A BRIEF SUMMARY OF MY VIEWS ON THE GEOMETRIC AND ARITHMETIC THEORIES

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This is a supplement to a paper in Russian and a lecture in Turkish, both available on this site and in this section of it. The paper contains a theorem expected to be valid in general in the geometric theory but proved in the paper only in a special but convincing case. The lecture in Turkish is a brief review, in a lecture for an audience of mathematicians with varied backgrounds, of the difference between the geometric theory and the arithmetic theory and a description, again brief, of what I believe is valid in the two contexts. In contrast to the geometric case in which the road to a complete theory is, even if long, quite clear, the arithmetic case, although its possibilities are familiar, seems far beyond our present abilities. I review briefly the geometric theory, in so far as it was treated in my Russian paper, and then pass to what I surmise to be the ultimate form of the arithmetic theory.

For the geometric theory, in the form discussed in that paper, the essential features are the introduction of Hecke operators on infinite dimensional spaces and their eigenvalues and a galoisian group. I only discuss one group apart from $GL(1)$, namely $GL(2)$, and only curves of genus one and took at the beginning, but with a great deal of hesitation, a bold and important step. According to the treatment of Atiyah, Bun_G is the union of a space of dimension one and one of dimension two. The Hecke operators, which are hermitian integral operators are defined to act separately on the functions on the two spaces. Otherwise it was not at all clear how they might be defined.[1] This led to eigenfunctions and eigenvalues that could be calculated and that could be compared with a different aspect of the theory, that given by a galoisian group. This group, similar to a galois group, is also similar to a group

[1]The reader who ignores these two sentences and believes that the definitions appropriate in a sheaf-theoretic context are also appropriate here will not understand this paper.

Indeed the essential elements of the paper are brief and are concentrated in two locations. The first is Section VI. In the Hecke theory, the essential elements are the Hecke operators and the eigenvalues. These are, in the arithmetic theory, eigenvalues of operators on finite-dimensional spaces. So their existence is not problematic. In the geometric theory the relevant operators act on infinite-dimensional spaces and the way to ensure that their eigenvalues and eigenvectors are defined, a condition essential to any Hecke theory, is to take them to be hermitian operators with respect to some metric on a Hilbert space. This I did, by taking what appears to be an arbitrary and brutal step. This I took in Section VI explaining as well as I could the reasons. The step is, however, only justified by its results and these appear only very much later in Section XI. The intermediate sections are concerned with, first of all, a calculation of the eigenvalues of the resulting Hecke operators, and then a review, at some length, of the relevant differential geometry and of the pertinent sections of the paper of Atiyah-Bott, essential prerequisites for an understanding of the procedure. Finally, two-dimensional connections are obtained as projections of one-dimensional connections on two-fold coverings and two-dimensional representations of the galoisian group of the paper or of a similar group introduced by Atiyah-Bott are obtained from one-dimensional representations by induction. These are all elementary, familiar operations. It is their meshing and their comparison with the earlier eigenvalue calculations to obtain the final result that is, as observed in Section XI when all the calculations—by and large elementary—are finally completed, so striking and so compelling. We have the first step towards a proof of the existence of what I refer to as the (geometric) galoisian group, a notion, in this strict form, peculiar to the geometric theory.

introduced by Atiyah-Bott to define parameters in the Yang-Mills theory on algebraic curves. The conclusion of my Russian paper is that for the group $GL(2)$ and for an elliptic curve its unitary representations in ${}^LG$ parametrize the eigenfunctions of the Hecke operators. There is every reason to expect that a similar statement—with similar although technically more difficult proofs—is valid for all groups and all curves. Most of the Russian paper is devoted to an introduction to the relevant geometric notions. The decisive steps are on p. 59, where I draw attention explicitly to the strange nature of the procedure, and on p. 86 and adjacent pages, in particular in the phrase 'Теперь это совершенно ясно.' on p. 86 and in the line preceding it. This is where the miracle occurs that allows a decisive comparison with the eigenvalues of Hecke operators calculated much earlier in the paper. This was also an essential aspect of the paper for it justifies our choice of function space and of the Hecke operators.

The lecture in Turkish is another matter. I was not presenting any novel theorems but simply my view of the theory as a whole. I discovered as I was preparing it that I laboured under a misconception. I thought that the notion of a galoisian group was appropriate not only for the geometric theory but also for the arithmetic theory. This appears to be not so, as I shall explain. There is, however, a general concept available that of a Tannakian category,[2] which, if I am not mistaken, can be regarded as a generalization of the object formed by the (unitary) representations of a group. In both, direct sums and tensor products are essential features. The most important feature is that the group is replaced by the category of its representations, whose properties are those of a Tannakian category. This allows, it appears, an algebraic variety, especially over a number field, to be replaced by a Tannakian category generated by its cohomology—of whatever sort, whether in a lump or divided by dimension I am not certain. Thus in the arithmetic theory by a Tannakian category to which L-functions are attached. So it is clear that what is appropriate for the arithmetic theory is not a galoisian group, which is not sufficiently flexible or large, but an appropriate Tannakian category.

It is moreover clear that this is the role of functoriality, for me one of the two goals of the arithmetic theory. Functoriality as it has been conceived until now, in particular as a technique for dealing with general automorphic L-functions, should perhaps be conceived as something slightly different, not only as a means of transferring automorphic functions rather Hecke eigenfunctions, more precisely, Hecke eigenvalues from one group to another but also as a means of constructing from automorphic representations a Tannakian category. This Tannakian category is then in the arithmetic theory a substitute for the galoisian group of the Russian paper in the geometric theory. It should then map to the Tannakian category associated to algebraic varieties.[3] All this is, of course, with respect to a given finite-dimensional field of algebraic numbers. That would be reciprocity and would, of course, be a long step forward in number theory, not something for my generation!

I add three obvious remarks. From this stance the difference between the arithmetic and the geometric theory is that the Tannakian category associated to the geometric theory is defined by a group, that associated to the arithmetic theory is not. Something slightly, but only slightly, different is needed. This was presumably clear to, say, Deligne and Milne, but for other reasons. Secondly the task with respect to functoriality, in which some progress has been made over the years, is not changed. Thirdly, the defining groups G and ${}^LG$ can be brought into play.

[2]P. Deligne and J. S. Milne, *Tannakian categories,* http://www.jmilne.org/math/xnotes/tc.pdf

[3]Rather the other way around!