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Singularities of Admissable Normal Functions

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Preface

This is an expanded version of a talk given at the International Colloquium on "Cycles, Motives and Shimura Varieties" held at the Tata Institute of Fundamental Research in January, 2008. In a general sense this paper is about algebraic cycles, and on a personal note I would like to observe that my first paper on cycles [G2] appeared in the volume arising from the International Conference on Algebraic Geometry held at the TIFR in January, 1968 — exactly 40 years ago. Although much has happened in the subject of algebraic cycles — on the arithmetic, Hodge theoretic and formal aspects — the fundamental problem — the *Hodge conjecture* and its generalizations — remains, and it has been joined by another fundamental problem — the *conjectures of Bloch and Beilinson*. These two problems reflect, in some sense, the geometric and arithmetic aspects of cycles. Although there have been some very interesting connections between these perspectives, my own view is that further fundamental progress will require an even deeper fusion between the two, perhaps complemented by new formal constructions.

This paper is largely an informal exposition of joint work with Mark Green, some of which was also joint with Matt Kerr, together with various speculations for which I alone am responsible.

I would like to thank the organizers, especially Vasudevan Srinivas, for their efforts in putting together this excellent colloquium and the TIFR for hosting it.

1 Introduction

In a general sense, we are interested in the *global* study of singularities of Hodge theoretic invariants arising from geometry. The connection with the title of this article is that, for an admissable normal function ν (defined below), its singular set $\operatorname{sing}\nu$ is one such.

By way of context, among the main techniques that have been developed to apply Hodge theory to geometric questions are:

- *Infinitesimal methods* (cf. [G2] and [GMV] for a general introduction and references), which might be thought of as variational at a generic point;
- *Degenerations of Hodge structures* (cf. [CKS] and [Ka] and the references cited therein), which might be though-of as variational at a special point;
- *Hodge theoretic aspects of the decomposition theorem* (cf. [BBD] and [dCM1], [Sa1]).

Of course there are others — curvature properties/stability of the Hodge bundles, global PDE techniques, etc. — but the above are the ones most relevant to this paper.

We are interested in geometric questions concerning the existence and structure of Hodge theoretic singularities in a global situation. This work is intended to be an overview of certain questions related to algebraic cycles and normal functions. However, the following "toy problem" illustrates the type of specific questions that have motivated the general considerations:

> *Can there exist a non-isotrivial family of* **smooth** *Calabi-Yau varieties of dimension n and parametrized by a* **complete** *curve?*

The answer is:

$n = 1$ *no*; use the j-function;

$n = 2$ *yes*; e.g., apply suitable semi-stable reduction to a generic pencil of quartic surfaces in $\mathbb{P}^3$;

$n = 3$ *no*, if the Hodge numbers $h^{p,q}$ of a fibre satisfy[1]

$$h^{2,1} < h^{1,1} + 12 .$$

To put the $n = 3$ case in context, we recall [Di] that there exist families of smooth principally polarized abelian varieties of dimension $g \geqq 3$ and parametrized by a complete curve, but they do not exist for $g = 1, 2$. If we consider Jacobians of curves of genus g and think of the genus as the dimension of the image of the Abel-Jacobi map, then for Calabi-Yau varieties the Hodge number $h^{2,1}$ is an upper bound on the dimension of the image of its Abel-Jacobi map, and the above might be thought of as an analogue of the curve result.

The plan of this paper is as follows (notations to be explained in the text):

[1] cf. [GGK3].

- We will give the definitions of an admissable normal function (ANF) $\nu \in \Gamma(S, \tilde{\mathcal{J}}_{e,\nabla})$ and its singular locus $\operatorname{sing}\nu \subset S$.[2]
- Given (X_0^{2n}, L, ζ) where $\zeta \in \mathrm{Hg}^n(X_0)_{\mathrm{prim}}$ and $S = |\check{L}|$ parametrizes the universal family of hyperplane sections $\{X_s\}$, there is an associated ANF ν_ζ and and

$$\operatorname{sing}\nu_\zeta = \left\{ \begin{array}{c} s_0 \in S : H^{2n}(X_{s_0}, \mathbb{Q}) \\ \text{has a "new" Hodge class} \end{array} \right\} .$$

The same result quite possibly holds for S when $S \to |\check{L}|$ parametrizes any semi-stable reduction (SSR) for the universal family of hyperplane sections.

- (work in progress) Admissable normal functions may be "graphed"; i.e., for a principally polarized variation of Hodge structure of weight $2n-1$ where the degenerate Hodge structures occur along a divisor $D \subset S$ with simple normal crossings, there exists

$$\tilde{J}_e \to S$$

such that

$$\tilde{\mathcal{J}}_{e,\nabla} = \mathcal{O}_S(\tilde{J}_e)_\nabla .$$[3]

- (work in progress)[4] There exists

$$\Xi_e \subset \tilde{J}_e$$

such that

$$\operatorname{sing}\nu = \nu^{-1}(\Xi_e)$$

- (work in progress)[5] $\tilde{J}_e$ and Ξ_e exist universally; i.e. there exists a diagram

$$\begin{array}{ccc} & & \tilde{J}_\Sigma \supset \Xi \\ & \overset{\nu}{\nearrow} & \downarrow \\ S & \xrightarrow{\ \tau\ } & \Gamma\backslash D_\Sigma \end{array}$$

[2] For a classical normal function, $\operatorname{sing}\nu = \emptyset$.

[3] This has been done when $\dim S = 1$ (cf. [GGK1]) and in the classical case $n = 1$ (cf. [Y]). For recent work see [Sch].

[4] This has been done when $n = 1, 2$ and when the VHS arises as in footnote 3; in these cases the construction of Ξ has been carried out over the locus in S where the X_{s_0} have at most nodal singularities.

[5] This has also been done under the same conditions as in footnote (3) above.

where τ is the Torelli map and such that

$$\operatorname{sing}\nu = \nu^{-1}(\Xi) .$$

If $\operatorname{codim}^I_{J_\Sigma}(\Xi) = d$, then this has the implication

(i) $$\operatorname{sing}\nu \neq \emptyset \Rightarrow \operatorname{codim}_S(\operatorname{sing}\nu) \leqq d ,$$

which is an existence result, and the additional implication

(ii) $$\nu^*([\Xi]) \neq 0 \Rightarrow \operatorname{sing}\nu \neq \emptyset .$$

The topological condition $\nu^*([\Xi]) \neq 0$ also gives an existence result; at present we have no ideas on how to prove the topological condition in non-classical cases.

2 Admissable Normal Functions and Their Singularities

Notations and assumptions

$(\mathcal{H}_\mathbb{Z}, \mathcal{F}^p, \nabla, S^*)$ is a principally polarized variation of Hodge structure (VHS) of odd weight $2n-1$ over a smooth, quasi-projective base space S^*. Here, $\mathcal{H}_\mathbb{Z}$ is a local system and the $\mathcal{F}^p$ give a filtration of $\mathcal{H} =: \mathcal{H}_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O}_X$ that induces a Hodge filtration on each fibre. We have

$$S^* = S \backslash D$$

where S is a smooth, projective variety and $D \subset S$ is a reduced, local normal crossing divisor (NCD). With this assumption, the local monodromies T_i around the local irreducible branches D_i of D are quasi-unipotent, and we assume further that they are unipotent; moreover, there are canonical extensions $\mathcal{H}_e, \mathcal{F}^p_e$ of $\mathcal{H}, \mathcal{F}^p$ and $\mathcal{H}_{\mathbb{Z},e} = j_*(\mathcal{H}_\mathbb{Z})$ where $j : S^* \to S$ is the inclusion, and where the Gauss-Manin connection is given by

$$\nabla : \mathcal{F}^p_e \to \mathcal{F}^{p-1}_e \otimes_{\mathcal{O}_S} \Omega^1_S(\log D) .$$

We define the sheaf over S

$$\mathcal{J}_e = \mathcal{F}^n_e \backslash \mathcal{H}_e / \mathcal{H}_{\mathbb{Z},e} \cong \check{\mathcal{F}}^n_e / \mathcal{H}_{\mathbb{Z},e}$$

where the isomorphism results from the principal polarization, and we set

$$\mathcal{J}_{e,\nabla} = \{\nu \in \mathcal{J}_e : \nabla\tilde{\nu} \in \mathcal{F}^{n-1}_e \otimes_{\mathcal{O}_S} \Omega^1_S(\log D)\}$$

where $\tilde{\nu}$ is any local lifting of ν to a section of $\check{\mathcal{F}}^n$.

