# *Some Remarks and Examples on Continuous Systems and Moduli*

PHILLIP A. GRIFFITHS*

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We are primarily concerned with some examples of continuous systems and moduli of general complex varieties. After preliminary remarks in Section 1, we give in paragraphs 2 and 3 examples of continuous systems whose parameter space is nowhere reduced. The first such example was given by Mumford (*Amer. J. Math.*, **85** (1962) 642–648); in our case, we compute explicitly the cohomological obstructions. In Section 4 we use the second example above to give a moduli space in higher dimensions. Paragraph 5 is devoted to some simple examples of continuous systems generated by curves in projective space; the point to be made here is that the available criterion for completeness of the characteristic system seldom applies in practice. In Appendix I we discuss the relation between continuous systems and deformations via monoidal transformations; and, in Appendix II, we discuss the Riemann–Roch theorem for prime divisors by using continuous systems.

**1. Characteristic systems of continuous systems.** Let $X$ be a compact, connected complex submanifold of a complex manifold $W$. The notion of a *continuous system* $\{X_\tau\}_{\tau \varepsilon A}$ of compact submanifolds $X_\tau \subset W$ and for which $X = X_o$ for some $o \varepsilon A$ has been defined by Kodaira [1]. Here $A$ is an analytic space and the $X_\tau$ are to depend holomorphically on $\tau \varepsilon A$. If $\mathbf{N}_\tau \to X_\tau$ is the normal bundle of $X_\tau \subset W$, then there is defined the *infinitesimal displacement mapping* [1]

$$\rho_\tau : \mathbf{T}_\tau(A) \to H^0(\mathbf{N}_\tau). \tag{1}$$

(If $\partial/\partial t^\alpha \varepsilon \mathbf{T}_0(A)$, then we may write

$$\rho_0\left(\frac{\partial}{\partial t^\alpha}\right) = \left.\frac{\partial X_\tau}{\partial t^\alpha}\right]_{\tau=0}$$

where $\tau = (t^1, \cdots, t^m) \varepsilon A$.)

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Here $\mathbf{T}_\tau(A)$ = Zariski tangent space to $A$ at $\tau$ (for our purposes we may always assume that $A \subset \mathbf{C}^m$) and we write $H^0(\mathbf{N}_\tau) = H^0(X_\tau, O(\mathbf{N}_\tau))$. The image $\rho_\tau(\mathbf{T}_\tau(A)) \subset H^0(\mathbf{N}_\tau)$ is the *characteristic system* cut out on $X_\tau$ by the continuous system $\{X_\tau\}$; the continuous system is said to be *complete* at $\tau$ if $\rho_\tau(\mathbf{T}_\tau(A)) = H^0(\mathbf{N}_\tau)$; the continuous system is said to be *effectively parametized* if $\rho_\tau$ is injective.

The concept of a *maximal continuous system* is defined in the obvious way—any other continuous system is contained in it—and the following theorem can be proven [2]:

***Theorem*** *Given $X \subset W$, there exists a maximal continuous system $\Sigma = \{X_\tau\}_{\tau \varepsilon A}$ containing $X = X_o$. We may assume that $A \subset H^0(\mathbf{N})$ where $\mathbf{N} \to X$ is the normal bundle, and $\rho_0$ is then the identity mapping. The continuous system is complete if $H^1(\mathbf{N}) = 0$.*

Because of this result, we may speak of *the* characteristic system of $X \subset W$. Perhaps the simplest incomplete characteristic system is the following: Let $X$ be a compact Riemann surface of genus $p \geqq 1$, let $\{U_\alpha\}$ be a coordinate covering of $X$, and suppose that $\lambda \,\varepsilon\, H^1(X, O)$ is given by a cocycle $\{\lambda_{\alpha\beta}\}$, $\lambda_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{C}$. We form a manifold $W$ from $\bigcup_\alpha U_\alpha \times \mathbf{C}$ by the equivalence relation: $(u_\alpha, \xi_\alpha) \sim (u_\beta, \xi_\beta)$ if, and only if, $u_\alpha = u_\beta$ and

$$\xi_\alpha = \frac{\xi_\beta}{1 + \lambda_{\alpha\beta}(u_\beta)\xi_\beta} = g_{\alpha\beta}(u_\beta, \xi_\beta).$$

From $\lambda_{\alpha\beta} + \lambda_{\beta\gamma} = \lambda_{\alpha\gamma}$ we find $g_{\alpha\beta}(g_{\beta\gamma}) = g_{\alpha\gamma}$ so that $W$ is a surface which is fibered over $X$ with $\mathbf{C}$ as fibre. We may embed $X \subset W$ by the local equation $\xi_\alpha = 0$, and the normal bundle $\mathbf{N} \to X$ has transition functions $\partial g_{\alpha\beta}/\partial \xi_\beta]_{\xi=0} = 1$; thus $\mathbf{N}$ is analytically trivial. On the other hand, the continuous system $\Sigma$ generated by $X$ consists of $X$ alone since, if $X' \,\varepsilon\, \Sigma$, $X' \neq X$, then $X \cap X' = \phi$ (since $\mathbf{N}$ is trivial) and so $X'$ would be given locally by $\xi_\alpha = \eta_\alpha(u_\alpha) \neq 0$. If $\varphi_\alpha = 1/\eta_\alpha$, then from $\eta_\alpha = \eta_\beta/(1 + \lambda_{\alpha\beta}\eta_\beta)$ it follows that $\varphi_\alpha - \varphi_\beta = \lambda_{\alpha\beta}$ or $\lambda = 0$ in $H^1(X, O)$. Thus $\Sigma$ is incomplete. This example is essentially due to Zappa [3].

**2. An everywhere obstructed family.** Let $X$ be a compact, complex manifold with $H^1(X, O) \neq 0$ and take $\varphi \,\varepsilon\, H^1(X, O)$. For $|\lambda| < \epsilon$, the line bundles $\mathbf{T}_\lambda = \exp(\lambda\varphi) \,\varepsilon\, H^1(X, O^*)$ may be assumed to satisfy $\mathbf{T}_\lambda \cong \mathbf{T}_{\lambda'}$ if, and only if, $\lambda = \lambda'$. Let $\mathbf{L} \to X$ be a line bundle, set $\mathbf{L}_\lambda = \mathbf{L} \otimes \mathbf{T}_\lambda$, and assume:

$$(2) \qquad H^0(X, O(\mathbf{L}_\lambda)) = 0 \quad \text{for} \quad \lambda \neq 0, \qquad H^0(X, O(\mathbf{L})) \neq 0.$$

($\mathbf{L}$ = trivial bundle will do.)

