"The History of the Prime Number Theorem"

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Monday, August 12, 1996, 5-6 pm

Opening lecture at

"In Celebration of the Centenary of the Proof of the Prime Number Theorem, A Symposium on the Riemann Hypothesis." Seattle, Washington, August 12-15, 1996. Sponsored by the American Institute of Mathematics.

Antiquity: Euclid,
Eratosthenes

L. Euler notes that

$$\prod_p (1-p^{-s})^{-1} = \sum n^{-s},$$

for $s>1$, concludes

$$\sum_p \frac{1}{p} \text{ diverges.}$$

C. F. Gauss 1792 or 1793
empirically arrives at

$$\pi(x) \approx \int_2^x \frac{dt}{\log t} = \mathrm{li}\, x$$

continues throughout his life as new more extensive prime number tables appear. Correspondence with Bessel 1810, letter to Encke 1849.

NACHLASS.

EINIGE ASYMPTOTISCHE GESETZE DER ZAHLENTHEORIE.

[I.]

[Handschriftliche Eintragung in dem Buche:] JOHANN CARL SCHULZE, Neue und erweiterte Sammlung logarithmischer .... Tafeln. I, Berlin 1778; [von GAUSS' Hand] Gauß. 1791.

[Auf der Rückseite des letzten Blattes.]

[1.]

Primzahlen unter a $(=\infty)$

$$\frac{a}{la}.$$

[2.]

Zahlen aus zwei Factoren

$$\frac{lla \,.\, a}{la},$$

(wahrsch.) aus 3 Factoren

$$\frac{\frac{1}{2}(lla)^2 a}{la}, \ldots$$

et sic in inf.

A. M. Legendre in his "Essai sur la théorie des nombres" 1st edition 1798 states that $\pi(x)$ probably can be approximated by an expression

$$\frac{x}{A \log x + B},$$

where A and B are constants.

In the second edition 1808 he gives the formula

$$\pi(x) \approx \frac{x}{\log x - 1.08366}.$$

Abels letter to Holmboe 1824.
Dirichlet 1839 on the arithmetic progression.

May 24, 1848 P.L. Chebyshev read a paper before the St. Petersburg Academy where he proved:

§ VIII. *D'une loi très-remarquable observée dans l'énumération des nombres premiers.*

(394) Quoique la suite des nombres premiers soit extrêmement irrégulière, on peut cependant trouver avec une précision très-satisfaisante combien il y a de ces nombres depuis 1 jusqu'à une limite donnée x. La formule qui résout cette question est

$$y = \frac{x}{\log. x - 1.08366},$$

log.x étant un logarithme hyperbolique. En effet, la comparaison de cette formule avec l'énumération immédiate faite dans les tables les plus étendues, telles que celles de Wéga, de Chernac ou de Burckhardt, donne les résultats suivants.

Limite x.	Nombre y		Limite x.	Nombre y	
	Par la formule.	Par les tables.		Par la formule.	Par les tables.
10000	1230	1230	200000	17982	17984
20000	2268	2263	250000	22035	22045
30000	3252	3246	300000	26023	25988
40000	4205	4204	350000	29961	29977
50000	5136	5134	400000	33854	33861
60000	6049	6058	500000	41533	41538
70000	6949	6936	600000	49096	49093
80000	7838	7837	700000	56565	56535
90000	8717	8713	800000	63955	63937
100000	9588	9592	900000	71279	71268
150000	13844	13849	1000000	78543	78493

If a very good simple approximation function to $\pi(x)$ exists, it has to be $\operatorname{li} x$.

More precisely he showed

$$\sum_p \frac{\log^k p}{p^s} - \sum_n \frac{\log^{k-1} n}{n^s} = O(1)$$

as $s \to 1+$, also

$$\int_2^\infty \frac{\pi(x) - \operatorname{li} x}{x^{s+1}} \log^k x \, dx = O(1)$$

as $s \to 1+$.

From this he concludes: for any given $\alpha > 0$ and N, we have

$$\left| \pi(x) - \operatorname{li} x \right| < \frac{\alpha x}{\log^N x},$$

for a sequence of x that tends to ∞.

