Some Observations on the Infinitesimal Period Relations for Regular Threefolds with Trivial Canonical Bundle

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1. Introduction

It is well-known that, aside from algebraic curves, abelian varieties, and a few other isolated cases such as K3 surfaces, the period matrices of a family of algebraic varieties satisfy non-trivial universal infinitesimal period relations. In this note we shall discuss some remarkable properties of *any* local solution to the differential system given by the infinitesimal period relation associated to polarized Hodge structures of weight three with Hodge number $h^{3,0} = 1$.

Motivation for the study of this special case arises from the following algebro-geometric considerations: As is well known, an interesting class of projective algebraic varieties consists of those smooth X satisfying

$$(1.1)\qquad \begin{cases} K_X \simeq \mathcal{O}_X \\ q(X) = 0 \qquad \text{if } n = \dim X \geq 2. \end{cases}$$

Examples include elliptic curves, algebraic K3 surfaces, and smooth hypersurfaces $X \subset \mathbf{P}^{n+1}$ of degree $n+2$. For these varieties it is clear that the differential

$$(1.2)\qquad \kappa\colon H^1(X,\Theta) \longrightarrow \bigoplus_{p+q=n} \mathrm{Hom}\big(H^{p,q}(X), H^{p-1,q+1}(X)\big)$$

of the period mapping is injective (cf. Chapter III in[6]). In fact, the "first piece"

$$(1.3)\qquad \kappa\colon H^1(X,\Theta) \longrightarrow \mathrm{Hom}\big(H^{n,0}(X), H^{n-1,1}(X)\big)$$

is an isomorphism. This implies the infinitesimal Torelli theorem (Chapter VIII in loc. cit.), and it seems reasonable to expect a global Torelli theorem for X satisfying (1.1). For example, in the case of polarized K3 surfaces there is the theorem of Piateski-Shapiro and Shafarevich [5]. Interestingly, the smooth hypersurfaces of degree $n+2$ in $\mathbf{P}^{n+1}$ are exceptional cases in the recent generic global Torelli theorem of Donagi [4].

When $n = 3$ all of $H^3(X)$ is primitive, and in this paper we shall be interested in variations of Hodge structure that "look like" period matrices of such an X. More precisely, let us make the reasonable assumption that all of $H^1(X,\Theta)$ is unobstructed, so that the local moduli space (Kuranishi space) $\{X_s\}_{s\in S}$ of $X = X_0$ is smooth of dimension $m = h^1(X,\Theta) = h^{2,1}(X)$ with all Kodaira-Spencer mappings

$$\rho_s\colon T_s(S) \longrightarrow H^1(X_s,\Theta)$$

being isomorphisms. The period mapping for this family then gives an m-dimensional integral manifold

$$\varphi\colon S \longrightarrow D \subset \check{D}$$

for the differential system I on $\check{D}$ given by the infinitesimal period relations (this terminology is explained below). It turns out that m is the maximal dimension of integral manifolds of I, and we are able to put the differential system I on $\check{D}$ in local (in the *Zariski topology*) normal form. This in turn gives information on all integral manifolds of I, and from this we may draw several conclusions, two of which are:

(1.4) *Let $\alpha_0,\dots,\alpha_m;\ \beta_0,\dots,\beta_m \in H_3(X,\mathbf{Z})$ be a canonical homology basis (i.e., the intersection numbers $\alpha_i\cdot\alpha_j = \beta_i\cdot\beta_j = 0$, $\alpha_i\cdot\beta_j = \delta_{ij}$), $\omega(s) \in H^{3,0}(X_s)$ a non-zero generator, and set*

$$\begin{cases} A_i(s) = \int_{\alpha_i}\omega(s) & (A-\text{periods}) \\ B_i(s) = \int_{\beta_i}\omega(s) & (B-\text{periods}) \end{cases}$$

Then the complete Hodge structure $\{H^{p,q}(X_s)\}$ is determined by the functions $A_i(s)$.

More precisely, we shall show that the $B_i(s)$ are canonically and explicitly expressible in terms of the functions $A_i(s)$, $\partial A_i(s)/\partial s^j$; and that

$$H^{3,0}(X_s) \oplus H^{2,1}(X_s)$$

is then expressible in terms of the functions

$$A_i(s), \quad \frac{\partial A_i(s)}{\partial s^j}, \quad \frac{\partial^2 A_i(s)}{\partial s^i \partial s^j}.$$

By (2.2) below we then know the complete Hodge structure.

For the next result we shall use the concept of the infinitesimal variation of Hodge structure associated to X [3], which is given by the linear algebra data (1.2) in the manner expained at the end of this section.

(1.5) *Let $J(X)$ be the intermediate Jacobian of X. Then the infinitesimal variation of Hodge structure associated to X gives an $\otimes^2 H^{0,3}(X)$-valued cubic form Ξ on $T^*_{(0)}(J(X))$. Moreover, the algebraic invariants of this infinitesimal variation of this Hodge structure are uniquely determined by Ξ.*

It is our feeling that, in general, this cubic form uniquely determines X.

The precise statement of (1.5) is given in §2, and the cubic form is further discussed in §§6, 7. The precise formulation of (1.4) is given in §7 (cf. diagram (7.1) and the ensuing interpretation of it).

In concluding this section we want to make some remaks on terminology. Let M be a complex manifold and $\Omega^*_M = \bigoplus_{q\geq 0} \Omega^q_M$ the sheaf of exterior algebras given by all holomorphic forms. *By a sheaf of differential ideals* or *differential system* we shall mean a subsheaf of graded ideals $I \subset \Omega^*_M$ that is closed under exterior differentiation. An *integral manifold* of I will be given by a complex manifold S together with a holomorphic immersion

$$f: S \longrightarrow M \tag{1.6}$$

satisfying

$$f^*(I) = 0.$$

An *integral element* of I is given by a linear subspace $E \subset T_p(M)$ such that

$$\theta(p)\,|_E = 0 \text{ for all } \theta \in I.$$

Thus (1.6) is an integral manifold if, and only if,

$$f_*(T_s(S)) \subset T_{f(s)}(M)$$

is an integral element for all $s \in S$.