Definition ([Z1]) A *normal function* is given by

$$\nu \in \Gamma(S, \mathcal{J}_{e,\nabla}) .$$

Example 2.1 Given a family

$$f : X \to S$$

of projective varieties $X_s = f^{-1}(s)$ whose total space X and parameter space S are smooth and where the singular fibres occur over a reduced, local normal crossing divisor $D \subset S$, there is a polarized VHS, which we assume to be principal, as above where

$$\mathcal{H}_\mathbb{Z} = R^{2n-1}_f \mathbb{Z}/\text{torsion} .$$

We set

$$Z^n(X)_{\text{HOM}} = \left\{ \begin{array}{l} Z \in Z^n(X) \\ [Z]_\mathcal{U} = 0 \text{ in } H^{2n}(X_\mathcal{U}, \mathbb{Z}) \end{array} \right\}$$

where $\mathcal{U}$ is a small neighborhood of any point $s \in S$ and $[Z]_\mathcal{U}$ is the fundamental class of Z in $X_\mathcal{U} = f^{-1}(\mathcal{U})$. Varying Z in its rational equivalence class we may assume that for $s \in S^*$ the intersection

$$Z_s = Z \cdot X_s \in Z^n(X_s)$$

is proper, and since $[Z_s] = 0$ in $H^{2n}(X_s, \mathbb{Z})$ the Abel-Jacobi image

$$\nu_Z(s) =: \mathrm{AJ}_{X_s}(Z_s) \in J(X_s)$$

is defined where

$$J(X_s) = F^n H^{2n-1}(X_s, \mathbb{C}) \backslash H^{2n-1}(X_s, \mathbb{C}) / H^{2n-1}(X_s, \mathbb{Z})$$

is the intermediate Jacobian of X_s. From the considerations in [EZ] we may infer that

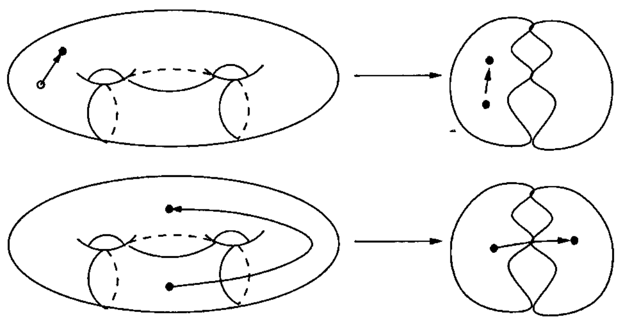
$$\nu_Z \in \Gamma(S, \mathcal{J}_{e,\nabla}) \textit{ defines a normal function.}$$

More precisely, ν_Z is defined over S^* and extends as indicated to give a normal function defined over S.

Remark In the two curve degenerations

(i) $X_s =$ ... $= X_{s_0}$

(ii) $X_s =$ ... $= X_{s_0}$



in the first case the normal function $\nu_Z = \mathrm{AJ}_{X_s}(s)$ extends across $s = s_0$ with limit

$$\lim_{s \to s_0} \mathrm{AJ}_{X_s}(Z_s) = \mathrm{AJ}_{X_{s_0}}(Z_{s_0}) \in J_e(X_{s_0})$$

given by the Abel-Jacobian image of Z_{s_0} in the generalized Jacobian $J_e(X_{s_0})$, but the normal function does not extend across $s = s_0$ in the second. Although this is a torsion phenomenon when the parameter space is a curve[6], when the parameter space arises by independently smoothing the nodes no multiple $m\nu_Z$ extends across $s = s_0$.

Using the exact cohomology sequence of

$$0 \to \mathcal{H}_{\mathbb{Z},e} \to \check{\mathcal{F}}^n_e \to \mathcal{J}_e \to 0$$

a normal function has a class

$$\delta(\nu) \in H^1(S, \mathcal{H}_{\mathbb{Z},e}) \,.$$

Theorem ([Z1]) *When* $\dim S = 1$ *the group* $H^1(S, \mathcal{H}_e)$ *has a Hodge structure of weight* $2n$ *and*

$$\delta(\nu) \in \mathrm{Hg}^n(S, \mathcal{H}_{\mathbb{Z},e}) \,.$$

Conversely, given $\zeta \in \mathrm{Hg}^n(S, \mathcal{H}_{\mathbb{Z},e})$ *there exists a normal function* ν *with* $\delta(\nu) = \zeta$.

Example 2.1 (continued, but assuming $\dim S = 1$**)** If we define, using the hopefully evident notation,

$$H^{2n}(X, \mathbb{Z})_{\mathrm{HOM}} = \{\zeta \in H^{2n}(X, \mathbb{Z}) : \zeta_{\mathcal{U}} = 0 \text{ in } H^{2n}(X_{\mathcal{U}}, \mathbb{Z})\}$$

then $\zeta \in \mathrm{Hg}^n(X, \mathbb{Z})_{\mathrm{HOM}}$ gives a normal function ν and

$$\delta(\nu) = \text{ Image of } \zeta \text{ in } H^{2n}(X, \mathbb{Z}) \to H^1(S, R_f^{2n-1}\mathbb{Z}) \,,$$

[6]In example (ii) above, the group of components of the Néron model is $\mathbb{Z}/2\mathbb{Z}$ so that $2\nu_Z$ extends.

where the RHS is defined using the Leray spectral sequence.

Due to the failure of Jacobi inversion in higher codimension, the above theorem has been of limited use in studying cycles. It does, however, suggest looking into the possible singular behaviour of ν_ζ in general situations as suggested by the particular example (ii) above. For example, setting $Z_\eta = Z \cdot X_\eta$ for η a generic point of S and defining

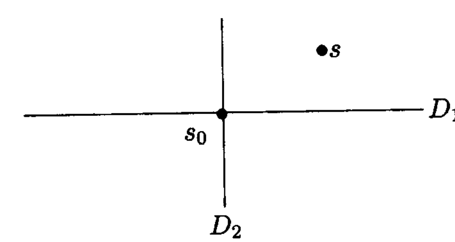
$$Z^n(X)_{\mathrm{hom}} = \left\{ \begin{array}{c} Z \in Z^n(X) : [Z_\eta] = 0 \\ \text{in } H^{2n}(X_\eta, \mathbb{Z}) \end{array} \right\}$$

then, for $Z \in Z^n(X)_{\mathrm{hom}}$, $\nu_Z(s)$ is defined on S^* and we may ask

What happens to $\nu_Z(s)$ *as* $s \to s_0 \in D$*?*

This leads to the notion of *admissable normal functions*. Denoting by Δ a disk in $\mathbb{C}$, in a neighborhood of $\mathcal{U}$ of $s_0 \in D$ with

$$\mathcal{U} \cap S^* =: \mathcal{U}^* \cong (\Delta^*)^k \times \Delta^l$$



we choose a local lifting $\tilde{\nu}$ of ν in a small neighborhood of s and analytically continue $\tilde{\nu}$ to a multi-valued section of $\check{\mathcal{F}}^n_e$ over $\mathcal{U}^*$. Denoting by $(T_i - I)$ analytic continuation around local branch D_i of D, we have

$$(T_i - I)\tilde{\nu} \in \mathcal{H}_{\mathbb{Z},s} \,.$$

Definition ν is an *admissable normal function* (ANF) if

(i) $\tilde{\nu}$ has moderate — i.e., logarithmic — growth along D; and

(ii) there exist non-zero integers m_i such that

$$m_i(T_i - I)\tilde{\nu} \in \mathrm{Ker}(T_i - I)^{\perp} \,.^7$$

[7]This definition appears in [GG1], where the ANF's were termed extended normal functions. It was pointed out to us by Greg Pearlstein that our extended normal functions were the same as admissable normal functions as defined earlier by M. Saito [Sa2].

Here, the $^\perp$ is with respect to the polarizing form.

Since $(T_i - I)\tilde{\nu}$ is well-defined modulo $\mathrm{Im}(T_i - I)$ and $\mathrm{Im}(T_i - I) \subseteq \mathrm{Ker}(T_i - I)^\perp$, condition (ii) is well-defined. For $\dim S = 1$, (ii) is automatically satisfied (cf. [GGK1]), and consequently if (i) is satisfied then there is an integer m such that $m\nu$ is a normal function in-the usual sense.

