

March 31, 1974

Dear Deligne:

For several reasons I have not responded to Dieudonné's request. I was in particular apprehensive that the result of collecting together the problems of twenty-five mathematicians would be tasteless hodge-podge. However, if several qualified mathematicians would take it upon themselves to give coherent accounts of the problems in the areas with which they are most concerned, the result might be very useful. You, for example, might think of doing this for  $L$ -functions, for it is not too great an exaggeration to suggest that it is a topic which contains the outstanding problems in algebraic geometry, in particular, those you describe to Dieudonné.

Although the present letter is principally a response to your invitation to comment on your list of problems, I have also included, as an appendix, some remarks on the Shimura varieties which I hoped you might find amusing.

I begin with an attempt to give a precise form to the problems of my Washington lectures.

**Local theory.** In *On the classification of irreducible representations of real algebraic groups* it is verified that to every class of mappings  $\varphi$  of the Weil group  $W_F$  ( $F = \mathbb{R}$  or  $\mathbb{C}$ ) satisfying some simple formal condition is associated a finite set  $\Pi(\varphi)$  of classes of irreducible quasi-simple representations of  $G(F)$ . The sets  $\Pi(\varphi)$  are disjoint and exhaustive.

There should be a similar statement for admissible representations of  $G(F)$  when  $F$  is non-archimedean. Let me give a conjecture. Given the finite Galois extension of  $K$  of  $F$  build the associate group

$$\hat{G} = \hat{G}^0 \times \mathfrak{G}(K/F)$$

and consider pairs  $(\varphi, Y)$  consisting of

(i) A continuous homomorphism  $\varphi$  of  $W_{K/F}$  into  $\hat{G}$  – such that for all  $w \in W_{K/F}$  the image  $\varphi(w)$  acts on the Lie algebra  $\mathfrak{g}$  of  $\hat{G}^0$  – such that

$$\begin{array}{ccc} W_{K/F} & \xrightarrow{\varphi} & \hat{G} \\ & \searrow & \downarrow \\ & & \mathfrak{G}(K/F) \end{array}$$

is commutative.

(ii) A nilpotent  $Y \in \hat{\mathfrak{g}}$  such that

$$\varphi(w)(Y) = q^{\alpha(w)}Y$$

for all  $w$ . Here  $\alpha$  is the standard map  $W_{K/F} \rightarrow \mathbb{Z}$  and  $q$  is the number of elements in the residue field of  $F$ .

The pair  $(\varphi_1, Y_1)$  associated to  $K_1$  and  $(\varphi_2, Y_2)$  associated to  $K_2$  will be regarded as equivalent if after lifting  $\varphi_1$  and  $\varphi_2$  to  $W_{L/F}$ , where  $K_1 \subseteq L, K_2 \subseteq L$ , we have

$$\varphi_2(w) = g\varphi_1(w)g^{-1} \quad Y_2 = \text{Ad } g(Y_1)$$

with  $g \in \hat{G}^0$ .

Recall that the family  $\hat{\mathfrak{p}}(G)$  of parabolic subgroups of  $\hat{G}$  was defined in the preprint referred to above. Let  $\Phi(G) = \Phi(G/F)$  consist of those classes  $(\varphi, Y)$  which satisfy the following conditions.

**If  $\hat{P}$  is a parabolic subgroup with Levi factor  $\hat{M}$ , if  $\varphi(W_{K/F}) \subseteq \hat{M}$ , and if  $Y$  lies in the Lie algebra of  $\hat{M}^0$ , then  $\hat{P}$  belongs to  $\hat{\mathfrak{p}}(G)$ .**

It should be possible to associate to each  $(\varphi, Y)$  which satisfy this condition a finite set  $\Pi(\varphi, Y)$  of classes of irreducible admissible representations of  $G(F)$ . The sets  $\Pi(\varphi, Y)$  should be disjoint, and their union should contain every irreducible admissible representation.

There are several properties which one expects the map  $(\varphi, Y) \rightarrow \Pi(\varphi, Y)$  to possess. For example, the elements of  $\Pi(\varphi, Y)$  should be square-integrable modulo the centre if and

only if  $(\varphi, Y)$  factors through a Levi factor of no proper parabolic subgroup of  $\hat{G}$ . However, there is no point in describing these properties now. I remind you, however, of the important normalizing property.