Let now $D = \{\lambda : |\lambda| < \epsilon\}$ and suppose that $f : D \to D$ is a holomorphic function with $f(0) = 0$. Define a complex manifold $W_f = \bigcup_{\lambda \varepsilon D} \mathbf{L}_{f(\lambda)}$ as follows: If $\{U_\alpha\}$ is a coordinate covering of $X$ relative to which $\mathbf{L}$ and $\varphi$ have transition functions $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{C}^*$, $\varphi_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbf{C}$, respectively, then $W_f$ is formed from $\bigcup_\alpha \{U_\alpha \times \mathbf{C} \times D\}$ by the equivalence relation: $(u_\alpha, \xi_\alpha, \lambda_\alpha) \sim (u_\beta, \xi_\beta, \lambda_\beta)$ if, and only if, $\lambda_\alpha = \lambda_\beta = \lambda$, $u_\alpha = u_\beta = u \in U_\alpha \cap U_\beta$, and $\xi_\alpha = \psi_{\alpha\beta}(u) \exp(f(\lambda)\varphi_{\alpha\beta}(u))\xi_\beta$. (We may think of $W_f$ as $\bigcup_{\lambda \in D} \mathbf{L}_{f(\lambda)}$.) Let $\sigma \in H^0(X, O(\mathbf{L}))$, $\sigma \neq 0$, and set $X_\sigma = \sigma(X) \subset W_f$.

***Lemma 1.*** *Let $A \subset H^0(X, O(\mathbf{L}))$ be a small neighborhood of $\sigma$. Then the maximal continuous system of $X_\sigma \subset W_f$ is given by $\{X_\tau\}_{\tau \in A}$ where $X_\tau = \tau(X)$.*

*Proof.* First observe that there are projections $\pi : W_f \to X$, $\theta : W_f \to D$ given locally by $\pi(u_\alpha, \xi_\alpha, \lambda_\alpha) = u_\alpha$, $\theta(u_\alpha, \xi_\alpha, \lambda_\alpha) = \lambda_\alpha$. Let $\{X_\tau\}_{\tau \in A}$ be the maximal continuous system of $X_\sigma \subset W_f$. Then $\pi(X_\tau) = X$ and $\theta(X_\tau) = \eta(\tau)$ where $\eta(\tau)$ is a holomorphic function from $A$ to $D$. Thus $X_\tau$ is a section of $\mathbf{L}_{\eta(\tau)}$ and so by (2) $\eta(\tau) = 0$, and the assertion is now obvious.

Let $\mathbf{N}_1 \to X_\sigma$ be the normal bundle of $X_\sigma \subset \mathbf{L}$; under the isomorphism $X_\sigma \cong X$, $\mathbf{N}_1$ corresponds to $\mathbf{L}$. Denote by $\mathbf{N}_2$ the normal bundle of $\mathbf{L} \subset W_f$. We have then the exact sequence:

$$0 \to \mathbf{N}_1 \to \mathbf{N} \to \mathbf{N}_2 \to 0, \tag{3}$$

where $\mathbf{N}$ = normal bundle of $X \subset W_f$. Obviously $\mathbf{N}_2$ is trivial and so, under the isomorphism $X \cong X_\sigma$, (3) is uniquely given by an element $e \in H^1(X, O(\mathbf{L}))$.

***Lemma 2.*** *$e = f'(0)(\varphi\cdot\sigma)$ where $\varphi\cdot\sigma$ is the cup product*

$$H^1(X, O) \otimes H^0(X, O(\mathbf{L})) \to H^1(X, O(\mathbf{L})).$$

*Proof.* Set $\zeta_\alpha = \xi_\alpha - \sigma_\alpha(u_\alpha)$ so that $X_\sigma$ is given locally by $\zeta_\alpha = 0 = \lambda_\alpha$. Then $\mathbf{N} \to X_\sigma$ has transition functions

$$N_{\alpha\beta} = \begin{bmatrix} \dfrac{\partial \zeta_\alpha}{\partial \zeta_\beta} & \dfrac{\partial \zeta_\alpha}{\partial \lambda_\beta} \\[2ex] \dfrac{\partial \lambda_\alpha}{\partial \zeta_\beta} & \dfrac{\partial \lambda_\alpha}{\partial \lambda_\beta} \end{bmatrix}_{\zeta = 0 = \lambda}.$$

Then

$$N_{\alpha\beta} = \begin{bmatrix} \psi_{\alpha\beta} & f'(0)\sigma_\alpha\phi_{\alpha\beta} \\ 0 & 1 \end{bmatrix}$$

and the Lemma follows.

***Theorem 1.*** *If $f'(0) = 0$, then $\{X_\tau\}_{\tau \in A}$ is everywhere incomplete. In fact, if $f^n(0) = 0$, $f^{n+1}(0) \neq 0$, then there is everywhere on $A$ an $n^{\text{th}}$ obstruction to completing the continuous system generated by $X_\tau$ ($\tau \in A$).*

*Proof.* If $f'(0) = 0$, then by Lemma 2, $\mathbf{N}_\tau = \mathbf{L} \oplus \mathbf{1}$ (under isomorphism $X_\tau \cong X$) and so $\{X_\tau\}$ is everywhere incomplete.

The assertion about the $n^{\text{th}}$ obstruction is straightforward to check.

**3. Another example of an obstructed family.** This example is more interesting than that of Sec. 2, although the geometric principle is essentially the same. Let $X$ be a curve of genus $p > 2$ and whose normalized period matrix is $(I, Z_0)$ where $Z_0 \in \mathbf{H}_p =$ Siegel's generalized upper half-space in genus $p$. Now each point $Z \in \mathbf{H}_p$ gives rise to a canonically polarized Abelian variety $A_Z$, and we let $U \subset \mathbf{H}_p$ be an open neighborhood of $Z_0$, $W = \bigcup_{Z \in U} A_Z$. Then $W$ is an open complex manifold of dimension $p(p+3)/2$ which contains $A_{Z_0}$. On the other hand, $A_{Z_0} = J$ is the Jacobian variety of $X$ and there is an embedding $X \subset J$.

