Singularities of Admissible Normal Functions

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Preface

This is an expanded version of a talk given at the International Colloquium on "Cycles, Motives and Shimura Varieties" held at the Tata Institute of Fundamental Research in January, 2008. In a general sense this paper is about algebraic cycles, and on a personal note I would like to observe that my first paper on cycles [G2] appeared in the volume arising from the International Conference on Algebraic Geometry held at the TIFR in January, 1968 — exactly 40 years ago. Although much has happened in the subject of algebraic cycles — on the arithmetic, Hodge theoretic and formal aspects — the fundamental problem — the Hodge conjecture and its generalizations — remains, and it has been joined by another fundamental problem — the conjectures of Bloch and Beilinson. These two problems reflect, in some sense, the geometric and arithmetic aspects of cycles. Although there have been some very interesting connections between these perspectives, my own view is that further fundamental progress will require an even deeper fusion between the two, perhaps complemented by new formal constructions.

This paper is largely an informal exposition of joint work with Mark Green, some of which was also joint with Matt Kerr, together with various speculations for which I alone am responsible.

I would like to thank the organizers, especially Vasudevan Srinivas, for their efforts in putting together this excellent colloquium and the TIFR for hosting it.

1 Introduction

In a general sense, we are interested in the global study of singularities of Hodge theoretic invariants arising from geometry. The connection with the title of this article is that, for an admissible normal function $\nu$ (defined below), its singular set $\text{sing} \nu$ is one such.

By way of context, among the main techniques that have been developed to apply Hodge theory to geometric questions are:
- Infinitesimal methods (cf. [G2] and [GMV] for a general introduction and references), which might be thought of as variational at a generic point;
- Degenerations of Hodge structures (cf. [CKS] and [Ka] and the references cited therein), which might be thought of as variational at a special point;
- Hodge theoretic aspects of the decomposition theorem (cf. [BBD] and [dCM1], [Sa1]).

Of course there are others — curvature properties/stability of the Hodge bundles, global PDE techniques, etc. — but the above are the ones most relevant to this paper.

We are interested in geometric questions concerning the existence and structure of Hodge theoretic singularities in a global situation. This work is intended to be an overview of certain questions related to algebraic cycles and normal functions. However, the following “toy problem” illustrates the type of specific questions that have motivated the general considerations:

Can there exist a non-trivial family of smooth Calabi-Yau varieties of dimension $n$ and parametrized by a complete curve?

The answer is:

$n = 1$ no; use the $j$-function;

$n = 2$ yes; e.g., apply suitable semi-stable reduction to a generic pencil of quartic surfaces in $\mathbb{P}^3$;

$n = 3$ no, if the Hodge numbers $h^{2,0}$ of a fibre satisfy\footnote{cf. [GGK3].}

$$h^{2,1} < h^{1,1} + 12 .$$

To put the $n = 3$ case in context, we recall [Dl] that there exist families of smooth principally polarized abelian varieties of dimension $g \geq 3$ and parametrized by a complete curve, but they do not exist for $g = 1, 2$. If we consider Jacobians of curves of genus $g$ and think of the genus as the dimension of the image of the Abel-Jacobi map, then for Calabi-Yau varieties the Hodge number $h^{2,1}$ is an upper bound on the dimension of the image of its Abel-Jacobi map, and the above might be thought of as an analogue of the curve result.

The plan of this paper is as follows (notations to be explained in the text):

- We will give the definitions of an admissible normal function (ANF) $\nu \in \Gamma(S, \delta, v)$ and its singular locus $\text{sing } \nu \subset S$;\footnote{For a classical normal function, $\text{sing } \nu = \emptyset$.}
- Given $(X^n_0, L, \zeta)$ where $\zeta \in H^0(X_0, \mathcal{O}_X)$ and $S = |L|$ parametrizes the universal family of hyperplane sections $(X_s)$, there is an associated ANF $\nu_\zeta$ and

$$\text{sing } \nu_\zeta = \\begin{cases} s_0 &\subset S : H^{2n}(X_{s_0}, \mathbb{Q}) \\ \text{has a "new" Hodge class} \end{cases} .$$

The same result quite possibly holds for $S$ when $S \to |L|$ parametrizes any semi-stable reduction (SSR) for the universal family of hyperplane sections.

- (work in progress) Admissible normal functions may be “graphed”; i.e., for a principally polarized variation of Hodge structure of weight $2n - 1$ where the degenerate Hodge structures occur along a divisor $D \subset S$ with simple normal crossings, there exists $\tilde{J}_{s} \to S$

such that $\tilde{J}_{s, v} = O_S(\tilde{J}_{s}) v$; \footnote{This has been done when $\dim S = 1$ (cf. [GGK1]) and in the classical case $n = 1$ (cf. [Y]). For recent work see [Sch].}

- (work in progress)\footnote{This has been done when $n = 1, 2$ and the VHS arises as in footnote 3; in these cases the construction of $\Xi$ has been carried out over the locus in $S$ where the $X_s$ have at most nodal singularities.} $\tilde{J}_{s}$ and $\Xi$ exist universally; i.e., there exists a diagram

$$\begin{array}{ccc}
J_{s} & \supset & \Xi \\
\downarrow & & \downarrow \\
S & \xrightarrow{\tau} & \Gamma \setminus D_{\Xi}
\end{array}$$

\footnote{This has also been done under the same conditions as in footnote (3) above.}
where \( \tau \) is the Torelli map and such that
\[
\text{sing} \nu = \nu^{-1}(\Xi).
\]
If \( \text{codim}_F(\Xi) = d \), then this has the implication
\[
(i) \quad \text{sing} \nu \neq \emptyset \Rightarrow \text{codim}_F(\text{sing} \nu) \leq d,
\]
which is an existence result, and the additional implication
\[
(ii) \quad \nu^*([\Xi]) \neq 0 \Rightarrow \text{sing} \nu \neq \emptyset.
\]
The topological condition \( \nu^*([\Xi]) \neq 0 \) also gives an existence result; at present we have no ideas on how to prove the topological condition in non-classical cases.

