kappa^2 - c^2} \end{pmatrix}.$$

Writing

$$h_2(s) = \begin{pmatrix} \dfrac{1}{2}\kappa^2/\sqrt{\kappa^2 - c^2} & -\dot{\kappa}/\sqrt{\kappa^2 - c^2} & 0 \\ \dot{\kappa}/\sqrt{\kappa^2 - c^2} & \dfrac{1}{2}\kappa^2/\sqrt{\kappa^2 - c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

we set

$$(e_1, e_2, x) = (f_1, f_2, y)R(\theta)h_1(s)h_2(s).$$

Taking (s, θ) as independent variables and κ, $\dot{\kappa}$ as functions of s determined by (4.a.7), this equation parametrizes the torus B_μ defined above.

The differential equation of motion is

$$\begin{pmatrix} 0 & \kappa ds & ds \\ -\kappa ds & 0 & 0 \\ ds & 0 & 0 \end{pmatrix} = (R(\theta)h_1(s)h_2(s))^{-1}d(R(\theta)h_1(s)h_2(s))$$

After a lengthy computation this becomes

$$\frac{d\theta}{ds} = \frac{\kappa^2 c}{2(\kappa^2 - c^2)}.$$

Write the right hand side as $f(s, c)$; note that $f(s, c) > 0$ by (4.9.7). We shall show that any O.D.E.

$$\frac{d\theta}{ds} = f(s, c),$$

on the torus with angular coordinates (θ, s), is linearizable. Suppose that s has period T (i.e., as s traverses the circle $\mathbf{R}/T \cdot \mathbf{Z}$, $(\kappa(s), \dot{\kappa}(s))$ goes once around the left hand oval in Fig. 1 in Section 3b)). Set

$$\nu = \int_0^T f(s, c)ds.$$

Then

$$\int_0^T \left(f(s, c) - \frac{\nu}{T}\right)ds = 0$$

so that

$$f(s, c) - \frac{s\nu}{T} = \frac{\partial g}{\partial s}(s, c)$$

for some function $g(s, c)$ satisfying $g(s + T, c) = g(s, c)$. We have

$$\frac{d}{ds}(\theta - g(s, c)) = \frac{\nu}{T}. \tag{4.b.4}$$

Set

$$\begin{cases} \varphi = \varphi(\theta, s, c) = \theta - g(s, c) \\ s = s. \end{cases}$$

In terms of the new angular coordinates (φ, s) on B_c the Euler-Lagrange system is by (4.b.4)

$$Td\varphi - \nu ds = 0.$$

This is a linear flow.

In the hyperbolic plane we may picture the solution curve as

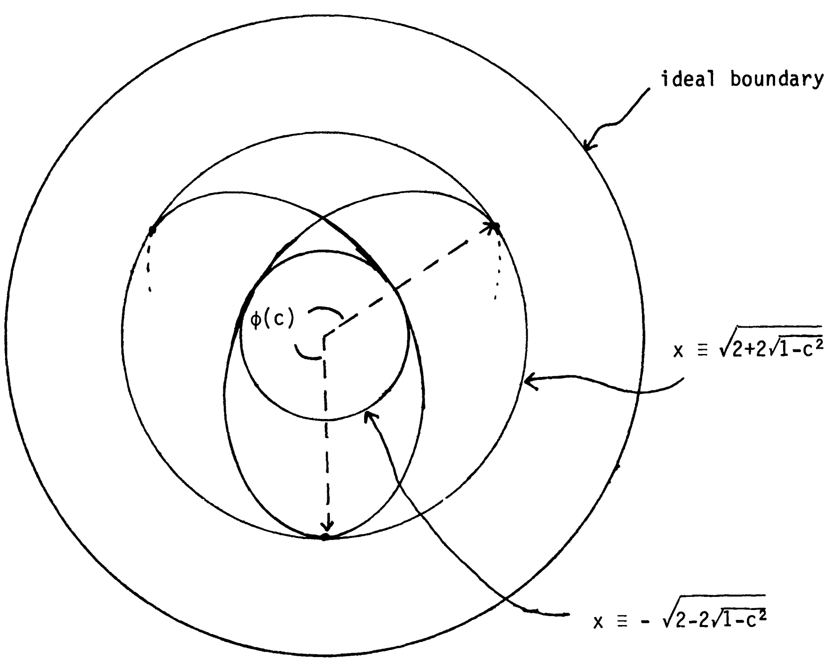

Figure 7

Here, $\varphi(c)$ is the "phase shift" when $(\kappa(s), \dot{\kappa}(s))$ traverses the oval once. The condition to get a periodic solution curve is $\varphi(c) \in \mathbf{Q}$. As c varies we find countably many periodic solutions.

Remark. In [5] Wilmore considered the functional

$$\tilde{\Phi}(S) = \frac{1}{2}\int_S \|\Pi\|^2 dA \tag{4.b.5}$$

defined on immersed surfaces $S \subset \mathbf{E}^3$, where $\|\Pi\|$ is the length of the 2nd fundamental form. Denoting by H and K the mean and Gaussian curvature, if we add to the integrand in (4.b.5) the divergence term KdA then we may equivalently study the functional

$$\Phi(S) = \frac{1}{2}\int_S (H^2 - K)dA. \tag{4.b.6}$$

The advantage of doing this is that $\Phi(S)$ is invariant under the full conformal group acting on $\mathbf{E}^3$.

