SOME TRANSCENDENTAL METHODS IN THE STUDY OF ALGEBRAIC CYCLES

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0. *Introduction*

The algebraic cycles which lie in a smooth, projective algebraic variety, together with the various equivalence relations (such as numerical or algebraic equivalence) which may be defined on such cycles, constitute a purely algebro-geometric concept. One may, however, attempt to study these by transcendental methods, and historically much of the structure of algebraic cycles (such as the notion of linear equivalence of divisors as related to the Jacobian and Picard varieties) was discovered in this way. Conversely, the questions in complex function theory which arose initially from algebraic geometry are frequently quite interesting in themselves. For example, the first and second Cousin problems for divisors in $\mathbb{C}^n$ were very much motivated by consideration of the Jacobian varieties of curves.

In this paper I shall give an expository account of a few transcendental methods for studying algebraic cycles of intermediate dimension. The main technique to be discussed is the use of *intermediate Jacobians*, and concerning these I shall focus on two things:

(i) how these complex tori may be used to give an interesting equivalence relation on algebraic cycles, which is between algebraic and rational equivalence, and which may well lead to a good generalization of the classical Picard variety[0].

[0] This is the so-called *incidence equivalence relation* which was introduced somewhat obscurely in [9]. An application of this equivalence relation is given in [4], where it is shown that it allows us to reconstruct a non-singular cubic-threefold from the singular locus of the canonical theta divisor on its intermediate Jacobian.

(ii) how the intermediate Jacobians may serve to detect the quotient of homological modulo algebraic equivalence.

The developments concerning (i) and (ii) are as yet incomplete, and one purpose of this paper is to formulate precise problems concerning intermediate Jacobians whose solution would very nicely yield the structure of the Picard ring (cf. section 1) of an algebraic variety. For all but one of these problems we can give a plausibility argument, and in all cases we can formulate and prove some analogue of the problem for varieties defined over function fields.

A second technique to be briefly discussed is the notion of *positivity* for general algebraic cycles, and we shall propose a definition which is suggested by looking at the cohomology classes on a smooth, projective variety which are represented using de Rham's theorem by positive differential forms.

Most of the material given below has already appeared in [9], [10], and [11]; however, here I have tried to give clearer and more precise formulations than before. The discussion of intermediate Jacobians has some points of contact with Lieberman's paper [22], and a few of the results on normal functions have appeared only in preprint form. In discussing positivity for algebraic cycles, I have used Hartshorne's definition of an ample vector bundle [14], and the presentation has been influenced by recent results of Bloch and Geiseker [3] concerning the numerical positivity of such bundles. Finally, the study of algebraic cycles via transcendental methods which we have tried to present is, in spirit, very much akin to the treatment given by Kodaira in [17], [18], [19] for the structure of the divisors on a smooth, projective variety.

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1. *Algebraic cycles and equivalence relations*

In this section we will give some concepts of a purely algebro-geometric nature. Then, in the remainder of the paper we will discuss some transcendental methods for studying these concepts.

We consider a complete, smooth, and projective algebraic variety V over the complex numbers, and we want to discuss the algebraic cycles, and the various equivalence relations among these, which lie on V (cf. Samuel [24]). For this we recall that an *algebraic cycle* on V is given by

$$Z = n_1 Z_1 + \dots + n_\mu Z_\mu \tag{1.1}$$

where the Z_j are irreducible algebraic subvarieties of V and the n_j are integers. We say that Z has codimension q if all the Z_j have codimension q, and will call Z *effective* if all n_j are non-negative. We may write

$$Z = Z_+ - Z_- \tag{1.2}$$

where Z_+ and Z_- are effective algebraic cycles, and this may be done in many different ways.

Two effective algebraic cycles Z and Z' are *strongly algebraically equivalent* if there is a connected algebraic variety T and an effective cycle $W \subset T \times V$ such that all intersections $W_t = (\{t\} \times V) \cdot W$ are defined and of the same dimension, and such that $Z = W_{t_1}$ and $Z' = W_{t_2}$ for $t_1, t_2 \in T$. A general algebraic cycle Z is *algebraically equivalent to zero*, written

$$Z \approx 0,$$

if we may write Z in the form (1.1) where Z_+ and Z_- are strongly algebraically equivalent. An equivalent definition is that we may write $Z = Z_+ - Z_-$ where Z_+ and Z_- are in the same (topological) component of the Chow variety of V.

We will say that Z is *rationally equivalent to zero*, written

$$Z \approxeq 0,$$

if Z is algebraically equivalent to zero as above with the parameter variety T being a rational variety. If we take all the algebraic cycles on V modulo rational equivalence, there results the graded *Chow ring*[1] (cf. [13])

$$C(V) = \bigoplus_{q=0}^{n} C_q(V)$$

where the grading is by codimension of cycles and where the product

$$C_p(V) \times C_q(V) \to C_{p+q}(V)$$

is induced by taking the intersection of cycles. Implicit in this definition is the assertion that any two algebraic cycles Z and W on V are rationally equivalent to cycles Z' and W' which intersect properly and the resulting product (1.2) should then be well-defined.

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[1] A graded ring will be a graded, commutative ring with unit $R = \bigoplus_{q=0}^{n} R_q$ such that the multiplication $R_p \times R_q \to R_{p+q}$ is compatible with the grading.

All equivalence relations we shall consider will be weaker than rational equivalence, and will therefore generate naturally an ideal in $C(V)$. We also recall that a map

$$f\colon V \to W$$

between smooth, projective varieties V and W induces additive homomorphisms

$$(1.3) \qquad \begin{cases} f_*\colon C(V) \to C(W) \\ f^*\colon C(W) \to C(V), \end{cases}$$

which are related to the product (1.2) by the formula

$$(1.4) \qquad f_*(X)\cdot Y = f_*[X\cdot f^*(Y)] \qquad (X\in C(V),\ \ Y\in C(W)).$$

Geometrically, the map f_* means "pushing cycles forward by f" and f^* means "lifting cycles back under f".

The cycles which are algebraically equivalent to zero generate a graded ideal

$$A(V) = \bigoplus_{q=0}^{n} A_q(V),$$

which is preserved under the maps f_* and f^* in (1.3). The quotient

$$NS(V) = C(V)/A(V)$$

will be called the *Neron-Severi ring* of V. For $q = 1$, $NS_1(V)$ is just the group of divisors modulo algebraic equivalence, and is the usual Neron-Severi group of V.

There is a homomorphism

$$(1.5) \qquad h\colon C(V) \to H^{\text{even}}(V, \mathbb{Z})$$

which sends a cycle $Z\in C_q(V)$ into its *fundamental class* $h(Z)\in H^{2q}(V,\mathbb{Z})$, the latter being the Poincaré dual of the homology class carried by Z. The kernel of h is the graded ideal

$$H(V) = \bigoplus_{q=0}^{n} H_q(V)$$

of cycles which are homologous to zero,[2] written

$$Z \sim 0.$$

We note that $A(V) \subset H(V)$; i.e., algebraic equivalence implies homological equivalence.

A map $f\colon V \to W$ induces on cohomology the usual maps $f_*\colon H^*(V, \mathbb{Z}) \to H^*(W, \mathbb{Z})$ and $f^*\colon H^*(W, \mathbb{Z}) \to H^*(V, \mathbb{Z})$, and the homomorphism h in (1.5) is functorial with respect to these induced maps.

Before defining the last equivalence relation, I should like to comment that there are in general two methods of studying algebraic subvarieties of arbitrary codimension: (i) by the divisors which pass through the subvariety, and (ii) by the subvarieties which meet the given subvariety. The first method (*method of syzygies*) has a global version in vector bundles and $K(V)$; the second method (*method of incidence*) leads to an equivalence relation on algebraic cycles which will now be defined.

Let Z be an algebraic cycle of codimension q on V and consider an algebraic family $\{W_s\}_{s \in S}$ of effective algebraic cycles of dimension $q-1$ whose parameter space S is smooth and complete. To be precise, we assume given an effective cycle

$$W \subset S \times V$$

such that all intersections $W_s = (\{s\} \times V)\cdot W$ are defined and of dimension $q-1$. By changing Z in its rational equivalence class, we may assume that the intersection $W\cdot(S \times Z)$ is defined and that

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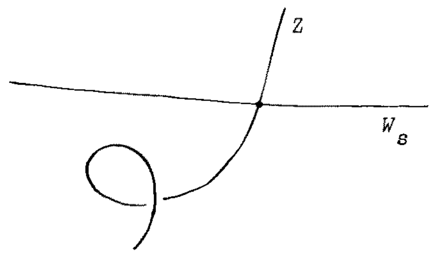
[2] Using l-adic cohomology, $H(V)$ may be defined purely algebraically.

$$D_Z = \mathrm{pr}_S[W\cdot(S \times Z)]$$

is a divisor on S. This *incidence divisor* D_Z is the set of all points $s \in S$ such that W_s meets Z, and where the points are of course counted with multiplicities ([9]).

Example. The first interesting case is when $\dim V = 3$ and Z is a curve. Then the W_s are also curves, so that the incidence relation may be pictured something like this:

(Fig. 1)



The linear equivalence class of the incidence divisor D_Z is well-defined by the rational equivalence class of Z, and we shall say that Z is *incidence equivalent to zero*, written $Z \overset{\sim}{\approx} 0$, if all such incidence divisors D_Z are linearly equivalent to zero on the parameter space S. The incidence equivalence relation again generates a graded ideal

$$I(V) = \bigoplus_{q=0}^{n} I_q(V),$$

and this ideal is preserved by the maps in (1.3).[3]

(1.6) *Definition. We define the Picard ring* Pic (V) *by*

$$\mathrm{Pic}\,(V) = C(V)/I(V),$$

and we also set $\mathrm{Pic}^0(V) = A(V)/I(V)$.