Suppose  $G$  is a quasi-split and splits over the unramified extension  $K$ . We consider pairs  $(\varphi, 0)$  where  $\varphi$  is of the form

$$\varphi(w) = \varphi_{\hat{g}}(w) = (\hat{g})^{\alpha(w)}.$$

$\hat{g}$  is some given semi-simple element of  $\hat{G}$  which maps to the Frobenius in  $\mathfrak{G}(K/F)$ . As in my Washington lecture  $\hat{g}$  determines a quasi-character  $\chi$  of  $T(F)$  if  $T$  is a Cartan subgroup contained in a Borel over  $F$ .  $\chi$  in turn determines the normalized principal series  $PS(\chi)$ , which of course may not be unitary.  $\Pi(\varphi, 0)$  consists of the **constituents** (not components) of  $PS(\chi)$  which contain the trivial representation of some special maximal compact. Observe that  $\Pi(\varphi, 0)$  may contain more than one class.

All the classes in a given  $\Pi(\varphi, Y)$  would be called  $L$ -indistinguishable. There are several cases which may be accessible enough that the conjecture can be further tested. The conjecture is therefore a little premature, for it would be best to examine these cases before making it. It is, by the way, my personal opinion that the local conjecture is, if correct, not too difficult, certainly not of the same order of difficulty as the global conjecture or the other problems on your list. The local theory is progressing very well in the hands of Casselman, Harish-Chandra, and Howe; I hope to see it pretty much completed within five to ten years.

**$\ell$ -adic motivation.** Ordinarily one introduces  $\hat{G} = \hat{G}^0 \times \mathfrak{G}(K/F)$  as a complex group, namely one takes  $\hat{G}^0 = \hat{G}^0(\mathbb{C})$ . However, we could take  $\hat{G} = \hat{G}^0(E)$  where  $E$  is a finite extension of  $\mathbb{Q}_\ell$ , with  $\ell$  prime to the residual characteristic of  $F$ . In the present paragraph we do this and write  $\hat{G} = \hat{G}(E)$ . There are moreover several ways to define  $\hat{G}$ :

$$\begin{aligned} \hat{G} &= \hat{G}^0 \times \mathfrak{G}(K/F) & \hat{G} &= \hat{G}^0 \times \varprojlim \mathfrak{G}(K/F) \\ \hat{G} &= \hat{G}^0 \times W_{K/F} & \hat{G} &= \hat{G}^0 \times \varprojlim W_{K/F} \end{aligned}$$

I shall use the first, a Galois form, which is simpler. However, it is sometimes important to use the last, a Weil form, because it is more flexible. In particular, non-trivial extensions of  $\mathfrak{G}(K/F)$  often split when inflated to  $W_{L/F}$ , if  $L$  is suitably large.

In any case suppose we are given a continuous homomorphism  $\psi : \mathfrak{G}(\overline{F}/F) \rightarrow \hat{G}$  such that

$$\begin{array}{ccc} \mathfrak{G}(\overline{F}/F) & \xrightarrow{\psi} & \hat{G} \\ & \searrow & \downarrow \\ & & \mathfrak{G}(K/F) \end{array}$$

is commutative and such that  $\psi(\sigma)$  is semi-simple if  $\sigma$  projects to the Frobenius. Let  $K^{\text{un}}$  be the maximal unramified extension of  $K$  and let  $K_\ell^+$  be the union of all tamely ramified extensions of  $K^{\text{un}}$  whose degree is a power of  $\ell$ .  $\mathfrak{G}(K_\ell^+/K)$  is an extension

$$1 \rightarrow Z_\ell \rightarrow \mathfrak{G}(K_\ell^+/K) \rightarrow \hat{\mathbb{Z}} \rightarrow 1.$$

This extension splits. Let  $U$  be the generator of  $\mathbb{Z}_\ell$  and  $\Psi$  the lifting of the generator of  $\hat{\mathbb{Z}}$ . If  $q^m$  is the order of the residue field of  $K$

$$\Psi U \Psi^{-1} = U^{q^m}.$$

We may, enlarging the finite Galois extension  $K$  if necessary, suppose that  $\psi$  factors through  $\mathfrak{G}(K_\ell^+/F)$  and that  $\psi(U)$  is unipotent

$$\psi(U) = \exp X, \quad X \in \hat{\mathfrak{g}}.$$

Then

$$\sigma U \sigma^{-1} = U^{\alpha(\sigma)}$$

if  $\sigma$  projects to the  $\alpha(\sigma)$  - power of the Frobenius.

