SOME REMARKS ON NEVANLINNA THEORY

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0. Nevanlinna theory is a beautiful subject whose results include some of the most striking and subtle in complex analysis. However, it is my feeling that most of the deeper theorems are essentially of a one-variable nature, and the most important problem in the subject is to examine those questions in several variables which are naturally posed rather than being analogues of the one-variable type results. In this paper I shall give a brief and incomplete survey of that part of Nevanlinna theory dealing with defect relations, the purpose being an attempt to substantiate my claim that the subject has a one-variable character. Then I shall turn to several variable questions, discussing some pitfalls, a few positive indications, and finally some naturally posed problems. As general references to value distribution theory, I suggest [7] for the classical case, [8] for First Main Theorems, [2] for defect relations in the equi-dimensional case, and [10] for the Ahlfors' theory of holomorphic curves.

1. In its most general setting, Nevanlinna theory deals with the global study of holomorphic mappings $f: U \to M$ between complex manifolds. Perhaps the most profitable case is when $U = \mathbb{C}^n$ and M is a projective algebraic variety. Then the growth of the mapping f has an intrinsic meaning, and in Nevanlinna theory one studies f by seeing how the image $f(\mathbb{C}^n)$ meets the subvarieties of M. The classical case is $f: \mathbb{C} \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, an *entire meromorphic function*. Here Nevanlinna theory deals with the distribution of the roots of the equation

$$f(z) = a \qquad (z \in \mathbb{C},\ a \in \mathbb{P}^1). \tag{1}$$

In brief outline, the highlights are the following (c.f. [7] for details):

Upper Bound. Let $n(a,r)$ be the number of solutions of (1) in the disc $|z| \leqq r$. Suppose first that f is holomorphic and let

$$M(f,r) = \max_{|z| \leqq r} \log |f(z)|$$

measure the growth of f. Then it follows, e.g., from *Jensen's theorem* that

$$n(a,r) \leqq CM(f,2r) + O(1,a). \tag{2}$$

[*Note:* For reasons arising from Jensen's theorem, it is convenient to consider the logarithmically averaged form of $n(a,r)$, called the *counting function*

$$N(a,r) = \int_0^r n(a,\rho)\frac{d\rho}{\rho}.$$

Then (2) follows from the inequality

$$N(a,r) \leqq M(f,r) + O(1,a).] \tag{3}$$

In case f is meromorphic, we define the *order function*

$$T(r) = \int_{a \in \mathbb{P}^1} N(a,r)d\mu(a)$$

where

$$d\mu(a) = (\sqrt{-1}/2\pi)\left[1 + |a|^2\right]^{-2} da \wedge d\bar{a}$$

is the non-Euclidean area element on the Riemann sphere. If f is holomorphic, then one proves the inequalities

$$T(r) \leqq M(f,r) + O(1)$$
$$M(f,r) \leqq CT(2r) + O(1),$$

so that the two ways of measuring the growth of f are essentially equivalent. The *First Main Theorem* (F.M.T.) gives

$$N(a,r) \leqq T(r) + O(1,a), \tag{4}$$

estimating the number of solutions of (1) by the growth of f.

Lower Bound. Historically, this proceeded in three stages:

(a) The *Liouville theorem**, which says that the image $f(\mathbb{C})$ does not omit an open set in $\mathbb{P}^1$. Thus (1) has a solution for almost all a;

(b) The *Picard theorem*, which says that (1) has solutions for all but at most *two* points $a \in \mathbb{P}^1$; and

(c) The *Nevanlinna defect relation*, which loosely stated says that for any *three* distinct points a_1, a_2, a_3 on $\mathbb{P}^1$,

$$N(a_1,r) + N(a_2,r) + N(a_3,r) \geqq T(r) + \varepsilon(r)$$
$$\text{where} \quad \lim_{r\to\infty}\left[\frac{\varepsilon(r)}{T(r)}\right] = 0. \tag{5}$$

On the whole, the situation is reminiscent of algebra: $T(r)$ plays the role of the degree of a polynomial, the upper bound (4) is similar to the (obvious) bound on the number of roots of a polynomial in terms of its degree, and the more subtle lower bound (5) is the analogue of the fundamental theorem of algebra. The only major difference is that *three* points, instead of *two* as in the polynomial case, are required in (5). As we shall see below, in several variables this similarity between algebraic and general holomorphic mappings is lost.

