*SOME RESULTS ON LOCALLY HOMOGENEOUS COMPLEX MANIFOLDS**

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In a previous issue of these Proceedings,[1] while discussing the periods of integrals on algebraic manifolds, we have indicated a construction of the space of all period matrices associated to a polarized algebraic variety. These period matrix varieties seem to lead to an interesting class of analytic spaces, and we shall outline here our results and additionally give some questions which have arisen.

1. *Construction of the Period Matrix Varieties.*—Let X be a polarized algebraic manifold and denote by $H_0^{r-q,q}$ the space of *primitive cohomology classes* of type $(r - q,q)$ on X. By the *Hodge decomposition:*

$$H^r(X)_0 = \sum_{q=0}^{r} H_0^{r-q,q} \qquad (H_0^{r-q,q} = \bar{H}_0^{q,r-q}),$$

where $H^r(X)_0 = W$ is the vector space, defined over $\mathbf{Q}$, of primitive cohomology classes in dimension q. There is defined on W a nonsingular rational quadratic form $Q:W \otimes W \rightarrow \mathbf{C}$ with the following properties:

$$\left.\begin{array}{ll} \text{(i)} & Q(w,w') = (-1)^r Q(w',w) \\ \text{(ii)} & Q(H_0^{r-q,q}, H_0^{p,r-p}) = 0 \qquad \text{for } p \neq q \\ \text{(iii)} & Q(H_0^{r-q,q}, H_0^{q,r-q}) > 0. \end{array}\right\} \tag{1.1}$$

Now let $V^q = \sum_{0 \leqq p \leqq q} H_0^{r-p,p}$ and $s = \left[\dfrac{r-1}{2}\right]$. Then $V^1 \subset V^2 \subset \ldots \subset V^s \subset W$ is a point in a flag manifold associated to W. In fact, if $V^1 \subset V^2 \subset \ldots \subset V^s \subset W$ is an arbitrary increasing sequence of subspaces with $\dim V^q = h_0^{0,r} + \ldots + h_0^{r-q,q}$, then there is defined a point $[V^1,\ldots,V^s]$ in a *flag manifold* $\mathbf{F}$.

Definition: We let D be the flags $[V^1,\ldots,V^s]$ which satisfy (1.1) above.

Thus D is an open domain in an intersection of quadrics on $\mathbf{F}$.

Theorem 1.1. *D is an open homogeneous complex manifold of the form $D = H\backslash G$ where G is a real, simple noncompact Lie group and, if G' is the compact form of G, $H \subset G'$ is a full centralizer of a torus.*

Examples: (*i*) When $r = 1$, $W = H^{1,0} + H^{0,1}$ and the form Q will be

$$\begin{pmatrix} 0 & \begin{matrix} \delta_2 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \delta_n \end{matrix} \\ \begin{matrix} -\delta_1 & & & \\ & \cdot & & \\ & & \cdot & \\ & & & -\delta_n \end{matrix} & 0 \end{pmatrix} \qquad 1 = \delta_1 | \delta_2 | \ldots | \delta_n$$

($n = h^{1,0}$). Then $D = U(n)\backslash Sp(n)$ is a Siegel upper half space $\mathbf{H}_n$.

(*ii*) When $r = 2$, $W = H^{2,0} + H_0^{1,1} + H^{0,2}$ and $D = H\backslash G$ where $G = SO(2h,k)$, $H = U(h) \times O(k)$ ($h = h^{2,0}$, $k = h_0^{1,1}$). Here $SO(2h,k)$ is the real orthogonal group of $\sum_{j=1}^{2h} x_j^2 - \sum_{i=1}^{k} x_i^2$. Note that D is a Cartan symmetric domain if and only if $h = 1$.

(*iii*) We note that if $\Gamma \subset G$ is the group of units of Q, then Γ acts properly discontinuously on D and $D/\Gamma = H\backslash G/\Gamma$ is an analytic space.

(*iv*) For r odd, the domain D is acted on by the symplectic group, but for $r \neq 1$, $2n - 1$, D is *not* the Siegel half space associated to Weil's higher Jacobians. For instance, if $\{X_t\}$ is a holomorphic family of varieties containing $X = X_0$, the mapping Φ: $\{t\} \to D$ given by $\Phi(t) = \{$Hodge decomposition of $H_0^r(X_t)\}$ is holomorphic, whereas Weil's Jacobians do not vary analytically with t in general.