[3] The incidence equivalence relation has been defined here by playing off Z against cycles whose codimension is $\dim(Z) - 1$. We could try to play off Z against all cycles whose dimension is $\leq \dim(Z) - 1$, but there is good evidence that this gives nothing beyond the case considered here.

As an initial justification for this definition, we observe that $\mathrm{Pic}_1(V)$ is the usual Picard variety of V because of the following easy

(1.7) *Lemma. If Z is a divisor on V, then Z is incidence equivalent to zero if, and only if, Z is linearly equivalent to zero.*

Let us define a *graded abelian variety* to be a direct sum

$$A = \bigoplus_{q=0}^{n} A_q$$

where each A_q is an abelian variety, and where A is made into a ring with a trivial multiplication. We shall say that A is *self-dual* if, up to isogeny, A_q is the dual abelian variety $\hat{A}_{n-q+1}$ to A_{n-q+1}. In section 2 below we shall discuss a transcendental proof of the

(1.8) *Proposition.* $\mathrm{Pic}^0(V)$ *is, in a natural way, a graded abelian variety.*[4]

We observe also that $\mathrm{Pic}^0(V)$ is functorially associated to V in that the maps in (1.3) induce maps between $A(V)$, $I(V)$ and $A(W)$, $I(W)$ which make the obvious diagrams commutative.

The outstanding question concerning $NS(V)$ and Pic (V) is the

Problem A. (i) *Is the Neron-Severi ring $NS(V)$ finitely generated?* (ii) *Is the graded abelian variety* $\mathrm{Pic}^0(V)$ *self-dual?*

The first part of this problem seems to be well known (I learned it from Mumford), and (1.8) together with the discussion in sections 2 and 3 below should lend credence to the second part, which at any

[4] Thus each graded component $\mathrm{Pic}^0_q(V)$ of $\mathrm{Pic}^0(V)$ is an abelian variety, and the induced product

$$A(V)/I(V) \otimes A(V)/I(V) \rightarrow A(V)/I(V)$$

is identically zero. I know of no algebro-geometric proof for either of these statements.

event is known in the following special cases:

a) $q = 1$ [5]

b) $\dim V = 2m - 1$ and $q = m$.

Observe that Problem A would very nicely give the structure of Pic (V) because of the exact sequence

$$(1.9) \qquad 0 \to \mathrm{Pic}^0(V) \to \mathrm{Pic}\,(V) \to NS(V) \to 0,$$

which results from the inclusions

$$(1.10) \qquad I(V) \subset A(V) \subset H(V) \subset C(V).$$

2. *Intermediate Jacobian varieties*

We want to discuss a transcendental method for studying the Chow ring $C(V)$ and its various ideals (1.10) introduced above. This is the use of the Jacobian variety $T(V)$ of the smooth, projective variety V, which will be a graded, self-dual complex torus functorially associated to V and whose definition we will now give.[6]

First we recall the *Hodge decomposition* [8]

$$(2.1) \qquad H^m(V,\mathbb{C}) = \bigoplus_{r+s=m} H^{r,s}(V) \qquad (H^{r,s}(V) = \overline{H^{s,r}(V)}),$$

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[5] In this case $\mathrm{Pic}^0_1(V)$ is the identity component of the usual Picard variety of V and $\mathrm{Pic}^0_n(V)$ is isogeneous to the *Albanese variety* of V.

[6] Thus $T(V) = \bigoplus_{q=0}^{n} T_q(V)$ is a direct sum of complex tori $T_q(V)$ where $T_q(V)$ is isogeneous to the dual torus $\hat{T}_{n-q+1}(V)$ of $T_{n-q+1}(V)$ and where a map $f\colon V \to W$ generates induced maps $f_*\colon T(V) \to T(W)$ and $f^*\colon T(W) \to T(V)$ which satisfy the usual functoriality properties.

and the associated *Hodge filtration*

$$F^{m,p}(V,\mathbb{C}) = \bigoplus_{\left\{\substack{r+s=m\\ s\leqq p}\right.} H^{r,s}(V). \tag{2.2}$$

There is a useful method for describing the Hodge filtration in terms of the de Rham description of $H^m(V,\mathbb{C})$ together with the complex structure on V, which is the following: Let $A^{m,p}(V)$ be the vector space of global C^∞ differential forms on V which have total degree m and have type $(m,0) + \ldots + (m-p,p)$ (these are the same indices as in the definition of $F^{m,p}(V,\mathbb{C})$). If $Z^{m,p}(V)$ are the d-closed forms in $A^{m,p}(V)$, then by de Rham's theorem there is a natural map

$$Z^{m,p}(V)/dA^{m-1,p-1}(V) \to H^m(V,\mathbb{C}).$$

Because V carries a Kähler metric, this map turns out to be injective with image $F^{m,p}(V,\mathbb{C})$, and so there results an isomorphism

$$F^{m,p}(V,\mathbb{C}) \cong Z^{m,p}(V)/dA^{m-1,p-1}(V). \tag{2.3}$$

We shall use the following notations:

$$\begin{aligned}
H_+^{2q-1}(V) &= \bigoplus_{\left\{\substack{r+s=2q-1\\ s\leqq q-1}\right.} H^{r,s}(V) = F^{2q-1,q-1}(V,\mathbb{C})\\
H_-^{2q-1}(V) &= \bigoplus_{\left\{\substack{r+s=2q-1\\ r\leqq 2q-1}\right.} H^{r,s}(V) \cong H^{2q-1}(V,\mathbb{C})/H_+^{2q-1}(V)\\
H_+^{\mathrm{odd}}(V) &= \bigoplus_{q=1}^{n} H_+^{2q-1}(V)\\
H_-^{\mathrm{odd}}(V) &= \bigoplus_{q=1}^{n} H_-^{2q-1}(V).
\end{aligned} \tag{2.4}$$

Observe the decomposition

$$H^{2q-1}(V,\mathbb{C}) = H_+^{2q-1}(V) + H_-^{2q-1}(V) \tag{2.5}$$

of $H^{2q-1}(V,\mathbb{C})$ into a direct sum of conjugate subspaces. The non-degenerate pairing

$$H_{-}^{2q-1}(V) \otimes H_{+}^{2n-2q+1}(V) \to H^{2n}(V,\mathbb{C}) \cong \mathbb{C}$$

induces a duality isomorphism

$$H_{+}^{2n-2q+1}(V) \cong \check{H}_{-}^{2q-1}(V). \tag{2.6}$$

Definition. The q^{th} intermediate Jacobian $T_q(V)$ is the complex torus $H_{+}^{2q-1}(V)\backslash H^{2q-1}(V,\mathbb{C})/H^{2q-1}(V,\mathbb{Z}) \cong H_{-}^{2q-1}(V)/H^{2q-1}(V,\mathbb{Z})$. The Jacobian variety $T(V)$ is the direct sum $\bigoplus_{q=0}^{n} T_q(V)$ of the intermediate Jacobians.

From (2.5) together with the definition of the intermediate Jacobian there results an $\mathbb{R}$-*linear* isomorphism

$$T_q(V) \cong H^{2q-1}(V,\mathbb{R})/H^{2q-1}(V,\mathbb{Z}). \tag{2.7}$$

The complex Lie algebra of $T_q(V)$ is $H_{-}^{2q-1}(V)$ and, using (2.6) and (2.4), the holomorphic differentials on $T_q(V)$ are given by the isomorphisms

$$H^{1,0}(T_q(V)) \cong H_{+}^{2n-2q+1}(V) \cong F^{2n-2q+1,n-q}(V,\mathbb{C}). \tag{2.8}$$

The intermediate Jacobian defined above is closely related to, but not the same as, the intermediate Jacobian variety $J_q(V)$ introduced by Weil [25] and studied by Lieberman [22]. The main points of comparison are: (i) there is a natural $\mathbb{R}$-linear isomorphism between the two complex tori; (ii) $T_q(V)$ varies holomorphically with V whereas $J_q(V)$ does not, and (iii) $J_q(V)$ is an abelian variety whereas $T_q(V)$ has an r-convex polarization (cf. [8]). A propos the point (iii) just made, we recall from [8] the

(2.9) *Lemma. Let S be a complex sub-torus of $T_q(V)$ whose complex Lie algebra is contained in the subspace $H^{q-1,q}(V)$ of the Lie algebra of $T_q(V)$. Then the r-convex polarization is 0-convex on S, so that in particular S is naturally an abelian variety.*

We have defined the direct sum $\bigoplus_{q=0}^{n} T_q(V)$ to be the (total) Jacobian variety of V by analogy with the classical definition of the Jacobian of a curve as being the complex torus associated to the period matrix of the differentials of odd degree on the curve. On the other hand, the Picard variety Pic (V), as defined in section 1, is a purely algebro-geometric concept arising out of the notion of linear equivalence of divisors. The relation between these falls under the general heading of "Abel's theorem" and will be discussed in section 3. The fact that the Jacobian variety $T(V)$ is a graded, self-dual complex torus functorially attached to V (cf. footnote [6]) follows immediately from the definitions, the duality (2.6), and the functorial behavior which the induced maps f_* and f^* on cohomology have with respect to the Hodge filtration (2.2).

The algebro-geometric importance of the Jacobian variety rests in the fact that there is an *Abel-Jacobi homomorphism*

$$\Phi: H(V) \to T(V) \tag{2.10}$$

whose definition is a generalization of the classical procedure for sending divisors of degree zero on a compact Riemann surface S into the Jacobian variety of S. To define Φ, it will suffice to give meaning to a symbol

$$\langle\phi, Z\rangle \qquad (\phi \in H^{1,0}(T_q(V)), \quad Z \in H_q(V)) \tag{2.11}$$

which has the properties of (i) being a complex number, (ii) being defined modulo periods, and (iii) being linear in each factor.