In general we have an exact sequence

$$0 \to \mathfrak{J}_{e,\nabla} \to \tilde{\mathfrak{J}}_{e,\nabla} \xrightarrow{\{\}} \mathcal{G} \to 0$$

where $\tilde{\mathfrak{J}}_{e,\nabla}$ is the sheaf of admissable normal functions and $\mathcal{G}$ is a sheaf of abelian groups such that setting $\mathcal{G}_{\mathbb{Q}} = \mathcal{G} \otimes_{\mathbb{Z}} \mathbb{Q}$,

$$\mathcal{G}_{\mathbb{Q}} \textit{ is supported in codimension } \geqq 2 \,.$$

Definition The *singular set* $\mathrm{sing}\,\nu$ is defined to be the support of

$$\{\nu\}_{\mathbb{Q}} \in \Gamma(S, \mathcal{G}_{\mathbb{Q}}) \,.$$

Remark Without assuming that the degeneracy locus $D \subset S$ of the VHS is a reduced divisor with local normal crossings, one may still define ANF's and their singular set will be defined to be

$$\left\{ \begin{array}{c} s_0 \in D: \text{ no multiple } m\nu \text{ has a single-valued lifting} \\ \text{in a punctured neighborhood } \mathcal{U}^* = \mathcal{U} \cap S^* \text{ of } s_0 \end{array} \right\}.$$

It then seems to be a likely result (cf. [BFNP] and [dCM2]) that, under dominant maps

$$\tilde{S} \xrightarrow{\pi} S \,,$$

ANF's pull back to ANF's and for $\nu \in$ an ANF defined on S

$$\mathrm{sing}\,(\pi^{-1}(\nu)) = \pi^{-1}(\mathrm{sing}\,\nu) \,;$$

i.e., *singularities do not disappear under branched coverings and blow-ups.*[8]

Example 2.1 (continued) We let $H^{2n}(X, \mathbb{Z})_{\mathrm{hom}}$ be the classes ξ such that, modulo torsion

$$\xi_\eta = 0 \text{ in } H^{2n}(X_\eta, \mathbb{Z}) \,.^9$$

[8]Footnote added in proof: It has been pointed out to me by Gregory Pearlstein that there are examples due to M. Saito and to Najmuddin Fakhrudding that this statement is false without further assumptions, perhaps to the effect that the parameters in S must be "essential".

Clearly we have

$$0 \to H^{2n}(X, \mathbb{Z})_{\mathrm{HOM}} \to H^{2n}(X, \mathbb{Z})_{\mathrm{hom}} \,,$$

and the cokernel may be roughly thought of as "classes supported over the discriminant locus D."

Theorem 2.2 *A class $\zeta \in \mathrm{Hg}^n(X)_{\mathrm{hom}}$ gives an admissable normal function ν_ζ.*

Example 2.3 Suppose given (X_0^{2n}, L, ζ_0) where X_0 is a smooth projective variety, $L \to X_0$ is a very ample line bundle and $\zeta_0 \in \mathrm{Hg}^n(X_0)_{\mathrm{prim}}$ is a primitive Hodge class. Setting $S = |\check{L}|$ we have the universal family of hyperplane sections

$$\begin{array}{c} X \subset X_0 \times S \\ \big\downarrow \\ S \end{array}$$

and ζ_0 pulls back to $\zeta \in \mathrm{Hg}^n(X)_{\mathrm{hom}}$.

Theorem 2.4 ([BFNP] and [dCM2]) *We have*

$$\mathrm{sing}\,\nu_\zeta = \{s_0 \in S : \zeta_{s_0} \neq 0 \textit{ in } \mathrm{Hg}^n(X_{s_0})/\mathrm{Hg}^n(X)_{\mathrm{hom}}\} \,.$$

Here, the singularities of X_{s_0} are arbitrary, including multiple components; $\mathrm{Hg}^n(X_{s_0})$ is defined as the Hodge classes in

$$\mathrm{Gr}_{2n}(H^{2n}(X_{s_0}^{\mathrm{red}}, \mathbb{Q}))$$

using the mixed Hodge structure on $H^{2n}(X_{s_0}^{\mathrm{red}}, \mathbb{Q})$.

Assuming inductively the Hodge conjecture (HC) in codimensions $\leqq n-1$, the above theorem has the following

Corollary 2.5 *The* HC *is equivalent to* $\mathrm{sing}\,\nu_\zeta \neq \emptyset$ *for* $L \gg 0$.

We may think of $\mathrm{sing}\,\nu_\zeta$ as the high degree hypersurfaces that support a cycle Z with $[Z] = \zeta$.

The proof of Theorem 2.4 uses the decomposition theorem (DT) for the special situation of the universal hyperplane section, together with a new ingredient — the "relative, weak hard Lefschetz theorem."

[9]In general, we let $\xi_s \in H^{2n}(X_s, \mathbb{Z})$ be the image of ξ under the restriction map $H^{2n}(X, \mathbb{Z}) \to H^{2n}(X_s, \mathbb{Z})$.

We remark that the argument of [BFNP] and [dCM2] seems to work if we only assume that

$$\zeta \in H^{n,n}(X)_{\text{hom}} ;$$

i.e., ζ need not be a Hodge class, and in fact could be transcendental. The conclusion is then that

$$\zeta_{s_0} \neq 0 \text{ in } H^{2n}(X)_{s_0}/H^{2n}(X)_{\text{hom}} . \tag{2.6}$$

If ζ is in fact transcendental, and if X_{s_0} is nodal, then

$$H^{2n}(X_{s_0})/H^{2n}(X)_{\text{hom}} = \text{Hg}^n(X_{s_0})/\text{Hg}^n(X)_{\text{hom}}$$

from which we infer that (2.6) does not hold. In fact, one may define the singluar locus of *any* class $\zeta \in H^{n,n}(X)_{\text{hom}}$; we suspect that $\text{sing}\,\zeta = \emptyset$ unless ζ is a Hodge class.

The construction of ν_ζ requires that ζ be in $H^{2n}(X,\mathbb{Z})$ as well as being in $H^{n,n}(X)_{\text{hom}}$. This suggests that any argument that

$$\text{sing}\,\nu_\zeta \neq \emptyset$$

will need to make use of the existence of ν_ζ as a *mapping* from S to some space. This will be further discussed in §§4 and 5 below.

As a general comment, in the absence of being able to construct cycles one may consider what the implications of the HC are. We here mention three:

(i) the HC has geometric consequences;

(ii) the HC has arithmetic consequences; [10]

(iii) the HC may have topological consequences.

The above corollary gives one instance of (i). Below we will discuss (iii), which may also be related to aspects of (ii).

3 Néron Models and Graphing ANF's

Classically, working analytically and denoting by $S = \Delta$ the disc with origin s_0 and $S^* = \Delta^* = \Delta\backslash\{s_0\}$ the punctured disc, given a family $J \to S^*$

[10]These include the *absolute Hodge condition*, as formulated by Grothendieck and Deligne (cf. [D-M]), and the field of definition of Noether-Lefschetz loci ([V]). These are further discussed in Section 5 below.

of principally polarized abelian varieties with unipotent monodromy T,[11] there exists a diagram of analytic fibre spaces of complex Lie groups

$$\begin{array}{ccc} J & \subset & \tilde{J}_e \\ \downarrow & & \downarrow \\ S^* & \subset & S \end{array} \tag{3.1}$$

with a number of properties,[12] including

- the diagram (3.1) is canonical and graphs admissable normal functions;[13]
- there is an exact sequence
$$0 \to J_e \to \tilde{J}_e \to G \to 0$$
where G is the finite group
$$G = (\text{Ker}(T-I))^{\perp}_{\mathbb{Z}} \,/\, \text{Im}(T_{\mathbb{Z}} - I)$$
sitting over the origin s_0;[14]
- the fibre $\tilde{J}_{e,s_0}$ is an extension of its identity component J_{e,s_0}, a semi-abelian variety, by G.

Two extensions of this construction have been given.

(i) *Néron models of intermediate Jacobians over* 1*-dimensional base spaces.*[15]

Here the same properties as above extend, with one significant difference:

[11]This restriction is not essential, but it simplifies the exposition.

[12]This is the classical *Néron model* — cf. [Ko] for the elliptic curve case and [BLR] for the general case.

[13]This means that ANF's are given by holomorphic sections of $\tilde{J}_e \to S$; we may express this by saying that

$$\tilde{\mathfrak{J}}_{e,\nabla} \cong \mathcal{O}(\tilde{J}_e)_\nabla ,$$

where we note that the condition on $\nabla\tilde{\nu}$ is automatically satisfied when $n = 1$. It is noteworthy that no compactification of the fibre $\tilde{J}_{e,s_0}$ over the origin is required to fill in the value $\nu(s_0) = \lim_{s\to s_0} \nu(s)$.