2. *Homogeneous Complex Domains and Their Duals.*—In a general fashion now, let Y be a homogeneous algebraic manifold with semisimple automorphism group $\tilde{G}$. Then it is well known that $Y = \tilde{G}/U$ where $U \subset \tilde{G}$ is a subgroup containing a Borel subgroup.

If now $G' \subset \tilde{G}$ is the maximal compact subgroup of $\tilde{G}$, then G' acts transitively on Y and $Y = G'/H$ where $H = G' \cap U$ is a compact subgroup of G'. In fact, $H = C(\tilde{T})$ is the full centralizer of a nontrivial torus $\tilde{T} \subset T \subset H$, T being a maximal torus in G'. In case $H = K \subset G'$ is a maximal compact subgroup, and only in this case, Y is a closed Hermitian symmetric space.

Suppose that now $G \subset \tilde{G}$ is a real form of $\tilde{G}$ which has no compact factors. If then $G \cap U = H$, we shall say that G is *admissible* for Y, and in this case the G-orbit of a general point in Y will be an open homogeneous domain $D \subset Y$ where $D = G/H$. Under these circumstances, we shall say that D and Y are *dual homogeneous complex manifolds.* For example, if Y is an Hermitian symmetric space, then D is (equivalent to) a bounded Cartan domain in $\mathbf{C}^n$.

Let now ρ: $H \to \mathrm{Aut}(E_\rho)$ be an irreducible representation of H. Then ρ extends naturally to a holomorphic representation $\rho: \tilde{H} \to \mathrm{Aut}\,(E_\rho)$, $\tilde{H}$ being the complexification of H. Since there is a normal analytic subgroup $N \subset U$ with $U/N \cong \tilde{H}$, from $1 \to N \to U \to \tilde{H} \to 1$ we may extend ρ to a holomorphic representation $\rho: U \to \mathrm{Aut}(E_\rho)$. The holomorphic bundle $\mathcal{E}_\rho = \tilde{G}X_U E_\rho$ will then be a *homogeneous vector bundle* over Y.

If $D \subset Y$ is dual to Y, then $\mathcal{E}_\rho$ restricts to a holomorphic bundle on D. Furthermore, if $\Gamma \subset G$ is a discrete group acting properly discontinuously and without fixed points on D, then $M = \Gamma\backslash D$ is a complex manifold and $\Gamma\backslash\mathcal{E}_\rho \to M$ is again a holomorphic vector bundle.

For example, the canonical bundles $\mathcal{K}_Y$, $\mathcal{K}_U$, $\mathcal{K}_D$ are all obtained in this fashion from a suitable character of H.

Suppose now that Σ, Λ, Ψ are the root systems relative to a fixed maximal torus $T \subset H \subset K \subset \tilde{G}$, of $\tilde{G}$, K, H, respectively, and let Σ^+, Λ^+, Ψ^+ be the positive roots in some lexicographic order. We assume that $n = \dim Y =$ number of roots in $\Sigma^+ - \Psi^+$, $n - m = \dim K/H =$ number of roots in $\Lambda^+ - \Psi^+$, and $m =$ number of roots in $\Sigma^+ - \Lambda^+$.

An irreducible representation $\rho: H \to \mathrm{Aut}(E_\rho)$ has highest weight $\rho \in T^*$ and we set:

$$\left.\begin{aligned} |\rho|_1 &= \{\text{number of } \alpha \in \Sigma^+ - \Lambda^+ \text{ such that } (\rho,\alpha) > 0\} \\ |\rho|_2 &= \{\text{number of } \alpha \in \Lambda^+ - \Psi^+ \text{ such that } (\rho,\alpha) < 0\} \end{aligned}\right\} \tag{2.1}$$

Then, if ρ is not singular,

$$|\rho_2| + m - |\rho|_1 = |\rho| \text{ is the } \textit{index } |\rho| \text{ of } \rho. \tag{2.2}$$

For example, for the character κ giving the canonical bundles, $|\kappa|_1 = 0$, $|\kappa|_2 = n - m$, $|\kappa| = n$.