Given a sub-bundle $W \subset T^*(M)$, the holomorphic sections of W generate a differential ideal I (take I to be the ideal generated algebraically by the forms θ^α and $d\theta^\alpha$ where $\theta^1, \dots, \theta^s$ is a local coframe for W). Given a sub-bundle $V \subset T(M)$, we may set $W = V^\perp$ and take the corresponding differential ideal.

Let D be a classifying space for polarized Hodge structures and $\check{D}$ the dual classifying space for weight n Hodge filtrations

$$F^n \subset F^{n-1} \subset \cdots \subset F^1 \subset F^0 = H$$

satisfying the 1st Hodge-Riemann bilinear relation (cf. Chapter I in [6]). A holomorphic mapping

$$\varphi \colon S \longrightarrow \check{D}$$

may be thought of as a holomorphically varying filtration $\{F^p(s)\}$ on H, and the *infinitesimal period relation*

$$dF^p(s) \subseteq F^{p-1}(s) \tag{1.7}$$

shall mean the following: For $s^1, \dots, s^m$ local holomorphic coordinates on S and $v(s) \in F^p(s)$, all derivatives

$$\frac{\partial v(s)}{\partial s^i} \in F^{p-1}(s).$$

Thinking of tangent vectors as tangents to curves, (1.7) defines a sub-bundle $T_h(\check{D}) \subset T(\check{D})$, and the corresponding sheaf of differential ideals $I \subset \Omega^*_{\check{D}}$ will be called the *differential system giving the infinitesimal period relation.*

This paper was motivated by trying to see if studying I via the general theory of differential systems would yield any insight into period matrices of algebraic varieties. Our basic point is that in the case $n = 3$ and $h^{3,0} = 1$, I on $\check{D}$ is essentially the 1st prologation of another differential system that is birationally equivalent to the canonical contact system. More precisely, *in this paper we agree that integral manifolds for I in the case $n = 3, h^{3,0} = 1$ shall have the additional property that the map $\psi \colon S \longrightarrow \mathbf{P}H$ given by $\psi(s) = F^3(s) \subset H$ be an immersion,* and then the statement about prolongations is correct. In this way we are able to determine all local maximal integral manifolds of I and to draw the conclusions mentioned above.

One final terminology we shall use is that of an *infinitesimal variation of Hodge structure* $\{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$. This is given by a polarized Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\} = p \in D$ together with an injective linear mapping

$$\delta\colon T \longrightarrow T_p(D) \tag{1.8}$$

whose image $E = \delta(T)$ is an integral element for the differential system giving the infinitesimal period relation (1.7). More concretely, (1.8) is given by

$$\delta\colon T \longrightarrow \bigoplus_{p+q=n} \operatorname{Hom}(H^{p,q}, H^{p-1,q+1}) \tag{1.9}$$

satisfying the conditions

$$\begin{cases} (i) & \delta(\xi_1)\delta(\xi_2) = \delta(\xi_2)\delta(\xi_1) \qquad \xi_1, \xi_2 \in T \\ (ii) & Q(\delta(\xi)\eta, \psi) + Q(\eta, \delta(\xi)\psi) = 0 \qquad \xi \in T \end{cases} \tag{1.10}$$

where $\eta \in H^{p,q}, \psi \in H^{n-p+1,q-1}$. We remark that if $\theta^1, \ldots, \theta^s$ are local holomorphic 1-forms on $\check{D}$ such that $\theta^\alpha = 0$ defines $T_h(\check{D}) \subset T(D)$, then (1.9) is equivalent to $\theta^\alpha(p)|_E = 0$ and (i) in (1.10) is equivalent to $d\theta^\alpha(p)|_E = 0$. Condition (ii) in (1.10) is the obvious one that the infinitesimal variation preserve the 1st bilinear relation.

2. Variation of Hodge Structure Associated to Certain 3-Folds

Let $H_{\mathbf{Z}}$ be a lattice of rank $2(m+2)$ (thus $H_{\mathbf{Z}} \simeq \mathbf{Z}^{2m+2}$) having a non-degenerate alternating bilinear form

$$Q\colon H_{\mathbf{Z}} \otimes H_{\mathbf{Z}} \longrightarrow \mathbf{Z}.$$

We will be concerned with polarized Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q\}$ of weight three on $H = H_{\mathbf{Z}} \otimes \mathbf{C}$ having Hodge numbers

$$h^{3,0} = 1,\ h^{2,1} = m.$$

This is given either by a Hodge decomposition

$$\begin{cases} H = H^{3,0} \oplus H^{2,1} \oplus H^{1,2} \oplus H^{0,3} \\ H^{p,q} = \overline{H^{q,p}} \end{cases}$$

satisfying the 1st and 2nd Hodge-Riemann bilinear relations, or by the corresponding Hodge filtration

$$(0) \subset F^3 \subset F^2 \subset F^1 \subset F^0 = H \tag{2.1}$$

where $F^p = \bigoplus_{k \geq 0} H^{p+k,3-p-k}$. We remark that the 1st bilinear relation is

$$\begin{cases} F^1 = F^{3\perp} \\ F^2 = F^{2\perp} \end{cases} \tag{2.2}$$

where $\perp$ denotes the orthogonal complement with respect to Q.

Motivated by the study of variation of Hodge structure for threefolds satisfying (1.1), we want to study holomorphic mappings,

$$\varphi\colon S \longrightarrow \check{D} \tag{2.3}$$

where S is a complex manifold and where the following conditions are satisfied:

(i) $\dim S = m = h^{2,1}$ and the composed mapping

$$S \overset{\varphi}{\to} \check{D} \overset{\pi}{\to} \mathbf{P}H \simeq \mathbf{P}^{2m+1} \tag{2.4}$$

is an immersion, where $\pi\{F^p\} = \{F^3 \subset H\}$ (this is a reflection of (1.3));

(ii) the infinitesimal period relations

$$\begin{cases} (i) & dF^3 \subseteq F^2 \\ (ii) & dF^2 \subseteq F^1 \end{cases} \tag{2.5}$$

are satisfied (this notation is explained in §1 – cf. just below (1.7)).