[14]The subscript "$\mathbb{Z}$" on $(\text{Ker}(T-I)^{\perp})_{\mathbb{Z}}$ means that we take the integral elements in the vector space $\text{Ker}(T-I)^{\perp}$.

[15]This is joint work with Mark Green and Matt Kerr, cf. [GGK1]. The Hodge theoretic aspects have been done independently by Patrick Brosnan and Gregory Pearlstein [BP1].

- $J_e \to S$ is a *slit analytic fibre space*[16] and

$$\dim J_{e,s_0} < \dim J_{e,\eta}$$

unless $(T-I)^2 = 0$ and the LMHS satisfies[17]

$$\mathrm{Gr}_{2n} \text{ is of Hodge-Tate type .}$$

The identity component of J_{e,s_0} of $\tilde{J}_{e,s_0}$ is what one might call a *semi-analytic complex torus*, meaning an abelian complex Lie group which is an iterated extension of compact, complex tori by $\mathbb{C}^*$'s. Note that there are no $\mathbb{C}$-factors.

In the localization of the geometric example to $S = \Delta$ and where X_{s_0} is a normal crossing divisor, we let $Z^n_{\#}(X_{s_0})$ denote the codimension n algebraic cycles on X_{s_0} which meet the strata properly. Then, for $Z_{s_0} \in Z^n_{\#}(X_{s_0})$ its fundamental *co*homology class

$$[Z_{s_0}] \in H^{2n}(X_{s_0}, \mathbb{Z})$$

may be defined ([GGK1]), and we denote by

$$Z^n_{\#}(X_{s_0})_{\mathrm{HOM}} \subset Z^n_{\#}(X_{s_0})$$

the subgroup of cycles Z_{s_0} with $[Z_{s_0}] = 0$. Using the MHS on $H^{2n-1}(X_{s_0}, \mathbb{C})$ one may define the *generalized Jacobian* $J(X_{s_0})$ and there is an Abel-Jacobi map

$$\mathrm{AJ}_{X_{s_0}} : Z^n_{\#}(X_{s_0})_{\mathrm{HOM}} \to J(X_{s_0}) .$$

Now let $Z \in Z^n(X)$ be a cycle such that the intersections

$$Z_s = Z \cdot X_s$$

are defined for all s and $Z_{s_0} \in Z^n_{\#}(X_{s_0})_{\mathrm{HOM}}$. Then there is defined a normal function $\nu_Z \in \Gamma(\Delta, \mathcal{J}_{e,\nabla})$, which may be represented by a section of

$$J_e \to S .$$

[16]Slit analytic fibre spaces have appeared in the work of Kato-Usui [KU] on partial compactifications of moduli spaces of polarized Hodge structures. There it is explained how on them one may "do geometry as usual relative to period maps". The situation of "doing geometry relative to ANF'S" seems to be similar, but the details are yet to be worked out.

A special feature of "doing geometry as usual" is given by the result [BP2] that when $\dim S = 1$ the zero locus of an ANF is an algebraic variety, and its extension to the case where $\dim S$ is arbitrary and locally D has one branch is given in [Z1, pp. 214–215] and in [Sa3].

[17]These conditions appeared first in [Cl] and were amplified in [Sa2].

We then have

$$\mathrm{AJ}_{X_s}(Z_s) = \nu_Z(s) \text{ for all } s \text{ and } ,$$

in a precise sense,

$$\lim_{s \to s_0} \mathrm{AJ}_{X_s}(Z_s) = \mathrm{AJ}_{X_{s_0}}(Z_{s_0}) .^{18} \tag{3.2}$$

Moreover, the limit (3.2) has the following properties, which are present but perhaps not so "visible" in the classical case

- the limiting MHS induces a filtration on J_{e,s_0} whose graded pieces are composed of compact, complex analytic tori and $\mathbb{C}^*$'s — no vector groups occur;
- the limit (3.2) is a mapping of filtered groups whose graded pieces are constructed from regulator maps defined on sub-quotients of higher Chow groups.[19]

As a closing remark, we note that the group G_{s_0} of components of the Néron model has a monodromy weight filtration — cf. the examples in [GGK2]. In general, it is my view that $\lim \mathrm{AJ}_{X_s}$ frequently has a richer geometric structure than AJ_{X_s} for a generic s, and in some ways even more than AJ_{X_s} in the classical $n = 1$ case.

(ii) *Néron models of PPAV's over higher dimensional base spaces.*[20]

In the local analytic situation where $S^* \cong (\Delta^*)^l \times \Delta^k$, we assume given a family of PPAV's $J \to S^*$ whose monodromies are unipotent. Then there is a diagram

$$\begin{array}{ccc} J & \subset & \tilde{J}_e \\ \downarrow & & \downarrow \\ S^* & \subset & S \end{array} \tag{3.3}$$

where

[18]Thus, except in the case where $(T-I)^2 = 0$ and Gr_n of the LMHS is of Hodge-Tate type, the limiting Abel-Jacobi map has geometric constraints not present classically. The RHS of (3.2) should be understood as the image of $\mathrm{AJ}_{X_{s_0}}(Z_{s_0})$ under the map induced from the Clemens-Schmid sequence.

[19]In the classical case $n = 1$, the $\mathbb{C}^*$'s in the generalized Jacobian of a nodal curve should be thought of as regulator images of $\mathrm{CH}^1(p, 1)$, where p is a node.

[20]cf. the Princeton thesis of Andrew Young. His result is a complex analytic toroidal construction.

- $\tilde{J}_e \to S$ is a *non-Hausdorff*[21] fibre space of complex Lie groups whose identity components are semi-abelian varieties of the same dimension as the generic fibre;
- (3.3) graphs ANF's, which we express as

$$\tilde{\mathcal{J}}_e \cong \mathcal{O}_S(\tilde{J}_e)$$

(here we omit the ∇-subscripts since the transversality condition is automatic in this case);
- If the local monodromies are denoted by T_i, then the group G_{s_0} of components of $\tilde{J}_{e,s_0}$ has a description

$$G_{s_0} \cong \left\{ \begin{array}{c} H^1(B^\bullet_{s_0}) \text{ where } B^\bullet_{s_0} \text{ is a Koszul-type} \\ \text{complex defined over } \mathbb{Z} \text{ and constructed} \\ \text{from the } T_i - I \text{ and the polarizing form}^{22} \end{array} \right\}.$$

In general, when $\dim S \geqq 2$ the group G_{s_0} is a finitely generated but in general *not* a finite abelian group.

Example 3.4 Referring to the curve degenerations pictured in §2, so that $S \cong \Delta^3$ is the parameter space obtained from smoothing the nodes independently, for the group G_s of components of $\tilde{J}_{e,s}$ we have

$$G_s = \begin{cases} 0 & s \neq (0,0,0) \\ \mathbb{Z} & s = (0,0,0)\,. \end{cases}$$

The ANF in (ii) has value "1" at the origin.

As an example of "doing geometry relative to ANF's", let

$$\mathcal{P} \to J \times_{S^*} J$$

be the Poincaré line bundle. Then:

- for an ANF ν, $\nu^*(\mathcal{P})$ initially defined over S^* has a canonical extension to S, even though $\mathcal{P}$ itself does not canonically extend to $\tilde{J}_e \times_S \tilde{J}_e$.[23]

[21] This non-separatedness causes no difficulty in doing geometry *relative to ANF's*; see below.
[22] This group $H^1(B^\bullet_{s_0}) \otimes_{\mathbb{Z}} \mathbb{Q}$ appears in [CKS] and is constructed from a Koszul-type complex constructed from the logarithms $N_i = \log T_i$ of monodromy.
[23] cf. [Y], where the precise meaning of canonical is explained.

What is obviously needed is to "amalgamate" the two constructions to have a Néron model for intermediate Jacobians over higher dimensional base spaces. This has been done for $n = 2$ in the nodal case (cf. [GG2]), which in some ways resembles the classical case since one has

$$\begin{cases} (T - I)^2 = 0 \\ \mathrm{Gr}_4 \text{ of the LMHS is of Hodge-Tate type.} \end{cases}$$

However, in other ways it is quite different in that the infinitesimal period relation enters in an essential way in dimension counts (loc. cit).