2. As for generalizations of the classical case to $f:\mathbb{C}^n \to M$, the situation regarding the position of $f(\mathbb{C}^n)$ vis à vis the *divisors* on M is in reasonably good shape. Now divisors on M are locally the zeroes of one holomorphic function, and it is pretty clear that the study of the zeroes of one function in several variables is about the same as in the good old days of one complex variable. From the analytic point of view, both are centered around the distributional equation (cf. [2])

$$\Delta \log |f| = \{f = 0\} \tag{6}$$

for an analytic function f and where Δ is the Laplacian and $\{f = 0\}$ is integration over the zero set of f. Thus my contention in the introduc-

* From our point of view, the Liouville theorem is obtained by integrating (4) over the image $f(\mathbb{C})$ in $\mathbb{P}^1$, and using that $T(r) \to \infty$ as $r \to \infty$ and $\int_{\mathbb{P}^1} d\mu = 1$.

tion should perhaps be amended to say that Nevanlinna theory in *codimension one* is in pretty good shape because the intuition and analytic methods from the one-variable case carry over, the new ingredient being mostly formalism.

3. Let me describe in outline what the situation regarding divisors is and mention one outstanding problem. To do this, it is convenient to use the relationship between divisors, line bundles, and Chern classes (this is the basis for the formalism mentioned above). The reference here is §0 of [5]. Given a *holomorphic line bundle* $L \to M$ and holomorphic section $\sigma \in \Gamma(M, \mathcal{O}(L))$, the zero set $\{\sigma = 0\}$ is a divisor D on M. Conversely, given D there is an associated line bundle $L = [D]$ and holomorphic section σ whose zero set is D. We denote by $|L|$ the *complete linear system* of all divisors D coming from sections σ of $L \to M$. Since σ and σ' determine the same divisor D exactly when $\sigma = \lambda\sigma'$ $(\lambda \in \mathbb{C}^*)$, $|L|$ is the projective space of lines in $\Gamma(M, \mathcal{O}(L))$. For example, if $M = \mathbb{P}^m$ and L is the hyperplane line bundle, then $|L| = (\mathbb{P}^m)^*$ is the dual projective space of hyperplanes in $\mathbb{P}^m$.

Now given a holomorphic line bundle $L \to M$, we choose a metric in the fibres and denote by $|\tau|^2$ the square length of a local section τ. Then

$$dd^c \log \frac{1}{|\tau|^2} = dd^c \log \frac{1}{|\tau'|^2}$$

for any two non-vanishing holomorphic sections τ, τ' of L over an open set on M, and thus

$$dd^c \log \frac{1}{|\tau|^2} = \omega$$

is a global C^∞ (1,1) form on M. Obviously ω is closed, and the deRham class of ω in $H^2(M, \mathbb{R})$ is the *Chern class*, $c_1(L)$, of $L \to M$. For any $D \in |L|$, ω is the Poincaré dual of the homology cycle $\{D\} \in H_{2m-2}(M, \mathbb{Z})$ carried by D. The line bundle $L \to M$ is *positive* in case there is a fibre metric such that ω is a positive (1,1) form on M; this means that locally

$$\omega = \frac{\sqrt{-1}}{2}\left[\sum_{i,j} g_{ij}\, dz_i \wedge d\bar{z}_j\right]$$

where the Hermitian matrix g_{ij} is positive definite; we write $\omega > 0$ in this situation.

The formalism of line bundles and Chern classes is introduced for two reasons: One is to have an analytic way of measuring the *size* of a divisor: Given line bundles L and L', we say that

$$c_1(L) > c_1(L')$$

in case there are fibre metrics whose Chern classes satisfy

$$\omega > \omega';$$

and then for divisors D and D' we say that

$$D > D'$$

in case $c_1([D]) > c_1([D'])$. The second reason is that the equation of currents [5]

$$dd^c \log \frac{1}{|\sigma|^2} = D \tag{7}$$

gives an analytic method of relating the size of a divisor to the defining equation. Note that (7) is the global intrinsic form of (6).