2 Admissible Normal Functions and Their Singularities

Notations and assumptions

\((\mathcal{H}_x, \mathcal{F}_x, \nabla, S^*)\) is a principally polarized variation of Hodge structure (VHS) of odd weight \(2n-1\) over a smooth, quasi-projective base space \(S^*\). Here, \(\mathcal{H}_x\) is a local system and the \(\mathcal{F}_x\) give a filtration of \(\mathcal{H} = H_x \otimes_{\mathcal{O}_X} \) that induces a Hodge filtration on each fibre. We have
\[
S^* = S \setminus D
\]
where \(S\) is a smooth, projective variety and \(D \subset S\) is a reduced, local normal crossing divisor (NCD). With this assumption, the local monodromies \(T_i\) around the local irreducible branches \(D_i\) of \(D\) are quasi-unipotent, and we assume further that they are unipotent; moreover, there are canonical extensions \(\mathcal{H}_x, \mathcal{F}_x\) of \(\mathcal{H}, \mathcal{F}\) and \(\mathcal{H}_{x,e} = j_*\mathcal{H}_x\) where \(j: S^* \to S\) is the inclusion, and where the Gauss-Manin connection is given by
\[
\nabla : \mathcal{F}_x^p \to \mathcal{F}_x^{p-1} \otimes_{\mathcal{O}_x} \Omega_S^1(\log D).
\]
We define the sheaf over \(S\)
\[
\mathcal{F}_x = \mathcal{F}_x^p \mathcal{H}_x/\mathcal{H}_{x,e} \cong \mathcal{F}_x^p/\mathcal{H}_{x,e}
\]
where the isomorphism results from the principal polarization, and we set
\[
\mathcal{G}_x, \nu = \{ \nu \in \mathcal{F}_x : \nabla \nu \in \mathcal{F}_x^{p-1} \otimes_{\mathcal{O}_x} \Omega_S^1(\log D) \}
\]
where \(\nabla\) is any local lifting of \(\nu\) to a section of \(\mathcal{F}_x\).

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Definition ([Z1]) A normal function is given by
\[
\nu \in \Gamma(S, \mathcal{G}_x, \nu).
\]

Example 2.1 Given a family
\[
f : X \to S
\]
of projective varieties \(X_s = f^{-1}(s)\) whose total space \(X\) and parameter space \(S\) are smooth and where the singular fibres occur over a reduced, local normal crossing divisor \(D \subset S\), there is a polarized VHS, which we assume to be principal, as above where
\[
\mathcal{H}_x = H_x^{2n-1}Z/\text{torsion}.
\]
We set
\[
\mathcal{Z}^n(X)_{\text{hom}} = \left\{ \begin{array}{l}
\mathcal{Z} \in \mathcal{Z}^n(X) \\
[Z]_U = 0 \text{ in } H^{2n}(X_U, \mathbb{Z})
\end{array} \right\}
\]
where \(U\) is a small neighborhood of any point \(s \in S\) and \([Z]_U\) is the fundamental class of \(Z\) in \(X_U = f^{-1}(U)\). Varying \(Z\) in its rational equivalence class we may assume that for \(s \in S^*\) the intersection
\[
Z_s = Z \cdot X_s \in \mathcal{Z}^n(X_s)
\]
is proper, and since \([Z]_s = 0\) in \(H^{2n}(X_s, \mathbb{Z})\) the Abel-Jacobi image
\[
\nu_Z(s) = \Lambda X_s(Z_s) \in J(X_s)
\]
is defined where
\[
J(X_s) = F^n H^{2n-1}(X_s, \mathbb{C}) \setminus H^{2n-1}(X_s, \mathbb{C}) / H^{2n-1}(X_s, \mathbb{Z})
\]
is the intermediate Jacobian of \(X_s\). From the considerations in [EZ] we may infer that
\[
\nu_Z \in \Gamma(S, \mathcal{G}_x, \nu) \text{ defines a normal function.}
\]
More precisely, \(\nu_Z\) is defined over \(S^*\) and extends as indicated to give a normal function defined over \(S\).

Remark In the two curve degenerations
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where the RHS is defined using the Leray spectral sequence.

Due to the failure of Jacobi inversion in higher codimension, the above theorem has been of limited use in studying cycles. It does, however, suggest looking into the possible singular behaviour of \( \nu \) in general situations as suggested by the particular example (ii) above. For example, setting \( Z_\eta = Z \cdot X_\eta \) for \( \eta \) a generic point of \( S \) and defining

\[
Z^n(X)_{\text{hom}} = \begin{cases} 
Z \in Z^n(X) : [Z_\eta] = 0 \\
in H^{2n} (X_\eta, Z)
\end{cases}
\]

then, for \( Z \in Z^n(X)_{\text{hom}} \), \( \nu_\eta(s) \) is defined on \( S^* \) and we may ask

*What happens to \( \nu_\eta(s) \) as \( s \to s_0 \not\in D? \)

This leads to the notion of *admissible normal functions*. Denoting by \( \Delta \) a disk in \( C \), in a neighborhood of \( U \) of \( s_0 \in D \) with

\[
U \cap S^* =: U^* \cong (\Delta^*)^k \times \Delta^l
\]

we choose a local lifting \( \bar{\nu} \) of \( \nu \) in a small neighborhood of \( s \) and analytically continue \( \bar{\nu} \) to a multi-valued section of \( \mathcal{H}^n \) over \( U^* \). Denoting by \( (T_1 - I) \)

analytic continuation around local branch \( D_i \) of \( D \), we have

\[
(T_1 - I) \bar{\nu} \in \mathcal{H}_{Z,s}.
\]

**Definition** \( \nu \) is an *admissible normal function* (ANF) if

(i) \( \bar{\nu} \) has moderate — i.e., logarithmic — growth along \( D \); and

(ii) there exist non-zero integers \( m_i \) such that

\[
m_i (T_1 - I) \bar{\nu} \in \text{Ker}(T_1 - I)\).
\]

---

\( ^6 \)In example (ii) above, the group of components of the Néron model is \( \mathbb{Z}/2\mathbb{Z} \) so that \( 2\nu_Z \) extends.
Here, the $\overline{\,}$ is with respect to the polarizing form.

Since $(T_{i} - I)v$ is well-defined modulo $\text{Im}(T_{i} - I)$ and $\text{Im}(T_{i} - I) \subseteq \text{Ker}(T_{i} - I)^{\perp}$, condition (ii) is well-defined. For $\dim S = 1$, (ii) is automatically satisfied (cf. [GGK1]), and consequently if (i) is satisfied then there is an integer $m$ such that $m\nu$ is a normal function in the usual sense.

In general we have an exact sequence

$$ 0 \to \mathcal{F}_{\nu} \to \mathcal{F}_{\nu} \to \mathcal{G} \to 0 $$

where $\mathcal{F}_{\nu}$ is the sheaf of admissible normal functions and $\mathcal{G}$ is a sheaf of abelian groups such that setting $\mathcal{G}_Q = \mathcal{G} \otimes \mathbb{Q}$, $\mathcal{G}_Q$ is supported in codimension \( \geq 2 \).