We remark that, in this case, $(i) \Rightarrow (ii)$ in (2.5). To see this let $s^1, \ldots, s^m$ be local holomorphic coordinates on S and write $\varphi(s) = \{F^p(s)\}$. If $0 \neq z(s) \in F^3(s)$, then (i) in (2.5) gives

$$\frac{\partial z(s)}{\partial s^i} \in F^2(s) \qquad i = 1, \ldots, m.$$

By (2.2) this implies that

$$(2.6)\qquad \begin{cases} (i) & Q\left(z(s), \dfrac{\partial z(s)}{\partial s^j}\right) = 0 \\ (ii) & Q\left(\dfrac{\partial z(s)}{\partial s^i}, \dfrac{\partial z(s)}{\partial s^j}\right) = 0 \end{cases}$$

The derivatives of (i) give, using (ii), that

$$(2.7)\qquad Q\left(z(s), \frac{\partial^2 z(s)}{\partial s^i \partial s^j}\right) = 0$$

On the other hand, since (2.4) is an immersion

$$\dim \operatorname{span}\left\{z(s), \frac{\partial z(s)}{\partial s^1}, \ldots, \frac{\partial z(s)}{\partial s^m}\right\} = m + 1.$$

This implies that

$$F^2(s) = \operatorname{span}\left\{z(s), \frac{\partial z(s)}{\partial s^1}, \ldots, \frac{\partial z(s)}{\partial s^m}\right\},$$

and then (2.2) and (2.7) give that

$$dF^2(s) \subseteq F^1(s).$$

Next we want to comment on an infinitesimal variation of Hodge structure $\{H_{\mathbb{Z}}, H^{p,q}, Q, T, \delta\}$ where $\{H_{\mathbb{Z}}, H^{p,q}, Q\}$ is a Hodge structure of the type we are considering and where

$$\delta\colon T \longrightarrow \operatorname{Hom}(H^{3,0}, H^{2,1})$$

is an isomorphism (here we may think of $T = T_s(S)$ and $\delta = \varphi_*$). Using the natural identifications

$$\begin{cases} H^{1,2} \simeq H^{2,1^*} \\ H^{0,3} \simeq H^{3,0^*} \end{cases}$$

the infinitesimal variation of Hodge structure induces maps

$$(2.8)\qquad \begin{cases} (i) & T \otimes H^{3,0} \xrightarrow{\sim} H^{2,1} \\ (ii) & T \otimes H^{2,1} \rightarrow H^{2,1^*} \\ (iii) & T \otimes H^{2,1^*} \xrightarrow{\sim} H^{3,0^*} \end{cases}$$

where (i) is an isomorphism, (ii) is a symmetric map, and (iii) is the dual of (i). These three combine to induce a map

$$\delta^{(3)}: T \otimes T \otimes T \longrightarrow \otimes^2 H^{0,3} \qquad (\simeq \mathbf{C}), \tag{2.9}$$

and (i) in (1.10) implies that this mapping is *symmetric;* i.e., we have an induced map

$$\Xi: \mathrm{Sym}^3 T \longrightarrow \otimes^2 H^{0,3} \tag{2.10}$$

Definition. We shall call (2.10) the *cubic form* Ξ associated to the infinitesimal variation of Hodge structure $\{H_{\mathbf{Z}}, H^{p,q}, Q, T, \delta\}$.

Given Ξ we may define $\delta^{(3)}$ in (2.9), and then if we set $H^{2,1} = T \otimes H^{0,3^*}$ we may define the maps (2.8) where (i), (iii) are the identity and (ii) is given by $\delta^{(3)}$. In this way, any algebraic invariant of the linear algebra data (1.2) (in this case) is uniquely determined by Ξ. This is our assertion (1.5).

For a mapping (2.3) satisfying the condition that (2.4) be an immersion, we have for each $s \in S$ a cubic form $\Xi(s)$ on $T_s(S)$ with values in $\otimes^2 H^{0,3}(s)$. To compute $\Xi(s)$ we let $s^1, \ldots, s^m$ be local holomorphic coordinates on S and $0 \neq z(s) \in F^3(s)$. For $\xi = \sum_i \xi^i \partial/\partial s^i \in T_s(S)$, it follows from (2.6) and (2.7) that

$$\Xi(s)(\xi) = Q\left(z(s), \sum_{i,j,k} \xi^i \xi^j \xi^k \frac{\partial^3 z(s)}{\partial s^i \partial s^j \partial s^k}\right) \tag{2.11}$$

is a well-defined function with values in $H^{0,3}(s)$, and this is the cubic form $\Xi(s)$.

For a geometrically given infinitesimal variation of Hodge structure (1.2) where X is a 3-fold satisfying (1.1), the cubic form is given as follows: Represent $H^1(X, \Theta)$ by Dolbeault cohomology $H^{0,1}_{\bar\partial}(X, \Theta)$ and define

$$\det: H^1(X, \Theta) \longrightarrow H^3(X, K_X^*) \tag{2.12}$$

by

$$\det\left(\sum_{i,j} \theta^i_{\bar j} \partial/\partial z^i \otimes d\bar z^j\right) = \det\|\theta^i_{\bar j}\| \cdot \wedge^i \partial/\partial z^i \otimes \wedge^j d\bar z^j.$$

Using the natural isomorphism

$$H^3(X, K_X^*) \simeq \otimes^2 H^{0,3}(X)$$

resulting from $K_X \simeq \mathcal{O}_X$, it follows from standard arguments (cf. [3]) that the cubic form is given by (2.12).

In the case of a smooth quintic hypersurface $X \subset \mathbf{P}^4$ given by $F(x) = 0$, we let

$$\begin{cases} S_d &= \text{forms } P(x) \text{ of degree } d \\ J_{F,d} &= \text{Jacobian ideal in degree } d. \end{cases}$$

Then (cf. [3]) the Grothendieck residue symbol gives an isomorphism

$$\text{Res} : S_{15}/J_{F,15} \simeq \mathbf{C},$$

and the cubic form is the natural mapping

$$\text{Sym}^3(S_5/J_{F,5}) \to S_{15}/J_{F,15} \simeq \mathbf{C}$$

indiced by multiplying polynomials (loc. cit.).