***Theorem 2.*** *(i) The continuous system $\Sigma$ generated by $X \subset J$ is complete if and only if $X$ is non-hyperelliptic; (ii) if $X$ is hyperelliptic, then $\Sigma$ is everywhere obstructed; (iii) the continuous system generated by $X \subset W$ is complete.*

*Proof.* If $\mathbf{T} =$ tangent bundle of $J$, then we have over $X$ the exact sheaf sequence

$$0 \to \Theta \to O_X(\mathbf{T}) \to O(\mathbf{N}) \to 0, \tag{4}$$

where $\Theta =$ tangent sheaf to $X$, $\mathbf{N} \to X$ is the normal bundle of $X \subset J$. If $\omega^1, \cdots, \omega^p$ are a basis for the Abelian differentials on $X$ whose period matrix is $(I, Z_0)$, then there is induced a trivialization $O_X(\mathbf{T}) \cong \{O_X\}^p$ and likewise $O_J(\mathbf{T}) \cong \{O_J\}^p$. (A germ $\theta \in O_J(\mathbf{T})$ is written $\theta = \Sigma f_i\, \partial/\partial w^i$ where $\langle \partial/\partial w^i, \omega^k \rangle = \delta_i^k$ and $f_i \in O_J$.) From (4) we find the exact cohomology diagram

$$\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & \{H^0(O_J)\}^p & & & & \{H^1(O_J)\}^p & & \\
 & & \downarrow & & & & \downarrow & & \\
0 \to H^0(\Theta) & \to & \{H^0(O_X)\}^p & \xrightarrow{\psi} & H^0(O(\mathbf{N})) & \xrightarrow{\delta} H^1(\Theta) \xrightarrow{\sigma} & \{H^1(O_X)\}^p & \to & \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array} \tag{5}$$

The dual space to $H^1(O_X)$ is $H^0(\mathbf{K})$ and so $\sigma$ in (5) induces a mapping:

$${}^t\sigma : H^0(\mathbf{K}) \otimes H^0(\mathbf{K}) \to H^0(\mathbf{K}^2), \tag{6}$$

(using $\{H^0(\mathbf{K})^p \cong H^0(\mathbf{K}) \otimes H^0(\mathbf{K})$ via $\omega^1, \cdots, \omega^p$).

***Lemma 3.*** *The mapping ${}^t\sigma$ in (6) is the cup product.*

*Proof of Lemma.* If $w^1, \cdots, w^p$ are Euclidean coordinates such that $J = \mathbf{C}^p/\Gamma$, $\Gamma$ being the lattice generated by the columns of $(I, Z_0)$, then $dw^\alpha \mid X = \omega^\alpha$ and the mapping $\Theta \to \{O_X\}^p$ is given by sending $\theta \to (f_1, \cdots, f_p)$ where

$$\theta = \sum_{\alpha=1}^{p} f_\alpha \frac{\partial}{\partial w^\alpha}$$

is a vector field along $X$.

Let now $\theta \in H^1(\Theta)$. In terms of a local coordinate $z$ on $X$, $\theta = g(z)\, \partial/\partial z \otimes d\bar{z}$ and $\sigma(\theta) = (f_1, \cdots, f_p)(g(z)d\bar{z})$ where $\partial/\partial z = \Sigma f_\alpha\, \partial/\partial w^\alpha$ and $(f_\alpha g)\, d\bar{z} \in H^1(O_X)$. Thus, if $\varphi = (\varphi^1, \cdots, \varphi^p) \in \{H^0(\mathbf{K})\}^p$ (= dual space of $\{H^1(O_X)\}^p$, then

$$\langle {}^t\sigma(\varphi), \theta\rangle = \langle \varphi, \sigma(\theta)\rangle$$
$$= \int_X (\Sigma f_\alpha \varphi^\alpha) \wedge g\, d\bar{z} = \int_X \Sigma \left\langle \omega^\alpha, \frac{\partial}{\partial z}\right\rangle \varphi^\alpha \wedge g\, d\bar{z} = \langle \Sigma\, \omega^\alpha \varphi^\alpha, \theta\rangle,$$

where $\Sigma\omega^\alpha\varphi^\alpha \in H^0(\mathbf{K}^2)$. Thus ${}^t\sigma(\varphi) = \Sigma\omega^\alpha\varphi^\alpha$ or ${}^t\sigma$ in (6) is the cup product, and this proves the Lemma.

Returning now to (5), we see that $\delta$ is zero if and only if the cup product

$$H^0(\mathbf{K}) \otimes H^0(\mathbf{K}) \xrightarrow{\mu} H^0(\mathbf{K}^2) \tag{7}$$

is onto; while, on the other hand, we have

***Noether's Theorem.*** (See [4]) *Assuming genus* $(X) > 2$, $\mu$ *in* (7) *is onto if and only if* $X$ *is non-hyperelliptic.*

(We may restate Noether's theorem geometrically as follows: if $X \subset P_{p-1}$ is a canonical curve, then the quadries on $P_{p-1}$ cut out, on $X$, a complete linear system and $X$ lies on $\frac{1}{2}(p-2)(p-3)$ such quadrics.)

Let now $\{X_\tau\}_{\tau \in A}$ be the continuous system generated by $X \subset J$. If $C$ is an analytic curve through the origin $0 \in A$, and if $\xi \in H^0(\mathbf{N})$ is the tangent to $C$ at 0, then, assuming that $X$ is general in $\{X_\tau\}_{\tau\in A}$, $\delta(\xi) = 0$ since, if this were not so, then $\{X_\tau\}_{\tau \in C}$ would be a family of non-singular curves $X_\tau \subset J$ and from $\delta(\xi) \neq 0$ it follows that not all the $X_\tau$ are biregularly equivalent. (This follows from the following result of Kodaira–Spencer: If $\{Y_t\}_{t\in B}$ is a complex analytic family with $\dim H^q(Y_t, \Theta_t) =$ constant for all $q$, and if all the $Y_t$ are biregularly equivalent, then the mappings $\rho_t : \mathbf{T}_t(B) \to H^1(Y_t, \Theta_t)$ are zero.) But then this contradicts Torelli's theorem [4].

The conclusion is then that the continuous system $\{X_\tau\}_{\tau\in A}$ consists of the translations of $X$ in $J$, and there is everywhere an obstruction if $X$ is hyperelliptic. This proves (i) and (ii) in Theorem 2.

The fact that the maximal continuous system $\{X_\sigma\}_{\sigma \in S}$ of $X \subset W$ is complete and $\dim S = 4p - 3$ is easy to verify using Teichmüller's theorem.

***Remark.*** If $X$ is non-hyperelliptic, $\dim H^0(\mathbf{N}) = p$ and $\dim H^1(\mathbf{N}) = p^2 - 3p + 3$; if $X$ is hyperelliptic, $\dim H^0(\mathbf{N}) = 2p - 2$ and $\dim H^1(\mathbf{N}) = p^2 - 2p + 1$. The obstruction in this case is of the same nature as that of theorem 1 above.

**4. An example of a moduli space.** An interesting question in the theory of moduli is the problem of *stability of analytic objects*: If $X$ is a compact, complex manifold, $\{X_t\}_{t\in B}$ a deformation of $X$, and $\gamma$ an analytic object on $X$, then when can $\gamma$ be continued analytically to $\gamma_t$ on $X_t$? For example, if $X$ is obtained from a variety $Y$ by blowing up along a submanifold $S \subset Y$, and if $Z \subset X$

is the total transform of $S$, then $Z$ is stable and so there exists $Z_t \subset X_t$, but it is not known if $X_t$ can be blown down along $Z_t$ (see [5] and Appendix I below).