Given $f: \mathbb{C}^n \to M$ and divisor D on M defined by the zeroes of $\sigma \in \Gamma(M, \mathcal{O}(L))$, we set (pardon the flood of notations)

$$\phi = \frac{\sqrt{-1}}{2\pi}\left[\sum_{i=1}^{m} dz_i \wedge d\bar{z}_i\right] \qquad (\textit{Kähler form on } \mathbb{C}^n)$$

$\phi^q = \phi\wedge\ldots\wedge\phi$ (q-times), $\Phi = \phi^n$

$B_r = \{z \in \mathbb{C}^n: |z| \leq r\}$

$D_f = f^{-1}(D)$ and $\omega_f = f^*\omega$

$$n(D,\rho) = \int_{D_f \cap B_\rho} \phi^{n-1}$$

$$N(D,r) = \int_0^r n(D,\rho)\rho^{1-2n}\, d\rho \qquad \textit{(counting function)}$$

$$t(L,\rho) = \int_{B_\rho} \omega_f \wedge \phi^{n-1}$$

$$T(L,r) = \int_0^r t(L,\rho)\rho^{1-2n}\, d\rho \qquad \textit{(order function)}$$

The F.M.T., which is proved using (7) exactly as in the one-variable case, gives

$$N(D,r) \leq T(L,r) + O(1,D), \tag{8}$$

which is an upper bound on the size of $f^{-1}(D)$ in terms of the average or expected value $T(L,r)$. From (8) we easily obtain* the *generalized Liouville theorem*: If $\omega > 0$ and f is non-constant, then the image $f(\mathbb{C}^n)$ meets almost all $D \in |L|$.

Consequently, for divisors we always have a good upper bound and at least a crude lower bound. The more refined Picard theorems and defect relations have been proved in essentially two cases (thus far, the latter holds whenever the former does):

(a) The *equidimensional case* of $f:\mathbb{C}^m \to M_m$ where the Jacobian $J(f) \not\equiv 0$; then the image $f(\mathbb{C}^m)$ meets any D such that [2]

$$c_1([D]) + c_1(K_M) > 0 \quad (K_M = \textit{canonical line bundle of } M); \text{ and} \tag{9}$$
$$D \text{ has simple normal crossings}$$

We remark that in case $M = \mathbb{P}^m$, the canonical line bundle

* The proof follows from (8) by the same argument as outlined in the first footnote.

$K_M = H^{-(m+1)}$ where $H \to \mathbb{P}^m$ is the hyperplane line bundle; for $m = 1$, this is the number *two* in the Picard theorem. As we shall see below, the canonical divisor also appears in higher codimensional questions.

(b) The case of a *non-degenerate holomorphic curve* $f: \mathbb{C} \to \mathbb{P}^m$. Here the *Borel theorem* states that the image $f(\mathbb{C})$ can miss at most $m + 1$ hyperplanes in general position. The corresponding defect relation is due to Ahlfors [1], whose proof has great geometric subtlety, while at the same time being based on the equation (6) above. The Ahlfors theorem has been generalized by Stoll to the position of the image of $f: \mathbb{C}^n \to \mathbb{P}^m$ relative to the linear hyperplanes in projective space. Stoll's results are given in his paper in Acta Math., Vol. 90 (1953), 1-115 and Vol. 92 (1954), 55-169. His methods are applied in his paper "Deficit and Bezout Estimates" which appears in Part B of these Proceedings.