**Definition** The singular set $\text{sing} \nu$ is defined to be the support of

$$ \nu|_Q \in \Gamma(S, \mathcal{G}_Q). $$

**Remark** Without assuming that the degeneracy locus $D \subset S$ of the VHS is a reduced divisor with local normal crossings, one may still define ANF's and their singular set will be defined to be

$$ \left\{ s_0 \in D : \text{no multiple } m\nu \text{ has a single-valued lifting} \right\} \subset \text{a punctured neighborhood } U^* = U \cap S^* \text{ of } s_0. $$

It then seems to be a likely result (cf. [BFNP] and [dCM2]) that, under dominant maps

$$ \tilde{\mathcal{G}} \to S, $$

ANF's pull back to ANF's and for $\nu$ an ANF defined on $S$

$$ \text{sing} (\pi^{-1}(\nu)) = \pi^{-1}(\text{sing} \nu); $$

i.e., singularities do not disappear under branched coverings and blow-ups.\(^8\)

**Example 2.1 (continued)** We let $H^{2n}(X, Z)_{\text{hom}}$ be the classes $\xi$ such that, modulo torsion

$$ \xi|_0 = 0 \text{ in } H^{2n}(X, Z). $$

Clearly we have

$$ 0 \to H^{2n}(X, Z)_{\text{hom}} \to H^{2n}(X, Z)_{\text{hom}}, $$

and the cokernel may be thought of as "classes supported over the discriminant locus $D".\(^8\)

**Theorem 2.2** A class $\xi \in H^{2n}(X)_{\text{hom}}$ gives an admissible normal function $\nu_{\xi}$.\(^8\)

**Example 2.3** Suppose given $(X^2, L, \zeta_0)$ where $X_0$ is a smooth projective variety, $L \to X_0$ is a very ample line bundle and $\zeta_0 \in H^{2n}(X_0)_{\text{prim}}$ is a primitive Hodge class. Setting $S = |L|$ we have the universal family of hyperplane sections

$$ X \subset X_0 \times S \to S $$

and $\zeta_0$ pulls back to $\xi \in H^{2n}(X)_{\text{hom}}$.\(^8\)

**Theorem 2.4** ([BFNP] and [dCM2]) We have

$$ \text{sing} \nu_{\xi} = \left\{ s_0 \in S : \zeta_{s_0} \neq 0 \in H^{2n}(X_{s_0})/H^{2n}(X)_{\text{hom}} \right\}. $$

Here, the singularities of $X_{s_0}$ are arbitrary, including multiple components; $H^{2n}(X_{s_0})$ is defined as the Hodge classes in

$$ \text{Gr}_{2n}(H^{2n}(X_{s_0}^{\text{red}}, \mathbb{Q})), $$

using the mixed Hodge structure on $H^{2n}(X_{s_0}^{\text{red}}, \mathbb{Q})$.\(^8\)

Assuming inductively the Hodge conjecture (HC) in codimensions $\leq n - 1$, the above theorem has the following

**Corollary 2.5** The HC is equivalent to $\text{sing} \nu_{\xi} \neq 0$ for $L \gg 0$.\(^8\)

We may think of $\text{sing} \nu_{\xi}$ as the high degree hypersurfaces that support a cycle $Z$ with $[Z] = \xi$.\(^8\)

The proof of Theorem 2.4 uses the decomposition theorem (DT) for the special situation of the universal hyperplane section, together with a new ingredient — the "relative, weak Lefschetz theorem."\(^8\)

Footnote added in proof: It has been pointed out to me by Gregory Pearlstein that there are examples due to M. Saito and to Najmuddin Fakhruddin that this statement is false without further assumptions, perhaps to the effect that the parameters in $S$ must be "essential".\(^8\)

\(^8\)Footnote added in proof: It has been pointed out to me by Gregory Pearlstein that there are examples due to M. Saito and to Najmuddin Fakhruddin that this statement is false without further assumptions, perhaps to the effect that the parameters in $S$ must be "essential".
We remark that the argument of [BFNP] and [dCM2] seems to work if we only assume that
\[ \zeta \in H^{n,n}(X)_{\text{hom}} ; \]
i.e., \( \zeta \) need not be a Hodge class, and in fact could be transcendental. The conclusion is then that
\[ \zeta_{s_0} \neq 0 \text{ in } H^{2n}(X_{s_0})/H^{2n}(X)_{\text{hom}} . \tag{2.6} \]

If \( \zeta \) is in fact transcendental, and if \( X_{s_0} \) is nodal, then
\[ H^{2n}(X_{s_0})/H^{2n}(X)_{\text{hom}} = H^{n,n}(X_{s_0})/H^{n,n}(X)_{\text{hom}} \]
from which we infer that (2.6) does not hold. In fact, one may define the singular locus of any class \( \zeta \in H^{n,n}(X)_{\text{hom}} \); we suspect that \( \text{sing } \zeta = \emptyset \) unless \( \zeta \) is a Hodge class.

The construction of \( \nu_\zeta \) requires that \( \zeta \) be in \( H^{2n}(X, \mathbb{Z}) \) as well as being in \( H^{n,n}(X)_{\text{hom}} \). This suggests that any argument that
\[ \text{sing } \nu_\zeta \neq \emptyset \]
will need to make use of the existence of \( \nu_\zeta \) as a mapping from \( S \) to some space. This will be further discussed in §§4 and 5 below.

As a general comment, in the absence of being able to construct cycles one may consider what the implications of the HC are. We here mention three:

(i) the HC has geometric consequences;
(ii) the HC has arithmetic consequences; \(^{10}\)
(iii) the HC may have topological consequences.

The above corollary gives one instance of (i). Below we will discuss (iii), which may also be related to aspects of (ii).

3 Néron Models and Graphing ANF’s

Classically, working analytically and denoting by \( S = \Delta \) the disc with origin \( s_0 \) and \( S^* = \Delta^* = \Delta \setminus \{ s_0 \} \) the punctured disc, given a family \( J \to S^* \)

\(^{10}\)These include the absolute Hodge condition, as formulated by Grothendieck and Deligne [cf. [D-M]], and the field of definition of Noether-Lefschetz loci ([V]). These are further discussed in Section 5 below.

