SOME LEMMAS TO BE APPLIED TO THE EISENSTEIN SERIES

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Let V be a Euclidean space and let V' be its dual. Let $\lambda^1, \ldots, \lambda^n$ be a basis for V' such that $\langle \lambda^i, \lambda^j \rangle \leqslant 0$ if $i \neq j$ and let $\mu^1, \ldots, \mu^n$ in V' be such that $\langle \lambda^i, \mu^j \rangle = \delta_{ij}$. If F is a subset of $\{1, \ldots, n\}$ let V'_F be the subspace of V' spanned by $\{ \mu^i \mid i \in F \}$. Let $\mu^i_F = \mu_F$ if i belongs to F and let μ'_F be the projection of μ^i on the orthogonal complement of V'_F if i is not in F. Let λ^i_F, $1 \leqslant i \leqslant n$, be such that $\langle \lambda^i_F, \mu^j_F \rangle = \delta_{ij}$. If i is not in F then $\lambda^i_F = \lambda^i$. Suppose i is in F and write $\lambda^i_F = \lambda^i + \sum_{k \notin F} c_{ik} \lambda^k$. Then, for k not in F, $\langle \lambda^i_F, \mu^k_F \rangle = \langle \lambda^i, \mu^j_F \rangle + c_{ik}$. It follows from Lemma 2.5 of E.S.[1] that $\langle \lambda^i, \mu^k_F \rangle \leqslant 0$[2] so $c_{ik} \geqslant 0$. Consequently if H in V is such that $\lambda^i(H) > 0$, $1 \leqslant i \leqslant n$, then $\lambda^i_F(H) > 0$, $1 \leqslant i \leqslant n$. Suppose i is not equal to j. If one of i and j is not in F then $\langle \lambda^i_F, \lambda^j_F \rangle \leqslant 0$. However if i and j belong to F then $\langle \lambda^i_F, \lambda^j_F \rangle = \langle \lambda^i_F, \lambda^j \rangle = \langle \lambda^i, \lambda^j \rangle + \sum_{k \notin F} c_{ik} \langle \lambda^k, \lambda^j \rangle \leqslant 0$. Consequently for all i and j, $\langle \mu^i_F, \mu^j_F \rangle \geqslant 0$. A hyperplane in V defined by an equation of the form $\mu^i_F(X) = 0$ **[B.2]** or $\lambda^i_F(X) = 0$ for some i and F will be called a special hyperplane. Let H be a point in V which does not lie on any special hyperplane. Given H we shall define for every subset F a function φ_F on V. φ_F will vanish identically unless $\lambda^i_F(H) < 0$ for all i in F. If $\lambda^i_F(H) < 0$ for all i in F then $\varphi_F(X)$ will be zero unless $\lambda^j(X)$ is different from zero for all j and $\mu^j_F(H)\lambda^j(X)$ is positive when j is not in F where it will be one. Let $a(F)$ be the number of negative numbers in $\left\{ \mu^j_F(H) \,\middle|\, j \notin F \right\}$. We claim that $\sum_F (-1)^{a(F)} \varphi_F(X)$ is zero unless $\lambda^j(X) > 0$ for all j when it equals one. If $\lambda^j(H) > 0$ for all j then $\lambda^j_F(H) > 0$ and $\mu^j_F(H) > 0$ for all j. Consequently $\phi_F(X)$ vanishes identically unless F is empty. Moreover $a(\phi) = 1$ and $\varphi_\phi(X)$ is zero or one according as all the numbers $\lambda^j(X)$, $1 \leqslant j \leqslant n$, are positive or not. Thus for such an H the assertion is true. Suppose that H' and H'' are two points which do not lie on any special hyperplane. It is enough to show that **[B.3]**

$$\text{(a)} \qquad \sum_F (-1)^{a'(F)} \varphi'_F(X) = \sum_F (-1)^{a''(F)} \varphi''_F(X).$$

H' and H'' can be joined by a polygonal path no segment of which lies in a special hyperplane and no point of which lies on the intersection of two distinct special hyperplanes. If this path meets no special hyperplane the equality (a) is obvious. If we can show that the equality is satisfied when only one point of the path lies on a special hyperplane it will follow that the equality (a) is always true. Let the one point lie on a special hyperplane defined by $\alpha(X) = 0$. If H' and H'' lie on the same side of this hyperplane then H' and H'' can be joined by a polygonal path which meets no special hyperplane so that the equality (a) will be satisfied. We suppose then that H' and H'' be on opposite sides of the hyperplane. That is, that $\alpha(H')$ and $\alpha(H'')$ are of opposite sign. If F is a subset of $\{1, \ldots, n\}$ and none of λ^i_F, $i \in F$ or μ^j_F, $j \in F$ is a multiple of α then $\lambda^i_F(H') = \lambda^i_F(H'')$, $i \in F$, and $\mu^j_F(H') = \mu^j_F(H'')$,

[1]Editorial comment: Langlands, Robert P., *On the Functional Equations Satisfied by Eisenstein Series.*

[2]λ^k, $k \notin F$ or the orthogonal complement of V'_F satisfies usual conditions plus $\{\mu^k_F\}$ is its dual since $\langle \lambda^i, \lambda^k \rangle \leqslant 0$, $k \notin F$.