As you know there is associated to  $X$  a parabolic subgroup  $\hat{P}^0$  of  $\hat{G}^0$ . Since  $\mathfrak{G}(K_\ell^+/K)$  and  $\mathfrak{G}(K_\ell^+/K^{\text{un}})$  are normal in  $\mathfrak{G}(K_\ell^+/F)$ ,  $\hat{P}^0$  is stable under  $\psi(\mathfrak{G}(K_\ell^+/F))$  and its normalizer  $\hat{P}$

in  $\hat{G}$  is parabolic. Let  $\hat{P}$  have the Levi factor  $\hat{M}$  and the unipotent radical  $\hat{N}$ . We may suppose  $\psi(\Psi) \in \hat{M}$ .

If  $\sigma \in \mathfrak{G}(K_\ell^+/F)$  write

$$\psi(\sigma) = n(\sigma)m(\sigma), \quad n(\sigma) \in \hat{N}, \quad m(\sigma) \in \hat{M}.$$

Let

$$\sigma\Psi\sigma^{-1} = \Psi U^{x(\sigma)}, \quad x(\sigma) \in \mathbb{Z}_\ell.$$

Then

$$\Psi^{-1}\sigma\Psi = U^{x(\sigma)}\sigma$$

and

$$\psi(\Psi^{-1})n(\sigma)\psi(\Psi) = U^{x(\sigma)}n(\sigma)$$

$$\psi(\Psi)^{-1}m(\sigma)\psi(\Psi) = m(\sigma).$$

It follows easily that

$$\exp\left(\frac{-x(\sigma)}{q^m - 1}\right)n(\sigma)$$

commutes with  $\psi(\Psi)$ .

Enlarging  $K$  once again if necessary, we may suppose that any eigenvalue of  $\psi(\Psi)$  on  $\hat{\mathfrak{G}}$  of which some power is an integral power of  $q$  is already an integral power of  $q$  and that the centralizer of  $\psi(\Psi)$  in  $\hat{G}^0$  is connected. Let  $\hat{\mathfrak{g}}_1$  be the Lie algebra spanned by

$$\{V \in \hat{\mathfrak{G}} \mid \psi(\Psi)V = q^r V, r \in \mathbb{Z}\}.$$

and let  $\hat{G}_1$  be the normalizer of  $\hat{\mathfrak{g}}_1$  in  $\hat{G}$ . Let  $\hat{\mathfrak{p}}_1$  be the parabolic sub-algebra of  $\hat{\mathfrak{g}}_1$  spanned by

$$\{V \in \hat{\mathfrak{g}} \mid \psi(\Psi)V = q^r V, r \in \mathbb{Z}, r \geq 0\}.$$

Let  $\hat{P}_1$  be the normalizer of  $\hat{\mathfrak{p}}_1$  in  $\hat{G}_1$ . The calculations above show that  $\psi(\mathfrak{G}(K_\ell^+/F)) \subseteq \hat{P}_1$ . Let  $\hat{M}_1$  be a Levi factor of  $\hat{P}_1$  and  $\hat{N}_1$  its unipotent radical. We may suppose that  $\psi(\Psi) \in \hat{M}_1$ . Since  $\hat{P}_1 = \hat{N}_1 \hat{M}_1$ , we may write

$$\psi(\sigma) = n_1(\sigma)m_1(\sigma).$$

Since  $\exp bX \in \hat{N}_1$ ,  $b \in \mathbb{Q}_\ell$ , and since the centralizer of  $\psi(\Psi)$  in  $\hat{N}_1$  is trivial we conclude by the same argument as before that

$$n_1(\sigma) = \exp\left(\frac{x(\sigma)}{q^m - 1}X\right).$$

The map

$$\sigma \longrightarrow m_1(\sigma)$$

is clearly a representation of  $\mathfrak{G}(K_\ell^+/F)$  by semi-simple matrices. Moreover

$$m_1(\sigma)(X) = q^{\alpha(\sigma)}X$$

and  $m_1(\sigma)$  factors through  $\mathfrak{G}(K^{un}/F)$ .