Using the isomorphism (2.8), we may consider ϕ as being a closed, C^∞ differential form of type $(2n-2q+1\,,\,0) + \ldots + (n-q+1\,,\,n-q)$ and which is defined modulo exact forms $d\eta$ where η is of type $(2n-2q\,,\,0) + \ldots + (n-q+1\,,\,n-q-1)$. Furthermore, $Z \in H_q(V)$ is represented by an algebraic cycle of codimension q which is the boundary of a $(2n-2q+1)$-chain Γ on V.[7] We then let

$$(2.12) \qquad \langle\phi,\Gamma\rangle = \int_\Gamma \phi \,.$$

This symbol is well defined modulo periods since $\int_\Gamma d\eta = \int_Z \eta = 0$ if $\phi = d\eta$ as above. Furthermore, it is clearly bilinear, and so we may then use it to define the Abel-Jacobi mapping (2.10).

The mapping Φ has the following basic properties:

(2.13) Φ is holomorphic on the ideal $A(V)$ of cycles algebraically equivalent to zero on V.

(2.14) The image $\Phi_q[A_q(V)]$ of codimension q cycles which are algebraically equivalent to zero is a complex sub-torus $I_q^0(V)$ whose Lie algebra is a subspace of $H_-^{q-1,q}(V)$. It follows then from (2.9) that $I_q^0(V)$ is an abelian variety.[8]

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[7] There is a foundational question here as to just what is meant by the equation "$\partial\Gamma = Z$". Moreover, we shall want to integrate over such "chains" Γ, and later on we shall want to let everything in sight depend on parameters. The foundational questions pose a significant problem which is resolved, using the theory of integral currents, in King's paper [16].

[8] The meaning of the notation $I_q^0(V)$ for the image in $T_q(V)$ of cycles algebraically equivalent to zero will be discussed in section 4 below.

(2.15) The Abel-Jacobi mapping Φ satisfies the hoped-for functorial properties. Thus, a holomorphic mapping $f\colon V \to W$ between smooth, projective varieties leads to commutative diagrams (cf. (1.3))[9]

$$\begin{array}{ccc} H(V) & \xrightarrow{\Phi_V} & T(V) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ H(W) & \xrightarrow[\Phi_W]{} & T(W) \end{array}$$

$$\begin{array}{ccc} H(W) & \xrightarrow{\Phi_W} & T(W) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ H(V) & \xrightarrow[\Phi_V]{} & T(V) \end{array}$$

(2.16) A noteworthy special case of (2.15) occurs when we consider a fixed cycle $Z \in C_q(V)$. Intersection with Z induces a map $Z\colon H_p(V) \to H_{p+q}(V)$, while cup-product with the fundamental class $h(Z) \in H^{2q}(V, \mathbb{Z})$ leads to a homomorphism $h(Z)\colon T_p(V) \to T_{p+q}(V)$, and we have a commutative diagram

$$\begin{array}{ccc} H_p(V) & \xrightarrow{\Phi_p} & T_p(V) \\ {\scriptstyle Z}\downarrow & & \downarrow{\scriptstyle h(Z)} \\ H_{p+q}(V) & \xrightarrow[\Phi_{p+q}]{} & T_{p+q}(V). \end{array}$$

Other basic properties of Φ occur when the complex structure of V is allowed to vary with parameters; these will be discussed in section 5 below.

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[9] These properties, which are heuristically quite reasonable, were first proved by Lieberman [22] and then independently by myself, using residues, in [9]. They also follow from the results in [16].

3. *Concerning Abel's theorem*

We continue discussing the Abel-Jacobi mapping (2.10), and in this section we are interested in the kernel of Φ; i.e., what is the equivalence relation on cycles given by Φ. In the classical case of divisors, the equivalence of Φ is linear equivalence,[10] from which it follows that

$$\Phi\colon \mathrm{Pic}_1^0(V) \to T_1(V)$$

is an isomorphism. In particular, this proves by transcendental methods that $\mathrm{Pic}_1^0(V)$, as defined algebro-geometrically in section 1 above, is an abelian variety. For general codimension, one-half of the above version of Abel's theorem can be proved:

(3.1) *Proposition* [9]. *If* $Z \in A(V)$ *is algebraically equivalent to zero and* $\Phi(Z) = 0$ *in* $T(V)$, *then* Z *is incidence equivalent to zero.*[11]

We want to restate (3.1) in a more suggestive manner. To do this, we let

$$K(V) = \bigoplus_{q=1}^{n} K_q(V)$$

be the kernel of the Abel-Jacobi mapping Φ on $A(V)$. It follows from (2.15) and (2.16) that $K(V)$ is a graded ideal in $A(V)$ which is functorially associated to V. The quotient

$$A(V)/K(V) \cong I^0(V)$$

[10] In the framework we are using, this result is proved by Kodaira in [18].

[11] In the case of curves, this proposition corresponds to what is ordinarily the "more difficult" half of Abel's theorem, which is the construction of a linear equivalence using differentials of the third kind.

is an abelian variety by virtue of (2.14). From (3.1) we have the inclusion

$$K(V) \subset I(V),$$

and this leads to the commutative diagram

$$\begin{tikzcd} A(V) \arrow[rr, "\Phi"] \arrow[dr] & & I^0(V) \subset T(V) \arrow[dl, "\Psi"] \\ & \mathrm{Pic}^0(V) & \end{tikzcd} \tag{3.2}$$

which is a direct sum of the commutative diagrams

$$\begin{tikzcd} A_q(V) \arrow[rr, "\Phi_q"] \arrow[dr] & & I^0_q(V) \subset T_q(V) \arrow[dl, "\Psi_q"] \\ & \mathrm{Pic}^0_q(V) & \end{tikzcd} \tag{3.2$_q$}$$

for each $q = 1,\dots,n$.

(3.3) *Corollary.* $\mathrm{Pic}^0_q(V)$ *is an abelian variety for all* $q = 1,\dots,n$.

(3.4) *Corollary. The induced pairing* $A(V)/I(V) \otimes A(V)/I(V) \to A(V)/I(V)$ *is zero, so that* $\mathrm{Pic}^0(V) = \bigoplus_{q=1}^{n} \mathrm{Pic}^0_q(V)$ *is a graded abelian variety according to the definition in section* 1.

Using recent results of Deligne [5], the diagram $(3.2)_q$ can be further understood in one important special case:

(3.5) *Proposition. Suppose that* $\dim V = 2m - 1$ *is odd and consider the diagram* $(3.2)_m$*. Then* (i) *the kernel of* Ψ_m *is finite, and* (ii) *the auto-duality* $T_m(V) \cong \check{T}_m(V)$ *induces an auto-duality* $I_m(V) \cong \check{I}^0_m(V)$ *(both dualities are up to isogenies).*[12]

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[12] See next page.

(3.6) *Corollary. In the diagram* $(3.2)_q$, *the mapping* Ψ_q *is an isogeny for* $q = 1, n$, *or in case* $q = m$ *and* $\dim V = 2m - 1$. *In both of these situations, the duality formula*

$$\mathrm{Pic}_q^0(V) \cong \check{\mathrm{Pic}}_{n-q+1}^0(V)$$

holds true (up to isogeny).

Problem B. (i) *Is the mapping* Ψ *in* (3.2) *an isogeny?* (ii) *Is the graded abelian variety* $I^0(V)$ *self-dual?*

An affirmative answer to this problem would settle part (ii) of problem A in section 1. Furthermore, it would identify the the equivalence relation on $A(V)$ induced by the Abel-Jacobi mapping as being the incidence equivalence relation (up to a finite group). As remarked in [9], both parts of problem B would have an affirmative answer if we knew the general Hodge conjecture [12].

Thus far we have only considered the kernel of Φ on the cycles algebraically equivalent to zero, whereas Φ is defined on the cycles which are homologous to zero. In this regard, let me propose

Problem C. *Is the induced mapping*

$$\Phi\colon H(V)/A(V) \to T(V)/I^0(V)$$

injective (up to a finite group)?

There are two bits of evidence for this problem. The first is that it would follow from a "relative version" of the Hodge conjecture, and the second is that one can prove an analogue of this problem over function fields. The precise statement of this latter result will be given in section 6 below.

[12] The first significant special case of this Proposition is when $m = 1$, which is the study of curves on a threefold. Applications of this are given in [4].

4. *Concerning the inversion theorem*

We continue to let V be a smooth, projective variety and denote by $\Phi\colon H(V) \to T(V)$ the Abel-Jacobi mapping defined in section 2. We further denote by $I(V)$ and $I^0(V)$ the images of

$$\begin{cases} \Phi\colon H(V) \to T(V) \\ \Phi\colon A(V) \to T(V), \end{cases}$$

so that $I(V)$ is the *group of invertible points* on the Jacobian variety $T(V)$ and $I^0(V)$ is the subgroup coming from cycles which are algebraically equivalent to zero. We recall from section 2 that $I^0(V)$ is an abelian subvariety of $T(V)$ whose Lie algebra is contained in the subspace $\bigoplus_{q=0}^{n} H^{q-1,q}(V)$ of $H^{\text{odd}}_{-}(V)$.[13] It is *not* possible to give any such (linear) restriction on the points in $I(V)$ ([10]).

The *inversion problem* is to describe the subgroups $I(V)$ and $I^0(V)$ in an a priori manner. For $I^0(V)$ there is a candidate, which is again suggested by the Hodge conjecture (cf. Lieberman [22]). To say what this is, we let $A_q(V)$ be the largest complex sub-torus of $T_q(V)$ whose *real* Lie algebra is a sub-Hodge structure of $H^{2q-1}(V,\mathbb{R})$ which is defined over $\mathbb{Q}$ and whose complexification is contained in $H^{q,q-1}(V) \oplus H^{q-1,q}(V)$. We let $A(V) = \bigoplus_{q=0}^{n} A_q(V)$ and observe from (2.14) the inclusion $I^0(V) \subset A(V)$ (cf. Grothendieck [12]).