3. *The Cohomology of Locally Homogeneous Bundles.*—Let $\rho: H \to \text{Aut}(E_\rho)$ be an irreducible representation with highest weight $\rho \in T^*$. We call ρ *good* if either $|\rho| = 0$ or $(\rho,\rho) > c_Y$ where c_Y is a suitable constant depending only on Y.

VANISHING THEOREM. (a) *If ρ is good, then $H^q(Y,\mathcal{O}(\mathcal{E}_\rho)) = 0$ for $q \neq |\rho|$ and $H^*(Y,\mathcal{O}(\mathcal{E}_\rho)) = 0$ if ρ is singular.* (b) *If $(\rho,\rho) > c_M$ where $M = \Gamma\backslash D$ is a compact, complex manifold, then $H^q(M,\mathcal{O}(\mathcal{E}_\rho)) = 0$ for $q \neq |\rho|_1 + |\rho|_2$.*

Examples: (*i*) The simplest case is where H has a 1-dimensional center, so that the line bundles over Y are $\mathcal{K}_Y$ and its powers. Let $\mathcal{K}_Y = \mathcal{O}(\mathcal{K}_Y)$, etc., and $\mathcal{K}_Y^\mu = \mathcal{O}(K_Y^\mu), \ldots$. Then we have

$$\left.\begin{aligned} H^q(Y,\mathcal{K}_Y^\mu) &= 0 \qquad \text{for } q < n,\ \mu > 0 \\ H^q(Y,\mathcal{K}_Y^{-\mu}) &= 0 \qquad \text{for } q > 0,\ \mu > 0. \end{aligned}\right\} \tag{3.1}$$

For $\Gamma\backslash G/H/ = M$, $\dim M = n = \dim Y$ and

$$\left.\begin{aligned} H^q(M,\mathcal{K}_M^\mu) &= 0 \qquad \text{for } \mu > \mu_0 > 0,\ q \neq n - m = \dim K/H \\ H^q(M,\mathcal{K}_M^{-\mu}) &= 0 \qquad \text{for } \mu > \mu_0 > 0,\ q \neq m = n - \dim K/H. \end{aligned}\right\} \tag{3.2}$$

(*ii*) This example, especially (3.2), shows that automorphic forms exists only if D is a symmetric domain—in general we shall find *automorphic cohomology* in dim K/H.

The proof of the theorem is done by using Kodaira's differential-geometric method together with a computation of the curvature in $\mathcal{E}_\rho$. By combining the vanishing theorem with the Atiyah-Singer-Hirzebruch theorem, we may show

EXISTENCE THEOREM. (a) *If ρ is good and is nonsingular, then $H^{|\rho|}(Y,\mathcal{O}(E))$ is an irreducible $\tilde{G}$-module W_ρ.* (b) *If $(\rho,\rho) > c_M$, then dim $H^{|\rho|_1+|\rho|_2}(M,\mathcal{O}(\mathcal{E}_\rho)) = c(Y)\cdot \dim W_\rho \cdot vol(M)$ where $c(Y) \neq 0$ depends on Y and $vol(M)$ is the volume of M.*

Example: Referring to (5) above, we have

$$\dim H^l(M,\mathcal{K}_M^\mu) = c_M\mu^n + (\cdots), \qquad l = \dim K/H, \tag{3.3}$$

where $c_M > 0$ and $(\cdots)$ are lower-order terms in μ.

4. *An Interpretation of Automorphic Cohomology.*—Let $Y = \tilde{G}/U$ and $M = \Gamma\backslash G/H$ be as in section 2, where we now assume that $\Gamma \cap K = \{e\}$, $K \subset G$ being the maximal compact subgroup. Then $S_0 = K/H$ is a compact, complex submanifold in D which projects isomorphically onto a subvariety of M. We let Σ' be the set of compact, complex submanifolds $S \subset D$, $\dim S = \dim S_0 = q_0$, and Σ the irreducible component of Σ' containing S_0. Over Σ we may define a bundle of vector spaces $\mathcal{E} \xrightarrow{\pi} \Sigma$ by $\mathcal{E}_S = \pi^{-1}(S) = H^{q_0}(S,\mathcal{K}_D^\mu)$. Since $\mathcal{K}_D^\mu|S_0$ is negative, $H^{q_0}(S_0, \mathcal{K}_D^\mu) = W^\mu$ will be an irreducible $\tilde{K}$-module with $\dim W^\mu \to \infty$ as $\mu \to \infty$.