It is natural to consider (4.b.6) defined on surfaces of revolution. This was done in [2] where it was pointed out that we obtain a classical 2nd order classical variational problem given by a functional

$$F(y) = \int_a^b f(y(x), y'(x), y''(x))dx \tag{4.b.7}$$

defined on functions $y \in C^2\,[a, b]$. It is straightforward that the Euler-Lagrange equations associated to (4.b.7) may be written in Hamiltonian form in $\mathbf{R}^5$ with coordinates $(x, y, y', \lambda, \lambda')$ and Hamiltonian $H = H(y, y', \lambda, \lambda')$ (loc. cit.). There the 2nd author conjectured that this Hamiltonian system was completely integrable.

Now invariance of the functional (4.b.6) under the dilation vector field $v = \Sigma_{i=1}^3 x^i\, \partial/\partial x^i$ in $\mathbf{E}^3$ gives a 1st integral $G = G(y, y', \lambda, \lambda')$ independent of H (actually, a slight variant of Noether's theorem must be used here since v is not invariant under translation along the axis of rotation). However, in [2] it was not noticed that (4.b.7) is also invariant under the 1-parameter group of inversions in spheres along the axis of revolution; this gives another 1st integral and leads to the complete integrability result.

The 1st author noted that, when written intrinsically, the functional (4.b.7) for surfaces of revolution reduces to the study of the functional (4.a.3). The consequence of complete integrability that the motion is equivalent to a linear flow on a 2-torus then follows from our result above. For more information on the variational equations associated to the Wilmore problem we refer to [1].

Case 4. $(\hat{\mu} \cdot \hat{\mu} > 0)$. In this case, G_μ is noncompact, and by analogy with (4.b.2) it will suffice to study the case $g\hat{\lambda} = c$ where $\Lambda = {}^t(0, 1, 0)$. Then (4.a.7) and (4.b.3) are

$$\dot{\kappa}^2 + \frac{\kappa^4}{4} - \kappa^2 = c^2, \qquad c > 0,$$

and

$$e_1\dot{\kappa} + e_2\left(\frac{1}{2}\kappa^2\right) + x(-\kappa) = cf_2.$$

In a purely analogous fashion to case 3 we have

$$(e_1, e_2, x) = (f_1, f_2, y)T(\tau)h_1(s)h_2(s)$$

where

$$T(\tau) = \begin{pmatrix} \cosh\tau & 0 & \sinh\tau \\ 0 & 0 & 1 \\ \sinh\tau & 0 & \cosh\tau \end{pmatrix}$$

$$h_1(s) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{c}\sqrt{\kappa^2 - c^2} & \kappa/c \\ 0 & \kappa/c & \frac{1}{c}\sqrt{\kappa^2 - c^2} \end{pmatrix}$$

$$h_2(s) = \begin{pmatrix} \frac{1}{2}\kappa^2/\sqrt{\kappa^2 - c^2} & -\dot{\kappa}/\sqrt{\kappa^2 - c^2} & 0 \\ \dot{\kappa}/\sqrt{\kappa^2 - c^2} & \frac{1}{2}\kappa^2/\sqrt{\kappa^2 - c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The differential equation of motion is

$$\begin{pmatrix} 0 & \kappa ds & ds \\ -\kappa ds & 0 & 0 \\ ds & 0 & 0 \end{pmatrix} = (T(\tau)h_1(s)h_2(s))^{-1}d(T(\tau)h_1(s)h_2(s)).$$

Again, after much computation this becomes

$$\frac{d\tau}{ds} = \frac{\kappa^2 c}{2(\kappa^2 + c)}.$$

In particular, τ is strictly increasing on solution curves and so there can be no periodic orbits of this type.

Case 5. ($\hat{\mu} \neq 0$ but $\hat{\mu} \cdot \hat{\mu} = 0$). In this case the Euler-Lagrange equations may be integrated in terms of elementary functions and we obtain curves which may be pictured as follows:

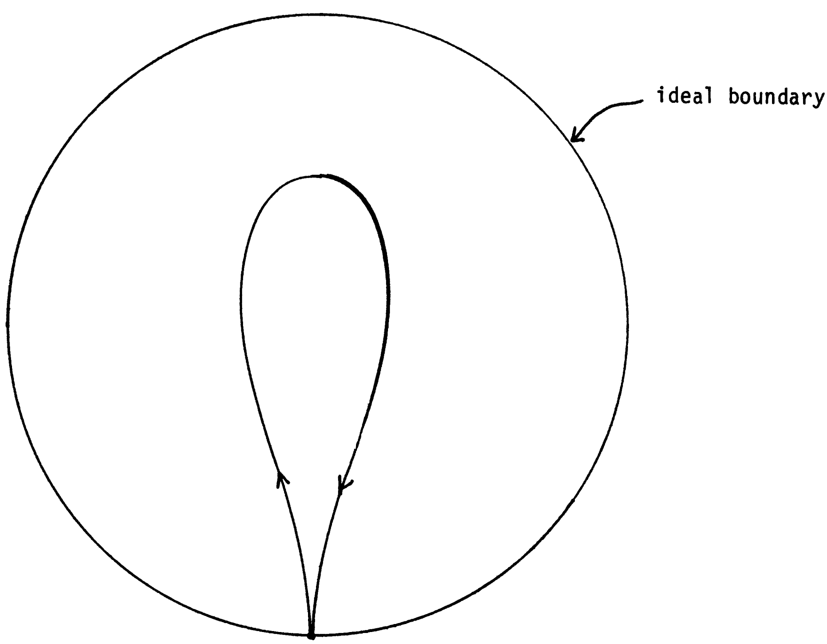

Figure 8

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