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of principally polarized abelian varieties with unipotent monodromy \( T \); \(^{11}\) there exists a diagram of analytic fibre spaces of complex Lie groups
\[ \begin{array}{c}
J \subset \tilde{J}_e \\
\downarrow \quad \downarrow \\
S^* \subset S
\end{array} \tag{3.1} \]
with a number of properties, \(^{12}\) including

- the diagram (3.1) is canonical and graphs admissible normal functions; \(^{13}\)
- there is an exact sequence
\[ 0 \to J_e \to \tilde{J}_e \to G \to 0 \]
where \( G \) is the finite group
\[ G = (\text{Ker}(T - I))_k / \text{Im}(T_k - I) \]
sitting over the origin \( s_0; \(^{14}\)
- the fibre \( \tilde{J}_{e,s_0} \) is an extension of its identity component \( J_{e,s_0} \), a semi-abelian variety, by \( G \).

Two extensions of this construction have been given.

(i) Néron models of intermediate Jacobians over 1-dimensional base spaces. \(^{15}\)

Here the same properties as above extend, with one significant difference:

\(^{11}\)This restriction is not essential, but it simplifies the exposition.
\(^{12}\)This is the classical Néron model — cf. [Ko] for the elliptic curve case and [BLR] for the general case.
\(^{13}\)This means that ANF’s are given by holomorphic sections of \( \tilde{J}_e \to S \); we may express this by saying that
\[ \tilde{J}_{e,v} \cong \mathcal{O}(\tilde{J}_e)_{\nu} , \]
where we note that the condition on \( \nu \mathcal{O} \) is automatically satisfied when \( n = 1 \). It is noteworthy that no compactification of the fibre \( \tilde{J}_{e,s_0} \) over the origin is required to fill in the value \( \nu(s_0) = \lim_{s \to s_0} \nu(s) \).
\(^{14}\)The subscript \( "T \) on \( (\text{Ker}(T - I))_k \) means that we take the integral elements in the vector space \( \text{Ker}(T - I) \).
\(^{15}\)This is joint work with Mark Green and Matt Kerr, cf. [GGK1]. The Hodge theoretic aspects have been done independently by Patrick Brosnan and Gregory Pearlstein [BPI].
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We then have

\[ AJ_{\mathcal{X}_s}(Z_s) = \nu_2(s) \text{ for all } s \text{ and } \]

in a precise sense,

\[ \lim_{s \to s_0} AJ_{\mathcal{X}_s}(Z_s) = AJ_{\mathcal{X}_{s_0}}(Z_{s_0}). \]

Moreover, the limit (3.2) has the following properties, which are present but perhaps not so "visible" in the classical case.

- the limiting MHS induces a filtration on \( J_{s,s_0} \) whose graded pieces are composed of compact, complex analytic tori and \( \mathbb{C}^* \)'s — no vector groups occur;
- the limit (3.2) is a mapping of filtered groups whose graded pieces are constructed from regulator maps defined on sub-quotients of higher Chow groups.

As a closing remark, we note that the group \( G_{s_0} \) of components of the Néron model has a monodromy weight filtration — cf. the examples in [GGK2]. In general, it is my view that \( \lim AJ_{\mathcal{X}_s} \) frequently has a richer geometric structure than \( AJ_{\mathcal{X}_s} \) for a generic \( s \), and in some ways even more than \( AJ_{\mathcal{X}_s} \) in the classical \( n = 1 \) case.

(ii) Néron models of PPAV's over higher dimensional base spaces.

In the local analytic situation where \( S^* \cong (\Delta)^d \times \Delta^k \), we assume given a family of PPAV's \( J \to S^* \) whose monodromies are unipotent. Then there is a diagram

\[
\begin{array}{ccc}
J & \subset & \tilde{J}_s \\
\downarrow & & \downarrow \\
S^* & \subset & S
\end{array}
\]

where
\( \tilde{J}_e \rightarrow S \) is a non-Hausdorff fibre space of complex Lie groups whose identity components are semi-abelian varieties of the same dimension as the generic fibre;

- (3.3) graphs ANF's, which we express as

\[ \tilde{\delta}_e \cong \mathcal{O}_S(\tilde{J}_e) \]

(here we omit the \( \nabla \)-subscripts since the transversality condition is automatic in this case);

- If the local monodromies are denoted by \( T_i \), then the group \( G_{S_0} \) of components of \( \tilde{J}_{e, S_0} \) has a description

\[ G_{S_0} \cong \left\{ \begin{array}{l}
H^1(\mathcal{B}_{S_0}^*) \text{ where } \mathcal{B}_{S_0}^* \text{ is a Koszul-type complex defined over } \mathbb{Z} \text{ and constructed from the } T_i - I \text{ and the polarizing form} \\
\text{from the } T_i - I \text{ and the polarizing form} \end{array} \right\} \]

In general, when \( \dim S \geq 2 \) the group \( G_{S_0} \) is a finitely generated but in general not a finite abelian group.

**Example 3.4** Referring to the curve degenerations pictured in §2, so that \( S \cong \Delta^3 \) is the parameter space obtained from smoothing the nodes independently, for the group \( G_s \) of components of \( \tilde{J}_{e,S} \) we have

\[ G_s = \begin{cases} 
0 & s \neq (0,0,0) \\
\mathbb{Z} & s = (0,0,0) 
\end{cases} \]

The ANF in (ii) has value "1" at the origin.

As an example of "doing geometry relative to ANF's", let

\[ \mathcal{P} \rightarrow J \times_S J \]

be the Poincaré line bundle. Then:

- for an ANF \( \nu, \nu^*(\mathcal{P}) \) initially defined over \( S^* \) has a canonical extension to \( S \), even though \( \mathcal{P} \) itself does not canonically extend to \( \tilde{J}_e \times_S \tilde{J}_e \)

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\(^{21}\)This non-separateness causes no difficulty in doing geometry relative to ANF's; see below.

\(^{22}\)This group \( H^1(\mathcal{B}_{S_0}^*) \otimes_{\mathbb{Z}} \mathbb{Q} \) appears in [CKS] and is constructed from a Koszul-type complex constructed from the logarithms \( T_i = \log T_i \) of monodromy.

\(^{23}\)cf. [Y], where the precise meaning of canonical is explained.

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**4 Universal Realization of the Singular Locus of an ANF**

What is obviously needed is to "amalgamate" the two constructions to have a Néron model for intermediate Jacobians over higher dimensional base spaces. This has been done for \( n = 2 \) in the nodal case (cf. [GG2]), which in some ways resembles the classical case since one has

\[ \begin{cases} 
(T-I)^2 = 0 \\
\text{Gr}_4 \text{ of the LMHS is of Hodge-Tate type.} 
\end{cases} \]

However, in other ways it is quite different in that the infinitesimal period relation enters in an essential way in dimension counts (loc. cit.).