If we have an imbedding  $\eta : E \rightarrow \mathbb{C}$  and if  $w \rightarrow \sigma$  is the standard map  $W_{K/F} \rightarrow \mathfrak{G}(K^{un}/F)$  we may define  $\varphi, Y$  by

$$\varphi : w \rightarrow \eta(m_1(\sigma)), \quad Y = \eta(X).$$

**Conclusion: An  $\ell$ -adic representation of  $\mathfrak{G}(\overline{F}/F)$  in  $\hat{G}(E)$  plus an imbedding  $E \rightarrow \mathbb{C}$  yields a pair  $(\varphi, Y)$ .**

You will find the imbedding  $\eta : E \rightarrow \mathbb{C}$  repugnant. I do not find it repugnant, merely perplexing. What I thoroughly detest is the introduction of  $\ell$ -adic representations of  $G(F)$ . The study of elliptic modular forms shows us already that we cannot pass from the  $\ell$ -adic theory to the complex theory, and therefore to  $L$ -functions, without imbeddings  $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ ,  $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ . One point which troubles me, but which I have not yet seriously pondered, is the following. If the

eigenvalues of all  $\psi(\sigma)$ , for any linear representation of  $\hat{G}(E)$ , lie in  $\overline{\mathbb{Q}}$  is  $(\varphi, Y)$  independent of  $\eta$ , provided  $\eta$  is the identity on  $\overline{\mathbb{Q}} \cap E$ .

A basic question, which you are best qualified to discuss, is: **What is a motive of type  $\hat{G}$  over a local field?**

Observe that it is clear how to define the local  $L$ -function  $L(s, \pi, \rho)$  for  $\pi \in \Pi(\varphi, Y)$ .

**Global theory.** If  $G$  is a connected reductive group over the global field  $F$  two representations  $\pi = \otimes \pi_v$  and  $\pi' = \otimes \pi'_v$  which occur as **constituents** of the space of automorphic forms on  $G(\mathbb{A})$  would be called  $L$ -indistinguishable if  $\pi_v$  and  $\pi'_v$  are  $L$ -indistinguishable for each  $v$ . The basic object in the theory of automorphic forms is a class of  $L$ -indistinguishable representations.

**Basic problem.** Suppose that  $H$  and  $G$  are given over the global field  $F$ . Take  $\hat{H}$  and  $\hat{G}$  in the Weil form. Suppose an  $L$ -homomorphism  $\psi$  (cf. the preprint referred to earlier)  $\hat{H} \rightarrow \hat{G}$  is given. Suppose  $\pi = \otimes \pi_v$  occurs as a constituent of the space of automorphic forms on  $H$ . Let  $\pi_v \in \Pi(\varphi_v, Y_v)$ . If  $\psi_v$  is the restriction of  $\psi$  to the associate group over  $F_v$ , suppose  $(\psi_v \circ \varphi_v, \psi_v(Y_v))$  lie in  $\Phi(G/F_v)$  for each  $v$ . Is there then a constituent  $\pi' = \otimes \pi'_v$  of the space of automorphic forms on  $G$  such that  $\pi'_v \in \Pi(\psi_v \circ \varphi_v, \psi_v(Y_v))$  for all  $v$ ?

An affirmative solution to this problem entails of course the solution to Artin's conjecture. The question can be motivated entirely within the theory of automorphic forms by the following two considerations:

- (a) **The Artin principle.** The only "natural" way to prove analytic continuation and the functional equation for an  $L$ -function is that of Hecke and Tate. This is effective only for certain  $L$ -functions associated to simple algebras. To handle any other  $L$ -function, one must prove a reciprocity law which shows it equals one of these.
- (b) **Examples.** A number of examples were known.
  - (i)  $H = 1, G = GL(1)$  - Class field theory.
  - (ii)  $H, G$  abelian - ditto

(iii)  $H$  the Levi factor of a parabolic subgroup of  $G$  over  $F$  and  $\psi : \hat{H} \rightarrow \hat{G}$  the associated imbedding - this is part of the theory of Eisenstein series.