4 Universal Realization of the Singular Locus of an ANF[23]

We will explain "what" is desired and "why" in the classical weight one case, where the required ingredients may be in place. Then we shall briefly discuss the higher weight case.

We denote as usual by $\mathcal{A}_g$ the moduli space for principally polarized abelian varieties of dimension g, realized as the quotient by the discrete subgroup $\Gamma = G_{\mathbb{Z}}$ of the Siegel upper-half-space

$$\mathcal{H}_g = G_{\mathbb{R}}/K$$

where $G_{\mathbb{R}} = Sp(g, \mathbb{R})$ and $K = \mathcal{U}(g)$. The Lie algebra $\mathcal{G}_{\mathbb{R}}$ is defined over $\mathbb{Q}$, and for each *fan* Σ there is defined a compactification

$$\mathcal{A}_g \subset \overline{\mathcal{A}}_{g,\Sigma}$$

of $\mathcal{A}_g$ (cf. [AMRT]). Recall that for general compactifications of quotients of Hermitian symmetric spaces, Σ is a union of rational, nilpotent cones $\mathbf{n}_\sigma$ which satisfy certain incidence relations and where $\mathrm{Ad}\Gamma$ permutes the $\mathbf{n}_\sigma$'s. In the case at hand each

$$\mathbf{n}_\sigma = \mathrm{span}\{N_1, \ldots, N_r\}$$

where the N_i are *integral* and satisfy

$$\begin{cases} [N_i, N_j] = 0 \\ N_i^2 = 0\,. \end{cases}$$

[23] This section is speculation on the potential outcome of work in progress.

Associated to $\mathbf{n}_\sigma$ is a set of nilpotent orbits whose limiting mixed Hodge structures, modulo reparametrizations, may be thought of as the boundary component $B_\sigma \subset \overline{\mathcal{A}}_{g,\Sigma}$ corresponding to $\mathbf{n}_\sigma$ (cf. the discussion in Cattani's article in [G2] and in [GGK2]).

What is desired is a *universal Néron model*

$$\tilde{J}_{g,\Sigma} \to \overline{\mathcal{A}}_{g,\Sigma}$$

together with a subvariety $\Xi \subset \tilde{J}_{g,\Sigma}$ such that, given an ANF $\nu \in \Gamma(S, \tilde{\mathcal{J}}_{e,\nabla})$, there is a diagram

$$\begin{array}{ccc} & & \tilde{J}_{g,\Sigma} \supset \Xi \\ & \overset{\nu}{\nearrow} & \downarrow \\ S & \xrightarrow{\ \tau\ } & \overline{\mathcal{A}}_{g,\Sigma} \end{array} \tag{4.1}$$

with

$$\operatorname{sing}\nu = \nu^{-1}(\Xi)\,. \tag{4.2}$$

Here we assume that the locus $D \subset S$ of degenerate Hodge structures is locally a NCD of the form

$$D \cap \mathcal{U} = D_1 \cup \cdots \cup D_r\,.$$

Then $\mathcal{U}$ will map under the Torelli map τ to a neighborhood of $\tau(s_0) \in B_\sigma$ where $\mathbf{n}_\sigma$ is the the rational nilpotent cone spanned by $N_i = \log T_i$ and where T_i is the local monodromy, assumed unipotent, around D_i. Intuitively

$$\tau(D\cap\mathcal{U}) = \left\{ \begin{array}{c} \text{limiting MHS's taken} \\ \text{along discs } \Delta \subset \mathcal{U} \\ \text{with } \Delta \cap D = \{s_0\} \end{array} \right\}.$$

With the caveat that we are ignoring important stack considerations, the constructions in Young's thesis serve to define $\tilde{J}_{g,\Sigma}$ as a set, so that at this level we have (4.1). Denoting by

$$\Xi = \left\{ \begin{array}{c} \text{union of non-torsion components} \\ \text{of fibres of } \tilde{J}_{g,\Sigma} \to \overline{\mathcal{A}}_{g,\Sigma} \end{array} \right\}$$

we then have, again at the set-theoretic level, (4.2). Defining the appropriate structures so that we properly have (4.1) and (4.2) is what is desired.

As to "why" (4.1) and (4.2) are desired, we give two préliminary reasons — both of which are existence results. Afterwards we will explain why, and possibly how, these must be modified. The two reasons are:

$$\left\{ \begin{array}{ll} \text{(i)} & \nu^{-1}(\Xi) \neq 0 \implies \operatorname{codim}_S(\operatorname{sing}\nu) \leqq \operatorname{codim}_{\tilde{J}_{g,\Sigma}}(\Xi) =: d \\ \text{(ii)} & \nu^*([\Xi]) \neq 0 \implies \nu^{-1}(\Xi) \neq \emptyset. \end{array} \right. \tag{4.3}$$

Below we shall see that (ii) is questionable, but if we let (X,ζ) vary in moduli over the Noether-Lefschetz locus then the extension of (ii) becomes at least plausible and would give an existence result.

Concerning (i), since codimension descreases under blowing up we cannot immediately conclude that

$$\dim_S(\operatorname{sing}\nu) \geqq \dim S - d. \tag{4.4}$$

We can, however, conclude this if — e.g. — we know that for a general point $s_0 \in \operatorname{sing}\nu$

$$\tau_* : T_{s_0}S \to T_{\tau(s_0)}\overline{\mathcal{A}}_{g,\Sigma} \text{ is injective.} \tag{4.5}$$

Suppose we are in the geometric case of Example 2.1 and X_{s_0} is a nodal curve of genus g. Then τ is the composition of

$$S \xrightarrow{\mu} \overline{\mathcal{M}}_g \xrightarrow{\sigma} \overline{\mathcal{A}}_{g,\Sigma}\,. \tag{4.6}$$

It is not hard to see that μ_* is injective on a general $T_{s_0}S$, so the question is one of σ_*.

In fact, to have the conclusion (4.4) it is enough to have that

$$\tau_* : T_{s_0}(\operatorname{sing}\nu) \to T_{\tau(s_0)}\overline{\mathcal{A}}_{g,\Sigma}$$

is injective. We factor as in (4.6) and denote by $A \subset \overline{M}_g$ the boundary component of stable, nodal curves of the same type as X_{s_0} and by $B \subset \overline{\mathcal{A}}_{g,\Sigma}$ the boundary component of LMHS's of the type of $\lim_{s\to s_0} H^1(X_s)$. The issue is then the injectivity of

$$\sigma_* : T_{m_0}A \to T_{\sigma(m_0)}B\,. \tag{4.7}$$

In the case of interest where Example 2.1 arises as in Example 2.3, as explained in [GG1] the injectivity of (4.7) in the situation of most interest is equivalent to the following geometric problem:

Let X_0 be a smooth algebraic surface, $C_0 \subset X_0$ a smooth curve, and $L \to X_0$ a line bundle that is sufficiently ample relative to X_0 and C_0. In particular, $L \otimes [-C_0]$ should be very ample. The linear system $|L|$ may be thought of as consisting of reducible curves

$$C + C_0 \tag{4.8}$$

where a general C is smooth and meets C_0 transversely. Each such curve has a mixed Hodge structure, which is part of the LMHS of a general smoothing ot (4.8). As shown to me by Mark Green, we then have

(4.9) *The differential (4.7) is injective if the differential of the map*

$$C \to \text{ MHS on } H^1(C + C_0)$$

is injective.

Mark has proved that this is the case if $L \gg 0$.

We do not know if the analogue of (4.9) holds when $n \geqq 2$. As discussed in [GG2], dimension counts must take into account the infinitesimal period relation; the issue would seem to be an interesting one.

Turning to (ii), we first observe that *it is unlikely to be true as stated.* To begin with, one must be more precise and consider the irreducible components of Ξ. Let us consider one, still denoted by Ξ, that lies over a boundary component B consisting of all LMH's corresponding to a nilpotent cone $\mathbf{n} = \operatorname{span}\{N_1, \dots, N_l\}$ where the $T_i = I + N_i$ are given by Picard-Lefschetz transformations

$$T_i(\gamma) = \gamma + (\gamma, \delta_i)\delta_i$$

where $\delta_1, \dots, \delta_l$ are primitive elements of $\mathcal{H}_{\mathbb{Z},s}$ with one relation

$$\delta_1 + \cdots + \delta_l = 0$$

among them. Then locally

$$\nu(S) \cap \Xi \textit{ lies one-to-one over } \tau(S) \cap B\,.$$

Although this certainly does *not* imply that

$$\tau^*([B]) = \nu^*([\Xi])\,,$$

what is definitely *not* true is

$$\tau^*([B]) \neq 0\,. \tag{4.10}$$

The reason is this: If (4.10) holds for X_0, then it would hold for any small deformation X_0' of X_0. This would then imply that X_0' contains reducible

sections of L of the type (4.8), which means that the Hodge class ζ_0 deforms to X_0. But in general the Hodge class will only deform over a proper subvariety — the *Noether-Lefschetz locus*

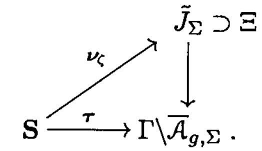
$$\mathcal{M}_\zeta \subset \mathcal{M}$$

of the moduli space $\mathcal{M}$ (assumed to exist) of X_0.