We will explain "what" is desired and "why" in the classical weight one case, where the required ingredients may be in place. Then we shall briefly discuss the higher weight case.

We denote as usual by \( \mathcal{A}_g \) the moduli space for principally polarized abelian varieties of dimension \( g \), realized as the quotient by the discrete subgroup \( \Gamma = G_Z \) of the Siegel upper-half-space

\[ \mathcal{H}_g = G_R / K \]

where \( G_R = Sp(g, \mathbb{R}) \) and \( K = \mathcal{U}(g) \). The Lie algebra \( \mathfrak{g}_R \) is defined over \( \mathbb{Q} \), and for each fan \( \Sigma \) there is defined a compactification

\[ \mathcal{A}_g \subset \overline{\mathcal{A}}_{g,\Sigma} \]

of \( \mathcal{A}_g \) (cf. [AMRT]). Recall that for general compactifications of quotients of Hermitian symmetric spaces, \( \Sigma \) is a union of rational, nilpotent cones \( \mathfrak{n}_e \) which satisfy certain incidence relations and where \( \mathfrak{Ad} \) permutes the \( \mathfrak{n}_e \)'s. In the case at hand each

\[ \mathfrak{n}_e = \text{span} \{ N_1, \ldots, N_r \} \]

where the \( N_i \) are integral and satisfy

\[ \begin{cases} 
[N_1, N_2] = 0 \\
N_1^2 = 0 
\end{cases} \]

\(^{23}\)This section is speculation on the potential outcome of work in progress.
Associated to $n_\sigma$ is a set of nilpotent orbits whose limiting mixed Hodge structures, modulo reparametrizations, may be thought of as the boundary component $B_\sigma \subset \overline{A}_{g,\Sigma}$ corresponding to $n_\sigma$ (cf. the discussion in Cattani's article in [G2] and in [GGK2]).

What is desired is a universal Néron model

$$\tilde{J}_{g,\Sigma} \to \overline{A}_{g,\Sigma}$$

together with a subvariety $\Xi \subset \tilde{J}_{g,\Sigma}$ such that, given an ANF $\nu \in \Gamma(S,\tilde{A}_{g,\Sigma})$, there is a diagram

$$\begin{array}{c}
S \\
\downarrow \tau \\
\overline{A}_{g,\Sigma}
\end{array} \xrightarrow{\nu} \tilde{J}_{g,\Sigma} \supset \Xi$$

with

$$\text{sing } \nu = \nu^{-1}(\Xi). \quad (4.2)$$

Here we assume that the locus $D \subset S$ of degenerate Hodge structures is locally a NCD of the form

$$D \cap U = D_1 \cup \ldots \cup D_r.$$ 

Then $U$ will map under the Torelli map $\tau$ to a neighborhood of $\tau(s_0) \in B_\sigma$.

Then $U$ is the nilpotent cone spanned by $N_i = \log T_i$ and $T_i$ is the local monodromy, assumed unipotent, around $D_i$. Intuitively

$$\tau(D \cap U) = \left\{ \begin{array}{l} \text{limiting MHS's taken} \\ \text{along discs } \Delta \subset U \\ \text{with } \Delta \cap D = \{s_0\} \end{array} \right. \quad (4.1)$$

With the caveat that we are ignoring important stack considerations, the constructions in Young's thesis serve to define $\tilde{J}_{g,\Sigma}$ as a set, so that at this level we have (4.1). Denoting by

$$\Xi = \left\{ \begin{array}{l} \text{union of non-torsion components} \\ \text{of fibres of } \tilde{J}_{g,\Sigma} \to \overline{A}_{g,\Sigma} \end{array} \right. \quad (4.2)$$

we then have, again at the set-theoretic level, (4.2). Defining the appropriate structures so that we properly have (4.1) and (4.2) is what is desired.

\section*{Singularities of Admissible Normal Functions}

As to "why" (4.1) and (4.2) are desired, we give two preliminary reasons — both of which are existence results. Afterwards we will explain why, and possibly how, these must be modified. The two reasons are:

\begin{align*}
(i) \quad \nu^{-1}(\Xi) \neq 0 & \implies \text{codim}_{\Sigma}(\text{sing } \nu) \leq \text{codim}_{\overline{A}_{g,\Sigma}}(\Xi) = d \quad (4.3) \\
(ii) \quad \nu^*(\Xi) \neq 0 & \implies \nu^{-1}(\Xi) \neq \emptyset.
\end{align*}

Below we shall see that (ii) is questionable, but if we let $(X,\zeta)$ vary in moduli over the Noether-Lefschetz locus then the extension of (ii) becomes at least plausible and would give an existence result.

Concerning (i), since codimension decreases under blowing up we cannot immediately conclude that

$$\text{dim}_{\Sigma}(\text{sing } \nu) \geq \text{dim } S - d. \quad (4.4)$$

We can, however, conclude this if — e.g. — we know that for a general point $s_0 \in \text{sing } \nu$

$$\tau_* : T_{s_0}S \to T_{\tau(s_0)}\overline{A}_{g,\Sigma} \text{ is injective.} \quad (4.5)$$

Suppose we are in the geometric case of Example 2.1 and $X_{s_0}$ is a nodal curve of genus $g$. Then $\tau$ is the composition of

$$S \xrightarrow{\mu} \overline{M}_g \xrightarrow{\tau} \overline{A}_{g,\Sigma}. \quad (4.6)$$

It is not hard to see that $\mu_*$ is injective on a general $T_{s_0}S$, so the question is one of $\sigma_*$. In fact, to have the conclusion (4.4) it is enough to have that

$$\tau_* : T_{s_0}(\text{sing } \nu) \to T_{\tau(s_0)}\overline{A}_{g,\Sigma} \text{ is injective.} \quad (4.7)$$

is injective. We factor as in (4.6) and denote by $A \subset \overline{M}_g$ the boundary component of stable, nodal curves of the same type as $X_{s_0}$ and by $B \subset \overline{A}_{g,\Sigma}$ the boundary component of LMHS's of the type of $\lim_{s \to s_0} H^1(X_s)$. The issue is then the injectivity of

$$\sigma_* : T_{s_0}A \to T_{\tau(s_0)}B. \quad (4.7)$$

In the case of interest where Example 2.1 arises as in Example 2.3, as explained in [GG1] the injectivity of (4.7) in the situation of most interest is equivalent to the following geometric problem:

Let $X_0$ be a smooth algebraic surface, $C_0 \subset X_0$ a smooth curve, and $L \to X_0$ a line bundle that is sufficiently ample relative to $X_0$ and $C_0$. In
particular, \( L \otimes [-C_0] \) should be very ample. The linear system \(|L|\) may be thought of as consisting of reducible curves

\[
C + C_0 \tag{4.8}
\]

where a general \( C \) is smooth and meets \( C_0 \) transversely. Each such curve has a mixed Hodge structure, which is part of the LMHS of a general smoothing of (4.8). As shown to me by Mark Green, we then have

\[
(4.9) \text{The differential (4.7) is injective if the differential of the map} \quad C \rightarrow \text{MHS on } H^1(C + C_0)
\]

is injective.