Problem D. *Do we have the equality* $I^0(V) = A(V)$?

If this question is answered affirmatively, then it follows that problem B in section 3 is also answered in the affirmative. In particular, $\operatorname{Pic}^0(V)$ as defined in section 1 would be a graded,

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[13] In particular, except for $q = 1, n$ together witn a few other special cases, we cannot expect $\Phi_q\colon H_q(V) \to T_q(V)$ to be surjective.

self-dual abelian variety and the notion of "incidence equivalence" would be justified.

The question of finding the whole group $I(V)$ of invertible points in the Jacobian is, to me, the most mysterious problem regarding algebraic cycles. At least for other questions, such as the Hodge conjecture or problems A, B, and D above, we have a plausible answer which has yet to fail (although there are precious few non-trivial examples).

Problem E. *Describe the subgroup $I(V)$ of invertible points on $T(V)$.*

In explicit terms, given a basis $\phi_1,\ldots,\phi_g$ of $H^{1,0}(T_q(V)) \cong Z^{2n-2q+1,n-q}(V)/dA^{2n-2q,n-q-1}(V)$ we want to know which points $(z_1,\ldots,z_g) \in \mathbb{C}^g$ are solutions of the inversion equations

$$(4.1)\qquad \left\{\begin{array}{lcl} z_1 & = & \int_\Gamma \phi_1 \\ \vdots & & \vdots \\ z_g & = & \int_\Gamma \phi_g \, , \end{array}\right.$$

where Γ is a $2n-2q+1$ chain whose boundary is an algebraic cycle.

To add to the mystery surrounding this question, we shall see in section 6 below (cf. Corollary 6.14) that a knowledge of $I(V)$ would have strong implications regarding the Hodge conjecture, but I don't see any reason that the converse statement should be true.

Our final problem regarding $I(V)$ and $I^0(V)$ was originally suggested to me by Mumford:

Problem F. *Is the quotient group $I(V)/I^0(V)$ finitely generated?*

To motivate this problem, we observe that $I(V)/I^0(V)$ is a *countable* subgroup of $T(V)/I^0(V)$, and so we have a Mordell-Weil type of question. In this form, an analogue over function fields of problem F can be proved (cf. Proposition 6.8). If both problems D and F are answered in the affirmative, then we would have a positive answer to problem A. In fact, being very optimistic, if problems B-D and F could be answered affirmatively, then we could draw two conclusions: (i) that the purely algebro-geometric problem A has a positive answer, and (ii) that the use of intermediate Jacobians provides a very strong transcendental method for studying algebraic cycles.

5. *Definition of normal functions*

We want to discuss how the intermediate Jacobians vary with parameters. More precisely, we will consider a situation

$$f\colon X \to S \tag{5.1}$$

where X and S are smooth, projective varieties and where the fibres $V_s = f^{-1}(s)$ are smooth, projective varieties for almost all $s \in S$. Letting $T(V_s)$ be the Jacobian variety of such V_s, we want to fit the $T(V_s)$ together and then discuss how the algebraic cycles on X relate to cross-sections of the resulting fibre space of commutative, complex Lie groups. In order to carry out this program we shall make the assumptions

$$\begin{cases} \dim S = 1, \text{ and} \\ f \text{ has only non-degenerate critical points.} \end{cases} \tag{5.2}$$

The first assumption is not too serious, but the second one is much too restrictive and it should be possible to eliminate it entirely, especially in light of recent results of P. Deligne and W. Schmid (cf. problem G below).

We shall analyze the intermediate Jacobians along the fibres of $f\colon X \to S$ in three steps.

(a) *Local theory around a non-critical value.* We let Δ denote the unit disc $\{s \in \mathbb{C} : |s| \leqq 1\}$ and $\Delta^* = \Delta - \{0\}$ the corresponding punctured disc. Assume given a situation

$$f\colon W \to \Delta$$

where W is a complex manifold (with boundary) and f is a proper, smooth, and projective holomorphic mapping. Thus the fibres V_s are all smooth, projective algebraic varieties. Topologically, W is diffeomorphic to the product $V_0 \times \Delta$ so that we may identify all of the cohomology groups $H^*(V_s, \mathbb{C})$ with $H^*(V_0, \mathbb{C})$[14]. When this is done, the Hodge filtration

$$F^{m,p}(V_s, \mathbb{C}) = H^{m,0}(V_s) + \dots + H^{m-p,p}(V_s)$$

gives a subspace of $H^m(V_0, \mathbb{C})$; this subspace has the two basic properties:

(i) $F^{m,p}(V_s, \mathbb{C})$ varies holomorphically with $s \in \Delta$;

(ii) the *infinitesimal bilinear relation* [8]

$$\frac{d}{ds}\{F^{m,p}(V_s, \mathbb{C})\} \subset F^{m,p+1}(V_s, \mathbb{C}) \tag{5.3}$$

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[14] To be precise, there is a retraction $r\colon W \to V_0$ and an inclusion $i\colon V_s \to W$. The composite map on cohomology

$$H^*(V_0, \mathbb{C}) \xrightarrow{r^*} H^*(W, \mathbb{C}) \xrightarrow{i^*} H^*(V_s, \mathbb{C})$$

is an isomorphism and gives the identification in which we are interested.

is satisfied.[15]

From (i) it follows that we may canonically construct an analytic fibre space of complex tori

$$\tilde{\omega}: T(W/\Delta) \to \Delta \tag{5.4}$$

with fibres $\tilde{\omega}^{-1}(s) = T(V_s)$. The holomorphic vector bundle of complex Lie algebras associated to (5.4) will be denoted by

$$\pi: \mathbb{L} \to \Delta.$$

The fibres $\mathbb{L}_s$ of π are given by

$$\mathbb{L}_s \cong H_-^{\mathrm{odd}}(V_s).$$

Letting $\mathcal{O}(T(W/\Delta))$ denote the group of holomorphic cross-sections of the fibre space (5.4), the exponential mapping gives an exact sheaf sequence

$$0 \to \Lambda \to \mathcal{O}(\mathbb{L}) \xrightarrow{\exp} \mathcal{O}(T(W/\Delta)) \to 0.$$

The sheaf Λ is a locally constant sheaf of $\mathbb{Z}$-modules, which is easily identified by the isomorphism (cf. (2.7))

$$\Lambda \cong R_{f_*}^{\mathrm{odd}}(\mathbb{Z}),$$

where $R_{f_*}^{\mathrm{odd}}(\mathbb{Z})$ is the *Leray direct image sheaf* for the constant sheaf $\mathbb{Z}$ on W relative to the map $f: W \to S$.

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[15] These two statements mean that we can choose vectors $e_1(s),\ldots,e_l(s)$ in $H^m(V_0, \mathbb{C})$ with the properties that (i) the $e_j(s)$ vary holomorphically with $s \in \Delta$ and $e_1(s) \wedge \ldots \wedge e_l(s) \neq 0$; (ii) the $e_j(s)$ give a basis for $F^{m,p}(V_s, \mathbb{C})$; and (iii) the vectors $de_j(s)/ds$ lie in $F^{m,p+1}(V_s, \mathbb{C})$.

We want now to discuss the implications of (5.3) on the fibre space (5.4). For this, we construct over Δ a holomorphic vector bundle

$$\mathbb{J} = \bigoplus_{q=1}^{n} \mathbb{J}_q$$

where the fibre $(\mathbb{J}_q)_s$ of $\mathbb{J}_q$ is given by

$$(\mathbb{J}_q)_s = H^{2q-1}(V_s, \mathbb{C})/F^{2q-1,q}(V_s, \mathbb{C}).$$

Then (5.3) implies that there is a homomorphism

$$(5.5) \qquad D\colon \mathcal{O}(T(W/\Delta)) \to \Omega^1(\mathbb{J}),$$

and the subsheaf of sections ν of $T(W/\Delta)$ which satisfy the equation

$$(5.6) \qquad D\nu = 0$$

will be denoted by $Hom\ (\Delta, T(W/\Delta))$. Thus we have

$$0 \longrightarrow Hom\ (\Delta, T(W/\Delta)) \longrightarrow \mathcal{O}(T(W/\Delta)) \longrightarrow \Omega^1(\mathbb{J}).$$

We will now explain the geometric meaning of the condition (5.6). For this, we let Z be an analytic cycle on W such that all intersections $Z_s = V_s \cdot Z$ are defined and induce an algebraic cycle Z_s which is homologous to zero on V_s. Using the Abel-Jacobi mappings

$$\Phi_{V_s}: H(V_s) \to T(V_s),$$

we may define a cross-section ν_Z of the fibre space (5.4) by the rule

$$\nu_Z(s) = \Phi_{V_s}(Z_s).$$

(5.7) *Proposition* [8]. *The cross section* ν_Z *is holomorphic and satisfies* $D\nu_Z = 0$.

(b) *Local theory around a non-degenerate critical value.* Now we assume given a situation

$$f\colon W \to \Delta$$

where W is a complex manifold and f is a proper, projective holomorphic mapping which is smooth except that $s = 0$ is a non-degenerate critical value for f. Thus the fibres V_s $(s \neq 0)$ are smooth, projective varieties while V_0 has an isolated, ordinary double point around which the mapping f has the local form

$$(z_1)^2 + \ldots + (z_{n+1})^2 = s$$

for suitable holomorphic coordinates $z_1, \ldots, z_{n+1}$ on W.