We shall prove that this is so in a special case, and the construction thus leads to a moduli space in higher dimensions.

Let $S$ be a curve of genus $p > 2$ and $J$ the Jacobian variety of $S$. Suppose that $X$ is the algebraic manifold obtained by blowing up $J$ along $S$; denote by $\pi : X \to J$ the canonical projection and set $Z = \pi^{-1}(S)$. Then $Z \xrightarrow{\pi} S$ is a fibre bundle with fibre the projective space $P = P_{p-2}$ and, if $\mathbf{L} \to Z$ is the normal bundle of $Z \subset X$, then $\mathbf{L} \mid P = -\mathbf{H}$ where $\mathbf{H} \to P$ is the hyperplane bundle.

***Theorem 3.*** *Let $\{X_t\}_{t \epsilon B}$ be an analytic deformation of $X = X_0$. Then there exists a family $\{J_t\}_{t \epsilon B}$ of Abelian varieties and a family $\{S_t\}_{t \epsilon B}$ of curves such that $S_t \subset J_t$, $J_t$ is the Jacobian of $S_t$, and $X_t$ = monoidal transform of $J_t$ along $S_t$.*

*Briefly: Moduli $(X) \cong$ Moduli $(S)$.*

*Proof.* Let $R^q_\pi(\mathbf{L})$ be the $q^{\text{th}}$ Leray sheaf of the pair $(O_Z(\mathbf{L}); Z \xrightarrow{\pi} S)$. Then, for $U \subset S$ a small disc,

$$H^0(U, R^q_\pi(\mathbf{L})) = H^q(\pi^{-1}(U), O_Z(\mathbf{L})) \cong \sum_{r+s=q} H^r(P, O_P(-\mathbf{H})) \hat{\otimes} H^s(U, O_U) = 0,$$

for all $q$. By the Leray spectral sequence, $H^q(Z, O_Z(\mathbf{L})) = 0$ for all $q$.

From $H^0(Z, O_Z(\mathbf{L})) = 0 = H^1(Z, O_Z(\mathbf{L}))$, we conclude that there exists a unique family $\{Z_t\}_{t \epsilon B}$ of compact submanifolds $Z_t \subset X_t$ such that $Z_0 = Z$. Furthermore, there exists a fibering of $Z_t$ by projective spaces $P = P_{p-2}$ such that $Z_t \to S_t (= Z_t/P)$ is a projective fibre bundle over a curve $S_t$ ([5], p. 87). If $\mathbf{L}_t \to Z_t$ is the normal bundle of $Z_t \subset X_t$, then clearly $\mathbf{L}_t \mid P = -\mathbf{H}$.

Now the normal bundle $\mathbf{N}$ of $S$ in $J$ is *positive* (the Euclidean second fundamental form of $S$ in $J$ is positive), and so we may blow down $X_t$ along $Z_t$ to obtain a compact, complex manifold $J_t$, which contains $S_t$, and is such that $X_t$ = monoidal transform of $J_t$ along $S_t$ [6]. Clearly $J_t$ = Jacobian variety of $S_t$.

In Appendix I we shall prove the following

***Lemma 4.*** *We have the following exact sequences:*

(8)
$$\begin{cases} 0 \to H^0(J, \Theta_J) \xrightarrow{\rho_0} H^0(S, O_S(\mathbf{N})) \xrightarrow{\delta} H^1(X, \Theta_X) \xrightarrow{\pi_*} H^1(J, \Theta_J) \xrightarrow{\rho_1} H^1(S, O_S(\mathbf{N})) \to 0, \\ \qquad 0 \to H^q(X, \Theta_X) \xrightarrow{\pi_*} H^q(J, \Theta_J) \to 0 \qquad (q \geqq 2) \end{cases}$$

*where $\rho_i$ is induced from $\Theta_J \mid S \to O_S(\mathbf{N})$ and $\pi_*$ is induced from $X \xrightarrow{\pi} J$.*

*The conclusions which we may draw are:*

(a) *$X$ has $3p - 3$ moduli and $X_t \cong X_{t'}$ if and only if $S_t \cong S_{t'}$;*

(b) *If $S$ is non-hyperelliptic, $\rho_0$ is onto and* $\dim H^1(X, \Theta_X) = \dim (\ker \rho_1) = 3p - 3$;

(c) *If $S$ is hyperelliptic, $\rho_0$ has* co-rank $p - 2$ *and* $\dim H^1(X, \Theta_X) = p - 2 + \dim (\ker \rho_1) = 3p - 3$;

(d) *For any $S$, $X$ has $3p - 3$* $= \dim H^1(X, \Theta_X)$ moduli *and* moduli $(X) \cong$ moduli $(S)$; *also* $\dim H^2(X, \Theta_X) = \frac{1}{2}p^2(p - 1)$.

**5. Curves in projective space.** Let $X$ be a curve of genus $p > 1$ non-singularly embedded in $P_N$. If $\mathbf{L} \to P_N$ is the canonical positive line bundle and $\mathbf{T} \to P_N$ is the tangent bundle, we recall the exact sequence:

$$0 \to O_{P_N} \xrightarrow{\lambda} O\{\mathbf{L}\}^{N+1} \xrightarrow{\pi} O(\mathbf{T}) \to 0, \tag{9}$$

where

$$\{L\}^{N+1} = \underbrace{L \oplus \cdots \oplus L}_{N+1}.$$

If $\xi_0, \cdots, \xi_N$ are homogeneous coordinates in $P_N$, then $\xi_\alpha \in H^0(P_N, O(\mathbf{L}))$ and $\lambda(f) = (f\xi_0, \cdots, f\xi_N)$, $\pi(\eta_0, \cdots, \eta_N) = \sum_{\alpha=0}^{N} \eta_\alpha\, \partial/\partial\xi_\alpha$. The exactness of (9) is equivalent to Euler's relation on derivatives of homogeneous polynomials. By restricting (9) to $X$, we find the exact sheaf diagram

$$\begin{tikzcd}
& & 0 \arrow[d] & & \\
& & O_X \arrow[d] & & \\
& & O_X\{\mathbf{L}\}^{N+1} \arrow[d] & & \\
0 \arrow[r] & \Theta \arrow[r] & O_X(\mathbf{T}) \arrow[r] \arrow[d] & 0_X(\mathbf{N}) \arrow[r] & 0 \\
& & 0 & &
\end{tikzcd} \tag{10}$$

where $\Theta$ = tangent sheaf of $X$ and $\mathbf{N} \to X$ is the normal bundle. The exact cohomology diagram of (10) gives

$$\begin{tikzcd}
& & 0 \arrow[d] & \\
& & H^0(O) \arrow[d] & \\
& & \{H^0(\mathbf{L})\}^{N+1} \arrow[d] & \\
& 0 \arrow[r] & H^0(O(\mathbf{T})) \arrow[r] \arrow[d] & H^0(\mathbf{N}) \arrow[ddd] \\
& & H^1(O_X) \arrow[d, "\psi"'] & \\
& & \{H^1(\mathbf{L})\}^{N+1} \arrow[d] & \\
0 & H^1(\mathbf{N}) \arrow[l] & H^1(\mathbf{T}) \arrow[l] \arrow[d] & H^1(\Theta). \arrow[l] \\
& & 0 &
\end{tikzcd} \tag{11}$$