This then suggests that, in Example 2.3, one consider normal functions associated to Hodge classes not only for a fixed (X_0, ζ_0), but rather one should allow this data to vary over $\mathcal{M}_\zeta$. We denote by $(X_t, \zeta_t, L_t)_{t \in \mathcal{M}_\zeta}$ this variation, together with that of the very ample line bundle. We set $S_t = |\check{L}_t|$ and

$$\mathbf{S} = \bigcup_{t \in \mathcal{M}_\zeta} S_t\,.$$

Then there are a Torelli map τ and normal function ν_ζ defined as in the diagram



In [GG2] it is proved that, subject to a technical assumption that we suspect is not essential,

$$\tau_\zeta^*([B]) \neq 0\,. \tag{4.11}$$

This is proved using the Lefschetz (1,1) theorem — i.e. the HC for $n = 1$. We also suspect that if there were an independent proof of the purely topological result (4.11), then the Lefschetz (1,1) theorem might follow.

The analogue of (4.11) for $n \geqq 2$ is not known.

Conclusion In the absence of being able to construct cycles in higher codimension, it seems of interest to examine consequences of the HC. The topological statement (4.11) could be one such.

5 Some General Observations and a Question

Above there has been discussion of some geometric consequences of the HC. There are several others, some of which have been proved, including

- the algebraicity of the Noether-Lefschetz loci (cf. [CDK]);

- the algebraicity of the zero locus of a normal function, done for $\dim S = 1$ in [BP1] and more recently by them for $\dim S$ arbitrary and where D locally has one branch (private communication, and also by M. Saito (private communication)). Further recent work is in [BP2] and [Sch].

There are also arithmetic-geometric consequences of the HC, two of which were mentioned in footnote (7). Roughly speaking, the second involves the following considerations: If we have a family of algebraic varieties

$$f : X \to S$$

where S (assumed irreducible) is defined over a number field — we simply write "X is defined over $\overline{\mathbb{Q}}$" — then as discussed in [V], one expects the Noether-Lefschetz loci

$$S_\zeta \subset S$$

where a class $\zeta \in \mathrm{Hg}^p(X_s)$, s a point of S, remains of Hodge class to also be defined over $\overline{\mathbb{Q}}$.

The reason is this: Any algebraic subvariety

$$V \subset S \tag{5.1}$$

may be thought of as given by an inclusion of abstract varieties

$$V(k) \subset S(k) \tag{5.2}$$

defined over an algebraically closed field k of characteristic zero, together with an embedding $\sigma : k \hookrightarrow \mathbb{C}$ giving rise to (5.1), which we may write as

$$V(k) \otimes_\sigma \mathbb{C} \subset S(k) \otimes_\sigma \mathbb{C} \,.$$

Varying σ gives the *spread* of (5.1). Since S is defined over $\overline{\mathbb{Q}}$, an irreducible component of the spread may be though of as a component of the spread

$$\overline{V} \subset S$$

of V in S.

Now, enlarging k if necessary, ζ may be thought of as giving a class

$$\zeta \in H^p(X_s(k), \Omega^p_{X_s(k)}) \tag{5.3}$$

where X_s is defined over k and we are using GAGA to identify analytic sheaf cohomology with its algebraic counterpart. We may take the spread

of (5.3), and *assuming absolute Hodge* an irreducible component of the parameter space of the spread will map to $\overline{S}_\zeta$; i.e. we have

$$\overline{S}_\zeta \subseteq S_\zeta \,,$$

or *S_ζ is defined over $\overline{\mathbb{Q}}$.*

The above is only heuristic — an invitation to [V]. The point, by no means new here, is to illustrate that arithmetic and geometric considerations should be considered together, not separately, in the study of cycles.

For example, one at least philosophical difficulty in the use of the Abel-Jacobi map and normal functions to study cycles is that *the Abel-Jacobi map loses too much geometry.* In [GG] it is shown that if one "enriches" the Abel-Jacobi map by taking spreads — in effect considering the well-defined part of all the Abel-Jacobi maps arising by varying the embeddings $\sigma : k \hookrightarrow \mathbb{C}$, then assuming the (generalized) Hodge conjecture and one of the Bloch-Beilinson conjectures, rational equivalence is captured (up to torsion). This at least suggests that one should consider adding spread considerations to the study of normal functions and their singularities. One possible outcome might be that, assuming absolute Hodge, the study and construction of algebraic cycles is in principle reduced to where everything is defined over $\overline{\mathbb{Q}}$ (cf. [V] for further discussion). At that point one might seek to combine classical and p-adic methods.

A further comment concerns the consequence Corollary 2.5 of the HC. There the operative phrase is "$L \gg 0$". *What exactly does this mean?* Suppose we are given a projective embedding of X with $\mathcal{O}_{X_0}(1)$ having the usual meaning. Taking $L = \mathcal{O}_{X_0}(m)$, in [GG1] we observed that

> *In general there cannot be a uniform bound on m in order to have* $\operatorname{sing} \nu_\zeta \neq 0$.

Here uniform bound means "as X_0 varies in moduli". Of course, one may reasonably expect a bound as (X, ζ) varies over the subvariety $\mathcal{M}_\zeta$ of its moduli space. In [GG1] it is noted that in the $n = 1$ case when X_0 is a surface, in order to be sure to have $\operatorname{sing} \nu_\zeta \neq \emptyset$ we must have

$$m \geqq c|\zeta|^2 \tag{5.4}$$

for some constant $c > 0$.

Question *In general does the HC imply the existence of a lower bound (5.4)?*

For surfaces, the estimate (5.4) is sufficient in that given X_0 there is a $C > 0$ such that

$$\operatorname{sing} \nu_\zeta \neq \emptyset \ \text{ for } \ m \geqq C|\zeta|^2 \,.$$

One may of course ask if, assuming the HC, the same holds in general.

Bounds such as (5.4) may be thought of as providing an *effective* HC.

As a reprise from these speculations/generalities, I would like to describe a concrete, geometric question, which will help frame another general consideration. Assume that we have the construction (3.3) for families of intermediate Jacobians as well as for families of PPAV's. Denote by ANF(S) the *group* of admissable normal functions. Then there is a diagram

$$\begin{array}{ccc} \mathrm{ANF(S)} \times H^*(\tilde{J}_e) & \longrightarrow & H^*(S) \\ \Cup & & \Cup \\ (\nu, \alpha) & \longrightarrow & \nu^*(\alpha)\,. \end{array} \tag{5.5}$$

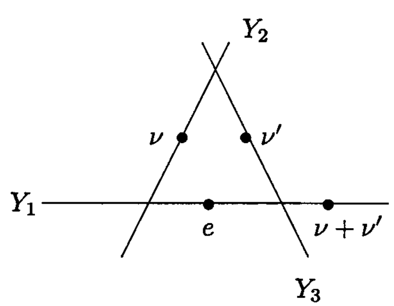
Question *What are the algebraic properties in the first factor of this map?*

Specifically, denoting by e the zero normal function we set

$$\Delta(\nu,\nu') = (\nu+\nu')^* - \nu^* - \nu'^* + e^*$$

and think of $\Delta(\nu,\nu')$ as the *derivation from linearity in the first factor of (5.5).*[24]

Example Let $f : X \to S$ be a minimal elliptic surface. If there is a fibre of type I_3 in Kodaira's notation, and if we have two sections ν and ν' that meet that fibre as pictured



then all

$$\Delta(\nu,\nu')[Y_i] \neq 0\,,$$

but it may be shown that

$$\Delta(\nu,\nu') = 0$$

[24] Equivalently, it is the deviation from the "theorem of the square."

for ordinary normal functions (their value at singular points is in the identity component).[25] Denoting by NF(S) the group of ordinary normal functions, this suggests the

Refined Question *(i) Is* $\Delta(\nu,\nu') = 0$ *on* NF(S) × NF(S)*? (ii) If so, then to what extent does the induced map*

$$\overline{\Delta} : \left(\frac{\mathrm{ANF(S)} \times \mathrm{ANF(S)}}{\mathrm{NF(S)} \times \mathrm{NF(S)}}\right) \to \mathrm{Hom}(H^*(\tilde{J}_e), H^*(S))$$

capture the singularities of admissable normal functions?