3. Contact Systems and Legendre Manifolds

A *contact manifold* is given by a complex manifold M of dimension $2m+1$ together with a holomorphic line sub-bundle $L \subset T^*(M)$ such that, if ω is a local generator of $\mathcal{O}(L) \subset \Omega^1_M$, then

$$\omega \wedge (d\omega)^m \neq 0. \tag{3.1}$$

It is easy to verify that this condition is independent of ω.

Given a contact manifold (M, L), a *Legendre manifold* is given by an m-dimensional complex manifold S together with an immersion

$$f\colon S \longrightarrow M \tag{3.2}$$

satisfying

$$f^*(\omega) = 0. \tag{3.3}$$

Put differently, let $I \subset \Omega^*_M = \bigoplus_{q \geq 0} \Omega^q_M$ be the sheaf of differential ideals generated over Ω^*_M by ω and $d\omega$. An integral manifold of I is given by a holomorphic immersion (3.2), where now S may be a complex manifold of any dimension, satisfying

$$f^*(I) = 0. \tag{3.4}$$

Since $f^*\colon \Omega^*_M \longrightarrow \Omega^*_S$ is a map of differential algebras, (3.3) and (3.4) are equivalent conditions, and so Legendre manifolds are simply m-dimensional integral manifolds of I.

Let $p \in M$ and $E \subset T_p(M)$ be an integral element of I. This is equivalent to

$$\omega(p)|_E = 0, \quad d\omega(p)|_E = 0.$$

It is well-known that any such integral element has $\dim E \leq m$. It follows that integral manifolds of I have dimension $\leq m$, and those of maximal dimension are exactly the Legendre manifolds.

Given a contact manifold (M, L) and local generator ω of $\mathcal{O}(L)$, by the well known theorem of Pfaff-Darboux (cf. [2]) we may choose local holomorphic coordinates $(x^1, \ldots, x^m, u, y_1, \ldots, y_m) = (x, u, y)$ for M such that

$$\omega = du - \sum_{i=1}^{m} y_i dx^i.$$

For a "general" Legendre manifold we will have $f^*(dx^1 \wedge \cdots \wedge dx^m) \neq 0$, and then locally the Legendre manifold is given parametrically by

$$x \to \big(x, u(x), y(x)\big).$$

The condition (3.3) is

$$y_i(x) = \frac{\partial u(x)}{\partial x^i},$$

so that the Legendre manifold is locally given by the 1-jet $\big(x, u(x), \partial u(x)/\partial x\big)$ of an arbitrary function $u(x)$.

In the next section we will give a global algebro-geometric version of this construction.

4. The Canonical Contact System and its Legendre Manifolds

We will describe a canonical example of a contact manifold and determine its Legendre manifolds. Let V be an $(m+2)$-dimensional vector space with coordinates $w^0, w^1, \ldots, w^{m+1}$; denote by $p_0, p_1, \ldots, p_{m+1}$ the dual coordinates in V^*. In $\mathbf{P}V \times \mathbf{P}V^*$ we consider the *incidence subvariety*

$$R \subset \mathbf{P}V \times \mathbf{P}V^*$$

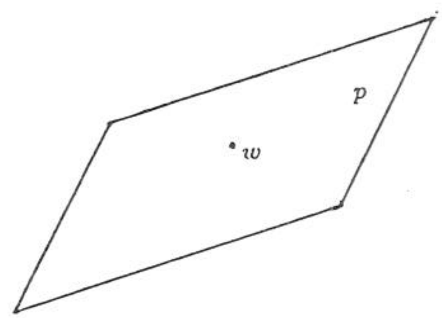
defined by

$$\sum_{\alpha=0}^{m+1} p_\alpha w^\alpha = 0.$$

We may also write this as

$$\langle p, w \rangle = 0, \tag{4.1}$$

and we think of points in R as pairs (w, p) where $w \in \mathbf{P}V$ and $p \in \mathbf{P}V^*$ is a hyperplane in $\mathbf{P}V$ containing w.



Such pairs (w, p) are sometimes called *contact elements.*

We shall give a canonical contact structure on R. For this we consider the standard projection

$$\begin{array}{c} (V \setminus \{0\}) \times (V^* \setminus \{0\}) \\ \downarrow \tilde{\omega} \\ \mathbf{P}V \times \mathbf{P}V^*, \end{array} \tag{4.2}$$

and on $V \times V^*$ we consider the tautological 1-form

$$\Omega = \langle p, dw \rangle = \sum_{\alpha} p_\alpha dw^\alpha. \tag{4.3}$$

For any holomorphic cross-section s of (4.2) over an open set $U \subset R$ we consider the 1-form on U given by

$$\omega = s^*\Omega = \langle p \circ s,\ d(w \circ s) \rangle. \tag{4.4}$$

Using (4.1) we may easily verify that ω is well-defined up to non-zero multiples.

We shall now check that

$$\omega \wedge (d\omega)^m \neq 0$$

by explicit computation. For this we will give a covering of R by open sets isomorphic to $\mathbf{C}^{2m+1}$ and over each open set a natural crosss-section s. Suppose that $(w,p) \in R$ and that $w^0 \neq 0$. Since $\langle p, w \rangle = 0$ it follows that one of $p_1, \ldots, p_{m+1}$ must be non-zero. Suppose that $p_{m+1} \neq 0$, and for $\mathbf{C}_R^{2m+1}$ with coordinates $(w^1, \ldots, w^m, p_0, \ldots, p_m)$ consider the diagram.

$$\begin{array}{ccc} & \overset{s}{\nearrow} & (V \setminus \{0\}) \times (V^* \setminus \{0\}) \\ & & \downarrow \tilde{\omega} \\ \mathbf{C}_R^{2m+1} & \underset{j}{\hookrightarrow} & R \subset \mathbf{P}V \times \mathbf{P}V^* \end{array} \tag{4.5}$$

where

$$s(w^1, \ldots, w^m, p_0, \ldots, p_m) = \left(1, w^1, \ldots, w^m, -\left(p_0 + \sum_{i=1}^{m} p_i w^i\right)\right) \times (p_0, \ldots, p_m, 1).$$