4. Turning now to higher codimension problems, we want to study a holomorphic mapping $f:\mathbb{C}^n \to M$ where M is a smooth projective variety. Let ω be a Kähler metric on M, ϕ the usual Kähler form on $\mathbb{C}^n$, and set (c.f. §5 of [5])

$$t_q(r) = \frac{1}{r^{2n-2q}} \int (f^*\omega)^q \wedge \phi^{n-q}$$
$$T_q(r) = \int_0^r t_q(\rho)\frac{d\rho}{\rho} \tag{10}$$

If $\lambda_1 \leq \ldots \leq \lambda_n$ are the eigenvalues of $f^*\omega$ relative to ϕ, then clearly

$$t_q(r) = \frac{1}{r^{2n-2q}} \int_{B_r} \sigma_q(\lambda_1,\ldots,\lambda_n)\cdot\Phi \tag{11}$$

where σ_q is the q^{th} elementary symmetric function of $\lambda_1,\ldots,\lambda_n$. The *order functions* $T_q(r)$ are the basic quantities regulating the growth of f. Given positive increasing functions $A(r)$, $B(r)$ we write

$$A \sim B$$

to mean that $A(r) \leqq CB(r) \leqq C'A(r)$ for positive constants C, C'. Then clearly

> $T_1(r) \sim T(L,r)$ where $L \to M$ is a positive line bundle; and the $T_q(r)$ are intrinsically defined in the sense of the equivalence relation $\sim$.

(Actually, the $T_q(r)$ are intrinsic in a somewhat more refined sense, but we won't be concerned with that here.) To see better what the $T_q(r)$ are, consider the case of

$$f: \mathbb{C}^2 \to \mathbb{P}^2$$

and let ω be the standard Kähler metric on $\mathbb{P}^2$. Then

$$\begin{aligned} T_1(r) &= \int_{D \in \mathbb{P}^{2*}} N(D,r)\, d\mu(D) \\ T_2(r) &= \int_{A \in \mathbb{P}^2} N(A,r)\, d\mu(A), \end{aligned} \tag{12}$$

where in the first equation D runs over the lines in $\mathbb{P}^2$, and in the second A varies over the points in $\mathbb{P}^2$.

If we now look to see what happens to the lower and upper bounds in the general case, then there is considerable trouble. To begin with, Cornalba and Shiffman [4] gave an f such that, for a suitable $A \in \mathbb{P}^2$,

$$\lim_{r \to \infty} \left[\frac{T_2(r)}{N(A,r)}\right] = 0 .$$

Thus, the lower bound (8) and the analogy with algebraic mappings are gone. Secondly, there is a famous example of Fatou-Bieberbach giving $f: \mathbb{C}^2 \to \mathbb{P}^2$ such that image $f(\mathbb{C}^2)$ omits an open set in $\mathbb{P}^2$, and consequently the Liouville theorem fails in higher codimension.

Roughly speaking, the reason for these troubles might be explained as follows: A map $f: \mathbb{C}^2 \to \mathbb{C}^2 \subset \mathbb{P}^2$ is given by

$$f(z_1, z_2) = (f_1(z_1,z_2),\ f_2(z_1,z_2))$$

where f_1, f_2 are two arbitrary entire functions on $\mathbb{C}^2$. As we go to infinity, these two functions may not "interact" properly, and the study of the common zeroes $\{f_1 = 0,\ f_2 = 0\}$ doesn't at first sight seem to be that much more promising than in the case of two real analytic functions on $\mathbb{R}^2$. Further evidence for this point of view is provided by the theorem of Chern [3] in the following form due to Wu:

> Given $f: \mathbb{C}^2 \to \mathbb{P}^2$, suppose that
>
> $$\lim_{r \to \infty} \left[\frac{t_1(r)}{T_2(r)}\right] = 0. \tag{13}$$
>
> Then the image $f(\mathbb{C}^2)$ is dense in $\mathbb{P}^2$.

Geometrically, $t_1(r)$ is the integral of $\lambda_1 + \lambda_2$ whereas $T_2(r)$ has to do with $\lambda_1\lambda_2$. Thus, if the mapping functions f_1, f_2 do interact to the extent that, e.g., $\lambda_2 \leqq c\,\lambda_1$ (so that f is *quasi-conformal*), then the Liouville theorem holds.

