THE RESIDUE CALCULUS AND SOME TRANSCENDENTAL RESULTS IN ALGEBRAIC GEOMETRY, II[*]

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We continue to outline certain developments in the analytical theory of algebraic manifolds which seem to have applications to moduli and other questions.[1]

5. *Some Applications.*—There appear to be two main applications to geometric problems, namely, to the problem of classifying integrals and to moduli. Neither of these is in definitive form, and so we shall only illustrate the main points.

(a) *Some questions on moduli:* To begin with, we must set up the problem correctly. We consider, then, a family $\{V_t\}_{t\epsilon B}$ of polarized algebraic manifolds; thus, there is given a distinguished class of positive line bundles $L_t \rightarrow V_t$. Instead of looking at all cohomology, we should take only the primitive classes $H_0{}^q(V_t) = \Sigma H_0{}^{q-r,r}(V_t)$. The polarization induces two bilinear relations, given by a quadratic form $Q:H_0{}^q(V_t) \otimes H_0{}^q(V_t) \rightarrow C$ satisfying

$$Q(H_0{}^{q-r,r}, \bar{H}_0{}^{q-r,r}) > 0 \tag{1}$$

$$Q(H_0{}^{q-r,r}, \bar{H}_0{}^{q-s,s}) = 0 \qquad \text{for } s \neq r. \tag{2}$$

Thus, we let $\mathcal{G}_q \subset \mathcal{F}_q$ be the flags

$$S_0 \subset S_0 + S_1 \subset \ldots \subset S_0 + \ldots + S_{q-1} \subset H_0{}^q(V),$$

where $\dim S_r = h_0{}^{q-r,r}$, $Q(S_r,\bar{S}_r) > 0$, $Q(S_r,\bar{S}_t) = 0$ for $t \neq r$. We must factor $\mathcal{G}_q$ by the group Γ_q of integral transformations $T:H_0{}^q(V) \rightarrow H_0{}^q(V)$ which preserve Q. In fact, it may be shown:

THEOREM. *$\mathcal{G}_q$ is a homogeneous complex manifold $H\backslash G$ where G is a real semi-simple Lie group without compact factors and $H \subset G$ is a compact subgroup. Furthermore, $\Gamma_q \subset G$ is a properly discontinuous group and so $M_q = \mathcal{G}_q/\Gamma_q = H\backslash G/\Gamma_q$ is an analytic space.*

For q odd, $\mathcal{G}_q$ is a bundle of algebraic manifolds over a bounded domain, but, e.g., $\mathcal{G}_2 = U(h)\times 0(l)\backslash SO(2h,l;R)$ $(h = h^{2,0}, l = h_0{}^{1,1})$, so that $\mathcal{G}_2$ is symmetric if and only if $h^{2,0} = 1$. A further difference with the classical theory is predicted by

THEOREM. *Let $\mathcal{L} \rightarrow M_2$ be the canonical sheaf (factor of automorphy). Then, for μ large,*

$$H^q(M_2, \mathcal{L}^\mu) = 0 \qquad \text{for } q \neq \frac{h^2 - h}{2}.$$

Let $M = M_1 \times \ldots \times M_n$; there is a well-defined holomorphic mapping $\Phi:(B - S) \rightarrow M$ given by $\Phi(t)$ = "period matrix of primitive integrals on V_t." By the results in part I, sections 1–3 of this paper,[1] we may locally write $\Phi(t)$ = "Plücker coordinates taken from the period matrix"

$$\pi(t) = \begin{pmatrix} \pi_{11}(t) & \ldots & \pi_{1b}(t) \\ \vdots & & \vdots \\ \pi_{h1}(t) & \ldots & \pi_{hb}(t) \end{pmatrix},$$

where each row in $\pi(t)$ is composed of segments $[\pi_{k_1}, \ldots, \pi_{k_m}]$ consisting of a basis of solutions to an equation (E).