It is clear that $s(\mathbf{C}_R^{2m+1}) \subset \tilde{\omega}^{-1}(R)$ and that $j = \tilde{\omega} \circ s$ is one-to-one. By (4.4)

$$\begin{aligned} \omega &= \langle p \circ s,\ d(w \circ s) \rangle \\ &= \sum_i p_i dw^i - d\left(p_0 + \sum_i p_i w^i\right) \\ &= -\left(dp_0 + \sum_i w^i dp_i\right). \end{aligned}$$

Thus

$$d\omega = -\sum dw^i \wedge dp_i$$

and

$$\omega \wedge (d\omega)^m = (-1)^{m+1} m! \quad dp_0 \wedge dw^1 \wedge dp_1 \wedge \cdots \wedge dw^m \wedge dp_m.$$

Definition. We shall call ths construction the *canonical contact structure*, and shall denote it by (R, J).

Here, $J \subset T^*(R)$ is the line bundle determined by the 1-form (4.3) using the prescription (4.4).

We shall now determine the Legendre manifolds

$$f: S \longrightarrow R \tag{4.6}$$

for the canonical contact structure. Assuming that S is connected and has local holomorphic coordinates $s^1, \ldots, s^m$, we may locally lift (4.6) to $(V \setminus \{0\}) \times (V^* \setminus \{0\})$ and give f by a vector-valued holomorphic function

$$s \to \big(w(s), p(s)\big)$$

where

$$\begin{cases} (i) & \sum_\alpha p_\alpha(s) w^\alpha(s) = 0 \\ (ii) & \sum_\alpha p_\alpha(s) dw^\alpha(s) = 0. \end{cases} \tag{4.7}$$

Consider the mapping

$$w: S \longrightarrow \mathbf{P}V$$

given by projecting f on the first factor, and suppose that this mapping has rank k at a general point. The image

$$X = w(S) \subset \mathbf{P}V$$

is then a k-dimensional piece of analytic variety. Denote by $X_{reg} \subset X$ the open dense set of smooth points, and for $w \in X_{reg}$ denote by $T_w(X) \subset \mathbf{P}V$ the tangent k-plane to X_{reg} at W.

Definition. We define the *Gauss correspondence*

$$\Gamma_X \subset R$$

to be the closure of the set

$$\{(w,p) : w \in X_{reg} \text{ and } T_w(X) \subseteq p\}.$$

It is clear that $\dim \Gamma_X = m$ and that we have a diagram

$$\begin{array}{ccc} & \Gamma_X \subset R \subset \mathbf{P}V \times \mathbf{P}V^* & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbf{P}V \supset X & & X^* \subset \mathbf{P}V^* \end{array}$$

where X^* is the *dual variety* of tangent hyperplanes to X. Equations (i) and (ii) in (4.7) say exactly that:

(4.8) *The image of a Legendre manifold for the canonical contact structure is a Gauss correspondence.*

To have the converse we must relax the condition that (4.6) be an immersion, and simply require that f have maximal rank at a general point of each irreducible component of S. With this technicality being understood, using resolution of singularities we may say that:

(4.9) *The images of Legendre manifolds for the canonical contact structure are exactly the Gauss correspodences of analytic subvarieties $X \subset \mathbf{P}V$.*

Remark. Using (4.1) we have on $\tilde{\omega}^{-1}(R)$

$$\Omega = \langle p, dw \rangle = -\langle dp, w \rangle.$$

This suggests that in the construction of the canonical contact system the roles of w and p (i.e., of $\mathbf{P}V$ and $\mathbf{P}V^*$) may be interchanged. In fact this is obviously the case, and using (4.9) reflects the well-known fact that

$$(X^*)^* = X;$$

i.e., for subvarieties $X \subset \mathbf{P}V$ the dual of the dual is the original X (no matter what $\dim X$ is).

5. The Canonical Contact System Associated to an Alternating Bilinear from Q

We will now describe another example of an algebraic contact manifold that on the one hand has to do with variation of Hodge structure, and that on the other hand may be mapped by a birational contact transformation into the canonical contact system.

Let H be a complex vector space of dimension $2(m+1)$ and

$$Q\colon H \otimes H \longrightarrow \mathbf{C} \tag{5.1}$$

a non-degenerate alternating bilinear form. Setting

$$P = \mathbf{P}H \simeq \mathbf{P}^{2m+1}$$

we shall canonically associate to (H,Q) a contact manifold (P,L). For this we let z denote a typical point of H and consider the projection

$$\begin{array}{c} H \setminus \{0\} \\ \downarrow \pi \\ P \end{array} \tag{5.2}$$

Given a local holomorphic section s of (5.2) we set

$$\begin{aligned} \omega &= s^* Q(dz, z) \\ &= Q\big(d(z \circ s), z \circ s\big). \end{aligned}$$

Since $Q(z,z) = 0$, it follows that ω is well-defined up to non-zero multiples.

To show that this gives a contact structure, we consider the alternating bilinear form (5.1) as an element

$$Q \in \wedge^2 H^*$$

and choose coordinates $z^0, z^1, \ldots, z^{2m+1} \in H^*$ for H so that

$$Q = z^0 \wedge z^{m+1} + \cdots + z^m \wedge z^{2m+1}. \tag{5.3}$$

Then

$$Q(dz, z) = \sum_{\alpha=0}^{m} z^{m+1+\alpha} dz^{\alpha} - z^{\alpha} dz^{m+1+\alpha}.$$

For $\mathbf{C}_P^{2m+1}$ with coordinates $(x^1, \ldots, x^m, y_0, y_1, \ldots, y_m) = (x, y)$ we consider the diagram

$$\begin{array}{ccc} & \overset{s}{\nearrow} & H\backslash\{0\} \\ & & \downarrow \pi \\ \mathbf{C}_P^{2m+1} & \hookrightarrow & P \end{array}$$

where

$$s(x, y) = (1, x^1, \ldots, x^m, y_0, \ldots, y_m)$$

Then for this cross-section s,

$$\begin{aligned} \omega &= Q\big(ds(x, y),\ s(x, y)\big) \\ &= -dy_0 + \sum_{i=1}^{m} y_i dx^i - x^i dy_i, \end{aligned}$$

and

$$d\omega = 2 \sum_i dy_i \wedge dx^i.$$

Consequently

$$\omega \wedge (d\omega)^m = (-1)^{m+1} 2^m m! \quad dy_0 \wedge dx^1 \wedge dy_1 \wedge \ldots dx^m \wedge dy_m,$$

so that we have in this way determined a contact manifold (P, L).