Now let A be the affine variety $\tilde{G}/\tilde{K}$ and $\mathcal{E}_\mu \to A$ the homogeneous bundle obtained from the $\tilde{K}$-module W^μ.

THEOREM 4.1. *There exists a G-equivariant embedding* $\varphi:\Sigma \rightarrow A$ *such that* $\mathcal{E} \rightarrow \Sigma$ *is the restriction of* $\mathcal{E}_\mu \rightarrow A$.

Remarks: Clearly G acts on Σ by $g \cdot S = g(S)$, and $G \subset \tilde{G}$ acts on $A = \tilde{G}/\tilde{K}$. Observe that there is a natural mapping

$$H^{q_0}(D,\mathcal{K}_D{}^\mu) \rightarrow H^0(\Sigma,\mathcal{E}). \tag{4.2}$$

Now we let $\Psi = \Gamma\backslash\Sigma$ be the irreducible analytic set of compact, complex subvarieties $S \subset M$, dim $S = q_0$, and with $S_0 \in \Psi$. The bundle $\mathcal{E} \rightarrow \Sigma$ projects to a bundle $\Gamma\backslash\mathcal{E} = \mathcal{E} \rightarrow \Psi$, and there is a natural mapping

$$H^{q_0}(M,\mathcal{K}_M{}^\mu) \xrightarrow{\rho} H^0(\Psi,\mathcal{E}). \tag{4.3}$$

THEOREM 4.4. *For* μ *sufficiently large, the image* $\rho\{H^{q_0}(M,\mathcal{K}_M{}^\mu\} \subset H^0(\Psi,\mathcal{E})$ *contains sections which give an embedding of* Ψ *in a Grassmann manifold.*

Remarks: These results lead to the following speculations:

(1) If $\mathcal{H}^q(D,\mathcal{K}_D{}^\mu)$ are the square-integrable harmonic forms on D, then $\mathcal{H}^q(D,\mathcal{K}_D{}^\mu) = 0$ for $q \neq q_0$;

(2) $\mathcal{H}^{q_0}(D,\mathcal{K}_D{}^\mu)$ is a finite sum of irreducible G-modules;

(3) $\Sigma \subset A$ is a bounded Stein domain and the G-module mapping

$$\mathcal{H}^{q_0}(D,\mathcal{K}_D{}^\mu) \rightarrow H^0(\Sigma,\mathcal{E})$$

is nontrivial;

(4) If $\Gamma \subset G$ is an arithmetic subgroup with $\Gamma \cap K = \{e\}$, then the Poincaré series operator

$$\pi:\mathcal{H}^{q_0}(D,\mathcal{K}_D{}^\mu) \rightarrow H^{q_0}(M,\mathcal{K}_M{}^\mu)$$

is onto for μ large; and

(5) The following diagram commutes:

$$\begin{array}{ccccc} 0 & \rightarrow & \mathcal{H}^{q_0}(D,\mathcal{K}_D{}^\mu) & \rightarrow & H^0(\Sigma,\mathcal{E}) \\ & & \downarrow \pi & & \downarrow \pi \\ 0 & \rightarrow & H^{q_0}(M,\mathcal{K}_m{}^\mu) & \rightarrow & H^0(\Psi,\mathcal{E}) \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

and Ψ is a Zariski open in an irreducible algebraic manifold.

5. *References.*—The material in section 1 arose out of our work on periods of integrals, as did also the interpretation of cohomology in section 4. Part (*a*) of the vanishing and existence theorems of section 3 were found some years ago by Borel-Weil, Bott, and Kostant [cf. *Ann. Math.*, **66**, 203–248 (1957)]. Part (*b*) of these theorems were found by Langlands and Ise for D a symmetric domain [cf. *Am. J. Math.*, **85**, 99–125 (1963)]. The uniform treatment of all of these results using a simple differential-geometric principle seems new, although the possibility of such was probably known.

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[1] Griffiths, P. A., these PROCEEDINGS, **55**, 1392 (1966).