$j \notin F$ so that $a'(F) = a''(F)$ **[B.4]** and $\varphi'_F(X) \equiv \varphi''_F(X)$. Let $\mathfrak{S}_1$ be the collection of those F such that, for some i in F, λ^i_F is a multiple of α and let $\mathfrak{S}_2$ be the collection of those F such that, for some i not in F, μ^i_F is a multiple of α. In either case the integer i is uniquely determined. Moreover the sets $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are disjoint. Suppose F_1 is in $\mathfrak{S}_1$. Then F_1 is not empty; suppose, for simplicity, that $F_1 = \{1, \dots, k\}$ and that λ^k_F is a multiple of α. Let $F_2 = \{1, \dots, k-1\}$. Then $\lambda^k_{F_1}$ and $\mu^k_{F_2}$ both lie in the space spanned by $\mu^1, \dots, \mu^k$ and are both orthogonal to $\mu^1, \dots, \mu^{k-1}$. Consequently $\mu^k_{F_2}$ is a multiple of $\lambda^k_{F_1}$ and hence of α. We remark for later use that, since $\langle \mu^k, \lambda^k_{F_1} \rangle = 1$, $\mu^k_{F_2}$ is a positive multiple of $\lambda^k_{F_1}$. In any case F_2 belongs to $\mathfrak{S}_2$, since $\{1, \dots, n\}$ does not belong to $\mathfrak{S}_2$. This process can be reversed. Thus, in this way, we can set up a one-to-one correspondence between $\mathfrak{S}_1$ and $\mathfrak{S}_2$. Let F_1 and F_2 be two corresponding elements, say the two above. The equality (a) will follow if it is shown that

$$(-1)^{a'(F_1)} \varphi'_{F_1}(X) + (-1)^{a'(F_2)} \varphi'_{F_2}(X) = (-1)^{a''(F_1)} \varphi''_{F_1}(X) + (-1)^{a''(F_2)} \varphi''_{F_2}(X).$$

[B.5] Since $\lambda^k_{F_1}(H')$ and $\lambda^k_{F_1}(H'')$ are of opposite sign, at most one of $\varphi'_{F_1}(\cdot)$ and $\varphi''_{F_1}(\cdot)$ does not vanish identically. An argument like that used above shows that for i in F_2, $\lambda^i_{F_2} = \lambda^i_{F_1} + c_i \lambda^k_{F_1}$ with $c_i \geqslant 0$. If both the functions $\varphi'_{F_1}(\cdot)$ and $\varphi''_{F_1}(\cdot)$ vanish identically then, for some i in F_2, $\lambda^i_{F_1}(H') = \lambda^i_{F_1}(H'')$ is positive. Suppose $\lambda^k_{F_1}(H')$ is positive, then $\lambda^i_{F_2}(H') = \lambda^i_{F_1}(H') + c_i \lambda^k_{F_1}(H')$ is positive. Consequently $\lambda^i_{F_2}(H')$ is also positive and the functions $\varphi'_{F_2}(\cdot)$ and $\varphi''_{F_2}(\cdot)$ both vanish identically. The only case with which we need to concern ourselves is that in which precisely one of the functions $\varphi'_{F_1}(\cdot)$ and $\varphi''_{F_1}(\cdot)$ does not vanish identically. If we take $\lambda^k_{F_1}(H'')$ positive it will have to be the first. If j is not in F_1 then $\mu^j_{F_1}(H') = \mu^j_{F_1}(H'')$ and $\mu^j_{F_2}(H') = \mu^j_{F_2}(H'')$. Let $\mu^j_{F_1} = \mu^j_{F_2} + d_j \mu^k_{F_2}$. Then $0 = \langle \mu^j_{F_1}, \mu^k_{F_2} \rangle = \langle \mu^j_{F_2}, \mu^k_{F_2} \rangle + d_j \langle \mu^k_{F_2}, \mu^k_{F_2} \rangle$ so that d_j is not positive. If $\mu^j_{F_2}(H')$ is positive then $\mu^j_{F_1}(H') = \mu^j_{F_2}(H') + d_j \mu^k_{F_2}(H')$ is positive because $\mu^k_{F_2}(H')$ is negative; if $\mu^j_{F_2}(H')$ is negative then $\mu^j_{F_1}(H') = \mu^j_{F_1}(H'') = \mu^j_{F_2}(H'') + d_j \mu^k_{F_2}(H'')$ **[B.6]** is also negative because $\mu^k_{F_2}(H'')$ is positive. In particular then $a'(F_1) = 1 + a'(F_2) = a''(F_2)$ and we are reduced to showing that

$$\varphi'_{F_1}(X) \equiv \varphi'_{F_2}(X) + \varphi''_{F_2}(X).$$

This equality follows from the definitions.