Postponing the Hodge-theoretic discussion until the two next sections, we shall give a birational contact transformation of (P, L) to the canonical contact manifold (R, J). For this we recall the two open sets

$$\begin{cases} \mathbf{C}_R^{2m+1} \subset R \\ \mathbf{C}_P^{2m+1} \subset P \end{cases}$$

given in (4.5) and (5.4). We shall define a one-to-one rational holomorphic mapping

(5.5) $$T \colon \mathbf{C}_R^{2m+1} \longrightarrow \mathbf{C}_P^{2m+1}$$

that preserves the contact forms. This mapping is given by the formulas

$$(5.6) \qquad \begin{cases} y_0 = p_0 + \frac{1}{2}\sum_{i=1}^{m} w^i p_i \\ y_i = \frac{1}{\sqrt{2}} p_i \\ x^i = \frac{1}{\sqrt{2}} w^i \end{cases}$$

Under this transformation

$$dy_0 + \sum_i (x^i dy_i - y_i dx^i) = dp_0 + \sum_i w^i dp_i,$$

so that (5.6) is indeed a birational contact transformation.

In concluding this section we want to discuss briefly the data needed to define (5.5). Given a filtration

$$F^3 \subset F^2 \subset F^1 \subset F^0 = H$$

with $\dim F^3 = 1, \dim F^2/F^1 = \dim F^1/F^2 = m$ and satisfying the 1st Hodge-Riemann bilinear relation

$$(5.7) \qquad \begin{cases} F^1 = F^{3\perp} \\ F^2 = F^{2\perp} \end{cases}$$

where $\perp$ is with respect to the bilinear form (5.1), we may choose coordinates $z^0, \dots, z^{2m+1}$ so that (5.3) holds and where

$$(5.8) \qquad \begin{cases} F^3 = \{z^1 = \cdots = z^{2m+1} = 0\} \\ F^2 = \{z^{m+1} = \cdots = z^{2m+1} = 0\} \\ F^1 = \{z^{m+1} = 0\} \end{cases}$$

Having choosen such a coordinate system we may define T by (5.6). Of course such a coordinate system is not unique. In particular, a transformation

$$(5.9) \qquad \begin{cases} \tilde{z}^0 = z^0 \\ \tilde{z}^i = z^i + \sum_{j=1}^{m} q^{ij} z^{m+1+j}, & q^{ij} = q^{ji}, \\ \tilde{z}^{m+i+\alpha} = z^{m+i+\alpha}, & 0 \le \alpha \le m, \end{cases}$$

leaves invariant the form (5.3) and sets of equations (5.8). Under a transformation (5.9) we have

$$\begin{cases} \tilde{x}^i = x^i + \sum_j q^{ij} y_j, & q^{ij} = q^{ji} \\ \tilde{y}_\alpha = y_\alpha \end{cases}$$

and, using (5.6),

$$\begin{cases} \tilde{w}^i = w^i + \sum_j q^{ij} p_j \\ \tilde{p}_i = p_i \\ \tilde{p}_0 = p_0 - \sum_{i,j} q^{ij} p_i p_j \end{cases}$$

Remark. One of our motivations for studying this particular birational contact transformation is that the case $m = 1$ has already proved useful in another context, which we now explain. For more details consult the paper by the first author [1].

The celebrated "twistor" map of Penrose is a smooth fibration $\tau: \mathbf{P}^3 \to S^4$ where the fibres are linear $\mathbf{P}^1$'s. It yields a method of transforming Riemannian geometry problems in S^4 into complex analysis problems in $\mathbf{P}^3$. In particular, the complex 2-plane field on $\mathbf{P}^3$ orthogonal (in the Fubini-Study metric) to the fibers of τ is dual to a holomorphic line bundle $L \subset T^*\mathbf{P}^3$ which furnishes a contact structure on $\mathbf{P}^3$ with the following remarkable property: the holomorphic integral curves of this contact structure project via τ to be minimal surfaces in S^4.

Using the above birational transformation in the case $m = 1$, we are able to transform algebraic curves in $\mathbf{P}^2$ into minimal surfacs in S^4. From the fact that every compact Riemann surface occurs in $\mathbf{P}^2$ as an algebraic curve and from an elementary general position construction, the first author then concludes that every compact Riemann surface immerses minimally and conformally in S^4.

6. How Period Mappings Uniquely Arise as 1st Prolongations of the Legendre Manifolds Associated to Q

We retain the notations of the preceeding sections. Let S be a k-dimensional complex manifold and

$$(6.1) \qquad \varphi_0: S \longrightarrow P$$

an immersion. Denote by $\mathcal{F}(1,k)$ the manifold of all flags

$$\begin{cases} F^3 \subset F^2 \subset H & \text{where} \\ \dim F^3 = 1 \text{ and } \dim F^3/F^2 = k \end{cases}$$

We define the *1st prolongation* of (6.1) to be the map

$$\varphi_0^{(1)}\colon S \longrightarrow \mathcal{F}(1,k)$$

given for $s \in S$ by

$$\begin{cases} F^3(s) = \varphi_1(s) \\ F^2(s) = \text{projective tangent space to } \varphi_0(S) \text{ at } \varphi_0(s). \end{cases}$$

If locally φ_0 is given by

$$(s^1, \dots, s^k) \to z(s^1, \dots, s^k) \in H \setminus \{0\},$$

then

$$\begin{cases} F^3(s) = \operatorname{span}\{z(s)\} \\ F^2(s) = \operatorname{span}\left\{z(s), \dfrac{\partial z(s)}{\partial s^1}, \dots, \dfrac{\partial z(s)}{\partial s^k}\right\} = \operatorname{span}\left\{z(s), \dfrac{\partial z(s)}{\partial s^i}\right\}. \end{cases}$$

If also the linear subspace

$$F^1(s) = \operatorname{span}\left\{z(s), \frac{\partial z(s)}{\partial s^i}, \frac{\partial^2 z(s)}{\partial s^i \partial s^j}\right\}$$

has constant dimension $k + l + 1$, then with the obvious notation we may define the *2nd prolongation*

$$\varphi_0^{(2)}\colon S \longrightarrow \mathcal{F}(1,k,l)$$

by

$$\varphi_0^{(2)}(s) = \{F^3(s) \subset F^2(s) \subset F^1(s)\}$$

(the reason for the indexing on the $F^p(s)$ will appear in a moment).