If now the degree $\delta$ of $X$ is $> 2p - 2$, then $H^1(\mathbf{L}) = 0 = H^1(\mathbf{N})$ and the continuous system $\{X_\tau\}_{\tau \varepsilon A}$ generated by $X \subset P_N$ is complete and,

$$\text{if } \dim H^0(\mathbf{L}) = m, \quad \dim A = (N-3)m + 4\delta \quad (m = \delta - p),$$
$$\text{if } N = 2, \text{ then } \dim A = 3\delta + p - 1, \tag{12}$$
$$\text{if } N = 3, \quad \dim A = 4\delta.$$

Consider now the case $\delta = 2p - 2$, $N = p - 1$ so that $\mathbf{L} = \mathbf{K}$ is the canonical bundle and $X \subset P_{p-1}$ is a *canonical curve.*

***Lemma 5.*** *The mapping $\psi : H^1(O) \to \{H^1(\mathbf{K})\}^p$ in* (11) *is an isomorphism.*

*Proof.* The mapping $\psi : O \to \{O(\mathbf{K})\}^p$ is given by $\psi(f) = (f\omega^1, \cdots, f\omega^p)$ where $f \varepsilon O$ and $\omega^1, \cdots, \omega^p$ are a basis for $H^0(\mathbf{K})$. The transpose of $\psi$ is ${}^t\psi : \{H^0(O)\}^p \to H^0(\mathbf{K})$ where ${}^t\psi(\lambda_1, \cdots, \lambda_p) = \Sigma\lambda_j\omega^j$ and thus $\psi$ is an isomorphism. Q.E.D.

It follows from Lemma 5 that $H^1(\mathbf{N}) = 0$ so that $\{X_\tau\}_{\tau\varepsilon A}$ is complete and

$$\dim A = (3p - 3) + (p^2 - 1). \tag{13}$$

For use below, we observe that the proof of Lemma 3 shows that, in (11) ${}^t\psi : \{H^1(\mathbf{L})^*\}^{N+1} \to H^1(O_X)^*$ or ${}^t\psi : \{H^0(\mathbf{K} - \mathbf{L})\}^{N+1} \to H^0(\mathbf{K})$ is given by

$${}^t\psi(\lambda_1, \cdots, \lambda_{N+1}) = \sum_{\alpha=1}^{N+1} \xi_\alpha \cdot \lambda_\alpha , \tag{14}$$

where $\lambda_\alpha \varepsilon H^0(\mathbf{K} - \mathbf{L})$, $\xi_\alpha \varepsilon H^0(\mathbf{L})$, and $\xi_\alpha \cdot \lambda_\alpha$ is the cup product in cohomology.

The condition $\delta > 2p - 2$ is too strong to have real interest; for example, if $C \subset P_3$ is the complete intersection of surfaces of degrees $n_1, n_2$ $(n_1 + n_2 > 4)$, then

$$n_1 n_2 = \delta = \frac{2p-2}{n_1 + n_2 - 4}$$

and so $\delta < 2p - 2$. In this case, the continuous system generated by $C$ is complete but $H^1(\mathbf{N}) \neq 0$ in general, as the following shows:

Let $S_1, S_2$ be non-singular surfaces in $P_3 = P$ of degrees $n_1, n_2$ respectively, and set $C = S_1 \cdot S_2$ where we suppose that $C$ is a non-singular curve. If $\mathbf{L} \to P$ is the hyperplane bundle, then $O_C(\mathbf{N}) = O_C(\mathbf{L}^{n_1}) \oplus O_C(\mathbf{L}^{n_2})$ where $\mathbf{N} \to C$ is the normal bundle of $C \subset P$.

***Lemma 6.*** *We have*

$$H^0(P, O_P(\mathbf{L}^{n_i})) \to H^0(C, O_C(\mathbf{L}^{n_i})) \to 0,$$

*for $i = 1, 2$ while $H^1(C, O_C(\mathbf{L}^{n_1})) \neq 0$ if $n_2 > 4$.*

*Proof.* From

$$0 \to O_{S_2} \to O_{S_2}(\mathbf{L}^{n_1}) \to O_C(\mathbf{L}^{n_1}) \to 0$$

and

$$0 \to O_P(\mathbf{L}^{n_1-n_2}) \to O_P(\mathbf{L}^{n_1}) \to O_{S_2}(\mathbf{L}^{n_1}) \to 0,$$

we find

$$H^0(O_{S_2}(\mathbf{L}^{n_1})) \to H^0(O_C(\mathbf{L}^{n_1})) \to 0$$

and

$$H^0(O_P(\mathbf{L}^{n_1})) \to H^0(O_{S_2}(\mathbf{L}^{n_1})) \to 0,$$

which gives

$$H^0(O_P(\mathbf{L}^{n_1})) \to H^0(O_C(\mathbf{L}^{n_1})) \to 0.$$

On the other hand, since $H^1(O_{S_2}(\mathbf{L}^{n_1})) = 0$, we have

$$0 \to H^1(O_C(\mathbf{L}^{n_1})) \to H^2(O_{S_2}) \to H^2(O_{S_2}(\mathbf{L}^{n_1}))$$

while

$$\dim H^2(O_{S_2}(\mathbf{L}^{n_1})) = \dim H^0(O_{S_2}(\mathbf{L}^{-n_1+n_2-4})) < \dim H^0(O_{S_2}(\mathbf{L}^{n_2-4})) = \dim H^2(O_{S_2}),$$

provided $n_2 > 4$.

This proves the Lemma.

If now $Q_k$ = homogeneous polynomials on $P_3$ of degree $k$, if $f_1$, $f_2$ are the defining equations of $S_1$, $S_2$, and if $n_1 > n_2$, then

$$H^0(O_C(\mathbf{N})) \cong \{Q_{n_2}/(f_2)\} \oplus \{Q_{n_1}/(f_2)Q_{n_1-n_2/(f_1)}\} \tag{15}$$

Thus the continuous system generated by $C \subset P$ is complete and is formed by perturbing $S_2$ and $S_1$ (mod $f_2$). On the other hand, $H^1(O_C(\mathbf{N})) \neq 0$ in case $n_1 > 4$ or $n_2 > 4$.