Mark has proved that this is the case if \( L \gg 0 \).

We do not know if the analogue of (4.9) holds when \( n \geq 2 \). As discussed in [GG2], dimension counts must take into account the infinitesimal period relation; the issue would seem to be an interesting one.

Turning to (ii), we first observe that it is unlikely to be true as stated. To begin with, one must be more precise and consider the irreducible components of \( \Xi \). Let us consider one, still denoted by \( \Xi \), that lies over a boundary component \( B \) consisting of all LMHS corresponding to a nilpotent cone \( n = \text{span}(N_1, \ldots, N_l) \) where the \( T_i = I + N_i \) are given by Picard-Lefschetz transformations

\[
T_i(\gamma) = \gamma + (\gamma, \delta_i)\delta_i
\]

where \( \delta_1, \ldots, \delta_l \) are primitive elements of \( \mathcal{H}(n) \), with one relation

\[
\delta_1 + \cdots + \delta_l = 0
\]

among them. Then locally

\[
\nu(S) \cap \Xi \text{ lies one-to-one over } \tau(S) \cap B.
\]

Although this certainly does not imply that

\[
\tau^*(\langle B \rangle) = \nu^*(\langle \Xi \rangle),
\]

what is definitely not true is

\[
\tau^*(\langle B \rangle) \neq 0. \tag{4.10}
\]

The reason is this: If (4.10) holds for \( X_0 \), then it would hold for any small deformation \( X'_0 \) of \( X_0 \). This would then imply that \( X'_0 \) contains reducible sections of \( L \) of the type (4.8), which means that the Hodge class \( \zeta_0 \) deforms to \( X_0 \). But in general the Hodge class will only deform over a proper subvariety — the Noether-Lefschetz locus

\[
M_\zeta \subset M
\]

of the moduli space \( M \) (assumed to exist) of \( X_0 \).

This then suggests that, in Example 2.3, one consider normal functions associated to Hodge classes not only for a fixed \( (X_0, \zeta_0) \), but rather one should allow this data to vary over \( M_\zeta \). We denote by \( (X_t, \zeta_t, L_t)_{t \in M_\zeta} \) this variation, together with that of the very ample line bundle. We set \( S_t = |L_t| \) and

\[
S = \bigcup_{t \in M_\zeta} S_t.
\]

Then there are a Torelli map \( \tau \) and normal function \( \nu_\zeta \) defined as in the diagram

\[
\begin{array}{ccc}
\nu_\zeta & \downarrow & 0 \\
\tau & \Downarrow & \Xi \\
S & \to & 1\setminus\Xi.
\end{array}
\]

In [GG2] it is proved that, subject to a technical assumption that we suspect is not essential,

\[
\tau^*([\xi]) \neq 0. \tag{4.11}
\]

This is proved using the Lefschetz (1,1) theorem — i.e. the HC for \( n = 1 \). We also suspect that if there were an independent proof of the purely topological result (4.11), then the Lefschetz (1,1) theorem might follow.

The analogue of (4.11) for \( n \geq 2 \) is not known.

**Conclusion** In the absence of being able to construct cycles in higher codimension, it seems of interest to examine consequences of the HC. The topological statement (4.11) could be one such.

## 5 Some General Observations and a Question

Above there has been discussion of some geometric consequences of the HC. There are several others, some of which have been proved, including

- the algebraicity of the Noether-Lefschetz loci (cf. [CDK]);
the algebraicity of the zero locus of a normal function, done for dim $S = 1$ in [BP1] and more recently by them for dim $S$ arbitrary and where $D$ locally has one branch (private communication, and also by M. Saito (private communication)). Further recent work is in [BP2] and [Sch].

There are also arithmetic-geometric consequences of the HC, two of which were mentioned in footnote (7). Roughly speaking, the second involves the following considerations: If we have a family of algebraic varieties

$$f: X \to S$$

where $S$ (assumed irreducible) is defined over a number field — we simply write "$X$ is defined over $\overline{Q}$" — then as discussed in [V], one expects the Noether-Lefschetz loci

$$S_\zeta \subset S$$

where a class $\zeta \in H^p(X_s)$, $s$ a point of $S$, remains of Hodge class to also be defined over $\overline{Q}$.

The reason is this: Any algebraic subvariety

$$V \subset S$$

may be thought of as given by an inclusion of abstract varieties

$$V(k) \subset S(k)$$

defined over an algebraically closed field $k$ of characteristic zero, together with an embedding $\sigma: k \hookrightarrow \mathbb{C}$ giving rise to (5.1), which we may write as

$$V(k) \otimes_\sigma \mathbb{C} \subset S(k) \otimes_\sigma \mathbb{C}.$$  

Varying $\sigma$ gives the spread of (5.1). Since $S$ is defined over $\overline{Q}$, an irreducible component of the spread may be thought of as a component of the spread

$$V_\zeta \subset S$$

of $V$ in $S$.

Now, enlarging $k$ if necessary, $\zeta$ may be thought of as giving a class

$$\zeta \in H^p(X_s(k), \Omega^p_{X_s(k)})$$

where $X_s$ is defined over $k$ and we are using GAGA to identify analytic sheaf cohomology with its algebraic counterpart. We may take the spread of (5.3), and assuming absolute Hodge, an irreducible component of the parameter space of the spread will map to $\overline{S}_\zeta$; i.e. we have

$$\overline{S}_\zeta \subset S_\zeta,$$

or $S_\zeta$ is defined over $\overline{Q}$.

The above is only heuristic — an invitation to [V]. The point, by no means new here, is to illustrate that arithmetic and geometric considerations should be considered together, not separately, in the study of cycles.