We let $W^* = f^{-1}(\Delta^*) = W - V_0$ so that $f\colon W^* \to \Delta^*$ is a differentiable fibre bundle. Consequently the fundamental group $\pi_1(\Delta^*, s_0)$ acts on the cohomology $H^*(V_{s_0}, \mathbb{Z})$, and we let $I^*(V_{s_0}, \mathbb{Z})$ be the subspace on which $\pi_1(\Delta^*, s_0)$ acts trivially (these are the so-called *local invariant cycles*). There is an obvious restriction map

$$H^*(W, \mathbb{Z}) \xrightarrow{r^*} I^*(V_{s_0}, \mathbb{Z}), \tag{5.8}$$

which turns out to be an isomorphism in our case where V_0 has ordinary double points.[16] From this it follows that the stalk of the Leray direct image sheaf for $\mathbb{Z}$ on W is given by

$$R^*_{f_*}(\mathbb{Z})_0 = I^*(V_{s_0}, \mathbb{Z}).$$

[16] This follows from an analysis of the topology of the degeneration $V_s \to V_0$ given by Lefschetz [21]. It has recently been proved by P. Deligne that r^* in (5.8) is *surjective over* $\mathbb{Q}$ where V_0 has arbitrary singularities; this is the *local invariant problem* which is discussed in §15 of [7].

The analysis of how the Hodge filtration $\{F^{m,p}(V_s, \mathbb{C})\}$ of $H^m(V_s, \mathbb{C})$ behaves as $s \to 0$ is not too difficult in the situation at hand, and this analysis leads to the construction of a *generalized Jacobian* $T(V_0)$ with the following properties [10]:

(i) $T(V_0)$ is a commutative complex Lie group which fits into an exact sequence

$$1 \to \mathbb{C}^* \to T(V_0) \to T(\tilde{V}_0) \to 0,$$

where $\tilde{V}_0$ is the standard desingularization of V_0.[17]

(ii) There exists a complex-analytic fibre space

$$\tilde{\omega}: T(W/\Delta) \to \Delta$$

of abelian complex Lie groups such that the fibres $\tilde{\omega}^{-1}(s) = T(V_s)$ for *all* $s \in \Delta$. Furthermore, letting $\pi: \mathbb{L} \to \Delta$ be the holomorphic vector bundle of complex Lie algebras, we have again the exponential sheaf sequence

$$0 \longrightarrow \Lambda \longrightarrow \mathcal{O}(\mathbb{L}) \xrightarrow{\exp} \mathcal{O}(T(W/\Delta)) \longrightarrow 0$$

where, because of the isomorphism (5.8), the isomorphism

$$\Lambda \cong R^{\mathrm{odd}}_{f_*}(\mathbb{Z})$$

of Λ with the Leray direct image sheaf holds just as before.

(iii) If Z is an analytic cycle on W such that all intersections

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[17] The "standard desingularization" of V_0 is obtained by blowing up the double point on V_0 to obtain a smooth, projective variety $\tilde{V}_0$ containing a non-singular quadric which may be contracted to yield the singular point on V_0.

$Z_s = Z \cdot V_s$ are defined and such that Z_s is homologous to zero on V_s for $s \neq 0$, then the Abel-Jacobi maps

$$\Phi_{V_s} : H(V_s) \to T(V_s) \qquad (s \neq 0)$$

induce a holomorphic cross-section ν_Z of $T(W/\Delta)$ over all of the disc Δ.

Before going on, I should like to discuss briefly the possibility of extending (i)-(iii) to the case of general f: $W \to \Delta$ where V_0 is allowed to have arbitrary singularities. The main tools used in (i)-(iii) were the isomorphism (5.8), which related the topology of V_{s_0} and W by means of the action of $\pi_1(\Delta^*, s_0)$ on $H^*(V_{s_0}, \mathbb{C})$, and the knowledge of how the Hodge filtration on $H^*(V_s, \mathbb{C})$ behaves as $s \to 0$. Now the former has been done in general by Deligne and Katz, with one conclusion being the local invariant cycle theorem over $\mathbb{Q}$ as discussed in footnote [16]. The latter has recently been done by W. Schmid, who has in particular verified the conjecture of Deligne as given in §9 of [7]. Thus, it seems that perhaps the time is ripe to work on the

Problem G. *Analyze the behavior of the Jacobian varieties along the fibres of* f: $W \to \Delta$ *where* V_0 *has arbitrary singularities. In particular, can we define a generalized Jacobian* $T(V_0)$ *so that the analogues of* (i)-(iii) *above will remain valid?*

A side condition on this problem is given by remark 6.9 below.

(c) *Global theory and definition of normal functions.* We assume now that we are given f: $X \to S$ where S is a compact Riemann surface, X is a smooth, projective algebraic variety, and f is a holomorphic mapping which has only non-degenerate critical points. Thus, localizing around a point $s_0 \in S$, we find either the situation in (a) or (b) above according as to whether s_0 is a regular or critical value of f. We denote the critical values of f by $\{s_1,\dots,s_N\}$ and set $S^* = S - \{s_1,\dots,s_N\}$, $X^* = X|S^*$. Then $f\colon X^* \to S^*$ is topologically a fibre bundle and so the fundamental group $\pi_1(S^*,s_0)$ acts on the cohomology $H^*(V_{s_0},\mathbb{Z})$.

Example. To see how to construct such a situation, we take an arbitrary smooth, projective variety X' embedded in a projective space $\mathbb{P}_m$. In $\mathbb{P}_m$ we consider a general pencil $|\mathbb{P}_{m-1}(s)|_{s\in\mathbb{P}_1}$ of linear hyperplanes, and we let $V_s = X'\cdot\mathbb{P}_{m-1}(s)$ be the residual intersection of the hyperplane $\mathbb{P}_{m-1}(s)$ with X'. If we blow up X' along the base locus $V_0\cdot V_\infty$ of the pencil $|V_s|_{s\in\mathbb{P}_1}$, then we obtain a smooth, projective algebraic variety X together with an obvious mapping $f\colon X\to\mathbb{P}_1$.[18] The point $s_0\in\mathbb{P}_1$ is a critical value for f if, and only if, the hyperplane $\mathbb{P}_{m-1}(s_0)$ is tangent to X'. In this case, the singular points of V_{s_0} occur along the locus of tangency, and to say that s_0 is a non-degenerate critical value for f means that V_{s_0} should have one isolated, ordinary double point at the place of tangency of $\mathbb{P}_{m-1}(s_0)$.[19] If all critical values for f are of this sort, then we shall say that $|V_s|_{s\in\mathbb{P}_1}$ is a *Lefschetz pencil* on X', and in this case the resulting fibration $f\colon X\to\mathbb{P}_1$ is of the type we want to consider.

Returning to the general case of $f\colon X\to S$ where f has non-degenerate critical points, we may combine the results of (a) and (b) to construct a complex-analytic fibre space of abelian complex Lie

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[18] Set theoretically, X is the disjoint union $\bigcup_{s\in\mathbb{P}_1} V_s$ of the hpperplane sections of X', and the map $f\colon X\to\mathbb{P}_1$ sends $x\in V_s$ onto $s\in\mathbb{P}_1$.

[19] There is a nice discussion of these matters in the paper of Andreotti-Frankel [1] and in the exposés of Katz [15].

groups

$$\tilde{\omega}: T(X/S) \to S$$

with $\tilde{\omega}^{-1}(s) = T(V_s)$ the Jacobian, or generalized Jacobian if s is a critical value for f, of V_s. The Lie algebras along the fibres of $\tilde{\omega}$ give a holomorphic vector bundle

$$\pi: \mathbb{L} \to S,$$

and we have the exponential sheaf sequence

$$(5.9) \qquad 0 \to \Lambda \to \mathcal{O}(\mathbb{L}) \to \mathcal{O}(T(X/S)) \to 0$$

where the sheaf Λ is described by the isomorphism

$$(5.10) \qquad \Lambda \cong R^{\mathrm{odd}}_{f_*}(\mathbb{Z})$$

with the Leray direct image sheaf. We shall continue to denote by $Hom\ (S,T(X/S))$ the subsheaf of $\mathcal{O}(T(X/S))$ of sections ν which satisfy the condition $D\nu = 0$ as explained in (a).

Definition. The group Hom $(S,T(X/S))$ *of global sections of* $Hom\ (S,T(X/S))$ *will be called the group of normal functions associated to* $f: X \to S$.

We now want to relate the algebraic cycles on X to these normal functions. For this we let $C(F)$ be the ideal in the Chow ring $C(X)$ of all rational equivalence classes of algebraic cycles on X which lie in a fibre of $f: X \to S$. Similarly, we let $H^{\mathrm{even}}(F,\mathbb{Z})$ be the ideal in $H^{\mathrm{even}}(X,\mathbb{Z})$ which is given by the Poincaré dual of the image

$$H_{\mathrm{even}}(V_s,\mathbb{Z}) \to H_{\mathrm{even}}(X,\mathbb{Z}) \qquad (s \in S^*).$$

The quotients by these two ideals will be denoted by

$$\begin{cases} C(X/S) = C(X)/C(F), & \text{and} \\ H^{\text{even}}(X/S,\mathbb{Z}) = H^{\text{even}}(X,\mathbb{Z})/H^{\text{even}}(F,\mathbb{Z}). & \end{cases}$$

The fundamental class mapping (1.5) and the restriction mapping $H^{\text{even}}(X,\mathbb{Z}) \xrightarrow{r^*} H^{\text{even}}(V_s,\mathbb{Z})$ induce maps

$$(5.11) \qquad \begin{cases} H^{\text{even}}(X/S,\mathbb{Z}) \xrightarrow{r^*} H^{\text{even}}(V_s,\mathbb{Z}), \quad \text{and} \\ \qquad\qquad\qquad\quad \rho^* \\ C(X/S) \longrightarrow H^{\text{even}}(X/S,\mathbb{Z}) \longrightarrow H^{\text{even}}(V_s,\mathbb{Z}). \end{cases}$$

Definition. The ideal Prim (X/S) *in* $C(X,S)$ *which is given by the kernel of* ρ^* *in* (5.11) *will be called the ring of primitive algebraic cycles for* $f\colon X \to S$.