An interesting special case is $n_1 = n_2 = 2$. Then $p = 1$, $H^1(O_C(\mathbf{N})) = 0$, and $\dim H^0(O_C(\mathbf{N})) = 18$. The resulting family of elliptic curves is classical. If $n_1 = 3$, $n_2 = 2$, then $p = 4$ and $C \subset P_3$ is a canonical curve (since $2p - 2 = 6 = n_1 n_2$) and $\dim H^0(O_C(\mathbf{N})) = 20 = (3p - 3) + (p^2 - 1)$. The fact that every $C$ lies on a unique quadric is a special case of Noether's theorem (*c.f.* [4] and the geometric statement of Noether's theorem above). This last remark generalizes to curves $C = S_1 \cdot S_2 \cdot S_3$ in $P_4$ where $n_1 = n_2 = n_3 = 2$. Then $\delta = 8$, $p = \frac{1}{2}(n_1 + n_2 + n_3 - 5)\,\delta + 1 = 5$ so that $C \subset P_4$ is a canonical curve of genus 5 which, as above, lies on $3 = \frac{1}{2}(p - 2)(p - 3)$ quadrics. In this case, $\dim H^0(O_C(N)) = 36 = (3p - 3) + (p^2 - 1)$.

As a final remark, if we assume that the hyperplane system is complete, then, by (14)

$${}^t\psi : H^0(\mathbf{L}) \otimes H^0(\mathbf{K} - \mathbf{L}) \to H^0(\mathbf{K})$$

is just the multiplication between a linear series and its adjoint; if ${}^t\psi$ is injective (*e.g.* $\deg \mathbf{L} = p - 1$ and $\mathbf{L}$ is non-special), then $H^1(O_X(\mathbf{N})) = 0$ and the continuous system generated by $X$ is complete. For example, if $C = S_1 \cdot S_2 \subset P_3$

is a complete intersection and $n_1 = n_2 = 3$, then

$$p - 1 = \tfrac{1}{2}(n_1 + n_2 - 4)n_1 n_2 = n_1 n_2$$

and, in fact, $\mathbf{K}_C = \mathbf{L}^2$. Thus $H^1(O_C(\mathbf{L})) \cong H^0(O_C(\mathbf{L}))$ and dualizing

$$H^1(O_C) \xrightarrow{\psi} \{H^1(\mathbf{L})\}^4 \xrightarrow{\pi} H^1(\mathbf{T}) \to 0$$

gives

$$(16) \qquad 0 \to \Lambda^2 H^0(O_C(\mathbf{L})) \xrightarrow{{}^t\pi} H^0(O_C(\mathbf{L})) \otimes H^0(O_C(\mathbf{L})) \xrightarrow{{}^t\psi} H^0(O_C(\mathbf{L}^2)) \to 0.$$

Thus $\psi$ is injective (which is equivalent to saying that the canonical series on $C$ is cut out by quadrics in $P_3$) and $\dim H^0(\mathbf{T}) = 15$, $\dim H^0(\mathbf{N}) = 36$, $\dim H^1(\Theta) = 27$, $\dim H^1(\mathbf{T}) = 6$, and $H^1(\mathbf{N}) = 0$. The mapping $H^0(\mathbf{N}) \to H^1(\Theta)$ has rank 21 and co-rank 6.

If $C = S_1 \cdot S_2$ where $n_1 = 4$, $n_2 = 2$, then again $\mathbf{K} = \mathbf{L}^2$, ${}^t\psi$ is onto, and $\dim H^0(\mathbf{T}) = 15$, $\dim H^0(\mathbf{N}) = 33$, $\dim H^1(\Theta)\ 1 = 24$, $\dim H^1(\mathbf{T}) = 7$, and $\dim H^1(\mathbf{N}) = 1$. The mapping $H^0(\mathbf{N}) \to H^1(\Theta)$ now has rank 18 and co-rank 6.

**Appendix I: The groups $\mathbf{H}^q(\Theta)$ and monoidal transformations.** Let $X$ be a compact, complex manifold and $Z \subset X$ a non-singular subvariety of codimension $r > 1$. We let $Y$ be the monoidal transform of $X$ along $Z$ and $\pi : Y \to X$ the birational projection; $W = \pi^{-1}(Z)$ is the total transform of $Z$. If $\mathbf{N}_Z$ is the normal sheaf of $Z$ in $X$, the restriction gives a map $\rho : \Theta_X \to \mathbf{N}_Z$ whereas, on $Y - W$, the projection gives $\pi_* : \Theta_Y \to \Theta_X$.

***Theorem 4.*** *We have the following exact sequence*:

$$(17) \qquad 0 \to H^0(Y, \Theta_Y) \xrightarrow{\pi_*} H^0(X, \Theta_X) \xrightarrow{\rho_0} H^0(Z, \mathbf{N}_Z)$$
$$\to H^1(Y, \Theta_Y) \xrightarrow{\pi_*} H^1(X, \Theta_X) \xrightarrow{\rho_1} H^1(Z, \mathbf{N}_Z) \to \cdots$$

*Proof.* Let $R^q_\pi(\Theta)$ be the $q^{\text{th}}$ Leray sheaf of the pair $(\Theta_Y : Y \xrightarrow{\pi} X)$; thus, $R^q_\pi(\Theta)$ comes from the presheaf $U \to H^q(\pi^{-1}(U), \Theta_Y)$ for $U \subset X$ an open set. It is clear that $R^q_\pi(\Theta)_x = 0$ for $q > 0$, $x \notin Z$ and $R^0_\pi(\Theta)_x \cong \Theta_x$ for $x \notin Z$. What we claim is that $R^q_\pi(\Theta) = 0$ for $q > 0$, and that $\pi : Y \to X$ induces an inclusion

$$0 \to R^0_\pi(\Theta) \xrightarrow{\pi_*} \Theta_X$$

with $\Theta_X / R^0_\pi(\Theta) \cong \mathbf{N}_Z$. Thus we will have:

$$(18) \qquad 0 \to R^0_\pi(\Theta) \xrightarrow{\pi_*} \Theta_X \xrightarrow{\rho} \mathbf{N}_Z \to 0;$$

and (17) follows then from (18) and the Leray spectral sequence [7].