Remark 6.2. (i) In general, making suitable constant rank assumptions, we may define the k^{th} *prolongation* to be the mapping given by the k^{th} *osculating flag*

$$\operatorname{span}\{z(s)\} \subset \operatorname{span}\{z(s), \partial z(s)\} \subset \cdots \subset \operatorname{span}\{z(s), \ldots, \partial^k z(s)\}$$

where $\partial^k z(s)$ denotes all vectors $\partial^k z(s)/\partial s^{i_1} \cdots \partial s^{i_k}$.

(ii) Denoting tangent vectors to S by

$$\xi = \sum_i \xi^i \partial/\partial s^i \in T_s(S),$$

the k^{th} *fundamental form* of (6.1) is by definition the symmetric k-linear function on $T(S)$ with values in $H/\operatorname{span}\{z(s), \ldots, \partial^{k-1} z(s)\}$ given by

$$\xi \to \sum_{i_1,\ldots,i_k} \xi^{i_1} \cdots \xi^{i_k} \frac{\partial^k z(s)}{\partial s^{i_1} \cdots \partial s^{i_k}} \in H/\operatorname{span}\{z(s), \ldots, \partial^{k-1} z(s)\}.$$

Equivalently, it is given by the linear function on

$$\left(H/\operatorname{span}\{z(s), \ldots, \partial^{k-1} z(s)\}\right)^*$$

defined for $\xi \in T_s(S)$ and $\lambda \in \left(H/\operatorname{span}\{z(s), \ldots, \partial^{k-1} z(s)\}\right)^*$ by

$$(\xi, \lambda) \to \langle \lambda, \sum_{i_1,\ldots,i_k} \xi^{i_1} \cdots \xi^{i_k} \frac{\partial^k z(s)}{\partial s^{i_1} \cdots \partial s^{i_k}} \rangle$$

(iii) We note the *expected dimension count*

$$\dim F^1(s) = k + 1 + k(k+1)/2 = (k+1)(k+2)/2.$$

To say that $\dim F^1(s) < (k+1)(k+2)/2$ means that the vector-valued function $z(s)$ satisfies a linear 2nd order PDE system.

Returning to the general discussion, we suppose that $k = m$ so that we have a canonical inclusion

$$\check{D} \subset \mathcal{F}(1, m)$$

(recall (cf. (2.2)) that $\{F^3 \subset F^2 \subset F^1 \subset H\} \in \check{D}$ is uniquely determined by $\{F^3 \subset F^2 \subset H\}$). We denote by $I \subset \Omega^*_{\check{D}}$ the sheaf of differential ideals given by the infinitesimal period relation (2.5). One of our main points is the following observation:

(6.3) *Let $\varphi_0\colon S \longrightarrow P$ be a Legendre manifold for the contact system on $P = \mathbf{P}H$ given by the alternating form Q. Then its 1st prolongation is a map*

$$\varphi_0^{(1)}\colon S \longrightarrow \check{D}$$

that is an m-dimensional integral manifold of the differential system I on $\Omega^*_{\check{D}}$. Coversely, any m-dimensional integral manifold of I is the 1st prolongation of such a Legendre manifold.

Proof. Let $\varphi_0\colon S \longrightarrow P$ be given locally by a vector-valued function

$$(s^1, \dots s^m) \rightarrow z(s) \in H \setminus \{0\}.$$

Then

$$\omega(s) = Q\big(dz(s), z(s)\big)$$

is the pullback under φ_0 of a local generator for the contact system on P. Using that $d\omega(s) = Q\big(dz(s), dz(s)\big)$, the conditions

$$\begin{cases} \omega(s) = 0 \\ d\omega(s) = 0 \end{cases}$$

give respectively

$$\begin{cases} (i)\ Q\left(\dfrac{\partial z(s)}{\partial s^i},\, z(s)\right) = 0 \\ (ii)\ Q\left(\dfrac{\partial z(s)}{\partial s^i},\, \dfrac{\partial z(s)}{\partial s^j}\right). \end{cases}$$

When combined, these imply that

$$\begin{cases} (i) \quad F^2(s) \subseteq F^2(s)^\perp \Rightarrow F^2(s) = F^2(s)^\perp \\ (ii) \quad dF^3(s) \subseteq F^2(s). \end{cases}$$

Condition (i) means that $\varphi_0^{(1)}(s) \in \check{D} \subset \mathcal{F}(1, m)$ (cf. (5.7)) while (ii) exactly means that $\varphi_0^{(1)}$ is an integral manifold of I.

To prove the converse we assume that $\varphi\colon S \longrightarrow \check{D}$ is an integral manifold of I. Writing $F^3(s)) = \mathrm{span}\{z(s)\}$, from $dF^3(s) \subseteq F^2(s)$ and using our blanket assumption that $s \to \{F^3(s) \subset H\}$ be an immersion we have

$$F^2(s) = \mathrm{span}\left\{z(s), \frac{\partial z(s)}{\partial s^i}\right\},$$

so that φ is the 1st prolongation of a mapping $\varphi_0\colon S \longrightarrow P$. The condition $F^2(s) = F^2(s)^{\perp}$ then implies that $\varphi_0\colon S \longrightarrow P$ is a Legendre manifold for the contact system on P given by Q.