Chebyshev first to utilize

$$\zeta(s) = \sum_n n^{-s} = \prod (1-p^{-s})^{-1}$$

for real $s > 1$ in this context. His proof depends on the identity

$$\int_2^\infty \frac{\pi(x) - \mathrm{li}\, x}{x^{s+1}}\, dx = \frac{1}{s}\sum_p p^{-s} - \frac{1}{s}\int_2^\infty t^{-s}\frac{dt}{\log t}.$$

The right hand side is

$$\frac{1}{s}\log\bigl((s-1)\zeta(s)\bigr) + g(s)$$

where $g(s)$ is regular at $s=1$, differentiating k times with respect to s and letting $s \to 1+$ one gets the required result.

In a second paper presented in 1850, Chebyshev obtains the first good bounds for

$\pi(x)$.

Writing: $\vartheta(x) = \sum_{p<x} \log p$,

$\psi(x) = \sum_{p^n<x} \log p$, so that

$$\psi(x) = \vartheta(x) + \vartheta(x^{\frac{1}{2}}) + \dots + \vartheta(x^{\frac{1}{n}}) + \dots,$$

Chebyshev considered

$$T(x) = \sum_m \psi\left(\frac{x}{m}\right) = \sum_m \sum_{p^n<\frac{x}{m}} \log p$$

$$= \sum_{p^n m<x} \log p = \sum_{m'<x} \sum_{p^n/m'} \log p$$

$$= \sum_{m'<x} \log m' = \log([x]!) =$$

$$= x(\log x - 1) + O(\log x).$$

He formed the linear combination

$$U(x) = T(x) - T\left(\frac{x}{2}\right) - T\left(\frac{x}{3}\right) - T\left(\frac{x}{5}\right) + T\left(\frac{x}{30}\right)$$

$$= Ax + O(\log x),\ A = \frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 5}{5} - \frac{\log 30}{30}$$

$A = 0.9212920229\ldots$

Inserting the expression for T by the ψ one gets

$$U(x) = \psi(x) - \psi\left(\tfrac{x}{6}\right) + \psi\left(\tfrac{x}{7}\right) - \psi\left(\tfrac{x}{10}\right) + \psi\left(\tfrac{x}{11}\right) - \psi\left(\tfrac{x}{12}\right) + \psi\left(\tfrac{x}{13}\right) - \psi\left(\tfrac{x}{15}\right) + \psi\left(\tfrac{x}{17}\right) - \psi\left(\tfrac{x}{18}\right) + \psi\left(\tfrac{x}{19}\right) - \psi\left(\tfrac{x}{20}\right) + \psi\left(\tfrac{x}{23}\right) - \psi\left(\tfrac{x}{24}\right) + \psi\left(\tfrac{x}{29}\right) - \psi\left(\tfrac{x}{30}\right) + \psi\left(\tfrac{x}{31}\right) - \cdots,$$

One sees that

$$\psi(x) - \psi\left(\tfrac{x}{6}\right) < U(x) < \psi(x),$$

from which

$$U(x) < \psi(x) < U(x) + U\left(\tfrac{x}{6}\right) + \cdots U\left(\tfrac{x}{6^n}\right) + \cdots$$

and so:

$$Ax - O(\log x) < \psi(x) < \tfrac{6}{5}Ax + O(\log^2 x)$$

$$= A'x + O(\log^2 x),$$

with $A' = 1.1055504275$.

Since $\vartheta(x) = \psi(x) + O(\sqrt{x})$,

and
$$\pi(x) = \int_2^x \frac{d\vartheta(t)}{\log t} =$$
$$= \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)\,dt}{t\log^2 t},$$

where
$$\int_2^x \frac{\vartheta(t)}{t\log^2 t}\,dt = O\left(\frac{x}{\log^2 x}\right),$$
we have similar bounds for $\vartheta(x)$ and $\pi(x)$.

Improvements, J.J. Sylvester 1881 and 1892; H. Poincaré 1891 analogue for "Gaussian integers".