(iv)  $H$  : the multiplicative group of quadratic extension,  $G = GL(2)$ , and  $\psi : \hat{H} \rightarrow \hat{G}$  the obvious imbedding - this is treated by Hecke and Maass.

(v)  $H = GL(2)$ ,  $G$  obtained from  $GL(2)$  over a quadratic extension by restriction of scalars,  $\psi : \hat{H} \rightarrow \hat{G}$ , the diagonal map on  $\hat{H}^0$  - this is Doi, Naganuma, Jacquet.

**Two vague problems.** Suppose  $\pi$  occurs in the space of automorphic forms on  $G$ .

- (i) Give a “reasonable” necessary and sufficient condition that  $L(s, \pi, \rho)$  be holomorphic for  $\text{Res} > 1$  and all representations  $\rho$  of the Galois form of  $\hat{G}$ . As you know, these are the  $\pi$  which will satisfy Ramanujan’s conjecture.
- (ii) Give a “reasonable” necessary and sufficient condition that there exist a group  $H$ , a  $\psi : \hat{H} \rightarrow \hat{G}$ , and a constituent  $\pi'$  of the space of automorphic forms on  $H$  such that  $\pi$  lies in the  $L$ -indistinguishable class on  $G$  obtained from  $\psi$  and  $\pi'$  and such that, for every representation  $\rho$  of  $\hat{H}$ ,  $L(s, \pi', \rho)$  is holomorphic for  $\text{Res} > 1$  with a pole at  $s = 1$  of order equal to the multiplicity with which the trivial representation occurs in  $\rho$ . These are the  $\pi$  for which one can expect to handle the Sato-Tate conjecture. Namely for almost all  $v$ ,  $\pi'_v \in \Pi(\varphi_{g_v}, 0)$  with  $g_v$  in a maximal compact  $\hat{H}_{\text{comp}}$  of  $\hat{H}(\mathbb{C})$ . The conjugacy classes  $\{g_v\}$  should be uniformly distributed in  $\hat{H}_{\text{comp}}$ .

**Motives.** Would you be willing to try to define a motive of type  $\hat{G}$  over a global field? One will of course want to know which  $L$ -indistinguishable classes of global  $\pi$  correspond to motives. I assume you believe that for a function field the necessary and sufficient condition is that for almost all  $v$ ,  $\pi_v \in \Pi(\varphi_{g_v}, 0)$  where  $\rho(g_v)$  has eigenvalues in  $\overline{\mathbb{Q}}$  for all representations  $\rho$  of the Galois form of  $\hat{G}$ .



Notice that if  $F$  is  $\mathbb{R}$  or  $\mathbb{C}$  then  $W_F$  contains  $\mathbb{C}^\times = S(\mathbb{R})$ . A homomorphism  $\varphi : W_F \rightarrow \hat{G}$  with

$$\begin{array}{ccc} W_F & \xrightarrow{\varphi} & \hat{G} \\ & \searrow \text{id} & \downarrow \\ & & W_F \end{array}$$

commutative could be said to be of type  $A_0$  if the restriction of  $\varphi$  to  $\mathbb{C}^\times$  factored as

$$\mathbb{C}^\times = S(\mathbb{R}) \hookrightarrow S(\mathbb{C}) \xrightarrow{\eta} \hat{G}$$

where  $\eta$  is algebraic over  $\mathbb{C}$ .

**Obvious guess.** For a number field,  $\pi$  corresponds to a motive if and only if for each infinite place  $v$ ,  $\pi_v \in \Pi(\varphi_v)$  where  $\varphi_v$  is of type  $A_0$ .

I suppose that the problem of defining a motive of type  $\hat{G}$  is intimately related to the Tate conjectures and therefore to such problems as the isogeny of two elliptic curves with the same zeta-function. Lang hoped to apply the methods of transcendental numbers to this problem. He has not yet succeeded. Nonetheless it is tantalizing to think that such methods may one day play a role.

All the best.

Yours,

R. Langlands

RL:MMM