Now, if $x \in Z$, we may take a polycylinder $U \subset X$, with center $x$, coordinates $z^1, \cdots, z^n$, and such that $Z \cap U$ is given by $z^{n-r+1} = \cdots = z^n = 0$. If $\theta \in \Gamma(\pi^{-1}(U), \Theta_Y)$, then, on $\pi^{-1}(U) - \pi^{-1}(U) \cap W$,

$$\pi_*(\theta) = \sum_{\alpha=1}^{n} \theta^\alpha(z^1, \cdots, z^n) \frac{\partial}{\partial z^\alpha}$$

where $\theta^{\alpha}$ are holomorphic functions on $U - U \cap Z$. Then, by Cauchy's formula, $\theta^{\alpha}$ is holomorphic in $U$ and so $\pi_*(\theta) \in \Gamma(U, \Theta_X)$. This gives the injection

$$0 \to R_{\pi}^{0}(\Theta) \xrightarrow{\pi_*} \Theta_X .$$

If now $\theta$ is a vector field in $\pi^{-1}(U)$, then it is clear that, along $W \cap \pi^{-1}(U)$, $\theta$ is tangent to $W$; and so, along $Z \cap U$, $\pi_*(\theta)$ is tangent to $Z$. More generally then we see that $\Gamma(U, R_{\pi}^{0}(\Theta)) =$ vector fields $\theta = \sum_{\alpha=1}^{n} \theta^{\alpha}\, \partial/\partial z^{\alpha}$ in $U$ which are tangent to $Z \cap U$. On the other hand, $\rho : \Theta_X \to \mathbf{N}_Z$ is given by

$$\rho(\theta) = \sum_{\lambda=n-r+1}^{n} \theta^{\lambda}(z^1, \cdots, z^{n-r}, 0, \cdots, 0) \frac{\partial}{\partial z^{\lambda}} ,$$

and the exactness of (18) is now obvious.

We now prove that $R_{\pi}^{q}(\Theta) = 0$ for $q > 0$. This is a local question around $x \in Z$ and, as above, we may assume that $x$ is the center of a polycylinder $U$ with coordinates $z^1, \cdots, z^n$ and with $Z \cap U$ given by $z^{n-r+1} = \cdots = z^n = 0$. Then $\pi^{-1}(U) \subset U \times P_{r-1}$ is given by the points $(z^1, \cdots, z^n; t^{n+r+1}, \cdots, t^n)$ with $z^{\lambda}t^{\mu} - z^{\mu}t^{\lambda} = 0$ for $\lambda, \mu = n - r + 1, \cdots, n$. Thus $\pi^{-1}(U) \cong V \times Q$ where $V$ is the unit polycylinder in $z^1, \cdots, z^{n-r}$ and where $Q$ is the quadratic transform of zero in the unit polycylinder in $\mathbf{C}^{n-r}$. Because both $V$ and $Q$ are pseudo-convex, we may use a Kunneth formula (This may be established by the results of [8].), and it will suffice to prove that $H^q(Q, \Theta_Q) = 0$ for $q > 0$. Thus, to simplify notations, we return to our previous case and assume $r = n$ and $W = \pi^{-1}(0) = P \subset V = \pi^{-1}(U)$, $P$ being a $P_{r-1}$ .

Now, if $\mathbf{L} =$ line bundle on $V$ determined by the divisor $P$, then $\mathbf{L} \mid P = -\mathbf{H}$ where $\mathbf{H}$ is the hyperplane bundle, and we have the exact sequences

$$0 \to O_V(\mathbf{L}^{-k-1})\Theta_V \to O_V(\mathbf{L}^{-k})\Theta_V \to O_P(\mathbf{H}^{k})\Theta_V \mid P \to 0$$

$$0 \to O_P(\mathbf{H}^{k})\Theta_P \to O_P(\mathbf{H}^{k})\Theta_V \mid P \to O_P(\mathbf{H}^{k-1}) \to 0$$

for $k = 0, 1, \cdots$. Since $H^q(V, O_V(\mathbf{L}^{-l})\Theta_V) = 0$ for $q > 0$, $l \gg 0$ [6], it will suffice to prove that $H^q(P, O_P(\mathbf{H}^{k})\Theta_P) = 0 = H^q(P, O_P(\mathbf{H}^{k-1}))$ for $q > 0$, $k \geqq 0$. But this result is well known.

*Proof of Lemma* 4. There we have the notation $Z =$ a curve $S$; $X = J$, the Jacobian of $S$; and $Y = X$, the blow up of $J$ along $S$. Since $H^0(S, \Theta_S) = 0$, from $0 \to \Theta_S \to O_S(\mathbf{T}) \to O_S(\mathbf{N}) \to 0$ we deduce

$$0 \to H^0(J, \Theta_J) \to H^0(S, O_S(\mathbf{N})) \to H^1(J, \Theta_J) \xrightarrow{\rho} H^1(S, O_S(\mathbf{N})) \to 0,$$

and

$$0 \to H^q(X, \Theta_X) \xrightarrow{\pi_*} H^q(J, \Theta_J) \to 0 \quad \text{for} \quad q \geqq 2.$$

These exact sequences plus (17) give (8).

***Corollary to Theorem 4.*** *If the mapping $\rho_1$ in* (17) *is zero, and if there are no obstructions to deformations of $X$, then any deformation of $Y$ is a monoidal transform. This happens if either $H^1(X, \Theta_X) = 0$ or $H^1(Z, \mathbf{N}_Z) = 0$.*

*Proof.* If $\rho_1$ is zero, then a refinement of the argument used in Kodaira [5], Theorem 1 shows that $Z$ is stable under the deformations of $X$. But then it is obvious from (17) that deformations $(Y) \cong \{\text{deformations } (X)\}$ plus $\{\text{continuous system generated by } Z \text{ in } X\}$, and so any deformation of $Y$ is a monoidal transform.

***Remark.*** The sequence (17) is suggestive for proving, in general, that any deformation of $Y$ is a monoidal transform. Indeed, *infinitesimally*, kernel of $\rho_1$ = tangents to deformations of $X$ where $Z$ is stable = tangent to deformations of $Y$/(continuous systems); and so any deformation of $Y$ should be a monoidal transform.

**Appendix II. Continuous systems generated by hypersurfaces.** Let $X$ be a closed algebraic manifold and $D \subset X$ an irreducible non-singular hypersurface. Then $D$ gives rise to a line bundle $\mathbf{L} \to X$ with $\mathbf{L} \mid D = \mathbf{N}$, the normal bundle of $D$ in $X$. In addition to the usual sequence

$$0 \to \Theta_D \to O_D(\mathbf{T}) \to O_D(\mathbf{N}) \to 0$$

we have now an exact sheaf sequence

$$0 \to O_X \xrightarrow{\sigma} O_X(\mathbf{L}) \to O_D(\mathbf{N}) \to 0, \tag{19}$$

where $\sigma \in H^0(O_X(\mathbf{L}))$ and $D = \{x \in X : \sigma(x) = 0\}$.