Similarly, we will denote by $\text{Prim}^{\text{even}}(X/S,\mathbb{Z})$ the kernel of r^* in (5.11) To understand the importance of primitive cycles, we refer to the example of a Lefschetz pencil discussed above. On X', there is a famous theorem of Lefschetz [21] which states that, over $\mathbb{Q}$, every cohomology class is the sum of a primitive class together with a class supported on a hyperplane section. In other words, the primitive cycles are the "building blocks" for all of the homology of X'.

Now we can come to the main point. Referring to (5.7) there will be a homomorphism

$$(5.12) \qquad \nu\colon \text{Prim}\,(X/S) \to \text{Hom}\,(S, T(X/S))$$

which assigns to each primitive algebraic cycle Z on X the cross-section ν_Z given as follows: By changing Z in its rational equivalence class, we may assume that all intersections $Z_s = Z\cdot V_s$ are defined and induce algebraic cycles which are homologous to zero on V_s $(s \in S^*)$. Then we have the formula

$$\nu_Z(s) = \Phi_{V_s}(Z_s) \tag{5.13}$$

where $\Phi_{V_s} : H(V_s) \to T(V_s)$ is the Abel-Jacobi mapping as given in section 2.

In the next section we will give some results and open problems concerning this homomorphism (5.12). In the case where $\dim X = 2$ and the base S is the projective line $\mathbb{P}_1$, the normal functions were introduced by Poincaré (1910) and used by Lefschetz [21] to give a complete analysis of the curves lying on an algebraic surface, including the results that: (i) homological and algebraic equivalence are the same for curves on a surface; (ii) the Abel-Jacobi map

$$\Phi\colon H_1(X) \longrightarrow \mathrm{Pic}_1^0(X)$$

is surjective (existence theorem for the Picard variety of an algebraic surface); and (iii) a homology class $\Gamma \in H_2(X, \mathbb{Z})$ is carried by an algebraic curve if, and only if, Γ is of type (1,1) (Lefschetz theorem). The proofs of these results were based on the formal properties of normal functions together with the Jacobi inversion theorem for the Jacobians $T(V_s)$. In the next section, we shall point out that the formal properties mostly go through, but, as discussed in section 4, the inversion theorem is completely missing and this is the hangup in trying to understand algebraic cycles on X by means of normal functions.

6. *Some results about normal functions*

We retain the notations of section 5, so that we are studying a situation

$$f\colon X \to S$$

where f has only non-degenerate critical points. In this paragraph we shall make the additional assumption that the base S is a projective line $\mathbb{P}_1$. With this assumption, the Leray spectral sequence for the constant sheaf $\mathbb{Z}$ on X and the mapping f degenerates at the E_2 term (cf. [1]), from which we may draw the following conclusions:

(i) There is a filtration on $H^{\text{even}}(X,\mathbb{Z})$ whose associated graded module is the direct sum

$$H^2(S\,,\,R^{\text{even}}_{f_*}\mathbb{Z}) + H^1(S\,,\,R^{\text{odd}}_{f_*}\mathbb{Z}) + H^0(S\,,\,R^{\text{even}}_{f_*}\mathbb{Z}).$$

(ii) The ideal $H^{\text{even}}(F,\mathbb{Z})$ of $H^{\text{even}}(X,\mathbb{Z})$ is $H^2(S\,,\,R^{\text{even}}_{f_*}\mathbb{Z})$, and the subgroup of $H^{\text{even}}(X,\mathbb{Z})$ of cycles which restrict to zero on the fibres of f is $H^2(S\,,\,R^{\text{even}}_{f_*}\mathbb{Z}) + H^1(S\,,\,R^{\text{odd}}_{f_*}\mathbb{Z})$.

iiii) From (i) and (ii) there results the isomorphism

$$\text{Prim}^{\text{even}}(X/S,\mathbb{Z}) \cong H^1(S\,,\,R^{\text{odd}}_{f_*}\mathbb{Z}). \tag{6.1}$$

We wish to study the group Hom $(S, T(X/S))$ of normal functions. For this we consider the cohomology sequence of the exponential sheaf sequence (5.9) together with the isomorphisms (5.10) and (6.7) to arrive at our basic diagram [20]

$$\begin{array}{ccccccc}
0 \longrightarrow & H^0(S,\mathcal{O}(\mathbb{L}))/H^0(S,\Lambda) & \longrightarrow & H^0(S,\mathcal{O}(T(X/S))) & \xrightarrow{\delta} & \text{Prim}^{\text{even}}(X,\mathbb{Z}) \\
 & \uparrow & & \uparrow & & \| \\
0 \longrightarrow & \text{Fix}\,(T(X/S)) & \longrightarrow & \text{Hom}\,(S,T(X/S)) & \xrightarrow{\delta} & \text{Prim}^{\text{even}}(X,\mathbb{Z}). \\
 & \uparrow & & \uparrow & & \\
 & 0 & & 0 & &
\end{array} \tag{6.2}$$

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[20] The subgroup Fix $(T(X/S))$ is defined to be the group of sections μ of $H^0(S,\mathcal{O}(\mathbb{L}))/H^0(S,\Lambda)$ whose projections into $\mathcal{O}(T(X/S))$ satisfy the equation $D\mu = 0$ as explained above (5.5).

We want to relate the diagram (6.2) to the normal functions which arise from algebraic cycles. For this we consider the diagram

$$\begin{tikzcd} \mathrm{Hom}\,(S, T(X/S)) \arrow[r, "\delta"] & \mathrm{Prim}^{\mathrm{even}}(X, \mathbb{Z}) \\ & \mathrm{Prim}\,(X/S) \arrow[ul, "\nu"] \arrow[u, "h"'] \end{tikzcd} \tag{6.3}$$

which arises from (6.2), (5.12), and (5.11). Speaking geometrically, ν assigns to a primitive algebraic cycle Z on X the normal function given by (5.13), and h assigns to Z its homology class in $H^{\mathrm{even}}(X, \mathbb{Z})/H^{\mathrm{even}}(F, \mathbb{Z})$.

(6.4) *Proposition* [10]. *The diagram* (6.3) *is commutative, so that the homology class* $h(Z) \in \mathrm{Prim}^{\mathrm{even}}(X, \mathbb{Z})$ *may be computed from the corresponding normal function* ν_Z.

To give the geometric interpretation of $\mathrm{Fix}\,(T(X/S))$, we observe from (2.15) that the inclusions $V_s \subset X$ induce a map

$$T(X) \xrightarrow{r^*} \mathrm{Fix}\,(T(X/S)). \tag{6.5}$$

Moreover, in case X arises from X' by the method of Lefschetz pencils as discussed in the example of section 5, (2.15) leads to a commutative diagram

$$\begin{tikzcd} T(X) \arrow[r, "r^*"] & \mathrm{Fix}\,(T(X/S)) \\ T(X'). \arrow[u] \arrow[ur, "\rho^*"'] & \end{tikzcd} \tag{6.6}$$

(6.7) *Proposition* [8]. *The restriction mapping* r^* *in* (6.5) *is onto. Moreover, in case* X *arises from* X' *by Lefschetz pencils, the mapping* ρ^* *in* (6.6) *is an isomorphism.*

There are two geometric conclusions which may be drawn from (6.4) and (6.7). The first is:

(6.8) *(Mordell-Weil for intermediate Jacobians).* The group $Hom\ (S, T(X/S))$ of normal functions is an extension of a finitely generated group by the "trace" or "fixed part" of the Jacobians along the fibres of the mapping $f\colon X \to S$.[21]

(6.9) *Remark.* This result has been proved in [8] when S is a curve of any genus but where always f is assumed to have only non-degenerate critical values. Referring to problem G in section 5 where we asked for an analysis of how the Jacobians $T(V_s)$ behave as s tends to a critical value s_0 for a general mapping $f\colon X \to S$, it should be the case that this analysis will lead to a proof of (6.8) for arbitrary mappings f.

The result (6.8) may be thought of as an analogue of problem F in section 4 over function fields.

The second consequence of (6.4) and (6.7) is

(6.10) *(Induction principle for Lefschetz pencils).* Suppose that X arises from X' by the method of Lefschetz pencils. Then the study of the (rational equivalence classes of) primitive algebraic cycles on X' as regards the homology class $h(Z')$ of such a cycle Z', or in case Z' is homologous to zero, the point $\Phi_{X'}(Z')$ in the Jacobian variety of X', may be done by studying the Jacobians of the hyperplane sections of X' and using the method of normal functions.

[21] By definition, the *fixed part* of the family $\{T(V_s)\}_{s \in S}$ of Jacobians along the fibres of $f\colon X \to S$ is the complex torus given by the image of the restriction mapping $T(X) \to T(V_s)$. The finitely generated group is identified in a special case by the exact sequence (6.13) below.