For example, one at least philosophical difficulty in the use of the Abel-Jacobi map and normal functions to study cycles is that the Abel-Jacobi map loses too much geometry. In [GG] it is shown that if one "enriches" the Abel-Jacobi map by taking spreads — in effect considering the well-defined part of all the Abel-Jacobi maps arising by varying the embeddings $\sigma: k \hookrightarrow \mathbb{C}$, then assuming the (generalized) Hodge conjecture and one of the Bloch-Belinson conjectures, rational equivalence is captured (up to torsion). This at least suggests that one should consider adding spread considerations to the study of normal functions and their singularities. One possible outcome might be that, assuming absolute Hodge, the study and construction of algebraic cycles is in principle reduced to where everything is defined over $\overline{Q}$ (cf. [V] for further discussion). At that point one might seek to combine classical and $p$-adic methods.

A further comment concerns the consequence Corollary 2.5 of the HC. There the operative phrase is "$L \gg 0$". What exactly does this mean? Suppose we are given a projective embedding $X$ with $\mathcal{O}_{X_0}(1)$ having the usual meaning. Taking $L = \mathcal{O}_{X_0}(m)$, in [GG1] we observed that

In general there cannot be a uniform bound on $m$ in order to have $\text{sing } \nu_\zeta \neq 0$.

Here uniform bound means "as $X_0$ varies in moduli". Of course, one may reasonably expect a bound as $(X, \zeta)$ varies over the subvariety $\mathcal{M}_\zeta$ of its moduli space. In [GG1] it is noted that in the $n = 1$ case when $X_0$ is a surface, in order to be sure to have $\text{sing } \nu_\zeta \neq 0$ we must have

$$m \geq c|\zeta|^2$$

for some constant $c > 0$.

**Question** In general does the HC imply the existence of a lower bound (5.4)?

For surfaces, the estimate (5.4) is sufficient in that given $X_0$ there is a $C > 0$ such that

$$\text{sing } \nu_\zeta \neq 0 \text{ for } m \geq C|\zeta|^2.$$
One may of course ask, assuming the HC, the same holds in general.

Bounds such as (5.4) may be thought of as providing an effective HC.

As a reprise from these speculations/generalities, I would like to describe a concrete, geometric question, which will help frame another general consideration. Assume that we have the construction (3.3) for families of intermediate Jacobians as well as for families of PPAV’s. Denote by ANF(S) the group of admissible normal functions. Then there is a diagram

\[
\begin{array}{ccc}
\text{ANF}(S) \times H^*(\tilde{J}_e) & \longrightarrow & H^*(S) \\
\psi & \\ (\nu, \alpha) & \longrightarrow & \nu^*(\alpha)
\end{array}
\]  
(5.5)

**Question**  
*What are the algebraic properties in the first factor of this map?*

Specifically, denoting by $e$ the zero normal function we set

\[\Delta(\nu, \nu') = (\nu + \nu')^* - \nu^* - \nu'^* + e^*\]

and think of $\Delta(\nu, \nu')$ as the derivation from linearity in the first factor of (5.5).**

**Example**  
Let $f : X \to S$ be a minimal elliptic surface. If there is a fibre of type $I_3$ in Kodaira’s notation, and if we have two sections $\nu$ and $\nu'$ that meet that fibre as pictured

\[
\begin{array}{c}
Y_1 \\
\nu \bullet \\
\nu + \nu' \bullet
\end{array}
\]

then all

\[\Delta(\nu, \nu')[Y_i] \neq 0 ,

but it may be shown that

\[\Delta(\nu, \nu') = 0

\]

for ordinary normal functions (their value at singular points is in the identity component).** Denoting by NP(S) the group of ordinary normal functions, this suggests the

**Refined Question** *(i) Is $\Delta(\nu, \nu') = 0$ on $\text{NP}(S) \times \text{NP}(S)$? (ii) If so, then to what extent does the induced map

\[
\bar{\Delta} : \frac{\text{ANF}(S) \times \text{ANF}(S)}{\text{NP}(S) \times \text{NP}(S)} \to \text{Hom}(H^*(\tilde{J}_e), H^*(S))
\]

**capture the singularities of admissible normal functions?**

To conclude, we want to pose an issue that combines this question with the preceding question centering around the lower bound (5.4). This discussion will be heuristic; we will pass over the significant technical issues that would be necessary to address to make it precise.

The idea is, as in proofs of the nullstellensatz, not try to initially deal with bounds of the type (5.4). Keeping the notation $(X_0, O_{X_0}(1))$ as above, we may assume $O_{X_0}(1)$ is sufficiently ample to have

\[H^0(0_{X_0}(k)) \otimes H^0(0_{X_0}(l)) \to H^0(0_{X_0}(k + l))

\]

for all $k, l$. We set

\[S_k = \text{PH}^0(0_{X_0}(k));

there are then spanning maps**

\[\mu_k,l : S_k \times S_l \to S_{k+l}

\]

given by

\[(X_s, X_t) \to X_s + X_t

\]

where $s \in S_k, t \in S_l$.

For the Néron models $\tilde{J}_{k,c} \to S_k$, in first approximation we have

\[
\tilde{J}_{k+1,c} \left|_{\text{Image } \mu_k,l} \right. \xrightarrow{\pi_{k+1}} \tilde{J}_{k,c} \times \tilde{J}_{l,c} \to 0
\]

\[
\downarrow
\]

\[
\text{Image } \mu_k,l \quad \longrightarrow \quad S_k \times S_l
\]

\[
(5.6)
\]

**The point is that the connecting map

\[H^0(S, \mathcal{O}_S) \to H^1(S, \mathcal{O}_S^2)

\]

is a homomorphism of groups.

**These issues center around the extent to which we need to use SSR to have unipotent monodromies and to have the discriminant loci with local normal crossings, which up until now are needed to construct the Néron models $\tilde{J}_{k,c} \to S_k$.

**A spanning map is one whose image linearly spans.
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The reason is that for general \((s,t) \in S_k \times S_l\) the identity component \(J_k(X_s + X_l)\) of the generalized intermediate Jacobian will map onto \(J_k(X_s) \times J_l(X_l)\), reflecting the fact that the MHS on \(H^{2n-2}(X_s \cup X_l)\) is an extension of \(H^{2n-2}(X_s) \oplus H^{2n-2}(X_l)\). Of course if \(X_s, X_l\) are singular or fail to meet transversely the situation is more complicated but, as suggested above, we will not address this.