Suppose now that $\Sigma = \{D_x\}_{\tau \in B}$ is the maximal continuous system generated by $D \subset X$. If $P(X)$ is the *Picard variety* [9] of $X$, then there is a holomorphic map $\varphi : B \to P(X)$ given by $\varphi(\tau)$ = algebraic equivalence class of $D_\tau - D_0$. The fibre $\varphi^{-1}(\varphi(\tau))$ is a Zariski open set in the complete linear system $|X_\tau|$, and the following is an easy computation:

***Lemma 7.*** *The following diagram commutes*:

$$\begin{tikzcd}
T_0(B) \arrow[rr, "\rho"] \arrow[dr, "\varphi*"'] & & H^0(O_D(N)) \arrow[dl, "\delta"] \\
& H^1(O_X) &
\end{tikzcd}$$

*where we have used the natural isomorphism* $\mathbf{T}_0(P(X)) = H^1(O_X)$.

***Corollary.*** *If* $H^1(O_X(\mathbf{L})) = 0$, *then* $\varphi(B)$ *is an open set in* $P(X)$.

The above Lemma serves as a basis for an elementary treatment of the *Riemann–Roch Theorem* on curves and surfaces. For example, let $X$ be a curve and $D = \Sigma n_i P_i$ ($P_i \in X$, $n_i > 0$) a divisor of degree $d$ on $X$. We may take $B$ as the $d$-fold symmetric product $X^{(d)}$ and $\Sigma$ = all divisors of degree $d$ on $X$ [4]. Furthermore, the Picard variety $P(X)$ is canonically isomorphic to the Jacobian variety $J = J(X)$ and $\varphi(D) = \Sigma n_i\varphi(P_i)$ where $\varphi : X \to J$ is the usual inclusion. If $\omega \in T_0(J)^*$ is a holomorphic differential, then $\varphi^*(\omega) = \Sigma n_i\omega(P_i) \in T_D(X^{(d)})^*$ and so $\varphi^*(\omega) = 0$ if and only if $(\omega) \geqq D$ and so dim ker $\varphi^* = i(D)$, the index of speciality of $D$.

Now, on the other hand, $O_D(\mathbf{N})$ is the sky-scraper sheaf

$$\Sigma\, \mathbf{L}_{p_i} \otimes \{O_{D,p_i}/(\mathrm{m}_{p_i})^{n_i}\}^*,$$

where $\mathrm{m}_{p_i} \subset O_{D,p_i}$ is the maximal ideal. There is a canonical section $\sigma \in H^0(O_X(\mathbf{L}))$ and dividing by $\sigma$ induces a natural isomorphism $H^0(O_D(\mathbf{N})) \cong T_D(X^{(d)})$. Also, $H^0(O_X(\mathbf{L}))/(\sigma) \cong T_0(|X|)$, and so the exact cohomology sequence of (19) becomes

$$0 \to T_0(|D|) \xrightarrow{i} T_D(X^{(d)}) \xrightarrow{\varphi^*} T_{\varphi(D)}(J) \to H^1(O_X(\mathbf{L})) \to 0, \tag{20}$$

where $i$ = differential of the inclusion $|D| \subset X^{(d)}$. From (20) we find

$$\dim |D| - i(D) = d - p \qquad (p = \dim J = \dim H^1(O_X)),$$

which is the Riemann–Roch theorem for positive divisors on curves. This approach is related to that given by Mattuck and Mayer, *Ann. Scuola Norm.*, Pisa, 1963.

The essential content of the above is to prove the *duality theorem*

$$(\dim H^1(O_X(\mathbf{L})) = i(D))$$

by using general remarks on continuous systems and the isomorphism $P(X) \cong J(X)$. This last isomorphism follows from the natural isomorphism $H^0(\Omega_X^1) \cong \overline{H^1(O_X)}$; which is a special and elementary case of the duality theorem.

In general (*i.e.*, $\dim X = n$), we may prove the formula $i(D) = \dim \overline{H^n(O_X(\mathbf{L}))}$ from the elementary isomorphism $H^0(\Omega_X^n) \cong \overline{H^n(O_X)}$ as follows ($H^n(O_X) = \overline{H^0(\Omega_X^n)}$ (Dolbeault theorem) and this gives the natural isomorphism $H^0(\Omega_X^n) = \overline{H^n(O_X)}$.): From $H^n(O_X) \xrightarrow{\sigma} H^n(O_X(\mathbf{L})) \to 0$ we deduce that $\overline{H^n(O_X(\mathbf{L}))} \cong$ holomorphic $n$-forms $\omega$ on $X$ such that $\omega/\sigma$ is holomorphic; thus $\dim H^n(O_X(\mathbf{L})) = i(D)$.

This is made quite transparent by dualizing (19) to

$$0 \to \Omega_X^n(-\mathbf{L}) \xrightarrow{\sigma} \Omega_X^n \xrightarrow{\mathrm{Res}} \Omega_D^{n-1}(-\mathbf{L}) \to 0, \tag{21}$$

where Res $\psi = \psi/d\sigma$ ($\psi \in \Omega_X^n$) is the Poincaré residue operator. The exact cohomology sequence

$$H^q(\Omega_X^n(-\mathbf{L})) \xrightarrow{\sigma} H^q(\Omega_X^n) \xrightarrow{\mathrm{Res}} H^q(\Omega_D^{n-1}(-\mathbf{L}))$$

of (21) is dual to the exact cohomology sequence

$$H^{n-q}(O_X(\mathbf{L})) \xleftarrow{\sigma} H^{n-q}(O_X) \xleftarrow{\delta} H^{n-q-1}(O_D(\mathbf{L}))$$

deduced from (19).

If $X$ is a surface and $D \subset X$ is an effective curve, let $\Sigma = \{D_\tau\}_{\tau \in B}$ be the continuous system generated by $D = D_0$. Then we have, assuming $\Sigma$ is complete,

$$0 \to \mathbf{T}_0(|D|) \xrightarrow{i} \mathbf{T}_0(\Sigma) \xrightarrow{\varphi_*} \mathbf{T}_{\varphi(D)}(P(X)) \to H^1(O_X(\mathbf{L})) \to H^1(O_D(\mathbf{N}))$$
$$\to H^2(O_X) \to H^2(O_X(\mathbf{L})) \to 0$$

which gives the Riemann–Roch inequality:

$$\dim |D| + i(D) \geqq \tfrac{1}{2}(D^2 - D\cdot K) - q + p_g \quad (q = \dim P(X),\, p_g = \dim H^2(O_X)).$$

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University of California
Berkeley


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