G.F.B. Riemann's note to Prussian Academy of Science in Berlin (of which he had just been elected a correspon-ding member) in 1859 finally brings in $\zeta(s)$ as a function of a complex variable. The motivation is inversion of the relation

$$\frac{1}{s}\log\zeta(s) = \int_2^\infty \frac{f(x)}{x^{s+1}}\,dx$$

where

$$f(x) = \pi(x) + \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \cdots + \tfrac{1}{n}\pi(x^{\frac{1}{n}}) + \cdots$$

which in essence is already present in Chebyshev's work. Considering the righthand expression as a fourier integral (writing $x = e^u$; $s = a+it$) he finds

$$f(x) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{x^s}{s}\log\zeta(s)\,ds,$$

for $a>1$.

Riemann writes $s = \frac{1}{2} + it$ (where t may be complex) and

$$\xi(t) = \frac{1}{2} s(s-1)\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s),$$

and shows that ξ is an integral function of t^2, all of whose zeros have imaginary parts between $-\frac{i}{2}$ and $\frac{i}{2}$. From growth considerations he concludes that

$$\xi(t) = \xi(0)\prod\left(1 - \frac{t^2}{\alpha^2}\right)$$

where α runs through the zeros of ξ, if α runs through the zeros of ξ with positive real part. He states that

$$N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + O(\log T)$$

if $N(T)$ denotes the number of zeros with real part in the interval $(0, T)$, and that there seems to be about that many real zeros there, so he conjectures that all zeros of $\xi(t)$ are real. (or all non-trivial zeros of $\zeta(s)$ on the line $\sigma = \frac{1}{2}$).

Using a rather reckless procedure of integrating termwise (after having expressed $\log \zeta(s)$ in terms of $\log \xi\left(\frac{s-\frac{1}{2}}{i}\right)$ and simple terms, and integrating first by parts) he arrives at the formula

$$f(x) = Li(x) - \sum_{\alpha} Li\left(x^{\frac{1}{2}+\alpha i}\right) + Li\left(x^{\frac{1}{2}-\alpha i}\right)$$
$$+ \int_x^{\infty} \frac{1}{t^2-1} \frac{dt}{t \log t} - \log 2 .$$

It is clearly a preliminary note, and might not have been written if L. Kronecker had not urged him to write up something about this work. (Letter to Weierstrass Oct 26, 1859). It is clear there are holes that need to be filled in, but also clear that he had a lot more material than is in the note.

What also seems clear:

Riemann is not interested in an asymptotic formula, not in the prime number theorem, what he is after is an exact formula!

In his introduction Riemann mentions Gauss and Dirichlet. It is known (letter from Schering) that he had read Legendre. He had undoubtedly also seen the work of Chebyshev which had been published in French. It is quite possible that it was Chebyshev's first paper referred to earlier, which inspired him to consider the zeta function. I am convinced that Riemann knew that $\zeta(s)$ has no zeros on the line $\sigma = 1$. If there were one it would have to be a simple zero since

$$\zeta(\sigma)\,|\zeta(\sigma+it)| > 1,$$

for $\sigma > 1$. If there were one say $1+it_0$, one gets by looking at the higher derivatives of

$$\frac{\zeta'}{\zeta}(s) + \frac{\zeta'}{\zeta}(s+it_0),$$

as $s \to 1+$, and taking the

real part that

$$\sum_p \frac{\log p}{p^\sigma}^{k+1} (1+\cos t_0 \log p) = O(1),$$

as $\sigma \to 1+$.

This means that

$$\sum_p \frac{\log p}{p}^{k+1} \cos^2 \frac{t_0 \log p}{2} < \infty,$$

since $\sum_p \frac{1}{p}$ diverges, contradiction.

Had Riemann's goal been the prime number theorem, he would probably have considered $\psi(x)$ instead of his $f(x)$, and used a smoothened expression like $\int_2^x \psi(t)dt \sim \int_2^x \frac{\psi(t)}{t} dt$, leading to factors $\frac{x^{s+1}}{s(s+1)} \sim \frac{x^s}{s^2}$ in his integrals instead of $\frac{x^s}{s}$. It is very likely that he would have succeeded had he tried.