The central problem here is first to construct a good compactification $\hat{M}$ of M, and then to show that Φ extends to a holomorphic mapping of B into $\hat{M}$. For the case of curves, take M = Siegel space, $\hat{M}$ = Satake compactification, and then Φ is holomorphic (as shown by Mumford and Mayer, unpublished result). For the case of surfaces with $p_g \leqq 4$, we have constructed $\hat{M}$ and proved that Φ is holomorphic.

In fact, the above suggests how one might compactify $M = \mathcal{G}/\Gamma$, and this checks known results for symmetric bounded domains of types III, IV: If Δ is the unit disk, then a holomorphic mapping $\pi: \Delta - \{0\} \rightarrow \hat{M}$ is holomorphic at zero if the row vectors of π are composed from the solutions of a regular differential equation.

For dim $V > 1$, the influence of periods on moduli depends on the extent to which the periods distinguish birationally inequivalent varieties. On the local question (when do the periods give local moduli?), we have a multiplicative problem in cohomology [by (C)] which may sometimes be reformulated as a multiplicative problem on linear systems. In a number of examples (nonsingular surfaces in P_3, hypersurfaces on abelian varieties, cubic threefolds, ...), the periods give local moduli generically. It seems plausible that this will be the case for surfaces with ample canonical series. Further evidence is given by the result: $\Phi(t) = \Phi(t')$ in M if and only if the homology class $[f] \epsilon H_{2n}(V_t \times V_{t'}, Z)$ of the graph of the homeomorphism $f: V_t \rightarrow V_{t'}$ is of type (n,n). Thus, if $\Phi(t) = \Phi(t')$, one might hope for a birational correspondence $f: V_t \rightarrow V_{t'}$.

We close with an application to the (algebraic) Kummer surface, which is the nonsingular surface of degree 4 in P_3. In this case, dim $M = 19$ and it is known that the set of $t \epsilon M$ cut out by period matrices of Kummer surfaces is open. Our results imply that every t is covered by a (perhaps singular) Kummer surface. Furthermore, $\dim(\hat{M} - M) = 1$ and the surfaces over $\hat{M} - M$ are cut out by a one-parameter family of Kummer surfaces having acquired a double curve.

(b) *Residue calculus and classification of integrals:* Let W be an algebraic n-manifold and ω a rational n-form on ω with polar locus $V \subset W$. If σ is an n-cycle in $W - V$, then we want to analyze the integral $\int_\sigma \omega$. Especially, there are two types of such integrals; those where $\sigma \sim 0$ in W (called *residues*), and the other periods (called *cyclic periods*). The cyclic periods should be thought of as given transcendental quantities, and what we want is an effective method of writing residues as integrals of rational $n - 1$ forms on a V'_{n-1}. Once this is done, then we will have a decomposition $\int_\sigma \omega = \Sigma \int_{\sigma_\alpha} \omega_\alpha$, where $\int_{\sigma_\alpha} \omega_\alpha$ is a cyclic period of a rational form on an algebraic manifold.

So there are two problems; one is to give a basis $\sigma_1, \ldots, \sigma_s$ for the n-cycles in $W - V$ which bound in W, and the other is to write $\int_{\sigma_\alpha} \omega$ as $\int_{\gamma_\alpha} \omega'$, where ω' is a rational $n - 1$ form.

For curves the solution is essentially trivial, and corresponds to writing $\omega = \omega_2 + \omega_3 + d\psi$ where ω_2 is a form of the second kind (giving the cyclic periods), ω_3 is of the third kind (giving the residues), and ψ is a function.

For surfaces, the solution is known by the methods of Picard and Lefschetz, and goes as follows (for $W = P_2$): A 2-cycle σ in $P - V$ is of the form $\sigma = \partial c_3$, c_3 being a 3-chain. If $S(V) \subset V$ is the singular locus, we may assume that c_3 meets V in a

1-cycle missing $S(V)$, as well as any finite number of intersections $C_j \cdot V$ of V with other curves in P. Then $c_3 \cdot V$ is a 1-cycle γ in $V - S(V) - \Sigma C_j \cdot V$ and $\sigma \sim \tau(\gamma)$, the tube lying over γ in $P - V$. Clearly, $\int_\sigma \omega = \int_{\tau(\gamma)} \omega$.