Given a Hodge class \(\zeta \in H^*(X_0)_{\text{prim}}\), for each \(k\) we have

\[
S_k \overset{\nu_k}{\rightarrow} \tilde{J}_{k,k}\,.
\]

It is reasonable to guess that the sequence of ANF's \(\nu_k,\zeta\) are compatible in the sense that

\[
\pi_{k,l} \left( \nu_k,\zeta \big|_{\text{image } \mu_{k,l}} \right) = \nu_k \times \nu_l\,.
\]

(5.7)

We will write this more suggestively as

\[
\nu_k \big|_{\text{image } \mu_{k,l}} \rightarrow \nu_k + \nu_l\,.
\]

where it is understood that the LHS is restricted to the image of \(\mu_{k,l}\).

A perhaps more subtle issue is the relation between \(\text{sing } \nu_k\) and \(\text{sing } \nu_l\). Again in first approximation we would guess that

\[
\text{sing } \nu_k = \big(\text{sing } \nu_k \times S_l\big) \cup \big(S_k \times \text{sing } \nu_l\big)\,.
\]

Geometrically, this says that "new" Hodge classes on \(X_k \cup X_l\) can only come from those on either \(X_k\) or \(X_l\). If correct, this leads to the reasonable conclusion that singularities of ANF's cannot be produced from Segre images of lower degree hypersurface sections.

Next, we have the maps

\[
\varphi_k : H^*(X_0)_{\text{prim}} \rightarrow \text{Hom} \left( H^*(\tilde{J}_{k,k}), H^*(S_k) \right)\,.
\]

(5.8)

There are possible compatibility relations among

\[
\varphi_{k+l} \quad \text{and} \quad \varphi_k, \varphi_l
\]

as suggested by (5.6). Moreover, as suggested above the failure of \(\varphi_k\) to be a homomorphism of groups is related to the presence of singularities of ANF's.

Two further extensions of the above construction are suggested by the discussion in Section 4. For this we fix a subgroup

\[
\Lambda \subset H^*(X_0)_{\text{prim}}
\]

and denote by \(\mathcal{M}_\Lambda \subset \mathcal{M}\) the Noether-Lefschetz locus of all pairs \((X_0, \Lambda)\) where \(X_0\) is a deformation of \(X_0\) and \(\Lambda \subset H^*(X_0)_{\text{prim}}\) corresponds to \(\Lambda\).

We then denote by \(S_{k,\Lambda}, J\) the above constructions of \(S_{k,\Lambda}, J\) varying over \(\mathcal{M}_\Lambda\).

Thus

\[
S \rightarrow \mathcal{M}_\Lambda \text{ is the projectification of a vector bundle } \mathcal{E}_\Lambda \text{ over } \mathcal{M}_\Lambda
\]

and thus

\[
H^*(S_{k,\Lambda}) = H^*(\mathcal{M}_\Lambda) \langle \xi_k \rangle / \left\{ \text{relations given the Chern classes of } \mathcal{E}_\Lambda \right\}
\]

where \(\xi_k \in H^2(S_{k,\Lambda})\). There are again compatibility relations among these constructions as \(k\) varies.

Finally, we let

\[
\Gamma_k \backslash D_{k,\Sigma_k}
\]

be the Kato-Ura spaces associated to relevant polarized Hodge structures. We also denote by

\[
\tilde{J}_{k,\Sigma_k} \rightarrow \Gamma_k \backslash D_{k,\Sigma_k}
\]

(5.9)

the universal Néron model. For each \(k, l\) one may imagine a boundary component

\[
B_{k,l} \subset \Gamma_{k+l} \backslash D_{k+l,\Sigma_{k+l}}
\]

consisting of LMHS's of the type that arise by smoothing \(X_k + X_l\) in \(\tilde{S}_{k+l} = [\mathcal{O}_{X_k}(k+l)]\), together with a surjective map

\[
B_{k,l} \rightarrow \left( \Gamma_k \backslash D_k \right) \times \left( \Gamma_l \backslash D_l \right)
\]

(5.10)

associating to a MHS its \(G_{2n-1}\) piece. The maps (5.10), together with the maps lying over them under (5.9), express the inter-relations among the universal Néron models.

Putting everything together, given \(\Lambda\) as above we have the family of ANF's \(\nu_{\Lambda,k}\) given by the \(\nu_{\Lambda,k}\)'s

\[
S_{k,\Lambda} \overset{\nu_{\Lambda,k}}{\rightarrow} \tilde{J}_{\Lambda,k,\Sigma_k}
\]

which induce maps

\[
\nu_{\Lambda,k}^* : \Lambda \rightarrow \text{Hom} H^*(\tilde{J}_{\Lambda,k,\Sigma_k}, S_{k,\Lambda})
\]

(5.11)

Conclusions and the central question (i) It is reasonable to expect that the deviations of (5.11) from being group homomorphisms may reflect the
singularities of the ANP $\nu_{A,k}$. (ii) The maps $\nu_{A,k}$ are related for different $k$ by the process described above. (iii) Let $\Xi_k \subset \tilde{J}_{A,k,\Sigma_k}$ be the non-torsion components of the universal Néron model. Then
\[ \text{sing } \nu_{A,k} = \nu_{A,k}^{-1}(\Xi_k). \]

(iv) Setting $d_k = \text{codim } \Xi_k$,
\[ \nu_{A,k}(\Xi_k) \in \Lambda \times H^{d_k}(\mathcal{M}_\Lambda|\xi_k)/(\text{relations}) \]
is a polynomial in $\xi_k$ whose coefficients $C(\xi, k)$ are functions of $\xi \in \Lambda$ and $k$ with values in the ring $H^*(\mathcal{M}_\Lambda)$.

Main question What can one say about these coefficients?

For example, what can be said about
\[ C(\xi + \xi', k) - C(\xi, k) - C(\xi', k) + C(e, k)? \]

What are the relations between
\[ C(\xi', k + l) \text{ and } C(\xi, k), C(\xi, l) \]
that arise from the above (and other) geometric constructions?
We note that
\[ \text{some } C(\xi, k) \neq 0 \implies \text{existence theorem}. \]

Interesting two cases might be

(a) $X_0$ is abelian surface with principal polarization and with an additional Hodge class $\xi$.

In this case, $\dim \mathcal{M}_\xi = 2$ and everything needed can be worked out explicitly, and

(b) $X_0 \subset \mathbb{P}^3$ is a quartic surfaces and $\xi$ is the class of a line in $X_0$.

In this case, $\dim \mathcal{M}_\xi = 18$ but much is known about it.

References