Some asymptotic relations involving primes were established in the following decades by F. Mertens who in 1874 proved

$$\sum_{p<x} \frac{\log p}{p} = \log x + O(1),$$

and

$$\sum_{p<x} \frac{1}{p} = \log\log x + c + O\left(\frac{1}{\log x}\right).$$

Mertens also conjectured based on empiric evidence that

$$\Big|\sum_{n<x} \mu(n)\Big| < \sqrt{x}.$$

Mertens first formula probably was known to Chebyshev, since it follows simply from

$$T(x) = \sum_{p^r<x} \log p \left[\frac{x}{p^r}\right] = x \log x + O(x)$$

T.J. Stieltjes in two C.R. notes 1885 claimed to have shown that the series

$$\sum \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)},$$

is convergent for $\sigma > \frac{1}{2}$, (which would clearly imply Riemanns statement about ~~the~~ the zeros of $\zeta(s)$ being on the line $\sigma = \frac{1}{2}$); from this he concluded

$$\psi(x) = x + O(x^{\frac{3}{4}+\varepsilon})$$

for any $\varepsilon > 0$!

G. Halphen in a C.R. note from 1883 states that $\vartheta(x) \sim x$ as $x \to \infty$. By some french authors this is later referred to as: la loi asymptotique d'Halphen ! (it surely was conjectured by Chebyshev if not earlier!).

1893 E. Cahen claims to prove $\vartheta(x) \sim x$ "Halphen's law" assuming the Riemann Hypothesis (as "proved" by Stieltjes).

Substantial progress was made when J. Hadamard in 1892 in connection with his work on entire functions, proved rigorously Riemann's assertion

$$\xi(t) = \xi(0) \prod \left(1 - \frac{t^2}{\alpha^2}\right),$$

he also showed

$$a\, T \log T < N(T) < A\, T \log T,$$

with positive constants a and A for $T > 15$.

Finally in 1896 Hadamard rigorously proved $\vartheta(x) \sim x$, "Halphen's law" (from which the prime number theorem follows, but he does not mention this at all!).

He bases his proof on the formula

$$\sum_{p^m < x} \log p \log^{\mu-1} \frac{x}{p^m} = -\frac{\Gamma(\mu)}{2\pi i} \int_{\alpha - i\infty}^{\alpha + i\infty} \frac{x^s}{s^\mu} \frac{\zeta'}{\zeta}(s)\, ds,$$

for $\mu > 1$. Using his results from 1892, and that $\zeta(1+it) \neq 0$ for t real. He proves that the left hand side is asymptotic to $\Gamma(\mu) x$. as $x \to \infty$. Taking $\mu = 2$, he gets $\vartheta(x) \sim x$ by a difference argument. He also sketches a proof of the analogous result for an arithmetic progression.

That $\zeta(1+it) \neq 0$, he concludes by showing that if $\zeta(1+it_0) = 0$ then $1+2it_0$ would be a pole of $\zeta(s)$ (an obvious contradiction).

The same year de la Vallée-Poussin independently, but building on Hadamard's 1892 paper, gives a proof along somewhat similar lines. He does state the prime number theorem in his paper! His paper treats not only the case of the ~~[illegible]~~ arithmetic progression but also that of a binary quadratic form.

de la Vallée Poussin concludes that $\zeta(1+it) \neq 0$ from the inequality

$$|\zeta^3(\sigma)\,\zeta^4(\sigma+it)\,\zeta(\sigma+2it)| \geq 1,$$

for $\sigma > 1$, based on the inequality

$$3 + 4\cos\varphi + \cos 2\varphi \geq 0.$$

A few years later he develops this idea, now applied to the logarithmic derivative as

$$R\left(-3\frac{\zeta'}{\zeta}(\sigma) - 4\frac{\zeta'}{\zeta}(\sigma+it) - \frac{\zeta'}{\zeta}(\sigma+2it)\right) \geq 0,$$

into an argument that shows $\zeta(s) \neq 0$ for

$$\sigma > 1 - \frac{a}{\log|t|}, \quad |t| > A,$$

where a and A are certain positive constants.

From this he concludes

$$\pi(x) = li\,x + O\left(x e^{-\alpha\sqrt{\log x}}\right),$$

for some constant $\alpha > 0$.

Later progress by J. E. Littlewood and I. Vinogradov and others in the direction of improving the remainder term is entirely based on improving estimates for certain exponential sums. Apart from that it is still de la Vallée Poussin's argument that is used. This can in principle never give us more than a zero free region which lies close to $\sigma = 1$, whose width tends to zero as $|t| \to \infty$.