Assume $\omega = g(x,y)dxdy/f^k$, $f(x,y)$ being irreducible. If $k = 1$,

$$\int_{\tau(\gamma)} \omega = \int_\gamma \frac{g(x,y)dx}{\partial f/\partial y} = -\int_\gamma \cdot \frac{g\,dy}{\partial f/\partial x}.$$

If $k > 1$, we may assume that γ misses the intersection of V and $\frac{\partial f}{\partial x} = 0$. Then $g\frac{dxdy}{f^k} - \frac{1}{k-1} g \frac{dxdy}{(\partial f/\partial x)^2 f^{k-1}} = d\left(\frac{h\,dy}{\partial f/\partial x \cdot f^{k-1}}\right)$, where $\frac{g}{(\partial f/\partial x)^?}$ is regular near $\tau(\gamma)$. Consequently,

$$\int_{\tau(\gamma)} \omega = \int_{\tau(\gamma)} \tilde{g}\frac{dxdy}{f^{k-1}} = \ldots = \int_{\tau(\gamma)} \frac{l(x,y)dxdy}{f},$$

where l is regular near $\tau(\gamma)$. We may evaluate this integral as before.

The case $\omega = (g\,dxdy)/(f_1^{\alpha_1}f_2^{\alpha_2}\ldots f_l^{\alpha_l})$, where $f_1,\ldots,f_l$ are irreducible and mutually prime, may be treated by partial fractions.

If dim $W = 3$, we can solve the problem but significant new phenomena appear, even in case V is nonsingular. For example, if $W = P_3$, $\omega = g\,\frac{dxdydz}{f^k}$ with $f(x,y,z) = 0$ being nonsingular, then every 3-cycle $\sigma \epsilon H_3(P - V)$ is a tube, $\sigma = \tau(\gamma)$. If $k = 1, \int_{\tau(\gamma)} \omega = \int_\gamma g\frac{dxdy}{\partial f/\partial z}$ as before. Especially, $\int_{\tau(\gamma)} \omega = 0$ if $\gamma \sim$ (curve) in V. But if $k > 1$, we cannot assume that γ misses the locus $\left(\frac{\partial f}{\partial z} = 0\right)\cdot V$. What must be done is the following: Write $H_2(V) = A + B$, where A is generated by H (= hyperplane section), $C_1,\ldots,C_{\rho-1}$, the C_j being effective algebraic curves giving a base on V; and where $B = A^\perp$. Then, if $\gamma \epsilon B$, we may evaluate $\int_{\tau(\gamma)} g\frac{dxdydz}{f^k} = \int_{\tau(\gamma)} l\frac{dxdydz}{f}$ as before; here the polar locus of γ misses $\tau(\gamma)$.

To evaluate $\int_{(C_j)} g\,\frac{dxdydz}{f^k}$, we assume for simplicity that $k = 2$ and that C_j meets the curve $(\partial f/\partial z = 0)\cdot V$ transversely at a point p, p being nonsingular on each curve. Let D_ϵ be a disk of radius ϵ around p on C_j. Since $g\,\frac{dxdydz}{f^2} + g\frac{dxdydz}{(\partial f/\partial z)^2 f} = d\left(g\,\frac{dxdy}{\partial f/\partial z f}\right)$,

$$\int_{\tau(C_j - D_\epsilon)} g\,\frac{dxdydz}{f^2} + \int_{\tau(C_j - D_\epsilon)} g\,\frac{dxdydz}{(\partial f/\partial z)^2 f} = \int_{\partial\tau(D_\epsilon)} \frac{g}{\partial f/\partial z \cdot f}\,dxdy,$$

and this last integral is of the form $\int_{\substack{|\xi| = \epsilon \\ |\eta| = \epsilon}} \frac{h(\xi,\eta)d\xi d\eta}{\xi^\alpha \eta^\beta}$ and so may be evaluated.