As mentioned at the end of section 5, this induction principle (6.10) may be thought of as giving the generalization of most of the formal properties possessed by the Jacobians of the curves in a Lefschetz pencil on an algebraic surface. However, the existence theorems are missing because we don't understand the group $I(V_s)$ of invertible points in $T(V_s)$, a state of affairs which is made even more frustrating by the following:

Let $f\colon X \to S$ be as above and define the group

$$\mathrm{Prim}^{\mathrm{Hodge}}(X, \mathbb{Z})$$

to be the subgroup of $\mathrm{Prim}^{\mathrm{even}}(X, \mathbb{Z})$ which comes from the subgroup $\bigoplus_{q=0}^{n} H^{q,q}(X)$ of $H^{\mathrm{even}}(X, \mathbb{C})$. Obviously the diagram (6.3) may be refined to a new diagram

$$\begin{array}{lccc} & \mathrm{Hom}\,(S, T(X/S)) & \xrightarrow{\ \delta\ } & \mathrm{Prim}^{\mathrm{even}}(X, \mathbb{Z}) \\ (6.11) & \Big\uparrow \nu & & \Big\uparrow i = \text{inclusion map} \\ & \mathrm{Prim}\,(X/S) & \xrightarrow{\ h\ } & \mathrm{Prim}^{\mathrm{Hodge}}(X, \mathbb{Z}). \end{array}$$

(6.12) *Proposition. Suppose that X arises from X' by the method of Lefschetz pencils and where X' is a smooth hypersurface in a projective space $\mathbb{P}_{2m+1}$. Then the images of i and δ in* (6.11) *coincide, so that we have the exact sequence*

$$(6.13) \qquad 0 \to \mathrm{Fix}\,(T(X/S)) \to \mathrm{Hom}\,(S, T(X/S)) \to \mathrm{Prim}^{\mathrm{Hodge}}(X, \mathbb{Z}) \to 0.$$

This proposition gives in a very special case the structure of the finitely generated group in the Mordell-Weil theorem (6.8). For the problem of constructing algebraic cycles, there is the following

(6.14) *Corollary. Let X' be a smooth hypersurface in $\mathbb{P}_{2m+1}$ and $\Gamma \in H^{m,m}(X') \cap \mathrm{Prim}^{2m}(X',\mathbb{Z})$ a primitive, integral homology class of type (m,m). Then Γ comes from an algebraic cycle if, and only if, the corresponding normal function $\nu_\Gamma \in \mathrm{Hom}\,(S,T(X/S))$ satisfies the inversion property that $\nu_\Gamma(s) \in I(V_s)$ for all $s \in S^*$.*

Thus, at least for smooth hypersurfaces in projective space, the construction of algebraic cycles in a given homology class is thrown back to the inversion problem as discussed in section 4.

Because of the lovely result by Gherardelli [6] on the intermediate Jacobian of the cubic threefold, we have the following

(6.15) *Corollary. Let V be a smooth hypersurface of degree three in $\mathbb{P}_5$ (V is a cubic fourfold). Then a class $\Gamma \in H^{2m}(V,\mathbb{Z})$ is algebraic if, and only if, Γ is of type (m,m) (for any m).*

7. *Positive algebraic cycles*

(a) *Preliminary comments on vector bundles.* Let V be a smooth, projective algebraic variety and $\mathbb{E} \to V$ an algebraic vector bundle of rank r. We denote by $P(\mathbb{E})$ the projective bundle of the dual $\check{\mathbb{E}}$ of $\mathbb{E}$, and shall use the notation $\mathbb{L} \to P(\mathbb{E})$ to denote the tautological line bundle over $P(\mathbb{E})$. The isomorphism of cohomology

$$H^k(V,\mathcal{O}(\mathbb{E}^{(\mu)})) \cong H^k(P(\mathbb{E}),\mathcal{O}(\mathbb{L}^{(\mu)}))\ [22]$$

may serve to eliminate confusion between bundles and their duals. Following Hartshorne [14], we say that $\mathbb{E} \to V$ is *ample* if the

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[22] $\mathbb{E}^{(\mu)}$ is the μ^{th} symmetric power of $\mathbb{E}$, and thus the fibre $\mathbb{E}_x^{(\mu)}$ is the vector space of homogeneous forms of degree μ on the projective space $P(\mathbb{E})_x$.

tautological line bundle $\mathbb{L} \to P(\mathbb{E})$ is ample, in the usual sense of the word for line bundles. An equivalent formulation is that the *vanishing theorem*

$$H^k(V, O(\mathbb{E}^{(\mu)}) \otimes S) = 0 \qquad (k > 0, \quad \mu \geqq \mu_0(S))$$

should hold for every coherent sheaf S on V.

We shall also use the notion of very ample, which deals with the vector space $H^0(\mathbb{E})$ of holomorphic cross-sections of $\mathbb{E} \to V$. First, we recall that $\mathbb{E}$ is said to be *generated by its sections* if the restriction mappings

$$H^0(\mathbb{E}) \to \mathbb{E}_x \qquad (x \in V)$$

are surjective for all points x. In this case there is an exact bundle sequence

$$0 \to \mathbb{F} \to V \times H^0(\mathbb{E}) \to \mathbb{E} \to 0$$

where the fibre $\mathbb{F}_x = \{\sigma \in H^0(\mathbb{E}) : \sigma(x) = 0\}$. For $\sigma \in \mathbb{F}_x$, the differential $d\sigma(x) \in \mathbb{E}_x \otimes \check{\mathbb{T}}_x(V)$ is well defined, and $\mathbb{E}$ is *very ample* if $\mathbb{E}$ is generated by its sections and if we also have the surjection

$$\mathbb{F} \xrightarrow{d} \mathbb{E} \otimes \check{\mathbb{T}}(V) \longrightarrow 0.$$

Geometrically, $\mathbb{E}$ is very ample if it is induced by a holomorphic immersion

$$f : V \to \text{Grassmannian}$$

where the image $f(V)$ is "sufficiently twisted". We recall that:

$$\begin{cases} \mathbb{E} \text{ very ample} \Longrightarrow \mathbb{E} \text{ ample,}^{23} \quad \text{and} \\ \mathbb{E} \text{ ample} \Longrightarrow \mathbb{E}^{(\mu)} \text{ very ample for } \mu \geqq \mu_0. \end{cases}$$

For an algebraic vector bundle $\mathbb{E} \to V$ of rank r the *total Chern class*

$$c(\mathbb{E}) \in C(\mathbb{E})$$

may be defined [13]. Writing

$$c(\mathbb{E}) = c_0(\mathbb{E}) + c_1(\mathbb{E}) + \ldots + c_n(\mathbb{E}),$$

we have $c_0(\mathbb{E}) = 1 \cdot V$ and $c_k(\mathbb{E}) = 0$ for $k > r$. Let $I = (i_0, \ldots, i_r)$ be an r-tuple of non-negative integers and $|I| = i_1 + 2i_2 + \ldots + ri_r$. We define the *Chern monomials*

$$c_I(\mathbb{E}) = c_1(\mathbb{E})^{i_1} \cdots c_r(\mathbb{E})^{i_r} \in C_{|I|}(\mathbb{E}).$$

From [11] we recall that there is a set $\mathbb{P}^+(r)$ of polynomials $P(c_1, \ldots, c_r)$ with rational coefficients, called *positive polynomials*, which has the properties:

(i) The polynomials in $\mathbb{P}^+(r)$ form a graded, convex cone over $\mathbb{Q}^+$;

(ii) all monomials c_I are in $\mathbb{P}^+(r)$, but for $r > 1$ these do not generate $\mathbb{P}^+(r)$; and

(iii) if $\mathbb{E} \to V$ is a very ample vector bundle, then any positive polynomial $P(c_1(\mathbb{E}), \ldots, c_r(\mathbb{E}))$ is *numerically positive* in the sense that the intersection number

[23] It is not the case that $\mathbb{E}$ is ample if it is induced from a holomorphic immersion $f : V \to$ Grassmannian, unless of course the rank of $\mathbb{E}$ is one.

$$(7.1) \qquad \deg\,[P(c_1(\mathbb{E}),\ldots,c_r(\mathbb{E}))\cdot Z] > 0$$

for every effective cycle Z of complementary dimension to $P(c_1(\mathbb{E}),\ldots,c_r(\mathbb{E}))$.

Remarks. For $r = 1$, the only positive polynomials of degree k are the obvious ones

$$\lambda c_1^k \qquad (\lambda \in \mathbb{Q}^+).$$

In this case, it is a theorem of Moishezon-Nakai [23] that the inequality (7.1) is both necessary and sufficient for $\mathbb{E} \to V$ to be ample.

For $r > 1$, there will be positive polynomials such as $c_1^2 - c_2$ which are not positive linear combinations of Chern monomials.

The positivity property (7.1) may well be true if we only assume that $\mathbb{E} \to V$ is ample. For the Chern classes themselves, this has recently been given a very nice proof by Bloch and Geiseker [3].

In order to state our main problem about ample vector bundles, we first give the

Definition. $\mathbb{E}$ *is numerically positive if, for each irreducible subvariety* W *of* V *and each quotient bundle* $\mathbb{Q}$ *of* $\mathbb{E}/W$*, the numerical positivity property* (7.1) *is valid.*

In [11] it was proved, by differential-geometric methods, that a sufficiently ample bundle is numerically positive in the above sense. Contrary to what mistakenly appeared in [11], the converse result does not seem to have been proved except in special cases; e.g., the result when $\dim V = 1$ is due to Hartshorne. I should like to offer my personal apology for any confusion which may have arisen from this mixup.

Problem H. *Find necessary and sufficient numerical conditions in order that a bundle* $\mathbb{E} \to V$ *should be ample. More specifically, is it true that* $\mathbb{E}$ *is ample if, and only if, it is numerically positive in the above sense?*

(b) *Positivity of divisors.* Let D be an effective divisor on a smooth, projective variety V. Any of the following four equivalent conditions may be taken as the definition of what it means for D to be *positive*:[24]

(i) There is a very ample line bundle $\mathbb{L} \to V$ such that D is a positive multiple of $c_1(\mathbb{L})$ in the rational Chow ring $C(V) \otimes_{\mathbb{Z}} \mathbb{Q}$.

(ii) For any effective algebraic q-cycle on V, the intersection number

$$D^{(q)} \cdot Z > 0.$$

(This is the Moishezon-Nakai criterion mentioned above.)

(iii) Denoting by $I(D)$ the ideal sheaf of D, we have the vanishing theorem

$$H^k(V, I(D)^{\mu} \otimes S) = 0 \qquad (k > 0, \quad \mu \geqq \mu(S)) \tag{7.2}$$

for any coherent sheaf S on V.

(iv) The complement $V - D$ is an *affine algebraic variety*, and is therefore *strongly-pseudo-convex* in the sense of complex function theory. (It is not the case that $V - D$ affine $\Longrightarrow$ D ample.)