To conclude, we want to pose an issue that combines this question with the preceding question centering around the lower bound (5.4). This discussion will be heuristic; we will pass over the significant technical issues that would be necessary to address to make it precise.[26]

The idea is, as in proofs of the nullstellensatz, not try to initially deal with bounds of the type (5.4). Keeping the notation $(X_0, \mathcal{O}_{X_0}(1))$ as above, we may assume $\mathcal{O}_{X_0}(1)$ is sufficiently ample to have

$$H^0(\mathcal{O}_{X_0}(k)) \otimes H^0(\mathcal{O}_{X_0}(l)) \twoheadrightarrow H^0(\mathcal{O}_{X_0}(k+l))$$

for all k, l. We set

$$S_k = \mathbb{P}H^0(\mathcal{O}_{X_0}(k))\,;$$

there are then spanning maps[27]

$$\mu_{k,l} : S_k \times S_l \;\to\; S_{k+l}$$

given by

$$(X_s, X_t) \;\to\; X_s + X_t$$

where $s \in S_k$, $t \in S_l$.

For the Néron models $\tilde{J}_{k,e} \to S_k$, in first approximation we have

$$\begin{array}{ccc} \tilde{J}_{k+l,e}\Big|_{\mathrm{Image}\ \mu_{k,l}} & \xrightarrow{\pi_{k,l}} & \tilde{J}_{e,k} \times \tilde{J}_{e,l} \to 0 \\ \downarrow & & \downarrow \\ \mathrm{Image}\ \mu_{k,l} & \longrightarrow & S_k \times S_l\,. \end{array} \tag{5.6}$$

[25] The point is that the connecting map

$$H^0(S, \mathcal{J}_e) \to H^1(S, R^1_f\mathbb{Z})$$

is a homomorphism of groups.

[26] These issues center around the extent to which we need to use SSR to have unipotent monodromies and to have the discriminant loci with local normal crossings, which up until now are needed to construct the Néron models $\tilde{J}_{k,e} \to S_k$.

[27] A spanning map is one whose image linearly spans.

The reason is that for general $(s,t) \in S_k \times S_l$ the identity component $J_e(X_s+X_t)$ of the generalized intermediate Jacobian will map onto $J_e(X_s)\times J_e(X_t)$, reflecting the fact that the MHS on $H^{2n-1}(X_s \cup X_t)$ is an extension of $H^{2n-2}(X_s \cap X_t)$ by $H^{2n-1}(X_s) \oplus H^{2n-1}(X_t)$. Of course if X_s, X_t are singular or fail to meet transversely the situation is more complicated but, as suggested above, we will not address this.

Given a Hodge class $\zeta \in \mathrm{Hg}^n(X_0)_{\mathrm{prim}}$, for each k we have

$$S_k \xrightarrow{\nu_{k,\zeta}} \tilde{J}_{e,k} .$$

It is reasonable to guess that *the sequence of ANF's* $\nu_{k,\zeta}$ *are compatible* in the sense that

$$\pi_{k,l}\left(\nu_{k+l,\zeta}\Big|_{\mathrm{Image}\ \mu_{k,l}}\right) = \nu_{k,\zeta} \times \nu_{l,\zeta} . \tag{5.7}$$

We will write this more suggestively as

$$\nu_{k+l,\zeta} \to \nu_{k,\zeta} + \nu_{l,\zeta} ,$$

where it is understood that the LHS is restricted to the image of $\mu_{k,l}$.

A perhaps more subtle issue is the relation between $\operatorname{sing}\nu_{k+l,\zeta}$ and $\operatorname{sing}\nu_{k,\zeta}, \operatorname{sing}\nu_{l,\zeta}$. Again in first approximation we would guess that

$$\operatorname{sing}\nu_{k+l,\zeta}\Big|_{\mathrm{Im}\,\mu_{k,l}} = (\operatorname{sing}\nu_{k,\zeta} \times S_l) \cup (S_k \times \operatorname{sing}\nu_{l,\zeta}) .$$

Geometrically, this says that *"new" Hodge classes on $X_k \cup X_l$ can only come from those on either X_k or X_l*. If correct, this leads to the reasonable conclusion that singularities of ANF's cannot be produced from Segre images of lower degree hypersurface sections.

Next, we have the maps

$$\varphi_k : \mathrm{Hg}^n(X_0)_{\mathrm{prim}} \to \mathrm{Hom}\left(H^*(\tilde{J}_{k,e}), H^*(S_k)\right) . \tag{5.8}$$

There are possible compatibility relations among

$$\varphi_{k+l} \text{ and } \varphi_k, \varphi_l$$

as suggested by (5.6). Moreover, as suggested above *the failure of φ_k to be a homomorphism of groups is related to the presence of singularities of ANF's.*

Two further extensions of the above construction are suggested by the discussion in Section 4. For this we fix a subgroup

$$\Lambda \subset \mathrm{Hg}^n(X_0)_{\mathrm{prim}}$$

and denote by $\mathcal{M}_\Lambda \subset \mathcal{M}$ the Noether-Lefschetz locus of all pairs (X_t, Λ_t) where X_t is a deformation of X_0 and $\Lambda_t \subset \mathrm{Hg}^n(X_t)_{\mathrm{prim}}$ corresponds to Λ. We then denote by $\mathbf{S}_k$, $\mathbf{J}$ the above constructions of S_k, $\tilde{J}$ varying over M_Λ. Thus

$\mathbf{S} \to \mathcal{M}_\Lambda$ *is the projectification of a vector bundle $\mathcal{E}_\Lambda$ over $\mathcal{M}_\Lambda$*

and thus

$$H^*(\mathbf{S}_k) = H^*(\mathcal{M}_\Lambda)[\zeta_k] \Big/ \left\{ \begin{array}{c} \text{relations given the} \\ \text{Chern classes of } \mathcal{E}_\Lambda \end{array} \right\}$$

where $\zeta_k \in H^2(\mathbf{S}_k)$. There are again compatibility relations among these constructions as k varies.

Finally, we let

$$\Gamma_k \backslash D_{k,\Sigma_k}$$

be the Kato-Usui spaces associated to relevant polarized Hodge structures. We also denote by

$$\tilde{J}_{k,\Sigma_k} \to \Gamma_k \backslash D_{k,\Sigma_k} \tag{5.9}$$

the universal Néron model. For each k, l one may imagine a boundary component

$$B_{k,l} \subset \Gamma_{k+l} \backslash D_{k+l,\Sigma_{k+l}}$$

consisting of LMHS's of the type that arise by smoothing X_k+X_l in $\check{S}_{k+l} = |\mathcal{O}_{X_0}(k+l)|$, together with a surjective map

$$B_{k,l} \to (\Gamma_k \backslash D_k) \times (\Gamma_l \backslash D_l) \tag{5.10}$$

associating to a MHS its Gr_{2n-1} piece. The maps (5.10), together with the maps lying over them under (5.9), express the inter-relations among the universal Néron models.

Putting everything together, given Λ as above we have the family of ANF's $\boldsymbol{\nu}_{\Lambda,k}$ given by the $\nu_{\Lambda,t}$'s

$$\mathbf{S}_{\Lambda,k} \xrightarrow{\boldsymbol{\nu}_{\Lambda,k}} \tilde{\mathbf{J}}_{\Lambda,k,\Sigma_k}$$

which induce maps

$$\boldsymbol{\nu}^*_{\Lambda,k} : \Lambda \to \mathrm{Hom} H^*\left(\tilde{\mathbf{J}}_{\Lambda,k,\Sigma_k}, \mathbf{S}_{\Lambda,k}\right) . \tag{5.11}$$

Conclusions and the central question *(i) It is reasonable to expect that the deviations of (5.11) from being group homomorphisms may reflect the* *singularities of the ANF* $\boldsymbol{\nu}_{\Lambda,k}$. *(ii) The maps* $\boldsymbol{\nu}_{\Lambda,k}$ *are related for different k by the process described above. (iii) Let* $\Xi_k \subset \tilde{\mathbf{J}}_{\Lambda,k,\Sigma_k}$ *be the non-torsion components of the universal Néron model. Then*

$$\operatorname{sing} \boldsymbol{\nu}_{\Lambda,k} = \boldsymbol{\nu}_{\Lambda,k}^{-1}(\Xi_k)\,.$$

(iv) Setting $d_k = \operatorname{codim} \Xi_k$,

$$\boldsymbol{\nu}_{\Lambda,k}^{*}([\Xi_k]) \in \Lambda \times H^{d_k}(\mathcal{M}_\Lambda)[\xi_k]/(\text{relations})$$

is a polynomial in ζ_k *whose coefficients* $C(\zeta,k)$ *are functions of* $\zeta \in \Lambda$ *and k with values in the ring* $H^*(\mathcal{M}_\Lambda)$.