Remarks 6.5. (i) Differentiation of (i) in (6.4) gives

$$Q\left(\frac{\partial^2 z(s)}{\partial s^i \partial s^j}, z(s)\right) = 0;$$

clearly this is equivalent to

$$dF^2(s) \subseteq F^3(s)^{\perp}.$$

It follows that

$$F^1(s) = F^3(s)^{\perp} = \mathrm{span}\left\{z(s), \frac{\partial z(s)}{\partial s^i}, \frac{\partial^2 z(s)}{\partial s^i \partial s^j}\right\}$$

is the 2nd osculating space to $\varphi\colon S \longrightarrow P$. In particular,

$$\dim F^1(s) = 2m + 1 = (m+1)(m+2)/2 - m(m-1)/2$$

so that $z(s)$ satisfies a system of $m(m-1)/2$ linear 2nd order P.D.E.'s

$$\sum_{i,j} q^{ij}(s) \frac{\partial^2 z(s)}{\partial s^i \partial s^j} \equiv 0 \bmod \mathrm{span}\left\{z(s), \frac{\partial z(s)}{\partial s^i}\right\}. \tag{6.6}$$

(ii) Referring to remark (ii) in (6.2) the 3rd fundamental form of $\varphi_0\colon S \longrightarrow P$ is the linear function on $(H/F^1(s))^* \simeq F^3$ given by the cubic form

$$\xi \to Q\left(z(s), \sum_{i,j,k} \xi^i \xi^j \xi^k \frac{\partial^3 z(s)}{\partial s^i \partial s^j \partial s^k}\right) \tag{6.7}$$

in the tangent bundle $T(S)$. Referring to (2.11), this is also the cubic form canonically associated to the image tangent spaces $\varphi_*(T_s(S)) \subset T(\check{D})$ (these integral elements of I are *1st order* invariants of φ).

The P.D.E. system (6.6) has the following interpretation: Locally consider the cubic form (6.7) as a section

$$\Xi(s) = \sum_{i,j,k} \Xi_{ijk}(s) ds^i ds^j ds^k$$

of $\mathrm{Sym}^3 T^*(s)$. The quadrics

$$\frac{\partial \Xi(s)}{\partial (ds^k)} = \sum_{i,j} \Xi_{ijk}(s) ds^i ds^j$$

span a subspace $II(s) \subset \mathrm{Sym}^2 T^*(S)$, and we let $II(s)^\perp \subset \mathrm{Sym}^2(S)$ be the annihilator. Then the P.D.E. system (6.6) corresponds to operators $q = \sum q^{ij}(s)\partial^2/\partial s^i \partial s^j$ whose symbol belongs to $II(s)^\perp$.

7. How Period Mappings Arise as 2nd Prolongations of Analytic Subvarieties $X \subset \mathbf{P}H^{2,1}$

We are now ready to put everything together. The situation thus far may be summarized by the diagram

$$\begin{array}{ccc} (\check{D}, I) & \overset{T^{(1)}}{\Leftrightarrow} & (R,J)^{(1)} \\ \downarrow & & \downarrow \\ (P,L) & \underset{T}{\Leftrightarrow} & (R,J) \\ & & \downarrow \\ & & PV \end{array} \tag{7.1}$$

that we may explain as follows:

(i) (P, L) is the contact manifold given by the alternating form Q on H (cf. §5);

(ii) $\check{D}$ is the dual classifying space for our particular Hodge structures, and I is the sheaf of differential ideals given by the infinitesimal period relation;

(iii) As explained in (6.3), $(\check{D}, I)$ may be considered as the 1st prolongation $(P, L)^{(1)}$ of the contact system (P, L) (Note: The general definition of the 1st prolongation of a sheaf of differential ideals is given, e.g., in [2]. All that we need to know here is that the integral manifolds of a differential system and of its 1st prolongation are in one-to-one correspondence.)

(iv) (R, J) is the canonical contact system as defined in §4. As explained there the Legendre manifolds for (R, J) are given by the Gauss correspondences of local analytic subvarieties $X \subset \mathbf{P}V$;

(v) $(R, J)^{(1)}$ is the 1st prolongation of (R, J) (loc. cit; we don't need to know explicitly what this is); and

(vi) T is the birational contact transformation given in §5), and $T^{(1)}$ is its 1st prolongation (we also don't need to know what this means).

Now let

$$\varphi\colon S \longrightarrow \check{D} \tag{7.2}$$

be a holomorphic immersion where $\dim S = m$ and $\varphi^*(I) = 0$, as might arise from the periods of a threefold satisfying (1.1). Then, by (6.3),

$$\varphi = \varphi_0^{(1)}$$

is the 1st prolongation of a Legendre manifold

$$\varphi_0\colon S \longrightarrow P. \tag{7.3}$$

Under the birational contact transformation T, which by the discussion at the end of §5) may be assumed to be well-defined in a *Zariski* neighborhood of a given point of P, we may consider (7.3) as a Legendre manifold

$$f\colon S \longrightarrow R \tag{7.4}$$

for the canonical contact system (R, J). Finally, (7.4) is given by the set of tangent hyperplanes to the image X of a holomorphic mapping

$$w\colon S \longrightarrow \mathbf{P}V \simeq \mathbf{P}^{m+1}. \tag{7.5}$$

Conclusion. *The assignment*

$$\varphi \to w$$

may be thought of as de-prolonging twice a variation of Hodge structure such as might arise from a family of 3-folds satisfying (1.1). The technical meaning of this statement has just been explained. *Intuitively, the period-type mapping (7.2) has been shown to arise from the 1st and 2nd derivatives* (2-jet) *of a holomorphic mapping (7.5).* In particular, referring to (5.3)-(5.6), in the case of the periods of a family $\{X_s\}_{s\in S}$ of 3-folds as discussed in §1 we may give (7.5) by the A-periods

$$w(s) = \left[\int_{\alpha_0} \omega(s), \int_{\alpha_1} \omega(s), \ldots, \int_{\alpha_m} \omega(s)\right], \tag{7.6}$$

and then the whole period mapping (7.2) is determined via (7.1) by the small piece (7.6). This establishes our assertion (1.4).

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