Suppose that $\mathfrak{p}$ is an ordered partition of $\{1, \dots, n\}$ into the non-empty sets $F_1, \dots, F_r$. If i belongs to F_1 we let $\mu^i_{\mathfrak{p}} = \mu^i$ and if i belongs to F_u and $1 < u \leqslant r$ we let $\mu^i_{\mathfrak{p}}$ be the projection of μ^i on the orthogonal complement of $\{\, \mu^i \mid i \in F_v,\ v < u \,\}$. We also let $\lambda^i_{\mathfrak{p}}$, $1 \leqslant i \leqslant n$, be such that $\langle \lambda^i_{\mathfrak{p}}, \mu^j_{\mathfrak{p}} \rangle = \delta_{ij}$. As above $\langle \lambda^i_{\mathfrak{p}}, \lambda^j_{\mathfrak{p}} \rangle \leqslant 0$ if $i \neq j$ and $\langle \mu^i_{\mathfrak{p}}, \mu^j_{\mathfrak{p}} \rangle \geqslant 0$ for all i and j. For the present purpose let us call a hyperplane defined by an equation of the form $\mu^i_{\mathfrak{p}}(X) = 0$ or the form $\lambda^i_{\mathfrak{p}}(X) = 0$ for some $\mathfrak{p}$ and some i a special hyperplane. Suppose H is a point which does not lie on any special hyperplane. Define the function $\varphi_{\mathfrak{p}}$ by the condition that $\varphi_{\mathfrak{p}}(X)$ is zero unless $\lambda^i_{\mathfrak{p}}(X) \mu^i_{\mathfrak{p}}(H) > 0$ **[B.7]** for all i when it is one. Define the function $\psi_{\mathfrak{p}}$ by the condition that $\psi_{\mathfrak{p}}(X)$ is zero unless $\lambda^i_{\mathfrak{p}}(X)$ is positive for i in F_1 and $\lambda^i_{\mathfrak{p}}(X) \mu^i_{\mathfrak{p}}(H)$ is positive for i not in F_1 when it is one. Let a_i be the number of elements in F_i and let $a(\mathfrak{p})$ be the sum of $\sum_{i=1}^r (a_i + 1)$ and the number of i such that $\mu^i_{\mathfrak{p}}(H)$ is positive. Let $b(\mathfrak{p})$ be the sum of $1 + \sum_{i=2}^r (a_i + 1)$ and the number of i in $\bigcup_{i=1}^r F_i$ such that $\mu^i_{\mathfrak{p}}(H)$ is positive. We must verify that, if X lies on no special hyperplane,

$$\sum_{\mathfrak{p}}(-1)^{a(\mathfrak{p})}\varphi_{\mathfrak{p}}(X)=\sum_{\mathfrak{p}}(-1)^{b(\mathfrak{p})}\psi_{\mathfrak{p}}(X)$$

when $\lambda^i(H)$ is positive for some i and

$$\sum_{\mathfrak{p}}(-1)^{a(\mathfrak{p})}\varphi_{\mathfrak{p}}(X)=1+\sum_{\mathfrak{p}}(-1)^{b(\mathfrak{p})}\psi_{\mathfrak{p}}(X)$$

when $\lambda^i(H)$ is negative for all i. These relations are easily verified when $n=1$ so suppose that $n>1$ and that the assertion is true when the dimension of V is less than n. If F is a subset of $\{1,\dots,n\}$ different from F_0, the null set, and **[B.8]** $F_1=\{1,\dots,n\}$ let $\mathfrak{S}(F)$ be the collection of $\mathfrak{p}=\{F_1,\dots,F_r\}$ such that F_r is the complement of F. Every partition except $\mathfrak{p}_0=\big\{\{1,\dots,n\}\big\}$ belongs to exactly one of the collections $\mathfrak{S}(F)$. It follows from the induction assumption that

$$\sum_{\mathfrak{p}\in\mathfrak{S}(F)}(-1)^{a(\mathfrak{p})}\varphi_{\mathfrak{p}}(X)-\sum_{\mathfrak{p}\in\mathfrak{S}(F)}(-1)^{b(\mathfrak{p})}\psi_{\mathfrak{p}}(X)=-(-1)^{a(F)}\varphi_F(X).$$

Thus

$$\sum_{\mathfrak{p}}(-1)^{a(\mathfrak{p})}\varphi_{\mathfrak{p}}(X)-\sum_{\mathfrak{p}}(-1)^{b(\mathfrak{p})}\psi_{\mathfrak{p}}(X)$$
$$=-\sum{}'(-1)^{a(F)}\varphi_F(X)+(-1)^{a(\mathfrak{p}_0)}\varphi_{\mathfrak{p}_0}(X)-(-1)^{b(\mathfrak{p}_0)}\psi_{\mathfrak{p}_0}(X)$$

where the sum on the right is over all F except F_0 and F_1. Since $\varphi_{\mathfrak{p}_0}(X)=\varphi_{F_0}(X)$, $a(\mathfrak{p}_0)=1+a(F_0)$, and $\varphi_{F_1}(X)$ is zero unless $\lambda^i(H)$ is negative for all i when it is one. The two relations reduce to the equality

$$\sum_F(-1)^{a(F)}\varphi_F(X)=\psi_{\mathfrak{p}_0}(X)$$

which was proved in the previous paragraph.

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