Main question *What can one say about these coefficients?*

For example, what can be said about

$$C(\zeta+\zeta',k) - C(\zeta,k) - C(\zeta',k) + C(e,k)?$$

What are the relations between

$$C(\zeta,k+l) \text{ and } C(\zeta,k), C(\zeta,l)$$

that arise from the above (and other) geometric constructions?
We note that

$$\text{some } C(\zeta,k) \neq 0 \implies \text{existence theorem.}$$

Interesting two cases might be

(a) X_0 is abelian surface with principal polarization and with an additional Hodge class ζ.

In this case, $\dim \mathcal{M}_\zeta = 2$ and everything needed can be worked out explicitly, and

(b) $X_0 \subset \mathbb{P}^3$ is a quartic surfaces and ζ is the class of a line in X_0.

In this case, $\dim \mathcal{M}_\zeta = 18$ but much is known about it.

References

[AMRT] A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactification of locally symmetric varieties*, in Lie Groups: History, Frontiers and Applications, IV, Math. Sci. Press, 1975.

[BP1] P. Brosnan and G. Pearlstein, personal communication.

[BP2] P. Brosnan and G. Pearlstein, *The zero locus of an admissible normal function*, Ann. of Math. (2) **170** (2009), no. 2, 883–897.

[BBD] A. A. Beilinson, J. N. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque **100** Paris Soc. Math. France, Paris, 1982.

[BFNP] P. Brosnan, H. Fang, Z. Nie, and G. Pearlstein, *Singularities of admissible normal functions*, With an appendix by Najmuddin Fakhruddin, Invent. Math. **177** (2009), no. 3, 599-629.

[BLR] S. Bosch, W. Lűtkebohmert, M. Raynaud, *Néron Models*, Springer-Verlag, 1990.

[CDK] E. Cattani, P. Deligne, and A. Kaplan, *On the locus of Hodge classes*, J. Amer. Math. Soc. **8** (1995), 483–506.

[CKS] E. Cattani, A. Kaplan, and W. Schmid, *Degenerations of Hodge structures*, Ann. of Math. **123** (1986), 457–535.

[Cl] C. H. Clemens, *The Néron model for families of intermediate Jacobians acquiring "algebraic" singularities*, Inst. Hautes Études Sci. Publ. Math. **58** (1983), 5–18.

[dCM1] M. de Cataldo and L. Migliorini, *The Hodge theory of algebraic maps*, Ann. Soc. École Norm. Sup. (4) **38** (2005) no. 5, 693–750.

[dCM2] M. de Cataldo and L. Migliorini, *A remark on singularities of primitive cohomology classes*, arXiv:0711.1307v1[math.AG] 8Nov2007.

[Di] S. Diaz, *Complete subvarieties of the moduli space of smooth curves*, Proc. Symp. in Pure Math. **46** (1987), 77–81.

[D-M] P. Deligne, J.S. Milne, Arthur Ogus, Kuang-yen Shih, *Hodge cycles and abelian varieties*, (notes of a seminar of P. Deligne by J. Milne), in Hodge cycles, motives, and Shimura varieties. Lecture Notes in Mathematics, **900**, Springer-Verlag, 1982.

[EZ] F. El Zein and S. Zucker, *Extendability of normal functions associated to algebraic cycles*, in Topics in Transcendental Algebraic Geometry, Ann. of Math. Stud., 269–288, Princeton Univ. Press, 1984.

[G1] P. Griffiths, *Some results on algebraic cycles on algebraic manifolds*, Proc. Internat. Conf. on Algebraic Geometry, Tata Institute (Bombay) (1968), 93–191, Oxford Univ. Press.

[G2] P. Griffiths, *Infinitesimal invariant of normal functions*, in Topics in Transcendental Algebraic Geometry, (Princeton, N.J., 1981/1982), 305–316, Ann. of Math. Stud. **106**, Princeton Univ. Press, 1984.

[GG] M. Green and P. Griffiths, *Hodge theoretic invariants for algebraic cycles*, Int. Math. Res. Not. 2003, **9**, 477–510.

[GG1] M. Green and P. Griffiths, *Algebraic cycles and singularities of normal functions*, in Algebraic Cycles and Motives, Vol. 1, 206–263, London Math. Soc. Lect. Note Series **343**, Cambridge Univ. Press, 2007.

[GG2] M. Green and P. Griffiths, *Algebraic cycles and singularities of normal functions, II*, in Inspired by S. S. Chern, A Memorial Volume of A Great Mathematician, (P. Griffiths, ed.), 176–268, Nankai Tracts Math., 11, World Sci. Publ., 2006.

[GGK1] M. Green, P. Griffiths, and M. Kerr, *Néron models and boundary components for degenerations of Hodge structure of mirror quintic type*, in *Curves and abelian Varieties*, 71–145, Contemp. Math., 465, Amer. Math. Soc. 2008.

[GGK2] M. Green, P. Griffiths, and M. Kerr, *Néron models and boundary components for degenerations of Hodge structure of mirror quintic type*, to appear.

[GGK3] M. Green, P. Griffiths, and M. Kerr, *Some enumerative global properties of variations of Hodge structures*, Mosc. Math. J. **9** (2009), no. 3, 469–530.

[GMV] M. Green, J. Murre, and C. Voisin, *Algebraic cycles and Hodge theory*, Lectures given at the Second C.I.M.E., session held in Torino, June 21–29, 1993, (A. Albano and F. Bardelli, eds.), Lecture Notes in Mathematics **1594**, Springer-Verlag, 1994.

[Ka] M. Kashiwara, *A study of variation of mixed Hodge structure*, Publ. Res. Inst. Math. Sci. **22** (1986), no. 5, 991–1024.

[Ko] K. Kodaira, *On compact analytic surfaces, II*, Ann. of Math. **77** (1968), 563–626.

[KP1] M. Kerr and G. Pearlstein, *Normal functions and the GHC*, at http://www.maths.dur.ac.uk/ dma0mk.

[KP2] M. Kerr and G. Pearlstein, *An exponential history of functions with logarithmic growth*, to appear in Topology of Stratified Spaces, MSRI Pub. **58**, Cambridge University Press, 2010.

[KU] M. Kato and S. Usui, *Classifying spaces of degenerating polarized Hodge structures*, Ann. of Math. Studies, **169**, Princeton University Press, Princeton, 2009.

[Sa1] M. Saito, *Decomposition theorem for proper Kähler morphisms*, Tohoku Math. J. **42** (1990), 127–147.

[Sa2] M. Saito, *Admissible normal functions*, J. Alg. Geom. **5** (1996), 235–276.

[Sa3] M. Saito, *Hausdorff property of the Néron models of Green, Griffiths and Kerr*, (arXiv:0803.2771).

[Sa4] M. Saito, *Variations of Hodge structure havin no cohomologically nontrivial admissible normal functions*, preprint, 2009.

[Sch] C. Schnell, *Complex-analytic Neron models for arbitrary families of intermediate Jacobians*, http://arxiv.org/abs/0910.0662.

[V] C. Voisin, *Hodge loci and absolute Hodge classes*, Compos. Math. **143** (2007), no. 4, 945–958.

[Y] A. Young, *Complex analytic Néron models for degenerating Abelian varieties over higher dimensional parameter spaces*, Ph.D. Thesis, Princeton University, 2008.

[Z1] S. Zucker, *Generalized intermediate Jacobians and the theorem on normal functions*, Invent. Math. **33** (1976), 185–222.

[Z2] S. Zucker, *Degeneration of Hodge bundles (after Steenbrink)*, in Topics in Transcendental Algebraic Geometry, (P. Griffiths, ed.), Ann. of Math. Stud. **106** (1984), 121–142, Princeton Univ. Press.

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