The condition (iv) implies the vanishing theorem

$$H^k(V - D, S) = 0 \qquad (k > 0) \tag{7.3}$$

[24] We shall give the definition of positivity for non-effective divisors in a little while. It may be that the adjective "ample" should be used rather than "positive" in the present context, but we prefer to stick to the older terminology.

in either the algebraic or analytic category. Conversely, the vanishing theorem (7.3) implies either that $V - D$ is an affine algebraic variety or is strongly-pseudo-convex according to the category in which we are given the result.

I should now like to mention two uses of positive (effective) divisors for problems in algebraic geometry. Both of these results are proved using the vanishing theorem (7.2), so that one might say that whereas the conditions (i) or (ii) are the more appealing geometrically, it perhaps is (iii) which is the most useful technically.

(a) If D is a positive divisor, then there is an integer μ_0 such that the set of all effective divisors which are algebraically equivalent in the strong sense to μD $(\mu \geqq \mu_0)$ generates the identity component of the Picard variety of V.[25]

(b) If $\{V_s\}_{s\in\Delta}$ is any variation of the complex structure of $V = V_0$ given by a situation

$$f: W \to \Delta$$

as discussed in section 4(a), if $\Gamma \in H^2(V_s, \mathbb{Z}) \cong H^2(V_0, \mathbb{Z})$ is the homology class of a divisor on V_s for all $s \in \Delta$, and if $h(D) = \Gamma$ in $H^2(V, \mathbb{Z})$, then for $\mu \geqq \mu_0$ there will exist divisors D_s' on V_s which vary holomorphically with $s \in \Delta$ and which specialize to μD at $s = 0$.[26]

[25] More precisely, we let $\Lambda(\mu)$ be the (normalized) component of the the Chow variety of V which contains the effective divisor μD. Then for $\mu \geqq \mu_0$ the Abel-Jacobi mapping

$$\Phi : \mathrm{Alb}(\Lambda(\mu)) \to T_1(V)$$

is an isomorphism. This is proved in Kodaira [19].

[26] This property might be stated as saying that "sufficiently positive divisors are stable under variation of structure".

We now discuss briefly the notion of positivity for non-effective divisors. For this we observe that either of conditions (i) or (ii) still makes sense and may be used to define what it means for a general divisor D to be positive. With this definition, there are two additional properties of positive divisors which I should like to mention:

(c) If D is a (not necessarily effective) divisor which satisfies (i) or (ii) above, then for $\mu \geqq \mu_0$ the divisor μD will be linearly equivalent to an effective divisor.

(d) If D is a positive divisor and D' is any divisor, then the divisor

$$\mu D + D'$$

will be positive for $\mu \geqq \mu(D')$.

(c) *Positivity of general cycles.* In view of the usefulness of positive divisors for problems in algebraic geometry, it is of importance to have a definition for positive algebraic cycles of any codimension. I should like to propose such a definition which is based on the incidence relation discussed in section 1.

To motivate this, we first give a transcendental condition which should certainly guarantee that an algebraic cycle be positive by any reasonable definition. Thus, suppose that Z is an algebraic cycle of codimension q on V and let $h(Z) \in H^{2q}(V, \mathbb{C})$ be the fundamental class of Z. The transcendental condition we have in mind is that, using the de Rham isomorphism, $h(Z)$ should be given by a *positive* (q,q) form Θ. This means locally

$$(7.4) \qquad \Theta = (\sqrt{-1})^{q(q-1)/2} \{\sum_{\alpha} \theta_\alpha \wedge \overline{\theta}_\alpha\}$$

where the θ_α are $(q,0)$ forms such that, for any set $\tau_1,\ldots,\tau_q$ of linearly independent $(1,0)$ tangent vectors, the contraction

$$< \theta_\alpha , \tau_1 \wedge \ldots \wedge \tau_q > \neq 0$$

for at least one index α. If this condition is satisfied, we shall say that *Z is positive in the differential-geometric sense.*[27]

(7.5) *Example.* Z is positive in the differential-geometric sense if, in the rational Chow ring $C(V) \otimes_{\mathbb{Z}} \mathbb{Q}$, Z is a positive polynomial in the Chern classes of a very ample vector bundle (cf. [11]).

We want to draw an algebro-geometric corollary of the notion of positive in the differential-geometric sense. For this we suppose that $\{W_s\}_{s\in S}$ is an algebraic family of effective algebraic $(q-1)$ cycles on V and assume that this family is *effectively parametrized* in the sense that

$$W_s = W_{s'} \Longrightarrow s = s'.$$

Then from [4] we have the easy

(7.6) *Proposition. If Z is positive in the differential-geometric sense, then the incidence divisor D_Z is positive on S.*

With (7.6) as motivation, we propose the

(7.7) *Definition. The algebraic cycle $Z \in C_q(V)$ is positive if, for any algebraic family $\{W_s\}_{s\in S}$ of effective algebraic $(q-1)$-cycles as above, the incidence divisor D_Z is positive on S.*

Using the Moishezon-Nakai numerical criterion for positivity of divisors recalled above, we want to give an equivalent numerical

[27] A divisor is positive in the algebro-geometric sense if, and only if, it is positive in the differential-geometric sense (Kodaira). If Z is smooth and is positive in the differential-geometric sense, then the *normal bundle* $\mathbb{N} \to Z$ is ample according to the definition in section 7(a).

formulation of (6.5). For this we consider the effective algebraic correspondences

$$W \subset S \times V$$

where $\dim [W \cdot \{s\} \times V] = q - 1$ for all $s \in S$. We shall use the notation V^k for the product $\underbrace{V \times \ldots \times V}_{k}$, Z^k for the diagonal cycle on V^k induced by Z on V, and $W_s = W \cdot \{s\} \times V$. Given a subvariety T of S, we define the effective algebraic cycle $\Delta_k(T)$ on the product V^k by the rule

$$\Delta_k(T) = \{(x_1, \ldots, x_k) \in V^k : \text{all } x_j \in W_s \text{ for some } s \in T\}.$$

Then $\dim [\Delta_k(T)] = k(q-1) + \dim T$ and $\operatorname{codim} [Z^k] = kq$. Using Moishezon-Nakai it follows that

(7.8) *Proposition. The cycle* $Z \in C_q(V)$ *is positive if, and only if, the numerical condition*

$$\deg [Z^k \cdot \Delta_k(T)] > 0$$

is satisfied for all k-dimensional subvarieties of S.

(7.9) *Corollary. The positive cycles form a graded, convex cone* $P(V) = \bigoplus_{q=0}^{n} P_q(V)$ *in the Chow ring* $C(V)$.

The analogues of the properties (i)-(iv) in section 7(b) for positive divisors are these:

(i) In $C(V) \otimes_{\mathbb{Z}} \mathbb{Q}$, Z is a positive linear combination of positive polynomials in the Chern classes of ample vector bundles.

(ii) The numerical condition (7.8) is satisfied.

(iii) In case Z is effective, we denote by $I(Z)$ the ideal sheaf of Z. Then we have the vanishing theorem

$$H^k(V, I(Z)^\mu \otimes S) = 0 \qquad (k \geqq q, \quad \mu \geqq \mu(S))$$

for any coherent sheaf S on V.

(iv) Again assuming Z is effective, the complement $V - Z$ is everywhere strongly $(q\text{-}1)$-convex and is strongly $(n\text{-}q\text{-}1)$-concave outside a compact set in $V - Z$.[28] In particular, for any coherent analytic sheaf S on $V - Z$ we have

$$(7.10) \qquad \begin{cases} H^k(V - Z, S) = 0 & (k > q) \\ \dim H^k(V - Z, S) < \infty & (k < q, \ S \text{ locally free}). \end{cases}$$

The vanishing-finite-dimensionality theorem (7.10) is due to Andreotti-Grauert [2] and has been stated analytically as I do not know if the corresponding algebraic theorem has been proved.[29]

Concerning these four properties, the main facts which I know are these:

Property (ii) is true if Z is positive in the differential-geometric sense (cf. Example (7.5)).

If Z is smooth and is the top Chern class of a very ample vector bundle, then all of the properties (i)-(iv) are true [11].

Problem I. *Are properties* (i)-(iv) *true for positive linear combinations of Chern classes of very ample vector bundles?*

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[28] In case Z is smooth, the $(q\text{-}1)$-convexity is roughly supposed to mean that the normal bundle of Z should be positive plus the assumption that Z should meet every subvariety of complementary dimension. The $(n\text{-}q\text{-}1)$-concavity is always present because Z is locally the zeroes of q holomorphic functions.

[29] To be specific, suppose that Z is smooth and the normal bundle $\mathbb{N} \to Z$ is ample. Then is it true that

$$\dim H^k(V - Z, S) < \infty \qquad (k > q)$$

for any coherent algebraic sheaf S on $V - Z$ and where the cohomology is in the Zariski topology? The corresponding analytic result is true by using (7.10).

The reason for stating this problem is two-fold: First, it may not be too difficult to prove it using the methods of [11] together with the analysis of the singularities of the basic Schubert cycles associated to a very ample vector bundle which has recently been given by Kleiman and Landolfi [20]. Secondly, in the rational Chow ring $C(V) \otimes_{\mathbb{Z}} \mathbb{Q}$ every cycle Z is a linear combination of these basic Schubert cycles, so that problem H might suffice for many applications of positive subvarieties to problems in algebraic geometry. In this connection we shall close by mentioning the

Problem J. *Which analogues of the properties* (a)-(d) *in section* 7(b) *for positive divisors are true for positive algebraic cycles?*

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Remark added in proof. A recent monograph by R. Hartshorne entitled *Ample subvarieties of algebraic varieties* (Springer-Verlag Lecture Notes #156) treats many questions closely related to our section 6.