Since $\int_{\tau(C_j - D_\epsilon)} g \frac{dxdydz}{(\partial f/\partial z)^2 f} = 0$, we may let $\epsilon \rightarrow 0$ to evaluate $\int_{\tau(C_j)} g \frac{}{f^2}$.

In case V is irreducible with singular curve C, we cannot write every $\sigma \epsilon H_3(W - V)$ as a tube. Indeed, $\sigma = \partial C_4$ and C_4 will meet C in a finite number of points. The trouble is similar to the above and may be overcome as follows: Let $D \subset V$ be a curve meeting C transversely at a finite number of points $p_1, \ldots, p_t$. For simplicity, assume $t = 1$, $p = p_1$, and let B be a small ball around p in W. We may assume that, locally, D is $V \cdot$(2-plane in 3-space) and we restrict our attention to $V \cdot$(this 2-plane). We may construct a tube τ_ϵ over $D - D \cap B_\epsilon$, and $\partial \tau_\epsilon$ will be a finite number (= number of local branches of V) of linked toral surfaces in a 3-sphere. If we take out the solid tori from the sphere, we are left with a 3-chain τ' such that $\tau_\epsilon + \tau' = \tau$ is a 3-cycle in $W - V$. Furthermore, $\tau = \partial c_4$ where $c_4 \cdot C = \{p\}$. Then, if $\sigma \epsilon H_3(W - V)$, $\sigma - k\tau$ will be a 3-cycle with $\sigma - k\tau = \partial c_4$ and $c_4 \cdot C = 0$. Thus, $\sigma - k\tau$ is a linear combination of tubes and, to evaluate $\int_\sigma \omega$, we may evaluate $\int_\tau \omega$. This integral may, as before, be written as a Cauchy integral.

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[1] Griffiths, P. A., these PROCEEDINGS, 55, 1303 (1966).

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*EXAMPLES OF SINGULAR NORMAL COMPLEX SPACES WHICH ARE TOPOLOGICAL MANIFOLDS**

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1. It has been proved by D. Mumford[2] that a 2-dimensional normal complex space which is a topological manifold is nonsingular. The following example shows that there is no corresponding theorem for dimensions higher than 2.

THEOREM. *Let X_k be the complex subspace of the $(k + 1)$-dimensional affine space C^{k+1} given by the equation $z_1^2 + \ldots + z_k^2 - z_0^3 = 0$ (k odd). Then the underlying topological space of X_k is a topological manifold.* (See *Note added in proof.*)

Remark: For $k = 2$, the rational double point of $x_1^2 + x_2^2 - x_0^3$ has a lens space $L(3,1)$ as neighborhood boundary (see ref. 1). The author does not know whether the theorem would also be true for even $k > 2$.

2. *Proof.*—The set of singular points of the hypersurface X_k consists only of the point $z = 0$; so it has codimension k and hence, by a result of Oka, X_k is normal for $k > 1$. The case $k = 1$ is well known, and therefore excluded in what follows. Let S^{2k+1} be the sphere $S^{2k+1} = \{z \epsilon C^{k+1} | z_0\bar{z}_0 + \ldots + z_k\bar{z}_k = 1\}$, and define W^{2k-1} as $W^{2k-1} = S^{2k+1} \cap X_k$. Then W^{2k-1} is a $(2k - 1)$-dimensional compact orientable differentiable manifold and the boundary of a neighborhood of 0 in X_k. The space $X_k - 0$ is homeomorphic to $W^{2k-1} \times (0, \infty)$; for instance, $\Phi(z_0, \ldots, z_k; t) = (t^2 z_0, t^3 z_1, \ldots, t^3 z_k)$ describes a homeomorphism Φ: $W \times (0, \infty) \rightarrow X - 0$. Therefore, to prove the theorem, one has to show that W^{2k-1} is a $(2k - 1)$-dimensional sphere.