After all these negative remarks, I should like to mention a little theorem which, it seems to me, bodes well for the study of value distribution theory in higher codimension, although not necessarily in the codimension one format. Namely, for a holomorphic mapping $f: \mathbb{C}^2 \to \mathbb{P}^2$, we may ask what, if any, relation holds between the two order functions $T_1(r)$ and $T_2(r)$? For $f: \mathbb{C}^2 \to \mathbb{C}^2 \subset \mathbb{P}^2$ of the form

$$f(z_1, z_2) = (z_1 + h(z_2),\ z_2),$$

the Jacobian $J(f) \equiv 1$ and it follows that

$$\begin{aligned} T_2(r) &= O(\log r) \\ T_1(r) &\sim M(h_2, r) \end{aligned} \tag{14}$$

Consequently, $T_1(r)$ is independent of $T_2(r)$, which may seem a little surprising in view of (12). On the other hand it may be proved that in case f omits a divisor D with $\mu c_1(D) + c_1(K_{\mathbb{P}^2}) \geqq 0$

$$\log T_2(r) \leqq 2 \log T_1(\theta r) + \mu T_1(\theta r) + O(1) \qquad (\theta > 1), \qquad (15)$$

and in case the image $f(\mathbb{C}^2)$ misses an ample divisor

$$T_2(r) \leqq C\, T_1(\theta r)^2 \qquad (\theta > 1) \qquad (16)$$

To me this result says two things:

(i) Holomorphic mappings do have in higher codimension a certain amount of symmetry, since an estimate (15) certainly doesn't hold in the real-analytic case.*

(ii) The canonical line bundle plays a very special role in the study of holomorphic mappings, not only for the study of divisors in the equi-dimensional case as mentioned in (9), but in higher codimensional questions also, as evidenced by (15).

5. As for how the study of Nevanlinna theory in higher codimension should proceed, I might offer the following suggestions: To begin with, value distribution theory in the classical situation of an entire meromorphic function was always balanced by, and frequently motivated by, a wealth of special functions whose behavior one wished to study. For example, the decisive upper bound (2) was originally found by Hadamard in connection with his study of the zeroes of $\zeta(s)$ vis à vis the distribution of prime numbers. Moreover, for such application it was obviously necessary to have an upper bound for every value a, and not just the average statements which may be proved in several variables (c.f. Stoll [9]).

Now in several variables it seems to me that the naturally posed problems deal more with *classes of holomorphic mappings* rather than with *specific functions*. As an example of this, suppose that A is an affine alge-

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* In general, the only relation between $T_1(r)$ and $T_2(r)$ arises from the inequalities

$$\int(\lambda_1 + \lambda_2) \leqq \left[\int \lambda_1 + \lambda_2)^2\right]^{\frac{1}{2}} \left(\int 1\right)^{\frac{1}{2}} \qquad \text{(Cauchy-Schwarz)}$$

$$\int(\lambda_1\lambda_2) \leqq \int(\lambda + \lambda_2)^2 \qquad \text{(arithmetic and geometric means)},$$

so that both $T_1(r)$ and $T_2(r)$ are dominated by the intrinsically defined, but non-geometric, quantity $\int_0^r (\int_{B_\rho} (\lambda_1 + \lambda_2)^2 \Phi) \frac{d\rho}{\rho}$. The proofs of (15) and (16) are given in the paper by J. Carlson and the author in these proceedings.

braic variety. Then the even rational cohomology ring $H^{ev}(A,\mathbb{Q}) = \sum_{q=0}^{n} H^{2q}(A,\mathbb{Q})$ is realized by homotopy classes of holomorphic mappings $f: A \to \text{Grass}(r,N)$ of A into a Grassmannian [6]. Such maps have an intrinsic notion of growth, and it is reasonable to ask how much growth must f have in order to realize a given class $\zeta \in H^{2q}(A,\mathbb{Q})$? For example, we cannot take f to be algebraic unless non-trivial conditions on the Hodge type of ζ are satisfied. Closely related to this existence question is the problem of uniqueness: If $f: A \to \text{Grass}(r,N)$ is algebraic and can be holomorphically deformed to a constant, then can this be done algebraically? In all these questions, value distribution theory in higher codimension should play an essential role, and conversely such algebra-geometric considerations furnish us with a wealth of naturally given classes of holomorphic mappings to study.

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